STOCHASTIC DOMINANCE AND THE MAXIMIZATION OF EXPECTED UTILITY

by

Leigh Tesfatsion

Discussion Paper No. 74 - 37, January 1974

Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

STOCHASTIC DOMINANCE AND THE MAXIMIZATION OF EXPECTED UTILITY*

by

Leigh Tesfatsion

In attempting to construct a general framework for the analysis of choice under uncertainty, researchers have long sought to establish reasonable criteria for the selection of one prospect over another. Among current researchers the concept of stochastic dominance[1] has attracted considerable attention. This paper attempts to clarify and generalize certain basic relationships between stochastic dominance and the maximization of expected utility.

The paper begins with a critique of an article by Giora Hanoch and Haim Levy [1]. Although Hanoch and Levy propose a series of interesting theorems relating stochastic dominance to the maximization of expected utility, errors appear in the statement and proof of these theorems which prevent (or should prevent) the researcher from using them directly. The necessary modifications are given in Part I below.

An undesirable feature of many articles in the area of stochastic dominance are the regularity conditions imposed on the utility functions (e.g., bounded, differentiable) and the random variables (e.g., absolutely continuous distribution function, nonnegative). The important 1971 article [2] by Josef Hadar and William Russell is marred by such restrictions. Using this paper as a base, a series of relatively general

---

*Research underlying this paper was supported by National Science Foundation Grant GS-3317 to the University of Minnesota

[1] First degree stochastic dominance is said to hold between two distribution functions F and G when $F(x) \leq G(x)\ \forall x \in R$; and second degree stochastic dominance when $\int_{-\infty}^{x} [F(t) - G(t)]\, dt \leq 0\ \forall x \in R$.

results have been obtained which appear especially interesting in their relation to the modified theorems of Part I. These results are presented below in Part II.

Notation

1. F and G for arbitrary right continuous distribution functions.
2. $U^{*} = \{u: R \to R \mid u$ nondecreasing and continuous$\}$.
3. $U^{*}(F, G) = \{u \in U^{*} \mid \int u dF - \int u dG$ well defined$\}$.
4. $U^{**} = \{u: R \to R \mid u$ nondecreasing and concave$\}$.
5. $U^{**}(F, G) = \{u \in U^{**} \mid \int u dF - \int u dG$ well defined and $\int_{-\infty}^{0} u dF > -\infty$, $\int_{-\infty}^{0} u dG > -\infty\}$.
6. [FDG wrt U] to mean $[\int u dF - \int u dG \geq 0$ $\forall\, u \in U$, with strict inequality holding for some $u \in U]$, where U is any specified class of functions.
7. $[F \leq G]$ to mean $[F(x) \leq G(x)\ \forall\, x]$.
8. $[F < G]$ to mean $[F(x) \leq G(x)\ \forall\, x,\ F \neq G]$.
9. $[F \underline{\ll} G]$ to mean $[\int_{-\infty}^{x} [F(t) - G(t)]\, dt \leq 0\ \forall\, x]$.
10. $[F \ll G]$ to mean $[\int_{-\infty}^{x} [F(t) - G(t)]\, dt \leq 0\ \forall\, x,\ F \neq G]$.
11. R will denote the real line, $(-\infty, +\infty)$.

12. The sign // will mark the end of major proofs, and the sign / the end of minor proofs.

13. The prefix "H-L" abbreviates "Hanoch-Levy", the prefix "H-R" abbreviates "Hadar-Russell".

I.

The theorems proposed by Hanoch and Levy in [1] relate stochastic dominance to a broader class of utility functions than considered by previous researchers working in the area. Unfortunately, they contain significant errors in both their statement and proof. As demonstrated below, modified versions of these theorems can be obtained which are almost as general as the original theorems.

All integrals below are to be interpreted as Riemann-Stieltjes integrals. In footnote four of their article (see [1], page 340) H-L declare that all integrals are to be considered Lebesgue.[2] However, the integral $\int_{-\infty}^{+\infty} [G - F](t)\, du(t)$ arising from integration by parts and basic to many of H-L's theorems may exist as an improper Riemann-Stieltjes integral and not as a Lebesgue integral.

A. <u>H-L Lemma 1</u>

If u is a nondecreasing, real-valued function with finite values for any finite x, then $\int udF - \int udG = \int [G - F]\, du$.

The statement and proof of this lemma contain three points of error:

1. Contrary to H-L's argument on pages 336-337, it is not true that $\int du(F - G) = 0$ for all u as given in

---

[2] "... unless specified otherwise." But they do not specify otherwise.

Lemma 1. Consequently, their proofs for both necessity and sufficiency, which they base on this argument, are incorrect.

Example: Let $u(x) = x \ \forall x$; $F(x) = 1 - 1/x$ if $x \geq 1$, $= 0$ if $x < 1$; $G(x) = 1 - 2/x$ if $x \geq 2$, $= 0$ if $x < 2$. Then for $N > 2$ and $M > 0$, $u(x)\ [F(x) - G(x)]\ \Big|_{-M}^{N} = N\ [2/N - 1/N] - [-M]\ [0 - 0] = 1$ $\Rightarrow \lim_{\substack{N \to +\infty \\ M \to +\infty}} U\ [F - G]\ \Big|_{-M}^{N} = 1$.

2. Integration by parts may not be valid if u and F or u and G have a common point of discontinuity.

Example: Let $u(x) = 1$ if $x \geq 1$, 0 if $x < 1$; $F(x) = 1$ if $x \geq 1$, 0 if $x < 1$; and $G(x) = 1$ if $x \geq 2$, $x/2$ if $0 \leq x < 2$, and 0 if $x < 0$. Suppose integration by parts were valid. Then for any $N > 2$, $M < 0$, $\int_{-M}^{N} u(x)\ dF\ (x) - \int_{-M}^{N} u(x)\ dG\ (x) = u(x)\ [F(x) - G(x)]\ \Big|_{-M}^{N} - \int_{-M}^{N} [F(x) - G(x)]\ du\ (x) \Rightarrow u(1) \cdot 1 - \int_{1}^{2} 1\ dx/2 = 1[1 - 1] - 0[0 - 0] + G[1] - F(1) \Rightarrow 1 - 1/2 = 0 - 0 + 1/2 - 1 \Rightarrow 1/2 = -1/2$

which is absurd.

3. $\int [G - F] du$ may exist while $\int ud\ F - \int ud\ G$ is not well defined.

Example: Let $u(x) = x$ for all x; $F(x) = G(x) = 1 - 1/x$ if $x \geq 1$, $= 0$ if $x < 1$. Then $\int [G - F]\ du = 0$, but $\int ud\ F = \int ud\ G = \int_{1}^{\infty} x(1/x^2)\ dx = +\infty$. Hence $\int ud\ F - \int ud\ G$ is not well defined.

One could avoid this problem by replacing $\int ud\ F - \int ud\ G$ with $\int ud\ \widetilde{[F - G]}$, where $\widetilde{F - G}$ is the Jordan decomposition of $F - G$. But for practical purposes, a statement and proof of the lemma in terms of a difference in expectations seem preferable.

Lemma 1* (Modification of H-L Lemma 1)

$$\int u\,dF - \int u\,dG = \int [G - F]\,du \quad \forall\, u \in U^*(F, G) \quad .$$

Proof

The proof of this equality for arbitrary $u \in U^*(F, G)$ will be given in three steps.

Step I

$[\int_{0-}^{\infty} u\,dF \text{ finite}] \Leftrightarrow [\int_{0-}^{\infty} [1 - F]\,du \text{ finite}]$, and if either side holds, then $\int_{0-}^{\infty} u\,dF = u(0)[1 - F(0^-)] + \int_{0-}^{\infty} [1 - F]\,du$.

Proof: Suppose $\int_{0-}^{\infty} u\,dF$ is finite. Then $\infty > \int_{N+}^{\infty} u\,dF \geq u(N)[1 - F(N^+)] \geq u(0)[1 - F(N^+)] \;\forall\, N \geq 0 \Rightarrow 0 = \lim_{N\to\infty} \int_{N+}^{\infty} u\,dF \geq \limsup_{N\to\infty} u(N)[1 - F(N^+)] \geq u(0) \lim_{N\to\infty} [1 - F(N^+)] = 0 \Rightarrow \lim_{N\to\infty} u(N)[1 - F(N^+)] = 0$. Using integration by parts, $\int_{0-}^{N+} u\,dF = uF\Big|_{0-}^{N+} - \int_{0-}^{N+} F\,du \equiv - u[1 - F]\Big|_{0-}^{N+} + \int_{0-}^{N+} [1 - F]\,du \Rightarrow \int_{0-}^{\infty} u\,dF = \lim_{N\to\infty} [- u [1 - F]\Big|_{0-}^{N+} + \int_{0-}^{N+} [1 - F]\,du] = 0 - [- u(0)[1 - F(0^-)] + \lim_{N\to\infty} \int_{0-}^{N+} [1 - F]\,du \Rightarrow \int_{0-}^{\infty} [1 - F]\,du$ is finite, and $\int_{0-}^{\infty} u\,dF = u(0)[1 - F(0^-)] + \int_{0-}^{\infty} [1 - F]\,du$.

Suppose $\int_{0-}^{\infty} [1 - F]\, du$ is finite. Then

$\int_{M-}^{N+} [1 - F]\, du \geq [1 - F(N^+)][u(N) - u(M)]$

$\geq 0 \ \forall\ N > M \Rightarrow \infty > \int_{M}^{\infty} [1 - F]\, du \equiv \lim_{N\to\infty}$

$\int_{M-}^{N+} [1 - F]\, du \geq \limsup_{N\to\infty} [1 - F(N^+)][u(N)$

$- u(M)] = \limsup_{N\to\infty} [1 - F(N^+)]\, u(N) \geq u(0)$

$\lim_{N\to\infty} [1 - F(N^+)] = 0$. Hence $0 = \lim_{M\to\infty}$

$\int_{M-}^{\infty} [1 - F]\, du \geq \lim_{M\to\infty} [\limsup_{N\to\infty} [1 - F(N^+)]$

$u(N)] \geq 0 \Rightarrow \lim_{N\to\infty} [1 - F(N^+)]\, u(N) = 0$.

Using integration by parts, it follows

that $\int_{0-}^{\infty} [1 - F]\, du \equiv \lim_{M\to\infty} \int_{0-}^{M+} [1 - F]\, du$

$= \lim_{M\to\infty} [[1 - F]\, u \Big|_{0-}^{M+} - \int_{0-}^{M+} u\, [-\, dF]]$

$= -\, u(0)[1 - F(0^-)] + \int_{0-}^{\infty} udF$. /

Step II

$[\int_{-\infty}^{0-} udF$ finite$] \Leftrightarrow [\int_{-\infty}^{0-} Fdu$ finite$]$,

and if either side holds, then $\int_{-\infty}^{0-} udF$

$= F(0^-)\, u(0) - \int_{-\infty}^{0-} Fdu$.

Proof: The proof of Step II is completely analogous to the proof of Step I, and therefore will be omitted.

Step III

$\int udF - \int udG = \int [G - F]\, du$.

Proof: Suppose $\int udF$ and $\int udG$ are finite.

Then $\int_{0-}^{\infty} udF$, $\int_{-\infty}^{0-} udF$, $\int_{0-}^{\infty} udG$ and $\int_{-\infty}^{0-} udG$ are finite, and by Steps I and II above,

$$\int udF - \int udG = \int_{0-}^{\infty} udF + \int_{-\infty}^{0-} udF - \int_{0-}^{\infty} udG$$

$$- \int_{-\infty}^{0-} udG = [u(0)\ [1 - F(0^-)] + \int_{0-}^{\infty} [1 - F]\, du]$$

$$+ [u(0)\ F\ (0^-) - \int_{-\infty}^{0-} Fdu] - [u(0)\ [1 - G(0^-)]$$

$$+ \int_{0-}^{\infty} [1 - G]\, du] - [u(0)\ G\ (0^-) - \int_{-\infty}^{0-} Gdu]$$

$$= \int_{0-}^{\infty} [1 - F]\, du - \int_{0-}^{\infty} [1 - G]\, du - \int_{-\infty}^{0-} Fdu$$

$$+ \int_{-\infty}^{0-} Gdu = \int_{-\infty}^{\infty} [G - F]\, du \quad .$$

Suppose $\int udF - \int udG = +\infty$. Since $\int_{0-}^{\infty} udG \geq u(0)\ [1 - G(0^-)] > -\infty$ and $\int_{-\infty}^{0-} udF \leq u(0)\ F\ (0^-) < +\infty$, $[\int udF - \int udG = +\infty] \Rightarrow [\int_{0-}^{\infty} udF \leq +\infty$, $\int_{-\infty}^{0-} udF$ finite, $\int_{0-}^{\infty} udG$ finite, $\infty > \int_{-\infty}^{0-} udG \geq -\infty]$, with either $\int_{0-}^{\infty} udF = +\infty$ or $\int_{-\infty}^{0-} udG = -\infty$, say the former. Then $\int_{-\infty}^{0-} Fdu$ and $\int_{0-}^{\infty} [1 - G]\, du$ are finite,

$\int_{0-}^{\infty} [1 - F]\, du = + \infty$ and $- \infty < \int_{-\infty}^{0} G du \leq + \infty$.

Therefore $\int_{0-}^{\infty} [1 - F]\, du - \int_{0-}^{\infty} [1 - G]\, du =$

$\int_{0-}^{\infty} [G - F]\, du = + \infty$, and $- \infty < \int_{-\infty}^{0-} G du - \int_{-\infty}^{0-} F du$

$= \int_{-\infty}^{0-} [G - F]\, du \leq + \infty \Rightarrow \int_{-\infty}^{\infty} [G - F]\, du = + \infty = \int u dF$

$- \int u dG$. Similarly for $\int_{-\infty}^{0-} u dG = - \infty$. /

The argument for $\int u dF - \int u dG = - \infty$ follows immediately from the above by symmetry. //

B. H-L Theorem 1

Define U to be the class of all nondecreasing, real-valued functions. Then [FDG wrt U] $\Leftrightarrow$ [F < G] .

This theorem is based on H-L Lemma 1, and thus is subject to the same criticisms. The class U must be modified accordingly.

Theorem 1*

$$[\text{FDG wrt } U^{*} (F, G)] \Leftrightarrow [F < G] \quad .$$

Proof

Suppose $F < G$. Then $G(x) - F(x) \geq 0 \; \forall \; x \Rightarrow \int [G - F]\, du \geq 0 \; \forall \; u \in U^{*} (F, G) \Rightarrow$ by Lemma 1*, $\int u dF - \int u dG \geq 0 \; \forall \; u \in U^{*} (F, G)$. Suppose $G(x') - F(x') > 0$. By right continuity of G and F , $\exists$ an interval $[x', \beta)$ over which $G > F$. Define $U_{\beta}(x) = x'$ if $x \leq x'$, $= x$ if $x' \leq x < \beta$, $= \beta$ if $x \geq \beta$. Then $U_{\beta} \in U^{*} (F, G)$ and by Lemma 1*, $\int u_{\beta}\, dF - \int u_{\beta}\, dG = \int [G - F]\, du_{\beta} = \int_{x'}^{\beta} [G(x) - F(x)]\, dx > 0$. Hence

$[F < G] \Rightarrow [\text{FDG wrt } U^* (F, G)]$.

Suppose for some x', $G(x') - F(x') < 0$. By right continuity of G and F $\exists$ an interval $[x', \partial)$ over which $G < F$. Define $U_\partial = x'$ if $x \le x'$, $= x$ if $x' \le x < \partial$, and $= \partial$ if $x \ge \partial$. Then $U_\partial \in U^* (F, G)$ and by Lemma 1*, $\int u_\partial dF - \int u_\partial dG = \int [G, F]\, du_\partial = \int_x^\partial [G(x) - F(x)]dx < 0$. Hence not $[G(x) - F(x) \ge 0 \;\forall\; x] \Rightarrow$ not $[\text{FDG wrt } U^* (F, G)]$. If $G \equiv F$ then $\int udF - \int udG = 0$ for all $u \in U^* (F, G)$, hence not $[\text{FDG wrt } U^* (F, G)]$. //

Corollary 1*

If $u \in U^* (F, G)$ and u is strictly increasing, then $[F < G] \Rightarrow [\int udF - \int udG > 0]$.

C. H-L Theorem 2

Let U_1 be the class of all nondecreasing, concave functions. Then $[\text{FDG wrt } U_1] \Leftrightarrow [F \ll G]$.

This theorem as given by H-L contains four points of error:

1. In both their argument for necessity (page 338) and for sufficiency (page 340, last statement in proof) H-L implicitly assume that $\int_{-\infty}^{x'} xdF - \int_{-\infty}^{x'} xdG$ is well defined for some (hence every) finite x' . Their hypotheses do not guarantee this. The following claim demonstrates that the assumption $[\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ well defined$]$ is needed.

   Claim

   $[\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ not well defined$] \Rightarrow [\int udF - \int udG$ not well defined for any nonconstant $u \in U_1]$.

Proof: $\left|\int_{-M}^{0} xdF\right|$ and $\left|\int_{-M}^{0} xdG\right|$ are finite for all $M \in R$ and $\int_{-\infty}^{0} xdF \leq 0 < \infty$, $\int_{-\infty}^{0} xdG \leq 0 < \infty$. Hence, for $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ not to be well defined, it must hold that $\int_{-\infty}^{M} xdF = \int_{-\infty}^{M} xdG = -\infty$ for all $M \in R$. For any nondecreasing, nonconstant, concave function u defined over R, $\exists$ constants b and c, with $b > 0$, s.t. $u(x) \leq bx + c$ for sufficiently small x. Hence for sufficiently small M, $-\infty = \int_{-\infty}^{M} [bx + c]\, dF \geq \int_{-\infty}^{M} u(x)\, dF(x)$, and similarly, for G in place of F. Thus $\int_{-\infty}^{\infty} udF - \int_{-\infty}^{\infty} udG$ is not well defined for any nonconstant $u \in U_1$. /

Hence, if $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ is not well defined, the sufficiency holds vacuously since not [FDG wrt U_1]; but the necessity condition will not imply FDG wrt either U_1 or the smaller class $U^{**}(F, G)$.

2. In their argument for sufficiency, H-L implicitly assume that $\int_{-\infty}^{x} |G - F|\, dt < \infty$ for some (hence all) finite x. There general hypotheses and sufficiency condition do not guarantee the existence of the integral in this sense (Lebesgue).

3. In both their necessity and sufficiency arguments, H-L apply Lemma 1 to functions in U_1. This would be invalid even for Lemma 1*, since function in U_1 are not necessarily

continuous. [For example, if $u(x) = 0$ when $x > x'$, and $-\infty$ when $x \le x'$, then u is nondecreasing, concave by H-L's definition (page 338), and discontinuous.] Since concavity is already assumed for $u \in U_1$, it is sufficient for continuity to further assume that the functions are finite-valued over R.

4. In their argument for sufficiency (page 339, (4)) H-L decompose $\int_{-\infty}^{x} [G - F](t)\, du(t)$ into its positive and negative parts for arbitrary $u \in U_1$. A necessary condition for this decomposition to be justified is that one of the parts be finite. Their hypotheses and sufficiency condition do not guarantee this for $u \in U_1$.

<u>Theorem 2*</u> [Modification of H-L Theorem 2]

Suppose $\int_{-\infty}^{0} x dF - \int_{-\infty}^{0} x dG$ is well defined and $\int_{-\infty}^{0} |G - F|\, dt < \infty$. Then [FDG wrt U** (F, G)] $\Leftrightarrow$ [$F \ll G$].

<u>Proof</u>

a. <u>Sufficiency</u>

Suppose $\exists\, x'$ s.t. $\int_{-\infty}^{x'} [G - F](t)\, dt < 0$. Define $u^*(x) = x'$ if $x \ge x'$, $= x$ if $x < x'$. $\int_{-\infty}^{0} |G - F|\, dt < \infty$ and $\int_{-\infty}^{0} x dF - \int_{-\infty}^{0} x dG$ well defined $\Rightarrow$ by Lemma 1*, $\int_{-\infty}^{0} x dG$ and $\int_{-\infty}^{0} x dF$ are finite. Hence $u^* \in U^{**}(F, G)$ and by Lemma 1*, $\int u^* dF - \int u^* dG = \int_{-\infty}^{x'} [G - F]\, du^* < 0$.

Thus not [FDG wrt U** (F, G)].

Suppose $F \equiv G$. Then $\int udF - \int udG = 0 \ \forall \ u \in U^{**} (F, G) \Rightarrow$ not [FDG wrt $U^{**} (F, G)$] .

b. Necessity

Assume $\int_{-\infty}^{x} [G - F] \, dt \geq 0 \ \forall \ x$. Then $\int_{-\infty}^{x} [G - F] \, I_A (t) \, dt + \int_{-\infty}^{x} [G - F] \, I_B (t) \, dt \geq 0 \ \forall \ x \Rightarrow \int_{-\infty}^{x} [G - F] \, I_A (t) \, dt \geq \int_{-\infty}^{x} [F - G] \, I_B (t) \, dt \ \forall \ x$, where $I_A (\cdot)$ is the indicator function for $A = \{x : G(x) \geq F(x)\}$, $I_B (\cdot)$ is the indicator function for $B = \{x : G(x) < F(x)\}$, and the decomposition is justified by the hypothesis $\int_{-\infty}^{0} | G - F | \, dt < \infty$. Since $0 \leq \int_{-\infty}^{0} [G - F] \, I_A (t) \, dt < \infty$, $\lim_{x \to -\infty} \int_{-\infty}^{x} [G - F] \, I_A (t) \, dt = 0$. The proof of necessity will now proceed in four steps.

Step I

One may define a function $T : R \rightarrow [-\infty, +\infty)$ s.t. the following hold:

1. For each x, * $\int_{-\infty}^{T(x)} [G - F](t) \, I_A (t) \, dt = \int_{-\infty}^{x} [F - G](t) \, I_B (t) \, dt$
2. $T(x) \leq x \ \forall \ x$
3. T is a monotone nondecreasing function, continuous and differentiable a.e. over its essential domain $D = \{x : T(x) > -\infty\}$.

Proof: By the above, for a value x' (possibly $-\infty$) to exist for each x s.t. $\int_{-\infty}^{x'}[G - F]\, I_A(t)\, dt = \int_{-\infty}^{x}[F - G]\, I_B(t)\, dt$, it is sufficient that $\int_{-\infty}^{x}[G - F]\, I_A(t)\, dt$ be a continuous function of x. But this is immediate, since for any x and y

$$\left| \int_{-\infty}^{x}[G - F]\, I_A(t)\, dt - \int_{-\infty}^{y}[G - F]\, I_A(t)\, dt \right| = \left| \int_{y}^{x}[G - F]\, I_A(t)\, dt \right| \le 2\,|x - y| .$$

Define

$$T(x) = \inf \{x' : \int_{-\infty}^{x'}[G - F]\, I_A(t)\, dt = \int_{-\infty}^{x}[F - G]\, I_B(t)\, dt\} .$$

Then T is a well-defined function of x, and, since $[G - F](t)\, I_A(t) \ge 0$ for all t, it is clear that $T(x) \le x$ for all x. By definition of T and positivity of $[F - G](t)\, I_B(t)$ for all t, $[x \ge x'] \Rightarrow [\int_{-\infty}^{x}[F - G](t)\, I_B(t)\, dt \ge \int_{-\infty}^{x'}[F - G](t)\, I_B(t)\, dt] \Rightarrow [T(x) \ge T(x')]$. Hence T is a monotone nondecreasing function of x, continuous and differentiable a.e. over its essential domain $D = \{x : T(x) > -\infty\}$. /

Step II

**$I_A(T(t))\,[G(T(t)) - F(T(t))]\,\tilde{T}(t) = I_B(t)\,[F(t) - G(t)]$ a.e. on R, where $\tilde{T}(t) = T'(t)$ if $T'(t)$ exists, $= 1$ if $T'(t)$ does not exist.

Proof: Let $t' = \sup \{t : T(t) = -\infty\}$. Then $T(t) = -\infty$ for all $t < t' \Rightarrow \int_{-\infty}^{s} [F(t) - G(t)] I_B(t)\, dt = 0$ for all $s < t' \Rightarrow [F(t) - G(t)] I_B(t) = 0$ a.e. over $\{t : t < t'\}$. Since $G(T(t)) - F(T(t)) = G(-\infty) - F(-\infty) = 0$ for all $t < t'$, ** holds a.e. over $\{t : t < t'\}$, and hence a.e. over $D^C = \{t : T(t) = -\infty\}$. Differentiating * (see Step I) at any point t where $T'(t)$ exists, one gets $I_A(T(t))[G(T(t)) - F(T(t))]\, T'(t) = I_B(t)[F(t) - G(t)]$. Hence ** holds a.e. over $D = \{t : T(x) > -\infty\}$. Since $D^C \cup D = R$, the proof is complete. /

Step III

For any $u \in U^{**}(F, G)$, $\int_{-\infty}^{x} [G(t) - F(t)]\, du(t) \geq 0 \ \forall x$; hence $\int u dF - \int u dG \geq 0$.

Proof: Let $u \in U^{**}(F, G)$. For any x, $\int_{-\infty}^{x} |G - F|\, du \leq \int_{-\infty}^{x} G du + \int_{-\infty}^{x} F du = -[\int_{-\infty}^{0} u dG + \int_{-\infty}^{0} u dF] + \text{constant} < +\infty$.

Hence the following decomposition is justified:

$$*** \int_{-\infty}^{x} [G - F]\, du = \int_{-\infty}^{x} [G - F](t)\, I_A(t)\, du(t)$$

$$+ \int_{-\infty}^{x} [G - F](t)\, I_B(t)\, du(t) = \int_{-\infty}^{x} [G - F](t)\, I_A(t)\, du(t)$$

$$- \int_{-\infty}^{x} [F - G](t)\, I_B(t)\, du(t) = \int_{-\infty}^{x} [G - F](t)\, I_A(t)\, du(t)$$

$$- \int_{-\infty}^{x} I_A(T(t))\, [G(T(t)) - F(T(t))]\, \tilde{T}(t)\, du(t),$$

where the last substitution follows from Step II. T monotone nondecreasing and u nondecreasing, concave and finite over R imply that $M_1 \equiv \{t : T'(t)$ does not exist and $T(t) > -\infty\}$ and $M_2 \equiv \{t : u'(t)$ or $u'(T(t))$ does not exist$\} \cap \{t : T'(t) > 0\}$ have Lebesgue measure zero, u is absolutely continuous, and $u'(t_2) \leq u'(t_1)$ whenever both derivatives exist and $t_1 \leq t_2$. Let $M_3 \equiv \{t : T(x) = -\infty$ or $T'(t) = 0\}$. Then $I_A(T(t))\,[G(T(t)) - F(T(t))]\,\tilde{T}(t) = 0$ whenever $t \in M_3$. Defining $S = \{t : t \leq x\} \cap [M_1 \cup M_2 \cup M_3]^c$, the following holds:

$$\int_{-\infty}^{x} I_A(T(t))\,[G(T(t) - F(T(t))]\,\tilde{T}(t)\,du(t)$$

$$= \int_S I_A(T(t))\,[G(T(t)) - F(T(t))]\,T'(t)\,u'(t)\,dt$$

$$\leq \int_S I_A(T(t))\,[G(T(t)) - F(T(t))]\,T'(t)\,u'(T(t))\,dt \quad .$$

Since the integrand and measure are positive, integrating over a larger set can only increase the value of the integral.

Hence,

$$\int_S I_A(T(t))\,[G(T(t)) - F(T(t))]\,T'(t)\,u'(T(t))\,dt$$

$$\leq \int_{T(S)} I_A(z)\,[G(z) - F(z)]\,u'(z)\,dz$$

$$\leq \int_{[z \leq T(x)]} I_A(z)\,[G(z) - F(z)]\,u'(z)\,dz$$

$$= \int_{[z \leq T(x)]} I_A(z)\,[G(z) - F(z)]\,du(z) \quad .$$

Substituting this result into ***,

$$\int_{-\infty}^{x} [G - F]\, du \geq \int_{-\infty}^{x} [G - F](t)\, I_A(t)\, du(t)$$

$$- \int_{[t \leq T(x)]} I_A(t)\, [G(t) - F(t)]\, du(t)$$

$$= \int_{[T(x) \leq t \leq x]} [G - F](t)\, I_A(t)\, du(t) \geq 0$$ since

$T(x) \leq x \ \forall\ x$. Hence by Lemma 1*, $\int udF - \int udG$

$$= \int_{-\infty}^{\infty} [G - F]\, du = \lim_{x \to \infty} \int_{-\infty}^{x} [G - F]\, du \geq 0 \quad . \qquad /$$

Step IV

If $F \not\equiv G$ in addition to $\int_{-\infty}^{x} [G - F]\, dt \geq 0 \ \forall\ x$,

then $\exists$ a $u \in U^{**}(F, G)$ s.t. $\int udF - \int udG > 0$.

Proof: $F \not\equiv G$ and $\int_{-\infty}^{x} [G - F]\, dt \geq 0 \ \forall\ x \Rightarrow \exists\ x' \in R$ s.t.

$G(x') > F(x') \Rightarrow$ (by rt. continuity of F and G) there

exists an interval $[x', \beta)$ over which $G(x) > F(x)$.

Define $u(x) = \beta$ if $x \geq \beta$, $u(x) = x$ if $x < \beta$. Then

$\int_{-\infty}^{0} | G - F |\, dt < \infty$ and $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ well defined

implies by Lemma 1* that $\int_{-\infty}^{0} xdG$ and $\int_{-\infty}^{0} xdF$ are

finite. Hence $u \in U^{**}(F, G)$ and by Lemma 1*,

$$\int udF - \int udG = \int [G - F]\, du = \int_{-\infty}^{\beta} [G - F]\, dt = \int_{x'}^{\beta} [G - F]\, dt$$

$$+ \int_{-\infty}^{x'} [G - F]\, dt > 0 \quad . \qquad /$$

Combining Steps III and IV, $[\int_{-\infty}^{x} [G - F]\, dt \geq 0 \ \forall\ x,\ G \not\equiv F]$

$\Rightarrow$ [FDG wrt $U^{**}(F, G)$] . //

D. H-L Theorem 3

Let F and G have mean values μ_1, μ_2 respectively. Suppose for some $x' < \infty$, $F(x) \leq G(x)$ for all $x \leq x'$, $F(x'') < G(x'')$ for some $x'' < x'$, and $F(x) \geq G(x)$ for all $x \geq x'$. Then FDG wrt concave utility functions if and only if $\mu_1 \geq \mu_2$.

Since this theorem is based on Lemma 1 and Theorem 2, it needs to be modified in accordance with the changes made there.

Theorem 3*

Suppose $\int xdF - \int xdG$ is well defined and $\int_{-\infty}^{0} |G - F| \, dt < \infty$. If $\exists \, x' \in R$ s.t. $F(x) \leq G(x) \; \forall \, x \leq x'$, $F(x) \geq G(x) \; \forall \, x \geq x'$, and $F \neq G$, then $[\text{FDG wrt } U^{**}(F, G)] \Leftrightarrow [\int xdF - \int xdG \geq 0]$.

Proof

Suppose $\int xdF - \int xdG \geq 0$. Then by Lemma 1* $\int xdF - \int xdG = \int [G - F] \, dx = \int_{-\infty}^{x'} [G - F] \, dt - \int_{x'}^{\infty} |G - F| \, dt \geq 0$. Hence $\int_{-\infty}^{x} [G - F] \, dt \geq 0$ for all x. Since $F \neq G$, $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ is well defined, and $\int_{-\infty}^{\infty} |G - F| \, dt < \infty$, by Theorem 2* $[\text{FDG wrt } U^{**}(F, G)]$.

Suppose not $[\int xdF - \int xdG \geq 0]$. Since $\int xdF - \int xdG$ is well defined by hypothesis, not $[\int xdF - \int xdG \geq 0] \Rightarrow \int xdF - \int xdG = \int_{-\infty}^{x'} [G - F] \, dt - \int_{x'}^{\infty} |G - F| \, dt < 0$. Hence for some finite $x'' > x'$, $\int_{-\infty}^{x''} [G - F] \, dt = \int_{-\infty}^{x'} [G - F] \, dt - \int_{x'}^{x''} |G - F| \, dt < 0$, whereas $\int_{-\infty}^{x'} (G - F) \, dt > 0$. By Theorem 2*, not $[\text{FDG wrt } U^{**}(F, G)]$. //

E. H-L Theorem 4

Let F and G be two distinct distributions with means μ_F and μ_G and variances σ_F^2 , σ_G^2 respectively, such that $F(x) = G(y)$ for all x and y which satisfy $x - \mu_F/\sigma_F = x - \mu_G/\sigma_G$. Let $\mu_F \geq \mu_G$ and $F(x_1) > G(x_1)$ for some x_1, (i.e., F and G intersect). Then $[\text{FDG wrt } U_1] \Leftrightarrow [\sigma_F^2 \leq \sigma_G^2]$.

Since H-L Theorem 4 is based on Theorem 1, Theorem 2 and Theorem 3, it needs to be modified in accordance with the changes made in those theorems. The proof of the modified theorem will not be given, since the only nontrivial change to be made is the replacement of U_1 with $U^{**}(F, G)$.

Theorem 4*

Let F and G have finite[3] means and variances (μ_F, σ_F^2), (μ_G, σ_G^2) respectively. Suppose $\mu_F \geq \mu_G$, $F(x) = G(y)\ \forall\ x$ and y satisfying $[x - \mu_F]/\sigma_F = [y - \mu_G]/\sigma_G$ and $F(x') > G(x')$ for some x' . Then $[\text{FDG wrt } U^{**}(F, G)] \Leftrightarrow [\sigma_F^2 \leq \sigma_G^2]$.

F. H-R Theorem 1[4]

Define U_1 to be the class of all bounded, strictly increasing, real-valued functions on R s.t. the first derivative exists and is continuous everywhere. Then $[\int udG - \int udF > 0\ \forall\ u \in U_1] \Leftrightarrow [G < F]$.

---

[3]Finite means implies $\int_{-\infty}^{0} xdF$ and $\int_{-\infty}^{0} xdG$ are finite, and hence by Lemma 1*, $\int_{-\infty}^{0} |G - F|\, dt \leq \int_{-\infty}^{0} Gdt + \int_{-\infty}^{0} Fdt = -\left[\int_{-\infty}^{0} xdG + \int_{-\infty}^{0} XdF\right]$ + constant $< +\infty$.

[4]The original H-R Theorems 1 and 2 are stated in terms of densities. This unnecessary restriction has been eliminated throughout the next two sections.

Corollary 1^* strengthens the necessity direction of H-R Theorem 1, since the class of functions appearing in the corollary is considerably larger. For this same reason, the sufficiency direction of H-R Theorem 1 is a strengthening of Theorem 1^*.

The proof of sufficiency in H-R Theorem 1 contains an error. H-R incorrectly take the converse of $[G(x) \le F(x) \ \forall \ x]$ to be $[\exists \ x' \in R$ s.t. $G(x) \ge F(x)$ for $x \le x'$, strict inequality holding for some $x'' < x'$, and $G(x) \le F(x)$ for $x \ge x']$. The correct converse is $[\exists \ x^* \text{ s.t. } G(x^*) > F(x^*)]$, which is much weaker.

The sufficiency still holds, however, as is proved below.

<u>Theorem 5*</u>

$[G(x^*) > F(x^*)$ for some $x^* \in R] \Rightarrow$ not $[\text{GDF wrt } U_1]$.

<u>Proof</u>

Suppose $G(x^*) > F(x^*)$. By right continuity of F and G $\exists$ an interval $[X^*, \beta)$ over which $G(x) > F(x)$. Let U' be the class of all real-valued, continuous, positive functions u' over R possessing a bell-shaped graph with peak at $([x^* + \beta]/2, \ u'([x^* + \beta]/2))$. By choosing $u' \in U'$ so that $u'([x^* + \beta]/2)$ is sufficiently large, $u'(x^*)$ and $u'(\beta)$ are sufficiently near zero, and $|u'(x)|$ approaches zero sufficiently rapidly as $|x| \to +\infty$, it is clear that the sign of $\int [F - G](x) u'(x) dx = \int_{\beta}^{\infty} [F - G](x) u'(x) dx + \int_{x^*}^{\beta} [F - G](x) u'(x) dx + \int_{-\infty}^{x^*} [F - G](x) u'(x) dx$ can be made negative.

It is also clear that one may require $|u'(x)|$ to approach zero so rapidly for $|x| \to +\infty$ that $u(x) = \int_{-\infty}^{x} u'(t) dt$ is a bounded function of x. In this case $u \in U_1$, $\int udF$ and $\int udG$ both exist and are finite, and by Lemma 1* $\int udG - \int udF = \int [F - G] (x) u'(x) dx < 0$. Hence $[G(x^*) > F(x^*)$ for some $x^*] \Rightarrow$ not [GDF wrt U_1]. //

H-R Corollary 1

If $G < F$, then all the odd moments of G are larger than all the respective odd moments of F. In the case $F(x) = G(x) = 0$ for $x < 0$, all the moments of G are larger than the respective moments of F.

The statement of this corollary needs to be made more precise. It can also be extended as follows.

1* Corollary B

Suppose $\int x^{2p+1} dG - \int x^{2p+1} dF$ is well defined for some integer $p \geq 0$. Then $[G < F] \Rightarrow [\int x^{2p+1} dG - \int x^{2p+1} dF > 0]$. If $[G < F]$ and $F(x) = G(x) = 0$ for $x < 0$, then $\int x^r dG - \int x^r dF > 0$ for all $r > 0$ for which $\int x^r dG - \int x^r dF$ is well defined.

Proof

Corollary 1*. /

H-R Theorem 2

Define $U_2 = [u \in U_1 \mid u''$ continuous and nonpositive]. Let F and G be the distribution functions for two nonnegative random variables with finite means. Then [GDF wrt U_2] $\Leftrightarrow [G \ll F]$.

The hypotheses made in H-R Theorem 2 are considerably stronger and the class of functions U_2 considerably smaller than the corresponding hypotheses and functions appearing in Theorem 2^*. Hence the necessity direction in Theorem 2^* strengthens that of H-R Theorem 2, whereas the sufficiency direction in H-R Theorem 2 is a strengthening of Theorem 2* only for distribution functions corresponding to non-negative random variables.

The proof of sufficiency in H-R Theorem 2 contains an error. H-R incorrectly take the converse of $[\int_0^x G(t)\,dt \le \int_0^x F(t)\,dt \ \forall\, x]$ to be $[\exists\, x'$ s.t. $\int_0^x G(t)\,dt \ge \int_0^x F(t)\,dt \ \forall\, x \le x']$. The correct converse is $[\exists\, x'$ s.t. $\int_0^{x'} G(t)\,dt > \int_0^{x'} F(t)\,dt]$ which is clearly weaker. The sufficiency can be proved as follows:

Theorem 6^*

$$\int_0^{x'} G(t)\,dt > \int_0^{x'} F(t)\,dt \Rightarrow \text{ not } [\text{GDF wrt } U_2] .$$

Proof

Suppose $\int_0^{x'} G(t)\,dt > \int_0^{x'} F(t)\,dt$. Let $\varepsilon, \Delta \in (0, 1)$, and let $u'_{\Delta,\varepsilon}(x)$ equal 1 for $x \in [0, x']$, have range in $(0, \Delta x'^2/x^2)$ for $x \in [x' + \varepsilon, +\infty)$ and be defined over $[x', x' + \varepsilon]$ in such a way that $u'_{\Delta,\varepsilon}$ is positive, differentiable, and nonincreasing over $(0, \infty)$. Then $\lim_{\substack{\Delta\to 0\\ \varepsilon\to 0}} u'_{\Delta,\varepsilon}(x) = 1$ if $0 \le x \le x'$, $= 0$ if $x > x'$, and for all $\varepsilon, \Delta \in (0, 1)$, $|[F(x) - G(x)]\, u'_{\Delta,\varepsilon}(x)| \le g(x)$, where $g(x) = 2x'^2/x^2$ if $x \ge x' + 1$, $g(x) = 2$ if $0 \le x < x' + 1$, and $g(x) = 0$ if $x < 0$.

It is clear that for each $\Delta, \varepsilon \in (0, 1)$, $u_{\Delta,\varepsilon}(x) = \int_0^x u'_{\Delta,\varepsilon}(t)\,dt$

is in U_2 . In particular, $u_{\Delta,\varepsilon}$ is bounded and hence $\int u_{\Delta,\varepsilon}\, dG - \int u_{\Delta,\varepsilon}\, dF$ is well defined. Using Lemma 1* and Lebesgue's Dominated convergence theorem, $\lim_{\substack{\Delta\to 0\\ \varepsilon\to 0}} [\int_0^\infty u_{\Delta,\varepsilon}\, dG - \int_0^\infty u_{\Delta,\varepsilon}\, dF] = \lim_{\substack{\Delta\to 0\\ \varepsilon\to 0}} [\int_0^\infty [F - G](t)\, u'_{\Delta,\varepsilon}(t)\, dt] = \int_0^\infty \lim_{\substack{\Delta\to 0\\ \varepsilon\to 0}} [[F - G](t)\, u'_{\Delta,\varepsilon}(t)]\, dt$ $= \int_0^{x'} [F - G](t)\, dt < 0$. One can therefore choose $\Delta^*, \varepsilon^* \in (0, 1)$ s.t. $\int_0^\infty u_{\Delta^*,\varepsilon^*}\, dG - \int_0^\infty u_{\Delta^*,\varepsilon^*}\, dF < 0$, and hence not [GDF wrt U_2] . /

II.

A problem which has often been considered by economists is the maximization of EU $(kX + (1 - k) Y)$ with respect to k, $k \in [0, 1]$, where X and Y represent the random "returns" for two different "assets," and U is the decision maker's utility function. It is useful to generalize this framework by replacing k and $1 - k$ with less restricted coefficients to allow, for example, the possibility of selling short.[5] This more general approach is employed below in Theorems 1′, 2′, and 3′, which are concerned with the ranking of mixtures of prospects in terms of stochastic dominance.

An undesirable feature of many existing theorems in the area of stochastic dominance is the regularity conditions researchers have thought necessary to place on the random variables; for example, possession of a density and nonnegativity. The first constraint is a nuisance. The second is a genuine restriction on the type of problem which can be considered. For example, under the assumption of non-negativity one cannot consider the important problem of buying insurance

---

[5]This approach is attempted by D. Cass and J. Stiglitz [see [3], Section 2, page 332]. Unfortunately their basic Theorem 1 [page 332] is incorrect. [Theorem 1 states that $d\hat{a}_M/dwo \gtreqless 0$ as $R' \gtreqless 0$, where $\hat{a}_M$ (unrestricted in sign or magnitude) is the optimal amount of initial wealth allocated to a safe asset, $1 - \hat{a}_M$ being allocated to a risky asset, and $R(x) = - x\varphi''(x)/\varphi'(x)$ is the Arrow-Pratt measure of relative risk aversion. An optimal point $\hat{a}_M$ may not exist under the Cass-Stiglitz hypotheses, and $R'(x) = 0 \ \forall\ x$ or $R'(x) < 0 \ \forall\ x$ is impossible since $R'(0) > 0$. The inequality appearing in the proof (top of page 333) is wrong for $\hat{a}_M \geq 1$. A direct counterexample to the proof is obtained by setting the expected return per dollar invested in the risky asset equal to the return of the safe asset. In this case $\hat{a}_M = 1$ and $d\hat{a}_M/dwo \equiv 0$ independently of R'.]

Y to complement one's current prospects X , where X may represent a fluctuating income and $EY \leq 0$.[6] All of the eight theorems presented below [except Theorem 1′, part i] are proved without the density and nonnegativity assumptions.

Finally, the conclusions of many of the theorems below are stated in terms of the stochastic dominance of one (set of) distribution function(s) by another. It is clear from Theorems 1* and 2* presented in Part II that such conclusions can be directly translated in terms of the maximization of expected utility over broad classes of utility functions.

In Theorem 1′ the decision maker is confronted with the choice of transforming his current portfolio containing a random prospect into a diversified portfolio containing a sure prospect and a specified amount of the original random prospect. Part ii), in particular, gives a necessary and sufficient condition for the second degree stochastic dominance of one portfolio over the other, assuming the diversified portfolio contains a positive amount of the random prospect.

<u>Theorem 1′</u>

Let X be a random variable with finite mean $\bar{x}$, and define $Y = a + bX$. Let F and G be the distribution functions of X and Y respectively. Then:

i) If $F(x) = 0$ for $x < 0$, $a \geq 0$ and $b \geq 1$, then $[G \leq F]$

ii) If $b \geq 0$, then $[a + b\bar{x} \geq \bar{x}] \Leftrightarrow [G \underline{<<} F]$.

[6] For an approach covering this latter problem where both Y and its coefficient are left unrestricted in sign, see [4].

Proof of i)

By definition of Y and right continuity of F, $G(y) = F([y - a]/b)$ if $y \geq a$, and $G(y) = 0$ if $y < a$. Since $[y - a]/b \leq y \ \forall\, y \geq a$, $G(y) = F([y - a]/b) \leq F(y) \ \forall\, y \geq a$. Hence $G(y) \leq F(y) \ \forall\, y$.

Proof of ii) Sufficiency

Case I: $b = 1$

$G(y) = F(y - a) \ \forall\, y \in R$. But $a + \bar{x} \geq \bar{x} \Rightarrow a \geq 0$. Hence $F(y) \geq G(y) \ \forall\, y$, and $[G \underline{<<} F]$.

Case II: $b \neq 1, 0$

$F(y) \gtreqqless F([y - a]/b) = G(y)$ for $y \gtreqqless [y - a]/b$, i.e. for $y \lesseqqgtr y^* = a/1 - b$. By hypothesis $\bar{y} = a + b\bar{x} \geq \bar{x}$, $\bar{x}$ finite. Hence by Lemma 1* $\bar{y} - \bar{x} = \int_{-\infty}^{\infty} x dG - \int_{-\infty}^{\infty} x dF = \int_{-\infty}^{\infty} [F - G]\, dx \geq 0$.

For any $x \geq y^*$, $\int_{-\infty}^{x} [F - G]\, dt = \int_{-\infty}^{\infty} [F - G]\, dt - \int_{x}^{\infty} [F - G]\, dt = \int_{-\infty}^{\infty} [F - G]\, dt + \int_{x}^{\infty} [G - F]\, dt \geq 0$ by the above. And similarly, for $x < y^*$, $\int_{-\infty}^{x} [F - G]\, dt \geq 0$. Hence $[G \underline{<<} F]$.

Case III: $b = 0$

In this case $Y \equiv a$. If $F \equiv G$ the corollary follows trivially. Assume $F \not\equiv G$. The hypotheses guarantee that

$\int_{-\infty}^{0} |G(t) - F(t)|dt < \infty$ and $\int xdF - \int xdG$ is well defined. Then $EY = a \geq \bar{x}$, $G(x) \geq F(x)$ for $x \geq a$, and $F(x) \geq G(x)$ for $x < a \Rightarrow$ by Theorem 3*, [GDF wrt U** (F, G)] . It then follows from Theorem 2* that [$G \underline{<<} F$] .

Proof of ii) Necessity

Suppose [$G \underline{<<} F$] . If $F \equiv G$, then $\int xdF = \int xdG$, i.e., $a + b\bar{x} = \bar{x}$. Suppose $F \not\equiv G$. The condition that $\bar{x}$ be finite guarantees that $\int_{-\infty}^{0} |G(t) - F(t)|\, dt < \infty$ and $\int_{-\infty}^{0} xdF - \int_{-\infty}^{0} xdG$ is well defined. Hence by Theorem 2*, [GDF wrt U** (F, G)] . The function $u: x \to x$ is in U** (F, G) . Hence $\int xdG - \int xdF \geq 0$ i.e., $a + b\bar{x} \geq \bar{x}$. //

1′ Corollary A

Let X be a random variable with distribution function F satisfying $1 - F(-z) \leq F(z)\ \forall\ z \leq 0$ (hence $\forall\ z$), and with finite mean $\bar{x}$. Let $Y = a + bX$ with distribution function G . Then [$a + b\bar{x} \geq -\bar{x}$] $\Rightarrow$ [$G \underline{<<} F$] .

Proof

$1 - F(-z) \leq F(z)\ \forall\ z \leq 0 \Rightarrow \bar{x} \leq 0$; for by Lemma 1*, $\int xdF = \int_{0}^{\infty} [1 - F]\, dx - \int_{-\infty}^{0} Fdx = \int_{-\infty}^{0} [1 - F(-x) - F(x)]\, dx \leq 0$. Suppose $b \geq 0$. Since $a + b\bar{x} \geq -\bar{x} \geq \bar{x}$, the corollary follows from Theorem

2′. Suppose $b < 0$. Define $\tilde{b} = -b$, $\tilde{X} = -X$. Then $Y = a + \tilde{b}\tilde{X}$, $\tilde{b} > 0$, and $a + \tilde{b}E\tilde{X} = a + b\bar{x} \geq E\tilde{X} = -\bar{x}$. Hence by Theorem 2′ $[G \underline{<<} \tilde{F}]$. But $\tilde{F}(z) = 1 - F(-z) \leq F(z)\ \forall\ z$. Hence $[G \underline{<<} F]$. /

1′ Corollary B

Let X be a random variable with distribution function F and finite mean $\bar{x}$, and let v be the value of a sure venture. Let $G(x) = 0$ for $x < v$, and $G(x) = 1$ for $x \geq v$. Then $[v \geq \bar{x}] \leftrightarrow [G \underline{<<} F]$.

Proof

Let $b = 0$ in Theorem 1′, ii). /

The next theorem extends and makes precise the often cited result that if X is "preferred" to Y , and W is independent of both X and Y , then $X + W$ will be "preferred" to $Y + W$.

Theorem 2′[7]

Let X, Y, and W denote random variables with distribution functions F, G, and H respectively. Assume that W is independent of X and Y . Let the distribution functions of the random variables $aX + bW$ and $aY + bW$ be denoted by $\hat{F}$ and $\hat{G}$ respectively, where $a \geq 0$. Then:

i) $[G < F] \Rightarrow [\hat{G} \leq \hat{F}]$, and $[\hat{G} < \hat{F}]$ if $dH(z) > 0$ for a.a.z

---

[7]This theorem corresponds to Theorem 5 in [2]. However, the hypotheses given for Theorem 5 are insufficient to guarantee the conclusions presented. Moreover, in Theorem 5 it is assumed that W has a density and $b \geq 0$. These restrictions are removed in Theorem 2′ above.

ii) $\int_{-\infty}^{\infty} |F(w) - G(w)| < \infty$ and $[G << F]$

$\Rightarrow [\hat{G} \underline{<<} \hat{F}]$, and $[\hat{G} << \hat{F}]$ if

$dH(z) > 0$ for a.a.z .

Proof of i)

If $a = 0$, $\hat{G} \equiv \hat{F}$ and $dH = 0$ a.e. and the Theorem holds. $a \neq 0$ will be assumed below.

Suppose $b = 0$. Then $\hat{F}(z) - \hat{G}(z) = F\left(\frac{z}{a}\right) - G\left(\frac{z}{a}\right) \geq 0 \ \forall z$, strict inequality holding for some z, so $[\hat{G} < \hat{F}]$.

Suppose $b > 0$. Then $\hat{F}(z) - \hat{G}(z) = F_{ax} * H_{bw}(z) - G_{ay} * H_{bw}(z)$ $= \int_{-\infty}^{\infty} F\left(\frac{z-t}{a}\right) dH\left(\frac{t}{b}\right) - \int_{-\infty}^{\infty} G\left(\frac{z-t}{a}\right) dH\left(\frac{t}{b}\right) = \int_{-\infty}^{\infty} \left[F\left(\frac{z-t}{a}\right) - G\left(\frac{z-t}{a}\right)\right] dH\left(\frac{t}{b}\right)$. Since $[G < F]$ by assumption, the integrand is everywhere nonnegative, and positive over some interval. Hence $[\hat{G} \leq \hat{F}]$, and $[\hat{G} < \hat{F}]$ if $dH(z) > 0$ for a.a.z.

Suppose $b < 0$. Define a new random variable $\tilde{W} = -W$. Then $aX + bW = aX + \tilde{b}\tilde{W}$, $aY + bW = aY + \tilde{b}\tilde{W}$, where $\tilde{b} = -b > 0$ and $\tilde{W}$ is independent of X and Y . Hence the proof goes through as above.

Proof of ii)

As in the proof of i), $a \neq 0$ will be assumed below.

Suppose $b = 0$. Then $\int_{-\infty}^{x} [\hat{F}(z) - \hat{G}(z)]\, dz = \int_{-\infty}^{x} \left[F\left(\frac{z}{a}\right) - G\left(\frac{z}{a}\right)\right] dz$ $\geq 0 \ \forall x$, strict inequality holding for some x . Hence $[\hat{F} << \hat{G}]$.

Suppose $b > 0$. Then $\int_{-\infty}^{x} [\hat{F}(z) - \hat{G}(z)]\,dz =$
$\int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} \left[F\left(\frac{z-t}{a}\right) - G\left(\frac{z-t}{a}\right)\right] dH\left(\frac{t}{b}\right)\right] dz = \int_{-\infty}^{\infty} dH\left(\frac{t}{b}\right)$
$\left[\int_{-\infty}^{x} \left[F\left(\frac{z-t}{a}\right) - G\left(\frac{z-t}{a}\right)\right] dz\right]$, where the interchange of integration is justified since $\int_{-\infty}^{\infty} | F(w) - G(w) |\,dw < \infty$ by hypothesis.
Since $[F << G]$ by hypothesis, $[\hat{F} \underline{<<} \hat{G}]$; and $[\hat{F} << \hat{G}]$ if $dH(w) > 0$ for a.a.w, as can be seen by a simple change of variables.

The case $b < 0$ follows immediately upon making the transformation introduced in the proof of i). //

The next theorem, a simple consequence of Theorem 2′, gives conditions under which a decision maker faced with two random, independent prospects will choose a diversified portfolio.

Theorem 3′

Let X_1 and X_2 be independent random variables with F_1 and F_2 their respective distribution functions. Let H_1 be the distribution function of $aX_1 + bX_2$, H_2 the distribution function of $aX_2 + bY_2$, where X_2 and Y_2 are independent and identically distributed, a and $b \in R$ with $a \geq 0$. Assume $[F_1 \underline{<<} F_2]$ and $[H_2 \underline{<<} F_1]$, and $\int_{-\infty}^{\infty} |F_1(w) - F_2(w)|\,dw < \infty$. Then $[H_1 \underline{<<} F_1]$ and $[H_1 \underline{<<} F_2]$.

Proof

By Theorem 2′, $[F_1 \underline{<<} F_2]$ and $\int_{-\infty}^{\infty} |F_1(w) - F_2(w)|\,dw < \infty \Rightarrow [H_1 \underline{<<} H_2]$.
Hence by the easily checked transitivity of $\underline{<<}$, $[H_1 \underline{<<} H_2 \underline{<<} F_1 \underline{<<} F_2]$ $\Rightarrow [H_1 \underline{<<} F_1]$ and $[H_1 \underline{<<} F_2]$. //

If a risk averter[8] is to choose between two portfolios, each containing a mixture of a sure prospect and a random prospect, and the value of the sure prospect exceeds the mean of the random prospect, it seems reasonable that he should prefer the portfolio containing the larger proportion of the sure prospect. This is demonstrated by Hadar-Russell [2] in the case where the random prospect is nonnegative and the utility function of the decision maker is bounded, strictly increasing, twice continuously differentiable, and concave. Theorem 4′ below demonstrates that it holds true under more general circumstances.

Theorem 4′

Let X be a random variable with finite mean $\bar{x}$ and v the value of a sure prospect satisfying $v \geq \bar{x}$. Let H_k be the distribution function for $kv + [1 - k]X$. Then $0 \leq k < k' \leq 1 \Rightarrow [H_k' \underline{\ll} H_k]$.

Proof

Let $k < k'$, $k, k' \in [0, 1]$, and define $k'' = [k' - k]/1 - k$, i.e., $k' = k + [1 - k]\, k''$. Then $k'v + [1 - k']X = [k + [1 - k]\, k'']v + [1 - k][1 - k'']X = k''v + [1 - k''][kv + [1 - k]X]$ $= a + bY$ where $a = k''v$, $b = 1 - k''$, and $Y = kv + [1 - k]X$. The mean of Y satisfies $\bar{y} = kv + [1 - k]\bar{x} \leq v = a/1 - b$. Hence by Theorem 1′, part ii, $[F_{a+bY} \underline{\ll} F_Y]$. But $F_{a+bY} \equiv H_k'$ and $F_Y \equiv H_k$. //

---

[8]Throughout this paper, "risk averter" is used as a synonym for "decision maker possessing a concave utility function."

The limitations of the mean-variance efficiency criterion[9] for portfolio selection are now common knowledge. It might still be asked whether this simple criterion is equivalent to stochastic dominance in certain frequently encountered situations. Theorem $5'''$ and its corollaries indicate that a direct relationship will exist between stochastic dominance and the mean-variance criterion only in highly special circumstances. [See also Part I, Theorems 3* and 4*.]

Theorem $5'$

Let F and G have finite means and variances (μ_F, σ_F^2), (μ_G, σ_G^2) respectively. Suppose there exists $x' \in (-\infty, +\infty]$, and $x'' \in (-\infty, x' \wedge 0]$ such that the following hold:

1. $F(x) \geq G(x)$ for all $x \geq x'$
2. $\int_{x''}^{x} [G - F](t)\, dt \geq 0$ for all $x \in [x'', x')$

   and $\int_{x''}^{x'} [1 - G](t)\, dt = \int_{x''}^{x'} [1 - F](t)\, dt$

   $= \Delta > 0$ if $x' \neq x''$

3. Either $x'' = 0$, or $F(x'') = G(x'')$ and $F(x''-) \leq G(x''-)$; either $x' = 0$, or $F(x')$ $= G(x')$ and $F(x'-) \leq G(x'-)$.

---

[9]The mean-variance criterion states that if a prospect X has larger mean and smaller variance than a prospect Y, the risk averter will prefer X to Y.

4. $\int_{-\infty}^{x} [G - F](t)\, dt \ge 0$ for all $x \le x''$

5. $\mu_F^2 \ge \mu_G^2$.

Then $\sigma_G^2 \ge \sigma_F^2$, strict inequality holding if strict inequality holds for some x in 1, 2, or 3, or $\mu_F^2 > \mu_G^2$.

Proof

$\sigma_G^2 - \sigma_F^2 \equiv \int x^2 dG - \int x^2 dF + \mu_F^2 - \mu_G^2$. Hence using 5., the theorem will be proved if it can be shown that $\int x^2 dG \ge \int x^2 dF$. The proof will be given in three steps.

Step I

$\int_{x'}^{\infty} x^2 dG \ge \int_{x'}^{\infty} x^2 dF$.

Proof: If $x' = +\infty$, the proof is done. Assume $x' < \infty$, and define $u(x) = [x - x']^2$ if $x \ge x'$, $u(x) = 0$ if $x < x'$. Then $u \in U^*(F, G)$, and by Lemma 1*

$\int_{x'}^{\infty} [x - x']^2\, dF - \int_{x'}^{\infty} [x - x']^2\, dG = 2 \int_{x'}^{\infty} [G - F](x)[x - x']\, dx \le 0$ by 1 . Hence $\int_{x'}^{\infty} x^2\, dF \le \int_{x'}^{\infty} x^2 dG + 2x' \int_{x'}^{\infty} [G - F]\, dx + [x']^2\, [F(x') - G(x')] \le \int_{x'}^{\infty} x^2 dG$ by 1., 3., and the definition of x' . /

Step II

$\int_{x''}^{x'-} x^2\, dG \ge \int_{x''}^{x'-} x^2\, dF$.

Proof: If $x' = x''$, the proof is done. Suppose $x' > x''$, and define $f^*(x) = 0$ if $x \ge x'$, $= [1 - F(x)]/\Delta$

if $x'' \le x < x'$, $= 0$ if $x < x''$. Define $g^*(x)$ similarly, with G in place of F. Then $F^*(x) = \int_{x''}^{x} f^*(t)\,dt$ and $G^*(x) = \int_{x''}^{x} g^*(t)\,dt$ are distribution functions, and $F^*(x) - G^*(x) = \int_{x''}^{x} [f^* - g^*](t)\,dt = \int_{-\infty}^{x} I_{[x'', x']}(t)[G - F](t)\,dt/\Delta \ge 0$ for all x by 2. Define $u(x) = x$ for all x. Then $u \in U^*(F^*, G^*)$ and by Theorem 1* $\int u\,dG^* - \int u\,dF^* = \int_{x''}^{x'} x\,[F - G](x)\,dx/\Delta \ge 0$. Using integration by parts, $\int_{x''}^{x'} 2x\,[F - G](x)\,dx = \int_{x''}^{x'-} x^2 dG - \int_{x''}^{x'} x^2 dF + [x']^2\,[F(x'-) - G(x'-)] - [x'']^2\,[F(x'') - G(x'')] \ge 0 \Rightarrow \int_{x''}^{x'-} x^2\,dG - \int_{x''}^{x'-} x^2\,dF \ge 0$ by 3. /

Step III

$$\int_{-\infty}^{x''-} x^2\,dG - \int_{-\infty}^{x''-} x^2\,dF \ge 0 \quad .$$

Proof

Define $u(x) = 0$ if $x \ge 0$, $u(x) = -x^2$ if $x < 0$; $\bar{G}(x) = 1$ if $x \ge x''$, $= G(x)$ if $x < x''$; $\bar{F}(x) = 1$ if $x \ge x''$, $= F(x)$ if $x < x''$. Then $u \in U^{**}(\bar{F}, \bar{G})$ and $\int_{-\infty}^{x} [\bar{G} - \bar{F}](t)\,dt = \int_{-\infty}^{x \wedge x''} [G - F](t)\,dt \ge 0\ \forall\, x$ by 4. Hence by Theorem 2* and 3., $0 \le \int u\,d\bar{F} - \int u\,d\bar{G} = \int_{-\infty}^{x''-} [-x^2]\,dF - \int_{-\infty}^{x''-} [-x^2]\,dG + [1 - F(x''-)]\,[-x''^2] - [1 - G(x''-)][-x''^2] \Rightarrow \int_{-\infty}^{x''-} x^2\,dG \ge \int_{-\infty}^{x''-} x^2\,dF$. /

Hence, combining Steps I, II, and III, $\int_{-\infty}^{\infty} x^2 dG \geq \int_{-\infty}^{\infty} x^2 dF$. It is clear from the proofs of Steps I, II, and III that strict inequality will hold if strict inequality holds for some x in 1, 2, or 3. If $\mu_F^2 > \mu_G^2$, strict inequality follows from the first statement of the proof. //

5′ Corollary A (H-R Theorem 3)

Let F and G be the distribution functions of two nonnegative random variables with a common mean and finite variances. Then $[F << G] \Rightarrow \sigma_G^2 > \sigma_F^2$.

5′ Corollary B

If F and G have finite means and variances (μ_F, σ_F^2), (μ_G, σ_G^2) respectively, and F and G intersect at $\tilde{x} \leq 0$ with $F \geq G$ for $x \geq \tilde{x}$, $F \leq G$ for $x \leq \tilde{x}$, and $\mu_F^2 \geq \mu_G^2$, then $\sigma_G^2 \geq \sigma_F^2$, and $\sigma_G^2 > \sigma_F^2$ if $F \neq G$.

5′ Corollary C

If F and G have finite means and variances (μ_F, σ_F^2), (μ_G, σ_G^2) respectively, with $\mu_G \geq 0$, and $\exists\, x^* \leq 0$ such that $F(x) \geq G(x)$ $\forall\, x \geq x^*$, $F(x) \leq G(x)\ \forall\, x < x^*$, then $[\mu_F \geq \mu_G] \Leftrightarrow$ [FDG wrt $U^{**}(F, G)$] $\Rightarrow [\sigma_F^2 < \sigma_G^2]$.

Proof

Combine Theorem 3* and 5′ Corollary B.

In Theorems 6′ and 7′ the choice is between mixtures of two independent and identically distributed prospects. Theorem 6′ demonstrates that diversification will be optimal for the risk averter given certain minor regularity conditions on the distribution function F . Theorem 7′ demonstrates that the risk averter will prefer the portfolio containing equal proportions of the two prospects if the distribution function has finite mean and variance.

Theorem 6′

Let X and Y be independent and identically distributed random variables with distribution function F satisfying $\int_{-\infty}^{0} F(x)\, dx < \infty$, and H_k the distribution function for $xK + [1 - k]\, Y$. Then $[H_k \underline{<<} F]\ \forall\, k \in [0, 1]$.

Proof

If $k = 0$ or 1 the theorem is immediate. The proof for $k \in (0, 1)$ will be given in two steps.

Step I

Assume F is absolutely continuous with density f . Let $k \in (0, 1),\ x \in R$. By definition, $H_k(z) = F_{kx} * F_{[1-k]Y}(z)$

$$= \int_{-\infty}^{\infty} F_{kx}\,(z - u)\ dF_{[1\, -\, k]Y}(u) = \frac{1}{1 - k}\int_{-\infty}^{\infty} F\left(\frac{y}{k}\right) f\left(\frac{z - y}{1 - k}\right) dy \ .$$

Integrating with respect to z , $\int_{-\infty}^{x} H_k\,(z)\ dz =$

$$\int_{-\infty}^{x} \frac{1}{1 - k}\left[\int_{-\infty}^{\infty} F\left(\frac{y}{k}\right) f\left(\frac{z - y}{1 - k}\right) dy\right] dz =$$

$$\int_{-\infty}^{\infty} F\left(\frac{y}{k}\right)\left[\frac{1}{1 - k}\int_{-\infty}^{x} f\left(\frac{z - y}{1 - k}\right) dz\right] dy = \int_{-\infty}^{\infty} F\left(\frac{y}{k}\right) F\left(\frac{x - y}{1 - k}\right) dy \ .$$

Letting $w = y/k$, $\int_{-\infty}^{x} H_k(z)\,dz = k \int_{-\infty}^{\infty} F(w)\, F\left(\frac{x - kw}{1 - k}\right) dw$

$= k \int_{-\infty}^{x} F(w)\, F\left(\frac{x - kw}{1 - k}\right) dw + k \int_{x}^{\infty} F(w)\, F\left(\frac{x - kw}{1 - k}\right) dw$

$= A + B$ (say). Letting $u = [x - kw]/1 - k$, $B = k$

$\int_{-\infty}^{x} F\left(\frac{x - (1 - k)\,u}{k}\right) F\left(\frac{x - [x - (1 - k)\,u]}{1 - k}\right) \left[\frac{1 - k}{k}\right] du$

$= [1 - k] \int_{-\infty}^{x} F\left(\frac{x - (1 - k)\,u}{k}\right) F(u)\,du$. Clearly $A \leq$ $k \int_{-\infty}^{x} F(w)\,dw$ and $B \leq [1 - k] \int_{-\infty}^{x} F(w)\,dw$. Hence $A + B$ $= \int_{-\infty}^{x} H_k(z)\,dz \leq \int_{-\infty}^{x} F(w)\,dw$. Since $x \in R$ and $k \in (0, 1)$ were chosen arbitrarily, and $\int_{-\infty}^{0} F(x)\,dx < \infty$ by hypothesis, $[H_k \underline{<<} F]\ \forall\, k \in (0, 1)$.

Step II

Let F be an arbitrary right continuous distribution function. Any distribution function is the limit in distribution of an increasing sequence of distribution functions with bounded densities. [Consider the convolution of F with a distribution function F_n corresponding to a non-negative random variable having a triangular density with base lying on $[0, 1/n]$. Let $n \to +\infty$]. Letting $\{F_n\}$ be such a sequence for F, it follows by Step I that

$$\int_{-\infty}^{x} H_{n,k}(z)\,dz = \int_{-\infty}^{\infty} F_n\left(\frac{y}{k}\right) F_n\left(\frac{x - y}{1 - k}\right) dy .$$

Given $\varepsilon > 0 \ \exists\ M$ s.t. $\forall\ n$,

$$\left| \int_{-M}^{x} H_{n,k}(z)\ dx - \int_{-M}^{M} F_n\left(\frac{y}{k}\right) F_n\left(\frac{x-y}{1-k}\right) dy \right| < \varepsilon .$$

[This holds since $\int_{-\infty}^{-M} H_{n,k}(z)\ dz = \int_{-\infty}^{\infty} F_n\left(\frac{y}{k}\right) F_n\left(\frac{-M-y}{1-k}\right) dy$

$= \int_{-\infty}^{-M/2} + \int_{-M/2}^{\infty} \left[F_n\left(\frac{y}{k}\right) F_n\left(\frac{-M-y}{1-k}\right)\right] dy \le \int_{-\infty}^{-M/2} F_n\left(\frac{y}{k}\right) dy$

$+ \int_{-M/2}^{\infty} F_n\left(\frac{-M-y}{1-k}\right) dy \le \int_{-\infty}^{-M/2} F\left(\frac{y}{k}\right) dy + \int_{-M/2}^{\infty} F\left(\frac{-M-y}{1-k}\right) dy$;

$\int_{M}^{\infty} F_n\left(\frac{y}{k}\right) F_n\left(\frac{x-y}{1-k}\right) dy \le \int_{M}^{\infty} F\left(\frac{x-y}{1-k}\right) dy$; and

$\int_{-\infty}^{-M} F_n\left(\frac{y}{k}\right) F_n\left(\frac{x-y}{1-k}\right) dy \le \int_{-\infty}^{-M} F\left(\frac{y}{k}\right) dy$] . Since

$\left| I_{[-M,x]}(z)\ H_{n,k}(z) \right| \le I_{[-M,x]}(z)\ \forall\ z$ and

$\left| I_{[-M,M]}(y)\ F_n\left(\frac{y}{k}\right) F_n\left(\frac{x-y}{1-k}\right) \right| \le I_{[-M,M]}(y)\ \forall\ y$ where

the indicator functions are clearly Lebesgue integrable and independent of n , it follows by Lebesgue's Dominated Convergence Theorem and definition of the F_n that

$$\lim_{n\to+\infty} \left| \int_{-M}^{x} H_{n,k}(z)\ dz - \int_{-M}^{+M} F_n\left(\frac{y}{k}\right) F_n\left(\frac{x-y}{1-k}\right) dy \right|$$

$= \left| \int_{-M}^{x} H_k(z)\ dx - \int_{-M}^{M} F\left(\frac{y}{k}\right) F\left(\frac{x-y}{1-k}\right) dy \right| < \varepsilon$. Since

this holds for arbitrarily small ε (hence arbitrarily large M) , $\int_{-\infty}^{x} H_k(z)\ dz = \int_{-\infty}^{\infty} F\left(\frac{y}{k}\right) F\left(\frac{x-y}{1-k}\right) dy$. The remainder of the proof follows as in Step I. //

Theorem 7′ [10]

Let H_k denote the distribution function of $kX + [1 - k]Y$, where X and Y are independent and identically distributed with distribution function F having finite mean and variance. Then $[H_{\frac{1}{2}} \leq\leq H_k] \; \forall \; k \in [0, 1]$.

Proof

Step I

Let $k \in (0, 1)$, and define $M(k, x) = \int_{-\infty}^{x} H_k(z)\, dz$.

By Step II in Theorem 6′, $M(k, x) = \int_{-\infty}^{\infty} F(t/k)\, F\left(\frac{x - t}{1 - k}\right) dt$.

Letting $y = x - t$, $M(k, x) = \int_{-\infty}^{\infty} F\left(\frac{x - y}{k}\right) F\left(\frac{y}{1 - k}\right) dy$

$= M(1 - k, x)$. Hence $M(\cdot\,, x)$ is symmetric over $(0, 1)$ about the point $k = 1/2$.

Step II

Assume X has a density f satisfying $f(x) < B < \infty \; \forall \; x \in R$.

By Step I in Theorem 6′, $M(k, x) = \int_{-\infty}^{\infty} F\left(\frac{t}{k}\right) F\left(\frac{x - t}{1 - k}\right) dt$.

Fix $k^* \in (0, 1)$ and $x \in R$. Let k_1, k_2 satisfy $0 < k_1 < k^* < k_2 < 1$. For $k \in [k_1, k_2]$, define $\Delta M(k) = [M(k, x) - M(k^*, x)]/k - k^*$. By the Mean Value

---

[10] Hadar-Russell propose a similar theorem (cf. [2], page 301). However, their hypotheses are neither necessary nor sufficient for the conclusions they present. They need additional hypotheses to justify differentiation under the integral. Their restrictions that X be nonnegative and possess a density are not essential, as Theorem 7′ above demonstrates.

Theorem applied to $\left[F\left(\frac{t}{k}\right) F\left(\frac{x-t}{1-k}\right)\right]$, $\exists\, k' = k'$ (k, k^*, t) lying between k and k^* such that

$$\left[F\left(\frac{t}{k}\right) F\left(\frac{x-t}{1-k}\right) - F\left(\frac{t}{k^*}\right) F\left(\frac{x-t}{1-k^*}\right)\right]$$

$$= \partial\left[F\left(\frac{t}{k}\right) F\left(\frac{x-t}{1-k}\right)\right] / \partial k\,(k')[k - k^*] \text{ . Hence}$$

$$\Delta M(k) = \int_{-\infty}^{\infty} \left[f\left(\frac{t}{k'}\right)\left(\frac{-t}{k'^2}\right) F\left(\frac{x-t}{1-k'}\right) + F\left(\frac{t}{k'}\right) f\left(\frac{x-t}{1-k'}\right)\left[\frac{x-t}{(1-k')^2}\right]\right] dt = \int_{-\infty}^{\infty} [I_1 + I_2]\, dt \quad \text{(say)} .$$

Define $G_1(t) = \sup\limits_{k\,\in[k_1,k_2]} f\left(\frac{t}{k}\right)$, $G_2(t) = \sup\limits_{k\,\in[k_1,k_2]} f\left(\frac{x-t}{1-k}\right)$. Then G_1 and G_2 are measurable and

$|I_1| \leq |t|\, G_1(t)/k_1^2$, $|I_2| \leq [[|x| + |t|]/[1-k_2]^2]\, G_2(t)$. Since $0 < k_1 < k_2 < 1$, $G_1(t) = 0[f(t)]$ as $|t| \to +\infty$ and $G_2(t) = 0[f(-t)]$ as $|t| \to +\infty$. Moreover, $G_1(t) \leq B\ \forall\, t$, $G_2(t) \leq B\ \forall\, t$. It follows, since $\int |t| f(t)\, dt < \infty$ by hypothesis, that $|t|\, G_1(t)/k_1^2$ and $[[|x| + |t|]/[1-k_2]^2\, G_2(t)$ are Lebesgue integrable functions, independent of k. Hence by Lebesgue's Dominated Convergence Theorem, $[\partial M/\partial k](k^*) = \lim\limits_{\substack{k \to k^* \\ k\,\in\,[k_1,k_2]}} \Delta M(k)$

$$= \int_{-\infty}^{\infty} \lim_{\substack{k \to k^* \\ k\,\in\,[k_1,\,k_2]}} \left[\left[F\left(\frac{t}{k}\right) F\left(\frac{x-t}{1-k}\right) - F\left(\frac{t}{k^*}\right) F\left(\frac{x-t}{1-k^*}\right)\right]/[k - k^*]\right] dt$$

$$= \int_{-\infty}^{\infty} f\left(\frac{t}{k^*}\right)\left[\frac{-t}{k^{*2}}\right] F\left(\frac{x-t}{1-k^*}\right) + F\left(\frac{t}{k^*}\right) f\left(\frac{x-t}{1-k^*}\right)\left[\frac{x-t}{(1-k^*)^2}\right] dt$$

$$= \int_{-\infty}^{\infty} wf\ (w) \left[F\left(\frac{x - (1 - k^*)w}{k^*}\right) - F\left(\frac{x - k^*w}{1 - k^*}\right)\right] dw = D\ (k^*) \quad \text{(say)}.$$

For $k \in [k_1, k_2]$, define $\Delta D(k) = [D(k) - D(k^*)]/k - k^*$

$= \int_{-\infty}^{\infty} wf\ (w)\ [[S(k, w) - S(k^*, w)]/k - k^*]\ dw$, where

$S(k, w) \equiv \left[F\left(\frac{x - (1 - k)w}{k}\right) - F\left(\frac{x - kw}{1 - k}\right)\right]$. Differentiation under the integral sign can now be justified using the same technique as used above for $\partial M/\partial k$, with the hypothesis of a finite variance replacing that of a finite mean. The result obtained is $\partial^2 M/\partial^2 k(k^*) = \frac{1}{k^{*2}} \int_{-\infty}^{\infty} wf(w)$

$f\left(\frac{x - (1 - k^*)w}{k^*}\right) [w - x]\ dw - \frac{1}{(1 - k^*)2} \int_{-\infty}^{\infty} wf(w)$

$f\left(\frac{x - k^*w}{1 - k^*}\right) [x - w]\ dw$. Letting $u = [x - (1 - k^*)w/k^*]$, the first integral becomes $\frac{1}{(1 - k^*)3} \int_{-\infty}^{\infty} [x - uk^*]\ f\left(\frac{x - uk^*}{1 - k^*}\right)$

$f\ (u)\ [x - u]\ du$. Combining this with the second integral,

one obtains:

$$\frac{1}{(1 - k^*)3} \int_{-\infty}^{\infty} f(w)\ f\left(\frac{x - k^*w}{1 - k^*}\right)[[x - w][x - wk^*] - [1 - k^*][w][x - w]]\ dw$$

$$= \frac{1}{(1 - k^*)3} \int_{-\infty}^{\infty} f\ (w)\ f\left(\frac{x - k^*w}{1 - k^*}\right) [x - w]^2\ dw$$

. The integrand is everywhere nonnegative. Since k^* was an arbitrary point in $(0, 1)$ and x an arbitrary point in R, $M(\cdot, x)$ is a convex function of k over $(0, 1)$ for each $x \in R$.

Step III

Let F be an arbitrary right-continuous distribution function. As discussed in the proof of Theorem 6′, F is the limit in distribution of an increasing sequence $\{F_n\}$ of distribution functions with bounded densities. For each F_n, by Step II above, $\int_{-\infty}^{x} H_{n,k}(z)\,dz$ is a convex function of k over (0, 1) $\forall x \in R$, where $H_{n,k}(z) = \frac{1}{1-k}\int_{-\infty}^{\infty} F_n\left(\frac{t}{k}\right) f_n\left(\frac{z-t}{1-k}\right) dt$. Since F_n converges to F in distribution, it is clear from a consideration of the corresponding characteristic functions that $H_{n,k}$ converges to H_k in distribution; i.e., $H_{n,k}(z) \to H_k(z)$ at all points of continuity of H_k, a set of Lebesgue measure one.

By Lemma 1*, $\int x dF$ finite $\Rightarrow \int_{-\infty}^{0} F(x)\,dx < \infty$. By Step I in Theorem 6′, for any $S \in R$, $\int_{-\infty}^{-S} H_{n,k}(z)\,dz$

$$= \int_{-\infty}^{\infty} F_n\left(\frac{y}{k}\right) F_n\left(\frac{-S-y}{1-k}\right) dy = \int_{-\infty}^{-S/2} + \int_{-S/2}^{\infty}\left[F_n\left(\frac{y}{k}\right) F_n\left(\frac{-S-y}{1-k}\right)\right] dy$$

$$\leq \int_{-\infty}^{-S/2} F\left(\frac{y}{k}\right) dy + \int_{-S/2}^{\infty} F\left(\frac{-S-y}{1-k}\right) dy,$$ and similarly for $\int_{-\infty}^{-S} H_k(z)\,dz$. Hence $M(k, x)$ is finite $\forall k \in (0, 1)$, $\forall x \in R$, and given $\varepsilon > 0$, $\exists S$ s.t. $\forall n$, $\int_{-\infty}^{-S} H_{n,k}(z)\,dz < \varepsilon$ and

$\int_{-\infty}^{-S} H_k (z)\, dz < \varepsilon$.

Fix $x \in R$. For any $M \geq -x$, $H_{n,k} (z)\, I_{[-M,x]}(z) \leq I_{[-M,x]}(z)\ \forall\ (z)$, where $I_{[-M,x]}(\cdot)$ is Lebesgue integrable and independent of n . Hence by Lebesgue's Dominated Convergence Theorem, $\lim_{n\to\infty} \int_{-M}^{x} H_{n,k} (z)\, dz = \int_{-M}^{x} \lim_{n\to\infty} H_{n,k} (z)\, dz = \int_{-M}^{x} H_k (z)\, dz$. Combining this result with the above inequalities on the tails, given $\Delta > 0\ \exists\ N$ s.t. $\forall\ n > N$, $\left| \int_{-\infty}^{x} H_{n,k} (z)\, dz - \int_{-\infty}^{x} H_k(z)\, dz \right| < \Delta$. Thus $\int_{-\infty}^{x} H_k (z)\, dz$, being the limit of convex functions, is itself convex.

Combining Steps I and III, $M(\cdot\,, x)$ is convex in k over $(0, 1)$ and symmetric over $(0, 1)$ about the point $k = \frac{1}{2}$; hence it attains a minimum over $(0, 1)$ at $k = \frac{1}{2}$. Since this is true for any $x \in R$, and $M(k, x)$ is finite $\forall\ k \in (0, 1)$, $\forall\ x \in R$, $[H_{\frac{1}{2}} \leq\leq H_k]\ \forall\ k \in (0, 1)$. By Theorem 6′, using $\int_{-\infty}^{0} F(x)\, dx < \infty$, $[H_k \leq\leq F]\ \forall\ k \in [0, 1]$, where $F = H_1 = H_0$. Thus $[H_{\frac{1}{2}} \leq\leq H_k]\ \forall\ k \in [0, 1]$. //

BIBLIOGRAPHY

[1] Hanoch, Giora and Haim Levy. "The Efficiency Analysis of Choices Involving Risk." Review of Economic Studies, 36, (1969).

[2] Hadar, Josef and William Russell. "Stochastic Dominance and Diversification." Journal of Economic Theory, 3, (1971).

[3] Cass, D. and J. E. Stiglitz. "Risk Aversion and Wealth Effects on Portfolios with Many Assets." Review of Economic Studies, July, 1972, pages 331-354.

[4] Hildreth, C. "Ventures, Bets and Initial Prospects." To appear in Decision Rules and Uncertainty: NSF-NBER Proceedings, M. Balch, D. McFadden, and S. Wu, Editors, North Holland, 1974.

SUMMARY

Summary of Theorems from Part I

Lemma 1*

$$\int u dF - \int u dG = \int [G - F] du \;\; \forall \; u \in U^* (F, G)$$

Theorem 1*

$$[\text{FDG wrt } U^* (F, G)] \Leftrightarrow [F < G]$$

1* Corollary A

If $u \in U^* (F, G)$ and u is strictly increasing, then

$[F < G] \Rightarrow [\int u dF - \int u dG > 0]$.

1* Corollary B

Suppose $\int x^{2p+1} dF - \int x^{2p+1} dG$ is well defined for some integer $p \geq 0$. Then $[F < G] \Rightarrow [\int x^{2p+1} dF - \int x^{2p+1} dG > 0]$. If $[F < G]$ and $F(x) = G(x) = 0$ for $x < 0$, then $\int x^r dF - \int x^r dG > 0 \;\; \forall \; r \geq 0$ for which $\int x^r dF - \int x^r dG$ is well defined.

Theorem 2*

If $\int_{-\infty}^{0} | G(t) - F(t) | dt < \infty$ and $\int_{-\infty}^{0} x dF - \int_{-\infty}^{0} x dG$ is well defined, then $[\text{FDG wrt } U^{**} (F, G)] \Leftrightarrow [F << G]$.

Theorem 3*

Suppose $\int xdF - \int xdG$ is well defined and $\int_{-\infty}^{0} | G(t) - F(t)| dt < \infty$. If $\exists\, x' \in R$ s.t. $F(x) \leq G(x)$ for $x \leq x'$, $F(x) \geq G(x)$ for $x \geq x'$, and $F \neq G$, then [FDG wrt U** (F, G)] $\Leftrightarrow$ $[\int xdF - \int xdG \geq 0]$.

Theorem 4*

Let F and G have finite means and variances (μ_F, δ_F) , (μ_G, δ_G) respectively. Suppose $\mu_F \geq \mu_G$, $F(x) = G(y)$ for all x and y satisfying $[x - \mu_F]/\delta_F = [y - \mu_G]/\delta_G$, and $F(x') > G(x')$ for some x' . Then [FDG wrt U** (F, G)] $\Leftrightarrow$ $[\delta_F^2 \leq \delta_G^2]$.

Theorem 5*[1]

Define $U_1 = \{u:R \to R | u$ bounded, strictly increasing, with continuous first derivative$\}$. Then $[\int udG - \int udF > 0 \ \forall\, u \in U_1] \Rightarrow [G < F]$.

Theorem 6*

Define $U_2 = \{u \in U_1 | u''$ continuous and nonpositive$\}$. Let F and G be the distribution functions for two nonnegative random variables with finite means. Then [FDG wrt U_2] $\Rightarrow$ [F << G] .

---

[1]Theorems 5* and 6* are modifications of two Hadar-Russell theorems which contain errors.

Summary of Theorems from Part II

Theorem 1′

Let X be a random variable with finite mean $\bar{x}$, and let $Y = a + bX$. Let F and G be the distribution functions of X and Y respectively. Then:

i) If $F(x) = G(x) = 0$ for $x < 0$, and $a \geq 0,\ b \geq 1$, then $[G \leq F]$.

ii) If $b \geq 0$, $[a + b\bar{x} \geq \bar{x}] \Leftrightarrow [G \underline{\leq\leq} F]$.

1′ Corollary A

Let X be a random variable with distribution function F satisfying $1 - F(-z) \leq F(z)\ \forall\ z \leq 0$ (hence $\forall\ z$), and with finite mean $\bar{x}$. Let $Y = a + bX$ with distribution function G. Then $[a + b\bar{x} \geq -\bar{x}] \Rightarrow [G \leq\leq F]$.

1′ Corollary B

Let X be a random variable with distribution function F and finite mean $\bar{x}$, and let v be the value of a sure prospect. Let $G(x) = 0$ for $x < v$, $G(x) = 1$ for $x \geq v$. Then $[v \geq \bar{x}] \Leftrightarrow [G \leq\leq F]$.

Theorem 2′

Let X, Y, and W denote random variables with distribution functions F, G, and H respectively. Assume that W is independent of X and Y. Let the distribution functions of the random variables $aX + bW$ and $aY + bW$ be denoted by $\hat{F}$ and $\hat{G}$ respectively, where $a \geq 0$. Then:

i) $[G < F] \Rightarrow [\hat{G} \leq \hat{F}]$, and $[\hat{G} < \hat{F}]$ if

$dH(z) > 0$ for a.a.z.

ii) $\int | F(w) - G(w) | < \infty$ and $[G \ll F] \Rightarrow [\hat{G} \underline{\ll} \hat{F}]$,

and $[\hat{G} \ll \hat{F}]$ if $dH(z) > 0$ for a.a.z.

Theorem 3′

Let X_1 and X_2 be independent random variables with F_1 and F_2 their respective distribution functions. Let H_1 be the distribution function of $aX_1 + bX_2$, H_2 the distribution function of $aX_2 + bY_2$, where X_2 and Y_2 are independent and identically distributed, $a, b \in R$, and $a \geq 0$. Assume $[F_1 \leq\leq F_2]$, $[H_2 \leq\leq F_1]$, and $\int | F_1 (w) - F_2 (w) | dw < \infty$. Then $[H_1 \leq\leq F_1]$ and $[H_1 \leq\leq F_2]$.

Theorem 4′

Let X be a random variable with finite mean $\bar{x}$ and v the value of a sure prospect satisfying $v \geq \bar{x}$. Let H_k be the distribution function for $kv + (1 - k) X$. Then

$0 \leq k < k' \leq 1 \Rightarrow [H_{k'} \ll H_k]$.

Theorem 5′

Let F and G have finite means and variances (μ_F, δ_F^2), (μ_G, δ_G^2) respectively. Suppose $\exists\, x' \in (-\infty, +\infty]$ and $x'' \in (-\infty, x' \wedge 0]$ such that the following hold:

1. $F(x) \geq G(x)$ for all $x \geq x'$.

2. $\int_{x''}^{x} [G - F]\, dt \geq 0$ for all $x \in [x'', x']$,

   and

   $$\int_{x''}^{x'} [1 - G](t)\, dt = \int_{x''}^{x'} [1 - F](t)\, dt$$

   $$= \Delta > 0 \text{ if } x' \neq x'' .$$

3. Either $x' = 0$, or $F(x') = G(x')$ and $F(x'-) \leq G(x'-)$; either $x'' = 0$ or $F(x'') = G(x'')$ and $F(x''-) \leq G(x''-)$.

4. $\int_{-\infty}^{x} [G - F]\,(t)\, dt \geq 0 \;\forall\, x \leq x''$.

5. $\mu_F^2 \geq \mu_G^2$.

Then $\delta_G^2 \geq \delta_F^2$, strict inequality holding if strict inequality holds for some x in 1, 2, or 3, or $\mu_F^2 > \mu_G^2$.

5′ Corollary A

Let F and G be the distribution functions of two non-negative random variables with a common mean and finite variances, δ_F^2 , δ_G^2 respectively. Then $[F << G] \Rightarrow [\delta_G^2 > \delta_F^2]$.

5′ Corollary B

Let F and G have finite means and variances (μ_F, δ_F^2), (μ_G, δ_G^2) respectively. If F and G intersect at $\tilde{x} \le 0$ with $F \ge G$ for $x \ge \tilde{x}$, $F \le G$ for $x \le \tilde{x}$, and $\mu_F^2 \ge \mu_G^2$, then $\delta_G^2 \ge \delta_F^2$ strict inequality holding if $F \ne G$.

5′ Corollary C

If F and G have finite means and variances (μ_F, σ_F^2), (μ_G, σ_G^2) respectively, with $\mu_G \ge 0$, and $\exists\, x^* \le 0$ such that $F(x) \ge G(x)\ \forall\, x \ge x^*$, $F(x) \le G(x)\ \forall\, x < x^*$, then $[\mu_F \ge \mu_G] \Leftrightarrow$ [FDG wrt u^{**} (F, G)] $\Rightarrow [\sigma_F^2 < \sigma_G^2]$.

Theorem 6′

Let X and Y be independent and identically distributed random variables with distribution function F satisfying $\int_{-\infty}^{0} F(x)\, dx < \infty$, and H_k the distribution function for $kX + [1 - k]Y$. Then $[H_k \le\le F]\ \forall\, k \in [0, 1]$.

Theorem 7′

Let H_k denote the distribution function of $kX + [1 - k]Y$, where X and Y are independent and identically distributed random variables with distribution function F having finite mean and variance. Then $[H_{\frac{1}{2}} \le\le H_k]\ \forall\, k \in [0, 1]$.