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SHORT NOTE

RAY TRACING IN THREE DIMENSIONS†

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Special methods for tracing rays through restricted earth models are reported in the literature. For example, Sattlegger (1965) has dealt with a two-dimensional model consisting of a sequence of constant-velocity layers separated by interfaces that may be dipping. More recently, Sorrells et al (1971) have considered three-dimensional (3D) models with plane interfaces of arbitrary strike and dip and with spherical surfaces of a special kind. In the earlier part of 1970, we were using a more general algorithm which allowed us to trace rays through a 3D model consisting of plane or curved surfaces of arbitrary nature. This algorithm was presented by the author at the 41st Annual SEG convention in November, 1971. A number of inquiries received by us since that time has encouraged us to summarize our results in this brief note.

Let (X_1, X_2, X_3) be a fixed, right-handed Cartesian coordinate system to which all surfaces and raypaths will be referred. Let us assume that we know the coordinates of a point S (s_1, s_2, s_3) and the direction cosines (i_1, i_2, i_3) of a unit ray vector **I** emerging from S (Figure 1). S can be either a source or a point on a previous reflector to which the ray has been traced. The aim is to trace a ray from S to a point P (p_1, p_2, p_3) on the next reflector; i.e., to find the coordinates of P, the length, and therefore the traveltime, from S to P, and the direction cosines of a ray (reflected or refracted) emerging from P so that we can repeat the procedure.

Step 1. Equation of the reflector

The first step in the procedure is obviously to define the reflector locally by an equation of the form $\phi(x_1, x_2, x_3) = 0$. This equation may be a simple linear equation for a plane dipping reflector, a higher-order polynomial equation, or any one of the spline functions representing more complex surfaces. We shall illustrate our calculations for a quadratic surface of the following type:

$$\begin{aligned} a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 \\ + 2a_{23}x_2x_3 + b_1x_1 + b_2x_2 \\ + b_3x_3 + c = 0. \end{aligned} \tag{1}$$

The coefficients of this equation may have been determined by fitting (1) to the coordinates of a number of points on the surface so that the sum of the squares of the fitting errors is minimum.

For the sake of abbreviated notation, we shall write the above equation in the form of a matrix equation:

$$\begin{aligned} \phi(x_1, x_2, x_3) \\ = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ + \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c = 0; \end{aligned}$$

or

$$\phi(x_1, x_2, x_3) = \mathbf{X}^T\mathbf{A}\mathbf{X} + \mathbf{B}^T\mathbf{X} + c = 0, \tag{2}$$

where **A** is a symmetric (not a requirement) matrix, **B** is a one-dimensional matrix of coefficients, **X** is a position vector, and c is a constant. A some-

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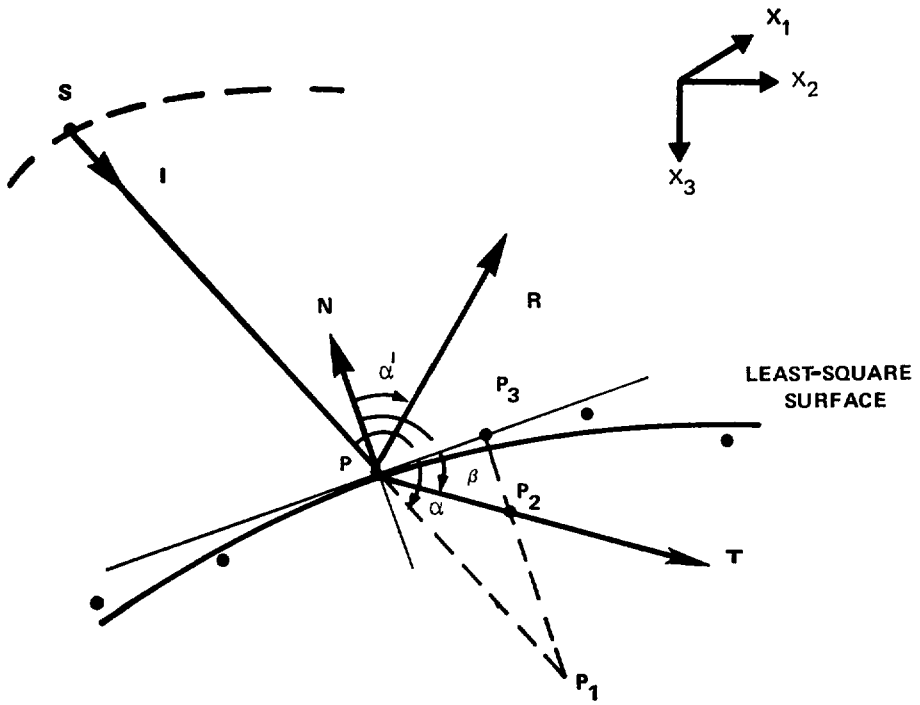

FIG. 1. View in the plane of incidence.

what simpler equation with $a_{13}=a_{23}=a_{33}=0$ may be sufficient for many three-dimensional problems. Notice that a plane dipping surface is obtained as a special case of the above equation when matrix **A** is identically zero.

Step 2. Length of the raypath from S *to* P

If we assume each layer is of uniform velocity, the point P (p_1, p_2, p_3) lies on a straight line from S in the direction of a unit vector **I**, and simultaneously on the reflector. In other words,

$$\mathbf{P} = \mathbf{S} + \mathbf{I}L, \tag{3a}$$

and

$$\mathbf{P}^T\mathbf{A}\mathbf{P} + \mathbf{B}^T\mathbf{P} + c = 0, \tag{3b}$$

where L is the length of the raypath from S to P. Substituting (3a) into (3b) gives

$$L^2(\mathbf{I}^T\mathbf{A}\mathbf{I}) + 2L(\mathbf{B}^T\mathbf{I}/2) + (\mathbf{S}^T\mathbf{A}\mathbf{S} + \mathbf{B}^T\mathbf{S} + c) = 0. \tag{4}$$

If C_1, C_2, and C_3 are used to represent the coefficients, the solution of this quadratic may be written as

$$L = -C_2/C_1 \pm (C_2/C_1)\sqrt{1 - C_1C_3}/C_2^2. \tag{5}$$

If the raypath really intersects the surface, both the roots are real. If the surface is convex as shown in Figure 1, both the roots are positive; the smaller of the two is the desired value. If the surface is concave, the positive root must be chosen and the negative root discarded. One note of caution is appropriate here. If C_1 is small (approaching a flat surface), equation (5) is not liable to give an accurate answer. Specifically for small values of $(C_1C_3)/(C_2^2)$, accurate answers can be obtained by substituting a series expression with a sufficient number of terms for the radical in equation (5).

Step 3. Coordinates of point P

Knowing the length L, the coordinates of the point P are given by (3a).

Step 4. Normal to the surface at P

The normal vector (**N**) to the surface at the point P is obtained by taking the gradient of the equation (2) and substituting the coordinates of P into it. We arbitrarily decide that **N** must also

point upward or that $\partial\phi/\partial x_3$ must always be negative.

$$\mathbf{GRAD}\,\phi = 2\mathbf{AP} + \mathbf{B}. \tag{6a}$$

Since $\mathbf{N}$ is a unit vector,

$$\mathbf{N} = \mathbf{GRAD}\,\phi \Big/ \sqrt{(\mathbf{GRAD}\,\phi)^T(\mathbf{GRAD}\,\phi)}. \tag{6b}$$

Step 5. Angle of incidence

The angle of incidence (α) is obtained from the dot product of unit vectors $\mathbf{I}$ and $\mathbf{N}$.

$$\cos\alpha = \mathbf{I}^T\mathbf{N}. \tag{7}$$

For the downgoing ray, $90° < \alpha \leq 180°$; and for the upcoming ray, $0° \leq \alpha < 90°$.

Step 6. Snell's law

The reflected or transmitted ray must lie in the plane of incidence defined by the incident vector and the normal vector. The angles that these rays make with the normal are given by

$$\frac{\sin\alpha}{V_i} = \frac{\sin\alpha'}{V_r} = \frac{\sin\beta}{V_t}, \tag{8}$$

where V_i is the velocity of the incident wavefront and V_r and V_t are the velocities of the reflected and transmitted wavefronts in the appropriate layers. In the case of two solid layers in contact, V_i, V_r, and V_t may assume the velocity of the compressional wave or the shear wave, depending upon the type of ray one is interested in tracing. For most geophysical applications, V_i and V_r will be the compressional wave velocity in the incident medium and V_t will be the compressional-wave velocity in the transmission medium. The sign of $\cos\alpha$ decides whether α' and β are greater or smaller than 90 degrees.

Step 7. Equation of a transmitted or reflected ray

In deriving this equation, we shall use the fact that these rays lie in the incident plane defined by two vectors $\mathbf{I}$ and $\mathbf{N}$. They can therefore be expressed as linear combinations of the vectors $\mathbf{I}$ and $\mathbf{N}$.

Consider first the transmitted ray $\mathbf{T}$ and the simple construction shown dotted in Figure 1.

$$\mathbf{T} = \frac{(PP_1)\mathbf{I} + (P_1P_2)\mathbf{N}}{(PP_2)}. \tag{9a}$$

If we let PP_1 = unit length, then

$$\mathrm{PP}_3 = \sin\alpha, \quad \mathrm{PP}_2 = \frac{\sin\alpha}{\sin\beta},$$

and

$$\mathrm{P_1P_2} = \mathrm{P_1P_3} - \mathrm{P_3P_2} = -\cos\alpha + \frac{\sin\alpha}{\sin\beta}\cos\beta = -\frac{\sin(\beta-\alpha)}{\sin\beta}.$$

Thus, equation (9a) becomes

$$\mathbf{T} = \frac{\sin\beta}{\sin\alpha}\mathbf{I} - \frac{\sin(\beta-\alpha)}{\sin\beta}\mathbf{N}, \tag{9b}$$

or, using equation (8), we find

$$\mathbf{T} = \frac{V_t}{V_i}\mathbf{I} - \left(\frac{V_t}{V_i}\cos\alpha - \cos\beta\right)\mathbf{N}. \tag{9c}$$

The equation for the reflected ray is obtained by simply noting that $\beta = \alpha'$ and replacing V_t by V_r.

$$\mathbf{R} = \frac{\sin\alpha'}{\sin\alpha}\mathbf{I} - \frac{\sin(\alpha'-\alpha)}{\sin\alpha}\mathbf{N} \tag{10a}$$

$$= \frac{V_r}{V_i}\mathbf{I} - \left(\frac{V_r}{V_i}\cos\alpha - \cos\alpha'\right)\mathbf{N}. \tag{10b}$$

When $V_r = V_i$ (compressional wave reflected as a compressional wave, for example), we get simplified equations.

Layer with continuously varying velocity

The above equations were derived assuming that the layer in question was of uniform velocity. They can be extended to the case of continuously varying velocity in any one or all of the layers by considering an infinitesimal layer. Notice (Figure 2) that the raypath will now be a continuous curve whose unit tangent at any point is the vector $\mathbf{I}$, which becomes $\mathbf{I}+\Delta\mathbf{I}$ (equivalent to vector $\mathbf{T}$ in the discrete case) after traveling a distance ds along the path. The gradient of the continuous velocity function $V(x_1, x_2, x_3)$ determines the normal vector at any point, or

$$\mathbf{N} = \pm\frac{\mathbf{GRAD}\,V}{\|\mathbf{GRAD}\,V\|};$$

the sign is chosen such that $\mathbf{N}$ is always pointing upwards according to our convention; i.e., we

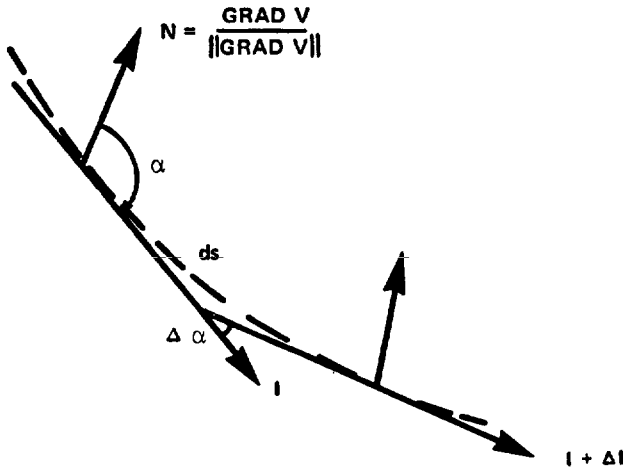

FIG. 2. Case of continuously varying velocity.

choose the negative sign if $\partial V/\partial X_3$ is positive and vice versa. The angle of incidence between **I** and **N** is α and it changes to $\alpha+\Delta\alpha$ (equivalent to β in the discrete case) in distance ds.

With these preliminary remarks, we can write (9b) after using Snell's law for the first term, as

$$\mathbf{I} + \Delta\mathbf{I} = \left(\frac{V+\Delta V}{V}\right)\mathbf{I} - \left(\frac{\sin\ \Delta\alpha}{\sin\ \alpha}\right)\mathbf{N}.$$

Because $\sin\Delta\alpha=\Delta\alpha$, if we divide by ds and take the limit, we obtain

$$\frac{d\mathbf{I}}{ds} = \left(\frac{1}{V}\frac{dV}{ds}\right)\mathbf{I} - \left(\frac{d\alpha}{ds}\frac{1}{\sin\alpha}\right)\mathbf{N}. \quad (11)$$

From differential geometry, $d\mathbf{I}/ds$ is the curvature vector whose magnitude K is equal to $d\alpha/ds$. The curvature vector also lies in the incident plane and is normal to the tangent vector **I**. Therefore,

$$\mathbf{I}\cdot\frac{d\mathbf{I}}{ds} = \frac{1}{V}\frac{dV}{ds} - \frac{d\alpha}{ds}(\mathbf{I}\cdot\mathbf{N})/\sin\alpha = 0.$$

Since $\mathbf{I}\cdot\mathbf{N}=\cos\alpha$, we get

$$K = \frac{d\alpha}{ds} = (\tan\alpha/V)\frac{dV}{ds}\cdot \quad (12)$$

However,

$$\frac{d\alpha}{ds} = \frac{d\alpha}{dV}\frac{dV}{ds} \quad \text{and} \quad \frac{d\alpha}{dV} = \frac{\tan\alpha}{V}$$

(from Snell's law); hence, equation (12) is verified.

Substituting equation (12) into equation (11) gives

$$\frac{d\mathbf{I}}{ds} = \frac{1}{V}\frac{dV}{ds}(\mathbf{I} - \mathbf{N}/\cos\alpha). \quad (13a)$$

Next, the coordinates of any point (defined by position vector **X**) are obtained by noting that

$$\frac{d\mathbf{X}}{ds} = \mathbf{I}. \quad (13b)$$

Since we are interested in total traveltime, it is convenient to write the above equations with time as the independent variable. In the final set of equations given below, we have also utilized the fact that

$$\mathbf{GRAD}\ V = \frac{dv}{ds}(\mathbf{N}/\cos\alpha)$$

and denoted the components of the vector **X** by the subscripts $j(1, 2, 3)$.

$$\frac{dx_j}{dt} = Vi_j, \quad (14a)$$

and

$$\frac{di_j}{dt} = \left(\frac{dV}{ds}\right)i_j - \frac{\partial V}{\partial x_j}\cdot \quad (14b)$$

Given the initial values of i_j, x_j and the velocity function $V(x_1, x_2, x_3)$ over the region of interest, we can integrate the above equations step by step. Customarily, the above six first-order differential equations [14(a) and 14(b)] are given as three second-order equations, and they are directly derived using either Fermat's or Huygen's principle (Officer, 1958).

As a final note, we have set up the equations for a continuous velocity case and for the discrete layer case as an initial value problem in which the initial values of x_j equal to s_j (the source coordinates) and i_j (the direction cosines of the ray emerging from the source) are required to be known. For solving the boundary-value problem in which s_j and the geophone location g_j on the surface of the ground are given, we have to alter our strategy slightly. The obvious strategy is iteration on i_j until the raypath arrives at g_j within a preset tolerance. If the first derivative of the time-distance curve is continuous in the neighborhood of the true answer, a satisfactory fit can be obtained with the Newton-Raphson technique. Notice that only two components of the initial ray vector **I** are unknown; the third component is fixed by requiring that **I** is a unit vector. Thus, we end up with the two-dimensional version of Newton's method.

REFERENCES

Officer, C. B., 1958, Introduction to the theory of sound transmission: New York, McGraw-Hill Book Company, Inc.

Sattlegger, J., 1965, A method of computing true interval velocities from expanding spread data in the case of arbitrary long spreads and arbitrarily dipping plane interfaces: Geophys. Prosp., v. 13, p. 306–318.

Sorrells, G. G., Crowley, J. R., and Veith, K. F., 1971, Methods for computing raypaths in complex geological structures: Bull. Seism. Soc. Am., v. 61, no. 1, p. 27–53.