An Inverse Problem for Two-Frequency Photon Transport in a Slab

Luigi Barletti
Dipartimento di Matematica "Ulisse Dini', Università di Firenze
Viale Morgagni 67/A, 50134 Firenze
barletti@math.unifi.it

Federica Dragoni
Scuola Normale Superiore
Piazza dei Cavalieri 7, 56126 Pisa
f.dragoni@sns.it

The subject of this communication is an inverse photon transport problem, motivated by astrophysics, which consists in obtaining the unknown densities of two different kinds of materials present in a dusty medium (say, and interstellar cloud) by flux measurements of photons with two different frequencies $\nu_1 > \nu_2$ (say, UV and IR). Clearly, the description of photon transport in an interstellar cloud requires a three-dimensional transport equation in a rather complicated geometry. Here, for the sake of simplicity, we shall set the problem in a space-homogeneous, slab geometry.
The mathematical model consists into a system of two stationary transport equations for the phase-space densities $f_1(x, \mu)$ and $f_2(x, \mu)$ of of photons with frequencies ν_1 and ν_2, respectively. Here, $x \in [0, l]$ is the position variable (l being the thickness of the slab) and $\mu \in (-1, 1)$ is the direction cosine. The stationary transport equations are assumed to have the following form,

$$\begin{aligned} \mu \frac{\partial f_1}{\partial x}(x,\mu) + \Sigma_1 f_1(x,\mu) &= \Sigma_{1\to 1} \int_{-1}^{1} p_{1\to 1}(\mu' \to \mu)\, f_1(x,\mu')\, d\mu' \\ \mu \frac{\partial f_2}{\partial x}(x,\mu) + \Sigma_2 f_2(x,\mu) &= \Sigma_{1\to 2} \int_{-1}^{1} p_{1\to 2}(\mu' \to \mu)\, f_1(x,\mu')\, d\mu' \\ &\quad + \Sigma_{2\to 2} \int_{-1}^{1} p_{2\to 2}(\mu' \to \mu)\, f_2(x,\mu')\, d\mu', \end{aligned} \tag{1}$$

where, $\Sigma_{1\to 1} \geq 0$, $\Sigma_{1\to 2} \geq 0$, $\Sigma_{2\to 2} \geq 0$ are the scattering cross-sections, $p_{1\to 1} \geq 0$, $p_{1\to 2} \geq 0$, $p_{2\to 2} \geq 0$ are the scattering probability densities and

$$\Sigma_1 := \Sigma_{1\to 1} + \Sigma_{1\to 2} + \Sigma_{1,c}, \qquad \Sigma_2 := \Sigma_{2\to 2} + \Sigma_{2,c},$$

are the total cross-sections ($\Sigma_{1,c} \geq 0$ and $\Sigma_{2,c} \geq 0$ are the capture cross-sections). Since we assume the medium to be space-homogeneous, then all the cross-sections and scattering probabilities are independent of x. Moreover, since $\nu_1 > \nu_2$, the energy-increasing scattering $2 \to 1$ is not considered for obvious physical reasons.
Assuming the scattering to be number-conservative, we have

$$\int_{-1}^{1} p_{i\to j}(\mu' \to \mu)\, d\mu = 1$$

for all $\mu' \in [-1,1]$ and for all $i,j = 1,2$ excluding $i = 1,\ j = 2$, because $p_{2\to 1} \equiv 0$. Moreover, we assume that the scattering is symmetric, i.e.

$$p(\mu' \to \mu) = p(\mu \to \mu').$$

Let us assume that the medium is homogeneous and composed by two kinds of dust, with different physical properties. Then, for $i,j = 1,2$ we put

$$\Sigma_{i\to j} = \rho_1\, \sigma^1_{i\to j} + \rho_2\, \sigma^2_{i\to j}, \qquad \Sigma_{i,c} = \rho_1\, \sigma^1_{i,c} + \rho_2\, \sigma^2_{i,c},$$

where $\rho_1 \geq 0$ and $\rho_2 \geq 0$ are the (constant) densities of the two dusts and the σ's are microscopic cross-sections. If the scattering properties are the same for the two kinds of dust, then the probabilities p's do not depend on the dust index.
The model is completed by the following boundary conditions of assigned inflow:

$$\begin{aligned} f_1(0,\mu) &= \varphi_1^+(\mu), \quad f_2(0,\mu) = \varphi_2^+(\mu), \quad \text{for } \mu \in (0,1) \\ f_1(l,\mu) &= \varphi_1^-(-\mu), \quad f_2(l,\mu) = \varphi_2^-(-\mu), \quad \text{for } \mu \in (-1,0), \end{aligned} \tag{2}$$

where $\varphi_1^\pm(\mu)$ and $\varphi_2^\pm(\mu)$ are known incoming photon distributions at both sides of the slab at the two frequencies .

The inverse problem consists in finding the unknown dust densities, ρ_1 and ρ_2, from the knowledge of the integrated right-outflows:

$$H_1 := \int_0^1 f_1(l,\mu)\mu\, d\mu, \qquad H_2 := \int_0^1 f_2(l,\mu)\mu\, d\mu.$$

Under fairly general conditions we can prove the well-posedness of the direct problem (1)+(2).

Moreover, in the following assumptions:

A1. the frequency-scattering vanishes, i.e. $\Sigma_{1\to 2} \equiv 0$;

A2. the left-inflow data φ_1^+ and φ_2^+ are positive on nonzero-measure sets;

A3. $\sigma_{i,c}^j > 0$ for $i, j \in \{1, 2\}$ and $\det(\sigma_{i,c}^j) \neq 0$;

we can prove that the mapping densities-to-outflows: $(\rho_1, \rho_2) \mapsto (H_1, H_2)$ is globally invertible and, therefore, that the inverse problem is well-posed.

Numerical experiments show that the two densities (ρ_1, ρ_2) can be computed from assigned outflows (H_1, H_2), by means of a simple bisection-like algorithm, over a range of several orders of magnitude.

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