Density of monodromy actions on non-abelian cohomology

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Abstract

In this paper we study the monodromy action on the first Betti and de Rham non-abelian cohomology arising from a family of smooth curves. We describe sufficient conditions for the existence of a Zariski dense monodromy orbit. In particular we show that for a Lefschetz pencil of sufficiently high degree the monodromy action is dense.

Contents

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1 Introduction

We work in the category of quasi-projective schemes over $\mathbb{C}$. Let $f : X \to B$ be a smooth projective morphism with connected fibers of dimesion one and genus at least two. Fix a base point $o \in B$ and let X_o be the corresponding fiber of f. In this paper we study the monodromy action of $\pi_1(B, o)$ on the degree one non-abelian Betti and de Rham cohomology of X_o. Let $\pi_1(X_o)$ denote the abstract fundamental group of X_o and let

$$\mathrm{mon} : \pi_1(B, o) \to \mathrm{Out}(\pi_1(X_o))$$

be the geometric monodromy representation of the family $f : X \to B$. For any positive integer n let

$$\mathrm{mon}^n_B : \pi_1(B, o) \longrightarrow \mathrm{Aut}(H^1_B(X_o, \mathrm{GL}(n, \mathbb{C})))$$

be the induced monodromy action on the non-abelian Betti cohomology with coefficients in $\mathrm{GL}(n, \mathbb{C})$.

The de Rham object which corresponds to mon^n_B is the Gauss-Manin connection ∇^n_{DR} on the relative de Rham cohomology stack $H^1_{DR}(X/B, \mathrm{GL}(n, \mathbb{C}))$. While the non-abelian Betti and de Rham cohomology are most naturally viewed as stacks, for the purposes of the present paper it suffices to work with the corresponding coarse moduli spaces. To indicate that we will write $M_B(X_o, n)$ and $M_{DR}(X/B, n)$ rather than $H^1_B(X_o, \mathrm{GL}(n, \mathbb{C}))$ and $H^1_{DR}(X/B, \mathrm{GL}(n, \mathbb{C}))$. Concretely $M_B(X_o, n)$ denotes the moduli space of (semisimplifications of) represenations of $\pi_1(X_o)$ in $\mathrm{GL}(n, \mathbb{C})$ and $M_{DR}(X_o, n)$ denotes the moduli space of rank n algebraic local systems of complex vector spaces on X_o. The total space $M_{DR}(X/B, n) \to B$ is a quasiprojective variety over B whose fiber over the point o is $M_{DR}(X_o, n)$.

For a loop $\gamma \in \pi_1(B, o)$ the action of $\mathrm{mon}^n_B(\gamma)$ on $M_B(X_o, n)$ is given by composing a n-dimensional represenation of $\pi_1(X_o)$ with some lift of the outer automorphism $\mathrm{mon}(\gamma)$ of $\pi_1(X_o)$. This gives a well defined action on conjugacy classes of representations of $\pi_1(X_o)$ and results in an algebraic automorphism $\mathrm{mon}^n_B(\gamma) : M_B(X_o, n) \to M_B(X_o, n)$.

There is an analytic action mon^n_{DR} of $\pi_1(B, o)$ on $M_{DR}(X_o, n)$ which is most naturally described through the Riemann-Hilbert correspondence (see e.g. [Deligne73], [Simpson95, Section 7]). It is shown in [Simpson95, Section 7] that the passage from a local system to its monodromy representation induces an isomorphism of analytic spaces

$$\tau : M_{DR}(X_o, n)^{\mathrm{an}} \widetilde{\to} M_B(X_o, n)^{\mathrm{an}}.$$

Now given γ we can define an analytic automorphism of $\mathrm{mon}^n_{DR}(\gamma)$ of $M_{DR}(X_o, n)$ by putting $\mathrm{mon}^n_{DR}(\gamma) = \tau^{-1} \circ \mathrm{mon}^n_B(\gamma) \circ \tau$. This analytic action is the monodromy of the algebraic non-abelian Gauss-Manin connection ∇^n_{DR} on the total space $M_{DR}(X/B, n)$ [Simpson95, Section 8].

It is natural to try to understand the complexity of the algebraic (respectively analytic) action of $\pi_1(B, o)$ on $M_B(X_o, n)$ (respectively $M_{DR}(X_o, n)$) by measuring in some way the size of the $\pi_1(B, o)$-orbits on $M_B(X_o, n)$ and $M_{DR}(X_o, n)$. Analogous questions concerning the monodromy action of $\pi_1(B, o)$ on spaces of special representations of $\pi_1(X_o)$ (e.g. real representations, representations with compact image, projective structures, etc.) have been the focus of active research in the recent years [Goldman97], [McMullen99], [Gallo *et al.*00]. In that direction the result most relevant to our setup is a theorem of W. Goldman who showed in [Goldman97] that the mapping class group acts ergodically on the space of all representations of $\pi_1(X_o)$ into $SU(2)$. Unfortunately, Goldman's proof does not generalize to the case of representations into $SU(n)$ for $n > 2$ and we do not know whether the mapping class group of X_o still acts ergodically on the space of such representations. Instead of pursuing the ergodicity question in its full generality we chose to work with non-abelian cohomology with complex coefficients. This allows us to use the algebraic (respectively analytic) nature of the monodromy action on $M_B(X_o, n)$ (respectively $M_{DR}(X_o, n)$) and to describe the size of the monodromy orbits on those spaces in geometric, rather than measure-theoretic terms.

Our first result is of essentially topological nature.

Theorem A *Assume that $g(X_o) \geq 3$ and that the image $\mathrm{mon}(\pi_1(B, o)) \subset \mathrm{Out}(\pi_1(X_o))$ of the geometric monodromy representation is of finite index in $\mathrm{Out}(\pi_1(X_o))$. Then:*

(i) *There is no meromorphic function on $M_B(X_o, n)^{\mathrm{an}}$ which is invariant under the action of $\mathrm{mon}^n_B(\pi_1(B, o))$ (equivalently there is no meromorphic function on $M_{DR}(X/B, n)^{\mathrm{an}}$ which is ∇^n_{DR}-invariant);*

(ii) *In the case of $M_B(X_o, n)$, considered with its natural structure of an affine algebraic variety, there exists a point $x_B \in M_B(X_o, n)$ so that the orbit*

$$\mathrm{mon}^n_B(\pi_1(B, o)) \cdot x_B \subset M_B(X_o, n)$$

is Zariski dense in $M_B(X_o, n)$; or equivalently, in the case of $M_{DR}(X/B, n) \to B$ there is a leaf of the foliation defined by ∇^n_{DR} which is Zariski dense in the algebraic Zariski topology.

This theorem suggests that for families $f : X \to B$ with a "large enough" geometric monodromy one should expect Zariski dense monodromy actions on non-abelian cohomology. Geometrically families with large monodromy naturally arise from hyperplane sections and Lefschetz fibrations. In this context we prove the following

Theorem B *Let Z be a smooth projective surface with $b_1(Z) = 0$. Let $\mathcal{O}_Z(1)$ be an ample line bundle on Z and let $n > 1$ be a fixed integer. Then there exists a positive integer*

ℓ (depending only on Z and $\mathcal{O}_Z(1)$), such that for every $k \geq \ell$ and for every Lefschetz fibration $f : \widehat{Z} \to \mathbb{P}^1$ in the linear system $|\mathcal{O}_Z(k)|$ we have:

(i) *There is no meromorphic function on $M_B(Z_o, n)^{\mathrm{an}}$ which is invariant under the action of $\mathrm{mon}_B^n(\pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_\mu\}, o))$ (equivalently there is no meromorphic function on $M_{DR}(\widehat{Z}/\mathbb{P}^1 \setminus \{p_1, \ldots, p_\mu\}, n)^{\mathrm{an}}$ which is ∇_{DR}^n-invariant);*

(ii) *In the case of $M_B(Z_o, n)$, there exist a point $x_B \in M_B(Z_o, n)$ so that the orbit*

$$\mathrm{mon}_B^n(\pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_\mu\}, o)) \cdot x_B \subset M_B(Z_o, n)$$

is Zariski dense in $M_B(Z_o, n)$; or equivalently, in the case of the space $M_{DR}(\widehat{Z}/\mathbb{P}^1 \setminus \{p_1, \ldots, p_\mu\}, n)$ the foliation defined by ∇_{DR}^n has a Zariski dense leaf.

Here as usual $\widehat{Z}$ is the blow-up of Z at the base points of the pencil and $p_1, \ldots p_\mu \in \mathbb{P}^1$ are the points where the map $\widehat{Z} \to \mathbb{P}^1$ is not submersive.

These statements can be viewed as nonabelian analogues of Deligne's irreducibility theorem [Deligne80, Section 4.4], which asserts that the monodromy group on the first cohomology of a Lefschetz pencil of curves is a subgroup of finite index in the full symplectic group of the lattice of vanishing cycles..

The paper is organized as follows. In Section 2.1 we examine the action of a finitely generated group on an affine algebraic variety. We show how the existence of a Zariski dense orbit can be deduced from the existence of an open orbit for the linearized action on the tangent space at a fixed point. Section 2.2 describes a particular point in the moduli space of representations of the fundamental group of a curve X_o, which corresponds to the Schrödinger representation of a suitably chosen finite Heisenberg group. This point is smooth and fixed by a subgroup of finite index in the monodromy. Moreover the tangent space of the moduli of representations at the 'Schrödinger point' is naturally identified with the first cohomology of an abelian etale cover Y_o of X_o. Section 2.3 recalls a powerful result of Looijenga which gives sufficient conditions for a subgroup in the mapping class group of X_o to map to a subgroup of finite index in the group $\mathrm{Sp}_{\mathrm{Gal}(Y_o/X_o)}(H_1(Y_o, \mathbb{Z}))$ of all symplectic transformations of $H_1(Y_o, \mathbb{Z})$ that commute with the action of the covering group of $Y_o \to X_o$. In Section 3.1 we study the problem of existence of an open orbit for the action of $\mathrm{Sp}_{\mathrm{Gal}(Y_o/X_o)}(H_1(Y_o, \mathbb{C}))$ on $H_1(Y_o, \mathbb{C})$ and in Sections 3.2 and 3.3 we show why Looijenga's conditions are satisfied under the hypothesis of Theorem A and B respectively. Finally in Section 4.1 we discuss the necessity of the hypotheses of Theorem A and B for the existence of a dense monodromy orbit. We conjecture that the density holds under very mild assumptions and give some additional evidence supporting the conjecture.

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2 Preliminary reductions

We start with some very general results about linear group actions on algebraic varieties, which will allow us to localize at a point the Zariski density property of an action.

2.1 Open orbits and dense actions

Suppose Γ is a finitely generated discrete group acting on an irreducible affine algebraic variety M. We suppose that for every point $x \in M$, the Zariski closure of the orbit denoted $Z(\Gamma x) \subset M$, is a proper closed subset of M. Suppose that k is an algebraically closed ground field such that M and the action are defined over k. For any point $x \in M$ (defined over a big algebraically closed field K), let $k(x)$ denote the algebraically closed field of definition of x with respect to k.

Lemma 2.1 *Suppose for some generic geometric point $x \in M(K)$, the closure $Z_K(\Gamma y)$ is a proper subset of M defined over K. Then the same is true for every other generic geometric point. In particular if there is one point whose orbit is Zariski dense in M, the same is true of every other generic point's orbit.*

Proof. Choose a point x such that

$$\mathrm{trdeg}(k(x)/k) = \dim(M).$$

Let $Z = Z(\Gamma x)$ and let $k(Z)$ denote the field of definition of Z.

Going to a subgroup of finite index in Γ we can assume that Z is (geometrically) irreducible. Noting that $k(x)$ is the field of definition of x as a point in the $k(Z)$-variety Z, we have

$$\mathrm{trdeg}(k(x)/k(Z)) \leq \dim(Z),$$

and so

$$\dim(Z) \cdot \mathrm{trdeg}(k(Z)/k) \geq \dim(M).$$

Let $f_1, \ldots, f_k$ be generators of the ideal of Z; they lie in the ring

$$k(Z)[M] := k[M] \otimes_k k(Z).$$

For a set of generators $\{\gamma\}$ of Γ, we have

$$\gamma f_i = \sum u_{ij}(\gamma) f_j$$

with $u_{ij}(\gamma) \in k(Z)[M]$. Choose a ring $A \subset k(Z)$ of finite type over k such that f_i and u_{ij} have coefficients in A and such that the above equations are true in $A[M]$. Let $S := \mathrm{Spec}(A)$. Then S is a scheme of finite type defined over k and the equations $f_i \in A[M]$ define a family

$$Z_S \subset S \times M.$$

By further localising A we may assume that the fibers Z_s for $s \in S$ are irreducible subvarieties of M all having the same dimension. There is a prefered generic point $q \in S$ which has field of definition $k(q) = k(Z)$ and with $\overline{Z_q} = Z$. Note also that by the equations we have assumed are true, the fibers Z_s are all closed Γ-invariant subsets of M.

The family Z_S is defined over k. By assumption its image in M contains at least one generic geometric point; therefore its image contains all generic geometric points and this shows that for any generic geometric point $y \in M$, the closure of the orbit $Z(\Gamma y)$ is a proper closed subset. Indeed if y is the image of a point $z \in Z_S$ then $Z(\Gamma y) \subset Z_t$ where t is the image of z in S.

A specialization argument shows (under the hypotheses of the previous paragraph) that for any point $y \in M$ (generic or not) the closure of the orbit is contained in a proper closed subset. The lemma is proven. □

We are now ready to prove the main result of this section

Theorem 2.2 *Suppose that Γ acts on an irreducible affine algebraic variety M. There are two possibilities: either*

(1) *there exists a nonconstant Γ-invariant meromorphic function; or*

(2) *there exists a point $x \in M$ with Γx Zariski-dense in M.*

Proof. Assume that there is a generic geometric point x with $Z_K(\Gamma x)$ a proper closed subset of M_K. Let Z_S be the Γ-invariant family defined over k that we constructed in the proof of Lemma 2.1.

Let $W \subset Z_S$ denote the set of all points which are contained in two or more distinct fibers Z_{s_1} and Z_{s_2} (i.e. two fibers with $Z_{s_1} \neq Z_{s_2}$). We claim that this is a constructible subset defined over k. To see this, look at the incidence subvariety of pairs of points whose images in M coincide, as a subvariety in $Z_S \times Z_S$. It is a closed subvariety. On the other hand the subset of $S \times S$ representing the condition that two fibers are equal as subsets of M, is constructible (see this by taking a compactification of M and compactifying the family then using Chow schemes), so the complement of the preimage of this subset, in the incidence variety, is constructible; the subset W is the projection of (the incidence variety minus the preimage of this subset of $S \times S$) on one of the factors Z_S so W is constructible; and the whole construction is defined over k.

Next we claim that W does not contain our original point x thought of as a point in the fiber over the original $q \in S$. For if it did, this would mean that there was a distinct $Z_s \neq Z_q$ containing x, then x would be contained in the Γ-invariant set $Z_s \cap Z_q$; but this

latter set is a proper subset of Z_q, contradicting the fact (by definition of our family) that Γx is Zariski-dense in Z_q.

It follows that there is an open set of the total space Z_S which does not meet W. Note that it is clear from the definition that W is the inverse image of a constructible subset in M; thus this subset does not contain the generic point of M so there is a closed set C such that W is contained in the preimage of C in Z_S. Since W is a Γ-invariant set, we may replace C here by the intersection of all of its translates so we can assume that C is Γ-invariant; finally then we can throw C out of M (i.e. replace M by $M - C$ in the whole discussion) so we may assume that W is empty.

By taking a closure of M and looking at the Chow scheme of subvarieties of this closure, we can replace our family by a family indexed by a new base scheme S where each fiber (considered as a subset of M) occurs exactly once.

Now by the above reduction the morphism $Z_S \to M$ is injective on points; also its image hits a generic point of M. Thus there is a largest open subset of M over which this is an isomorphism and we can replace M by this open subset (which is Γ-invariant). Hence we obtain a Γ-invariant fibration $M \to S$. Now any meromorphic function on S pulls back to give a Γ-invariant meromorphic function on M. This essentially proves the theorem. The only problem we need to address is that in the construction of the family $Z_S \to S$ we had to pass to a finite index subgroup of Γ and so the function just constructed may be invariant only under a subgroup of finite index of Γ. This however is easily remedied - by taking the different invariant polynomials in the Galois translates of our meromorphic function we obtain a meromorphic function invariant by the full Γ. □

The essential consequence of this theorem that we need is the following localization statement.

Corollary 2.3 *Suppose that $y \in M$ is a point in the smooth set of M, fixed by the action, so that Γ acts linearly on the tangent space $T(M)_y$. Let $G \subset GL(T(M)_y)$ be the Zariski closure of* $\mathrm{im}[\Gamma \to GL(T(M)_y)]$. *Assume that G acts on $T(M)_y$ with an open orbit, and that the connected component G^o of G has no nontrivial characters. Then there exists a point $x \in M$ with Γx Zariski-dense in M.*

Proof: By the previous theorem, we have to rule out the possibility that M admits a nonconstant Γ-invariant meromorphic function. Also from the proof of Theorem 2.2 we see that replacing Γ by a subgroup of finite index will not affect the alternatives (1) and (2). So without a loss of generality we may assume that $G = G^o$.

Suppose that h is such a function, and write the germ of h at y as f/g with $f, g \in \mathcal{O}_{M,y}$ relatively prime. Then for any $\gamma \in \Gamma$,

$$\frac{f}{g} = h = \gamma^* h \frac{\gamma^* f}{\gamma^* g},$$

which implies that there is a unit $u(\gamma) \in \mathcal{O}_{M,y}^{\times}$ with

$$\gamma^* f = u(\gamma) f,$$
$$\gamma^* g = u(\gamma) g.$$

One can note that $\gamma \mapsto u(\gamma)$ is a cocycle for Γ acting on the multiplicative group of units $\mathcal{O}^{\times}_{M,y}$. In particular we get that the value $u(\gamma)(y)$ is a character of Γ.

Without a loss of generality we may assume that h is not an invertible function at o (otherwise subtract off something) hence we may assume that one of f or g has a nontrivial leading term of some degree. We may suppose that f has such (otherwise replace h by h^{-1}). Let f_m be the leading term of f. Note that f_m (of degree m)—modulo higher order terms—is an element of $\mathrm{Sym}^m T^{\vee}(M)_y$. The action of Γ on this leading term factors through the group G. Our previous formula gives

$$\gamma^*(f_m) = u(\gamma)(y) f_m.$$

This shows that $\gamma \mapsto u(\gamma)(y)$ comes from a character of G; but by assumption there are no such characters Therefore we get $\gamma^*(f_m) = f_m$, so f_m is a G-invariant homogeneous form of degree m.

In particular, we can think of f_m as a G-invariant polynomial function on the tangent space $T(M)_y$. This contradicts the supposed existence of an open orbit in the action of G on $T(M)_y$, giving the desired contradiction. □

It is clear that the argument we used to prove Corollary 2.3 applies equally well to the analytic situation. More precisely the same reasoning gives the following

Corollary 2.4 *Suppose that N is an irreducible analytic space on which a finitely generated group Γ acts by analytic automorphisms. Suppose that $y \in N$ is a point in the smooth set of N, fixed by the action, so that Γ acts linearly on the tangent space $T(N)_y$. Let $G \subset GL(T(N)_y)$ be the Zariski closure of* $\mathrm{im}[\Gamma \to GL(T(N)_y)]$. *Assume that G acts on $T(N)_y$ with an open orbit, and that the connected component G^o of G has no nontrivial characters. Then there is no Γ-invariant analytic-meromorphic function on N*

Theorem 2.2 and Corollaries 2.3 and 2.4 not only provide a convenient localization criterion for the density of an action but also suggest another geometric notion of 'largeness' of the Γ-orbits. Motivated by Theorem 2.2, we define various degrees of analytic generic Zariski denseness as follows:

Definition 2.5 *Suppose that a finitely generated group Γ acts by analytic automorphisms on an irreducible analytic space N. Let $m : \Gamma \times N \to N$ be the action map.*

- *We say that the action m is* **AGZD1** *if there is no Γ-invariant analytic meromorphic function f on N.*
- *We say that the action m is* **AGZD2** *if there is no pair (U, f), where $U \subset N$ is an analytically Zariski dense open subset of N and $f : U \to Z$ is a holomorphic map from U to a complex analytic space Z with* $\dim Z < \dim N$.

- *We say that the action m is* **AGZD3** *if there is a point $x \in N$ such that $m(\Gamma)x$ is analytically Zariski-dense in N.*
- *We say that the action m is* **AGZD4** *if there is an analytically Zariski open subset $U \subset N$ such that for every $x \in U$ the $m(\Gamma)$ orbit of x is analytically Zariski-dense in N.*

Clearly for an analytic action m one has the implications:

$$\mathbf{AGZD4} \Rightarrow \mathbf{AGZD3} \Rightarrow \mathbf{AGZD2} \Rightarrow \mathbf{AGZD1},$$

but we don't think that the converse implications are true. Similarly, it is clear that if m is actually an algebraic action, then **AGZD1** implies that m is Zariski-dense in the algebraic sense of Theorem 2.2.

Suppose now that B is a base scheme and that $p : M \to B$ is a morphism equipped with a connection ∇ (by which we mean a stratification over the crystalline site of S [Grothendieck68], [Simpson95]). For the following definition it is not necessary to assume that ∇ is integrable.

Definition 2.6 *Suppose that B and the generic geometric fiber of M/B are irreducible. We say that $(p : M \to B, \nabla)$ is* generically Zariski dense *(or* **GZD***) if there is no algebraic meromorphic function f on the total space M which is invariant under ∇.*

If the connection ∇ is integrable, then the corresponding analytic family is associated to a local system of complex analytic spaces over B, which in turn corresponds to the monodromy action m of $\Gamma := \pi_1(B, o)$ on a fiber M_o^{an}. It is clear that if m is **AGZD1**, then $(p : M \to B, \nabla)$ is **GZD**.

In particular, the **AGZD1** property for $M_B(X_o, n)$ or equivalently $M_{DR}(X_o, n)$ implies the algebraic generic Zariski-denseness property **GZD** for the Gauss-Manin connection ∇^n_{DR}.

Consider now a family of smooth projective connected curves $f : X \to B$ and let $o \in B$ be a base point. We will show that when the geometric monodromy of $f : X \to B$ is of finite index in the mapping class group or when f comes from a Lefschetz pencil as in Theorem B, then the monodromy action of $\pi_1(B, o)$ on $M_B(X_o, n)$ is **AGZD1**. In combination with Theorem 2.2 this fact yields statement (i) of Theorems A and B. As explained above this automatically gives the analytic statement (ii) in both theorems. In fact, it follows from the above considerations that $M_{DR}(X/B, n) \to B$ together with the non-abelian Gauss-Manin connection is **GZD** in the sense of Definition 2.6.

In view of all this it only remains to show that under the hypothesis of Theorems A and B the algebraic action

$$\mathrm{mon}^n_B : \pi_1(B, o) \to \mathrm{Aut}(M_B(X_o, n)).$$

on the affine variety $M_B(X_o, n)$ is **AGZD1**. (Note that since X_o is a smooth curve the variety $M_B(X_o, n)$ is irreducible by [Simpson95, Section 11].) In view of Corollary 2.3 to achieve this

we only need to find a smooth point $\rho \in M_B(X_o, n)$ which is fixed by the monodromy group $\mathrm{mon}_B^n(\pi_1(B,o))$ and for which the Zariski closure of $\mathrm{mon}_B^n(\pi_1(B,o)) \subset GL(T(M_B(X_o,n))_\rho)$ acts on $T(M_B(X_o,n))_\rho$ with an open orbit.

In the next section we describe a proposal for such a point ρ which utilizes the Schrödinger representation of a finite Heisenberg group. Later on, we will show in Sections 3.2 and 3.3 that the open orbit property for the monodromy action on the tangent space at ρ holds, provided that the geometric monodromy of $f : X \to B$ is large enough.

We have framed the additional properties **AGZD2-4** in order to pose the question: which of these properties hold for families whose monodromy has finite index in the mapping class group? For (sufficiently ample) Lefschetz pencils?

2.2 The Schrödinger representaion

Since ρ is supposed to be fixed by the monodromy a natural choice would be to take ρ to be the trivial representation of $\pi_1(X_o)$ in $\mathrm{GL}(n,\mathbb{C})$. However the trivial representation is a singular point of $M_B(X_o,n)$ and so is unsuitable for our purposes. On the other hand any representation

$$\rho : \pi_1(X_o) \to \mathrm{GL}(n,\mathbb{C})$$

which has finite image will be fixed under some finite index subgroup of $\mathrm{mon}_B^n(\pi_1(B,o))$. Furthermore, the properties **AGZD1-AGZD4** and **GZD** are obviously stable under passage to a finite index subgroup of Γ. Hence we are free to replace B by any finite etale cover of B, and so it is enough to find a finite representation ρ which satisfies the open orbit condition.

In order to apply Corollary 2.3 we also need to choose $\rho \in M_B(X_o,n)$ to be a smooth point. This is equivalent to choosing ρ to be an irreducible representation.

To construct such a representation we proceed as follows. Let $\boldsymbol{\mu}_n \subset \mathbb{C}^\times$ be the group of all n-th roots of unity. Let $\widehat{\mathbb{Z}/n} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{C}^\times)$ denote the group of characters of the cyclic group $\mathbb{Z}/n$. Consider the finite Heisenberg group H_n. By definition H_n is the central extension

$$0 \to \boldsymbol{\mu}_n \to H_n \to \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \to 0$$

corresponding to the cocycle $e : \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \to \boldsymbol{\mu}_n \subset \mathbb{C}^\times$, $e(a,\alpha) = \alpha(a)$. Explicitly H_n can be identified with the set $\boldsymbol{\mu}_n \times \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ with a group law given by

$$(\lambda; a, \alpha) \cdot (\lambda'; a', \alpha') = (\lambda\lambda'\alpha'(a); a + a', \alpha\alpha').$$

Let $\phi_n : H_n \to GL(V_n)$ be the Schrödinger representation of H_n. By definition ϕ_n is the unique n-dimensional irreducible representation of H_n which has a tautological central character. One way to construct ϕ_n is to observe that the natural injective map

$$\boldsymbol{\mu}_n \times \widehat{\mathbb{Z}/n} \hookrightarrow H_n, \qquad (\lambda, \alpha) \mapsto (\lambda; 0, \alpha)$$

is a group monomorphism, i.e H_n contains $\boldsymbol{\mu}_n \times \widehat{\mathbb{Z}/n}$ as an abelian subgroup. Let $\mathbb{T}$ be the one dimensional complex representation of $\boldsymbol{\mu}_n \times \widehat{\mathbb{Z}/n}$ which corresponds to the pullback of the tautological character of $\boldsymbol{\mu}_n$ under the projection $\boldsymbol{\mu}_n \times \widehat{\mathbb{Z}/n} \to \boldsymbol{\mu}_n$. In other words $\mathbb{T} = (\mathbb{C}, \tau)$ where $\tau : \boldsymbol{\mu}_n \times \mathbb{Z}/n \to \mathbb{C}^\times$ is given by $\tau(\lambda, \alpha) = \lambda$. In terms of $\mathbb{T}$ then we have

$$(V_n, \phi_n) = \mathrm{Ind}_{\boldsymbol{\mu}_n \times \widehat{\mathbb{Z}/n}}^{H_n}(\mathbb{T}).$$

Explicitly we can identify V_n with the vector space of all complex valued functions on the finite set $\mathbb{Z}/n$ and the action ϕ_n by the formula

$$[\phi_n(\lambda; a, \alpha)f](x) = \lambda\alpha(x)f(x + a),$$

for all $x \in \mathbb{Z}/n$, $f \in V_n$ and $(\lambda; a, \alpha) \in H_n$.

The irreducibility of the representation ϕ_n follows from Frobenius reciprocity or directly by noticing that V_n has a basis consisting of the characteristic functions of the elements in $\mathbb{Z}/n$ and that the subgroup $\mathbb{Z}/n \subset H_n$ acts transitively on the elements of this basis.

Recall next that if C is any smooth curve of genus g, then there is a surjective homomorphism $\pi_1(C) \twoheadrightarrow \boldsymbol{F}_g$ onto a free group of g generators. This homomorphisms is obtained by moding $\pi_1(C)$ out by the normal subgroup generated by the a-cycles for a standard basis in the first homology of C. In particular since H_n is generated by three elements we can find a surjective homomorphism $\pi_1(C) \to \boldsymbol{F}_g \to H_n$ as long as $g \geq 3$.

By hypothesis the genus of X_o is at least three and so we can find surjective homomorphism $\psi_n : \pi_1(X_o) \twoheadrightarrow H_n$. Let $\rho : \pi_1(X_o) \to \mathrm{GL}(V_n)$ denote the composition $\rho := \psi_n \circ \phi_n$. By construction ρ is irreducible and so represents a smooth point of the moduli of representations. Moreover by a standard deformation theory argument (see e.g. [Lubotzky-Magid85]) we can identify the Zariski tangent space $T(M_B(X_o, n))_\rho$ with the group cohomology $H^1(\pi_1(X_o), \mathrm{ad}(\rho))$, where

$$\mathrm{ad}(\rho) : \pi_1(X_o) \to \mathrm{GL}(\mathrm{End}(V_n))$$

is the natural representation induced from ρ. Explicitly $\mathrm{ad}(\rho) = (\phi_n^\vee \otimes \phi_n) \circ \psi_n$ and since $\mathrm{ad}(\phi_n) = \phi_n^\vee \otimes \phi_n$ has a trivial central character we see that $\mathrm{ad}(\rho)$ factors trough the abelian group $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$.

It is not hard to calculate $H^1(\pi_1(X_o), \mathrm{ad}(\rho))$ in terms of geometric data on the curve X_o.

Lemma 2.7 *The tangent space to $M_B(X_o, n)$ at the Schrödinger representation ρ is given by*

$$T(M_B(X_o, n))_\rho = H^1(X_o, \mathrm{ad}(\rho)) = \oplus_{\chi \in \widehat{\mathbb{Z}/n} \times \mathbb{Z}/n} H^1(X_o, \mathbb{L}_\chi) \tag{2.1}$$

where $\mathbb{L}_\chi$ is the rank one local system on X_o corresponding to the character

$$\pi_1(X_o) \xrightarrow{\rho} H_n \to \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \xrightarrow{\chi} \mathbb{C}^\times.$$

Proof. Let

$$\mathrm{ad}(\phi_n)^{\mathrm{ab}} : \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \to \mathrm{GL}(\mathrm{End}(V_n))$$

denote the abelianization of the representation $\mathrm{ad}(\phi_n)$. The representation $\mathrm{ad}(\phi_n)^{\mathrm{ab}}$ fits in the commutative diagram

$$\begin{array}{ccc} \pi_1(X_o) & \xrightarrow{\mathrm{ad}(\rho)} & \mathrm{GL}(\mathrm{End}(V_n)) \\ {\scriptstyle \psi_n}\downarrow & & \uparrow{\scriptstyle \mathrm{ad}(\phi_n)^{\mathrm{ab}}} \\ H_n & \longrightarrow & \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \end{array}$$

and so understanding $\mathrm{ad}(\rho)$ is equivalent to understanding $\mathrm{ad}(\phi_n)^{\mathrm{ab}}$. But $\mathrm{ad}(\phi_n)^{\mathrm{ab}}$ is just the regular representation of the abelian group $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$. To see this note first that $\dim \mathrm{End}(V_n) = n^2 = \dim \mathbb{C}[\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}]$. Now since every irreducible representation of $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ occurs in $\mathbb{C}[\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}]$ with multiplicity one we need only to check that for every character $\chi : \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \to \mathbb{C}^\times$ we have

$$\dim \mathrm{Hom}_{(\mathbb{Z}/n \times \widehat{\mathbb{Z}/n})-\mathrm{mod}}(\chi, \mathrm{ad}(\phi_n)^{\mathrm{ab}}) \geq 1.$$

But the group of characters of $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ is naturally isomorphic to $\widehat{\mathbb{Z}/n} \times \mathbb{Z}/n$ and so each character χ as above is given by a pair $(\xi, x) \in \widehat{\mathbb{Z}/n} \times \mathbb{Z}/n$ via the formula $\chi(a, \alpha) = \xi(a) \cdot \alpha(x)$. Therefore we only need to show that for any pair (ξ, x) there exists a non zero element $A_{(\xi,x)} \in \mathrm{End}(V_n) = V_n^\vee \otimes V_n$ so that

$$(a, \alpha) A_{(\xi,x)} = \xi(a)\alpha(x) A_{(\xi,x)}$$

for all $(a, \alpha) \in \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$.

To construct the element $A_{(\xi,x)}$ recall that the vector space $V_n = \mathbb{C}[\mathbb{Z}/n]$ has a natural basis $\{e_0, e_1, \ldots, e_{n-1}\}$ consisting of characteristic functions of elements of $\mathbb{Z}/n$, i.e. $e_i(j) := \delta_{ij}$. Let $\{e_0^\vee, e_1^\vee, \ldots, e_{n-1}^\vee\}$ be the dual basis of $V_n^\vee$. Then in terms of the basis $\{e_i^\vee \otimes e_j\}_{i,j=0}^{n-1}$ of $V_n^\vee \otimes V_n$ the representation $\mathrm{ad}(\phi_n)^{\mathrm{ab}}$ is given by the formula

$$[\mathrm{ad}(\phi_n)^{\mathrm{ab}}(a, \alpha)](e_i^\vee \otimes e_j) = \alpha(j - i) e_{i-a}^\vee \otimes e_{j-a}.$$

In view of this we may take

$$A_{(\xi,x)} := \sum_{i=0}^{n-1} \xi(i) e_i^\vee \otimes e_{i+x}$$

which is obviously a non-zero eigenvector corresponding to the character $\chi = (\xi, x)$.

This shows that

$$(\mathrm{End}(V_n), \mathrm{ad}(\phi_n)^{\mathrm{ab}}) = \mathbb{C}[\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}] = \oplus_{\chi \in \widehat{\mathbb{Z}/n} \times \mathbb{Z}/n} (\mathbb{C}, \chi),$$

and so the lemma is proven. □

Let us now go back to the problem of checking whether $\mathrm{mon}(\pi_1(B,o))$ has a dense orbit on $M_B(X_o,n)$. As mentioned above, the fact that ρ has a finite image implies that the conjugacy class of ρ will be fixed by some finite-index subgroup of $\mathrm{mon}(\pi_1(B,o))$. In particular, applying Corollary 2.3 to this subgroup, it follows that in order to show the existence of a dense $\mathrm{mon}(\pi_1(B,o))$-orbit on $M_B(X_o,n)$, it suffices to check the following two items:

(i) The Zariski closure G of

$$\mathrm{im}[\mathrm{mon}(\pi_1(B,o)) \to GL(H^1(X_o,\mathrm{ad}(\rho)))]$$

in $GL(H^1(X_o,\mathrm{ad}(\rho)))$ has an open orbit on $H^1(X_o,\mathrm{ad}(\rho))$.

(ii) The identity component G^o of G does not have any non-trivial characters.

Condition **(ii)** follows easily from the isomorphism (2.1):

Lemma 2.8 *The identity component G^o of the Zariski closure of*

$$\mathrm{im}[\mathrm{mon}(\pi_1(B,o)) \to GL(H^1(X_o,\mathrm{ad}(\rho)))]$$

in $GL(H^1(X_o,\mathrm{ad}(\rho)))$ has no non-trivial characters.

Proof. Indeed, let $p : Y_o \to X_o$ be the abelian Galois cover of X_o corresponding to the surjection

$$\pi_1(X_o) \stackrel{\rho}{\to} H_n \to \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}. \tag{2.2}$$

Then Y_o is a smooth connected curve and the pushforward of the trivial one dimensional local system $\mathbb{C}_{Y_o}$ via p is precisely

$$p_*\mathbb{C}_{Y_o} = \oplus_{\chi \in \widehat{\mathbb{Z}/n} \times \mathbb{Z}/n} \mathbb{L}_\chi.$$

Thus we can identify $H^1(X_o,\mathrm{ad}(\rho))$ with $H^1(Y_o,\mathbb{C})$.

Next observe that without a loss of generality we may assume that the family $f : X \to B$ has an algebraic section $\sigma : B \to X$. Indeed, since X is quasi-projective a generic hyperplane section on X will be a multisection of f. But replacing B by an étale cover of a Zariski open set of B will replace $\mathrm{mon}(\pi_1(B,o))$ by a subgroup of finite index. In particular such areplacement will not affect the property that $\mathrm{mon}(\pi_1(B,o))$ fixes the conjugacy class of ρ. In fact, by taking another étale cover if necessary we can ensure that not only the conjugacy class of ρ is fixed under $\mathrm{mon}(\pi_1(B,o))$ but that the actual representation

$$\rho : \pi_1(X_o,\sigma(o)) \to GL(V_n)$$

remains fixed under $\mathrm{mon}(\pi_1(B,o))$.

Assume that we are in this situation. Then ρ lifts to a well defined representation of $\pi_1(B,o) \ltimes_{\mathrm{mon}} \pi_1(X_o,\sigma(o))$ and so defines a $\mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$-cover of X. In particular under the identification $H^1(X_o,\mathrm{ad}(\rho)) \cong H^1(Y_o,\mathbb{C})$ the action of $\mathrm{mon}(\pi_1(B,o))$ on $H^1(X_o,\mathrm{ad}(\rho))$ becomes just the monodromy action of $\pi_1(B,o)$ on $H^1(Y_o,\mathbb{C})$ corresponding to the family of curves $Y \to B$. But by Deligne's semisimplicity theorem [Deligne72, Corollaire 4.2.9] the monodromy action on the middle dimensional cohomology of any smooth projective family over a quasi-projective base has a semisimple Zariski closure. Thus G^o must be a connected semisimple algebraic group and so has no non-trivial characters. The lemma is proven. $\square$

In order to check that condition **(i)** is satisfied we need to invoke a powerful result of Looijenga [Looijenga97, Theorem 2.5], which will be done in the next section.

Before we explain Looijenga's theorem we need to introduce some notation. On the way we will also rephrase the condition **(i)** in a slightly more general context.

Let $f : X \to B$ be a smooth family of connected curves of genus $g \geq 3$. Let as before $o \in B$ be a fixed base point and let

$$\mathrm{mon} : \pi_1(B,o) \to \mathrm{Map}(X_o) \subset \mathrm{Out}(\pi_1(X_o))$$

be the corresponding geometric monodromy representation. Here $\mathrm{Map}(X_o)$ denotes the mapping class group of X_o. By definition $\mathrm{Map}(X_o)$ is the group of connected components of the group of orienation preserving diffeomorphisms of X_o. Also, $\mathrm{Map}(X_o)$ can be indentified with the index two subgroup of $\mathrm{Out}(\pi_1(X_o))$, consisting of all outer automorphisms which act trivially on $H^2(\pi_1(X_o),\mathbb{Z})$.

If $f : X \to B$ has a holomorphic section $\sigma : B \to X$ the representation $\mathrm{mon} : \pi_1(B,o) \to \mathrm{Map}(X_o)$ can be lifted to a geometric monodromy representation respecting the base point:

$$\mathrm{mon}^\sigma : \pi_1(B,o) \to \mathrm{Map}^1(X_o) \subset \mathrm{Aut}(\pi_1(X_o,\sigma(o))).$$

Here $\mathrm{Map}^1(X_o)$ denotes the mapping class group of the once punctured surface $X_o - \{\sigma(o)\}$.

Fix a finite abelian quotient $H_1(X_o,\mathbb{Z}) \twoheadrightarrow A$ and let $p^A : Y_o^A \to X_o$ be the corresponding Galois cover. Let $\mathrm{Map}(X_o,A)$ be the group of p-liftable mapping classes, i.e.

$$\mathrm{Map}(X_o,A) = \left\{ \varphi \in \mathrm{Map}(X_o) \,\middle|\, \begin{array}{l} \varphi \text{ preserves } \ker[H_1(X_o,\mathbb{Z}) \twoheadrightarrow A] \text{ and } \varphi \text{ induces} \\ \text{the identity on } A \end{array} \right\}$$

In exactly the same way we can define a subgroup $\mathrm{Map}^1(X_o,A) \subset \mathrm{Map}^1(X_o)$.

Clearly $\mathrm{Map}(X_o,A) \subset \mathrm{Map}(X_o)$ is of finite index and consists precisely of the mapping classes on X_o which lift to mapping classes on Y_o^A. Furthermore, since by definition each $\varphi \in \mathrm{Map}(X_o,A)$ induces the identity on A it follows that any lift $\tilde{\varphi} \in \mathrm{Map}(Y_o^A)$ of φ commutes with the action of A on Y_o^A, and that A acts transitively on the set of all such lifts. Thus, if we define $\mathrm{LMap}(X_o,A) \subset \mathrm{Map}(Y_o^A)$ to be the subgroup consisting of all lifts of elements in $\mathrm{Map}(X_o)$ we see that $\mathrm{LMap}(X_o,A)$ fits in a short exact sequence of groups

$$1 \to A \to \mathrm{LMap}(X_o,A) \to \mathrm{Map}(X_o,A) \to 1.$$

Assume now that $\mathrm{mon}(\pi_1(B,o)) \subset \mathrm{Map}(X_o, A)$. Since the natural homomorphism

$$\mathrm{Aut}(\pi_1(X_o, \sigma(o))) \to GL(H_1(X_o, \mathbb{Z}))$$

factors through $\mathrm{Out}(\pi_1(X_o))$ it follows that we will also have $\mathrm{mon}^\sigma(\pi_1(B,o)) \subset \mathrm{Map}^1(X_o, A)$. In particular $\mathrm{mon}^\sigma(\pi_1(B,o))$ preserves $\pi_1(Y_o^A)$ and so we get a short exact sequence of groups

$$1 \to \pi_1(B,o) \times_{\mathrm{mon}^\sigma} \pi_1(Y_o^A) \to \pi_1(X) \to A \to 0.$$

Let $Y^A \to X$ denote the A-Galois cover of X corresponding to the homomorphism $\pi_1(X) \twoheadrightarrow A$. By construction $\pi_1(Y^A) \cong \pi_1(B,o) \times_{\mathrm{mon}^\sigma} \pi_1(Y_o^A)$ and the corresponding monodromy representation

$$\mathrm{mon}_A : \pi_1(B,o) \to \mathrm{Map}(Y_o^A) \subset \mathrm{Out}(\pi_1(Y_o^A))$$

lands in $\mathrm{LMap}(X_o, A)$.

Furthermore, note that from the viewpoint of the density properties we are interseted in, the conditions that $f : X \to B$ has a section and that $\mathrm{mon}(\pi_1(B,o)) \subset \mathrm{Map}(X_o, A)$ are harmless. Indeed, as explained in the proof of Lemma 2.8, if $f : X \to B$ is an arbitrary smooth projective family of curves with B smooth and connected and X quasi-projective, then we can always find a Zariski open set $U \subset B$ containing the point $o \in B$, and a finite étale cover $(B', o') \to (U, o)$, so that the pulled-back family $X \times_B B' \to B'$ has a holomorphic section, and a geometric monodromy which is contained in $\mathrm{Map}(X_o, A)$. Since $\pi_1(U,o) \twoheadrightarrow \pi_1(B,o)$ is surjective and $\pi_1(B',o') \subset \pi_1(U,o)$ is a subgroup of finite index, it follows that the geometric monodromy $\mathrm{mon}(\pi_1(B',o'))$ of the family $X \times_B B' \to B'$ is a subgroup of finite index in $\mathrm{mon}(B,o)$. In particular, any density statement we can make for the action of $\pi_1(B,o)$ will be equivalent to the corresponding density statement for the action of $\pi_1(B',o')$.

The previous reasoning also shows that for any smooth family of curves $f : X \to B$, such that B is smooth and X is quasi-projective, and any surjective homomorphism $H_1(X_o, \mathbb{Z}) \twoheadrightarrow A$, there is an appropriate $(B',o') \to (B,o)$ and an A-Galois cover $Y^A \to X \times_B B'$, so that:

- The image of the monodromy representation $\mathrm{mon}_A : \pi_1(B',o') \to \mathrm{Map}(Y_o^A)$ is contained in $\mathrm{LMap}(X_o, A)$;
- The natural map $\mathrm{mon}_A(\pi_1(B',o')) \to \mathrm{mon}(\pi_1(B,o)$ has finite kernel and cokernel.

Motivated by the discussion in Sections 2.1 and 2.2 we make the following definition:

Definition 2.9 *Let A be a finite abelian group. A pair $(f : X \to B, H_1(X_o, \mathbb{Z}) \twoheadrightarrow A)$ is called* good *if the Zariski closure of*

$$\mathrm{im}\left[\pi_1(B',o') \overset{\mathrm{mon}_A}{\to} \mathrm{LMap}(X_o, A) \subset \mathrm{Map}(Y_o^A) \to \mathrm{Sp}(H_1(Y_o^A, \mathbb{Z}))\right]$$

in $\mathrm{Sp}(H_1(Y_o^A, \mathbb{C}))$ acts on $H_1(Y_o^A, \mathbb{C})$ with an open orbit.

Clearly now, the condition **(i)** is equivalent to the statement that if $A = \mathbb{Z}/n \times \widehat{\mathbb{Z}/n}$ and if the homomorphism $\pi_1(X_o) \to H_1(X_o, \mathbb{Z}) \to A$ is induced from a surjective homomorphism $\pi_1(X_o) \to H_n$, then the pair $(f : X \to B, H_1(X_o, \mathbb{Z}) \twoheadrightarrow A)$ is good.

In other words we need to find geometric restrictions on a family $f : X \to B$ and a homomorphism $H_1(X_o, \mathbb{Z}) \twoheadrightarrow A$, which will guarantee that the pair $(f : X \to B, H_1(X_o, \mathbb{Z}) \twoheadrightarrow A)$ is good.

As a first approximation one has to understand the image of $\mathrm{LMap}(X_o, A)$ into the symplectic group $\mathrm{Sp}(H_1(Y_o, \mathbb{Z})$. This is precisely the content of Looijenga's theorem [Looijenga97, Theorem 2.5].

2.3 A theorem of Looijenga

Let X_o be a smooth curve of genus $g \geq 3$ and let $H_1(X_o, \mathbb{Z}) \twoheadrightarrow A$ be a fixed homomorphism onto a finite abelian group A. Let $p^A : Y_o^A \to X_o$ be the corresponding A-Galois cover.

Consider the group

$$\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z})) \subset \mathrm{Sp}(H_1(Y_o^A, \mathbb{Z}))$$

consisting of all symplectic transformations of $H_1(Y_o^A, \mathbb{Z})$ which commute with the action of A. By definition the image of the natural map

$$\mathrm{LMap}(X_o, A) \to \mathrm{Sp}(H_1(Y_o^A, \mathbb{Z}))$$

is contained in $\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$. Also by definition, the subgroup $A \subset \mathrm{LMap}(X_o, A)$ maps to a central subgroup in $\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$ and so if we set

$$\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z})) := \mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))/Z(\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z})))$$

we get a well defined homomorphism:

$$\mathrm{Map}(X_o, A) \to \mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z})).$$

With this notation, Looijenga's theorem reads:

Theorem([Looijenga97, Theorem 2.5 and Corollary 2.6]) **(i)** *The image of the homomorphism*

$$\mathrm{Map}(X_o, A) \to \mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$$

is a subgroup of finite index in $\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$.

(ii) *There exists a subgroup* $\mathrm{Sp}_A^\sharp(H_1(Y_o^A, \mathbb{Z})) \subset \mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$ *which maps onto the group* $\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$ *with finite kernel. Furthermore* $\mathrm{LMap}(X_o, A)$ *maps into* $\mathrm{Sp}_A^\sharp(H_1(Y_o^A, \mathbb{Z}))$, *and the image of the homomorphism*

$$\mathrm{LMap}(X_o, A) \to \mathrm{Sp}_A^\sharp(H_1(Y_o^A, \mathbb{Z}))$$

is of finite index in $\mathrm{Sp}_A^\sharp(H_1(Y_o^A,\mathbb{Z}))$

The proof of this theorem is based on a detailed analysis of the structure of the arithmetic group $\mathrm{Sp}_A(H_1(Y_o^A,\mathbb{Z}))$ and of its relationship to the mapping class groups of the intermediate covers of $p^A : Y_o^A \to X_o$. In fact, in the process of proving this theorem, Looijenga isolates a fairly simple list of topological conditions that need to be imposed on a subgroup $\Gamma \subset \mathrm{Map}(X_o)$, so that it will map to a finite index subgroup in $\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A,\mathbb{Z}))$.

Our strategy in gauging the goodness of a pair $(f : X \to B, H_1(X_o,\mathbb{Z}) \twoheadrightarrow A)$ is to first show that $\mathrm{Sp}_A^\sharp(H_1(Y_o^A,\mathbb{C}))$ acts on $H_1(Y_o^A,\mathbb{C})$ with an open orbit, and then to find reasonable geometric restrictions on the pair, which will guarantee that the conditions from the Looijenga's list will be satisfied by the geometric monodromy of the family $f : X \to B$. To that end we recall briefly Looijenga's analysis of the structure of $\mathrm{Sp}_A(H_1(Y_o^A,\mathbb{Z}))$ and describe the conditions in Looijenga's list.

2.3.1. Skew-Hermitian modules. Following Looijenga [Looijenga97], the first step in understanding the arithmetic group $\mathrm{Sp}_A(H_1(Y_o^A,\mathbb{Z}))$ is to understand the almost-simple constituents of the semisimple algebraic group $\mathrm{Sp}_A(H_1(Y_o^A,\mathbb{C}))$.

For a finite abelian group A denote by $\mathfrak{C}(A)$ the set of all cyclic quotient groups of A. As usual two cyclic quotients $A \twoheadrightarrow C'$ and $A \twoheadrightarrow C''$ are considered to be the same iff

$$\ker[A \twoheadrightarrow C'] = \ker[A \twoheadrightarrow C''] \subset A.$$

A character of A is a homomorphism $\chi : A \to \overline{\mathbb{Q}}^\times \subset \mathbb{C}^\times$. Note that each character $\chi \in \widehat{A}$ extends uniquely to a ring homomorphism $\widetilde{\chi} : \mathbb{Z}[A] \to \overline{\mathbb{Q}} \subset \mathbb{C}$, given by $\widetilde{\chi}(\sum_{g\in A} c_g g) = \sum_{g\in A} c_g \chi(g)$. Clearly $\mathrm{im}(\chi) \subset \overline{\mathbb{Q}}$ is a cyclic subgroup. If $|\,\mathrm{im}(\chi)| = d$, then $\mathrm{im}(\chi) = \boldsymbol{\mu}_d \subset \overline{\mathbb{Q}}$ and $\mathrm{im}(\widetilde{\chi}) = \mathbb{Z}[\boldsymbol{\mu}_d] \subset \mathbb{Q}$. For any $\alpha \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have $\alpha(\chi) = \chi^k$ for some $k \in \mathbb{Z}$ with $(k,d) = 1$. Hence the $\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit of $\chi \in \widehat{A}$ consists of all generators of the cyclic subgroup $\langle\chi\rangle \subset \widehat{A}$. In particular both the kernel

$$I := \ker\left[\mathbb{Z}[A] \xrightarrow{\widetilde{\chi}} \overline{\mathbb{Q}}\right]$$

and the image $\mathrm{im}(\widetilde{\chi}) \subset \overline{\mathbb{Q}}$ are independent of the specific choice of $\chi \in \widehat{A}$, and depend only on the cyclic subgroup $\langle\chi\rangle \subset \widehat{A}$. Equivalently we may say that I and $\mathrm{im}(\widetilde{\chi})$ depend only on the choice of a cyclic quotient group $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$.

In other words, for any cyclic quotient $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$, we can define $I_C = \ker(\widetilde{\chi})$ for some generator $\chi \in \widehat{C} \subset \widehat{A}$ and $R_C = \mathbb{Z}[A]/I_C = \mathrm{im}(\widetilde{\chi}) \subset \overline{\mathbb{Q}} \subset \mathbb{C}$.

Consider the number field $K_C := R_C \otimes_{\mathbb{Z}} \mathbb{Q}$. By definition K_C is non-canonically isomorphi to the subfield of $\overline{\mathbb{Q}}$ generated by $\boldsymbol{\mu}_d \subset \overline{\mathbb{Q}}$, and R_C is just the ring of integers of K_C. We

have obvious homorphisms of rings

$$\begin{aligned} \mathbb{Z}[A] &\to \bigoplus_{[A\twoheadrightarrow C]\in\mathfrak{C}(A)} R_C \\ \mathbb{Q}[A] &\to \bigoplus_{[A\twoheadrightarrow C]\in\mathfrak{C}(A)} K_C . \end{aligned} \tag{2.3}$$

By the Chinese remainder theorem the second map is an isomorphism. Hence the first map is injective and has a finite cokernel.

To understand the field K_C a little better note that we have a natural involution '$\bar{\bullet}$', defined by the formula $\overline{\sum_{g\in A} c_g g} := \sum_{g\in A} c_g g^{-1}$. Clearly, on each K_C this involution induces the standard complex conjugation coming from $K_C \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. Let $K'_C := \{x \in K_C | \bar{x} = x\}$ denote the maximal real subfield of K_C. Note that $K'_C \cong \mathbb{Q}(\cos(2\pi/d))$ (non-canonically), and that $K'_C = K_C$ for $d = 1, 2$ and K_C is a purely imaginary quadratic extension of K'_C for $d > 2$.

Suppose now that W is an abelian group equipped with an action of A. For a cyclic quotient group $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$ we put

$$W_C := W \otimes_{\mathbb{Z}[A]} R_C .$$

Note that if W is a $\mathbb{Q}$-vector space, then W_C is naturally a K_C-vector space.

In particular, if W is a finitely generated free abelian group, then the map

$$W \otimes \mathbb{Q} \to \bigoplus_{[A\twoheadrightarrow C]\in\mathfrak{C}(A)} W_C \otimes \mathbb{Q} \tag{2.4}$$

is an isomorphism of $\mathbb{Q}$-vector spaces, which is equivariant with respect to the isomorphism of algebras $\mathbb{Q}[A] \cong \oplus_{C\in\mathfrak{C}(A)} K_C$. Similarly, the homomorphism

$$W \to \bigoplus_{[A\twoheadrightarrow C]\in\mathfrak{C}(A)} W_C$$

is an injective homomorphism of abelian groups, wich has a finite cokernel, and which is equivariant with respect to the morphism of rings $\mathbb{Z}[A] \to \oplus_{C\in\mathfrak{C}(A)} R_C$.

Assume now that W is a free abelian group of finite rank, equipped with an A-action and a non-degenerate $\mathbb{Q}$-valued symplectic form $(\bullet, \bullet)$ which is preserved by A. Then $(\bullet, \bullet)$ induces a $\mathbb{Q}[A]$-valued form $\langle \bullet, \bullet \rangle$, which is skew-Hermitian over $\mathbb{Z}[A]$. This means that $\langle \bullet, \bullet \rangle$ is $\mathbb{Z}[A]$-linear in its first argument and that $\overline{\langle x, y \rangle} = -\langle y, x \rangle$ for all $x, y \in W$. Explicitly

$$\langle x, y \rangle := \sum_{g\in A} (x, gy) g \in \mathbb{Q}[A].$$

After tensoring with R_C, the form $\langle \bullet, \bullet \rangle$ induces a K_C-valued Hermitian form $\langle \bullet, \bullet \rangle_C$ on W_C, defined over R_C.

Let $U(W_C)$ (respectively $U(W_C \otimes \mathbb{Q})$) be the group of R_C-linear automorphisms of W_C (respectively of K_C-linear automorphisms of $W_C \otimes \mathbb{Q}$) which preserve $\langle \bullet, \bullet \rangle_C$.

By construction $U(W_C \otimes \mathbb{Q})$ is the group of K_C'-rational points of a reductive algebraic group defined over K_C'. If $|C| = 1$ or 2, then $U(W_C) = \mathrm{Sp}(W_C, (\bullet, \bullet)_C)$. If $|C| \geq 3$, then $U(W_C)$ is the unitary group of the skew-Hermitian lattice $(W_C, \langle \bullet, \bullet \rangle_C)$.

With this setup we can now describe the structure of the group $\mathrm{Sp}_A(W)$ consisting of all isomorphisms of $(\bullet, \bullet)$ which commute with A. Observe that by construction the natural homomorphism

$$(W, \langle \bullet, \bullet \rangle) \to \oplus_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} (W_C, \langle \bullet, \bullet \rangle_C)$$

is a homomorphism of skew-Hermitian modules. In particular, from the above discussion it follows that

$$\mathrm{Sp}_A(W \otimes \mathbb{Q}) \to \prod_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} U(W_C \otimes \mathbb{Q}),$$

is an isomorphism, and that

$$\mathrm{Sp}_A(W) \to \prod_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} U(W_C)$$

is injective and has a finite cokernel.

We are now ready to describe the subgroup $\mathrm{Sp}_A^\sharp(W)$ which appears in the statement of Looijenga's theorem.

Let again W denote a finitely generated free abelian group equipped with an action of A. Let $T : W \otimes \mathbb{Q} \to W \otimes \mathbb{Q}$ be an A-equivariant endomorphism of the rational vector space $W \otimes \mathbb{Q}$. Then for every $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$ we get a K_C-linear endomorphism $T_C : W_C \otimes \mathbb{Q} \to W_C \otimes \mathbb{Q}$. Let $\det_{K_C}(T_C) \in K_C$ denote its K_C-determinant. Similarly we can consider the R_C-determinant of a R_C-linear automorphism of W_C. In particular we get a group homomorphism $\det_{R_C} : U(W_C) \to R_C^\times$ to the group of units in the ring R_C.

Let $SU(W_C) \subset U(W_C)$ (respectively $SU(W_C \otimes \mathbb{Q}) \subset U(W_C \otimes \mathbb{Q})$) denote the subgroup of unitary transformations of R_C-determinant (respectively K_C-determinant) one. By the Dirichlet unit theorem, the group $R_C^\times$ is a finitely generated abelian group of rank $\max(0, \phi(|C|)/2 - 1)$ (where ϕ is the Euler function) and with torsion equal to the group of roots of unity in R_C, which we may identify with C.

Let $C^{(2)}$ denote the group of squares in C and let $U^\sharp(W_C) := \det_{R_C}^{-1}(C^{(2)})$. We have a short eaxt sequence

$$1 \to SU(W_C) \to U^\sharp(W_C) \to C^{(2)} \to 1.$$

Define $\mathrm{Sp}_A^\sharp(W)$ (respectively $\mathrm{Sp}_A^\sharp(W \otimes \mathbb{Q})$) to be the preimage of $\prod_{C \in \mathfrak{C}(A)} U^\sharp(W_C)$ (respectively $\prod_{C \in \mathfrak{C}(A)} U^\sharp(W_C \otimes \mathbb{Q})$) in $\mathrm{Sp}_A(W)$ (respectively $\mathrm{Sp}_A(W \otimes \mathbb{Q})$). Clearly $\mathrm{Sp}_A^\sharp(W)$ surjects onto $\mathbb{P}\,\mathrm{Sp}_A(W) = \prod_{C \in \mathfrak{C}(A)} \mathbb{P}U(W_C)$ with a finite kernel.

In Section 3.1 we will analyze in more detail the complex group $\mathrm{Sp}_A^\sharp(W\otimes\mathbb{C})$ and will show that it acts on $W\otimes\mathbb{C}$ with an open orbit.

Before doing that however, we finish the summary of Looijenga's results by listing his sufficient conditions for a subgroup $\Gamma\subset \mathrm{Map}(X_o,A)$ to map to a subgroup of finite index in $\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A,\mathbb{Z}))$.

2.3.2. Looijenga's list. Let X_o be a smooth curve and let $H_1(X_o,\mathbb{Z})\twoheadrightarrow A$ be a fixed finite quotient. As we saw in the previous section, there exists an isogeny of arithmetic lattices:

$$\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A,\mathbb{Z}))\longrightarrow \prod_{[A\twoheadrightarrow C]\in\mathfrak{C}(A)}\mathbb{P}U(H_1(Y_o^A,\mathbb{Z})_C)$$

where $p^A: Y_o^A\to X_o$ is the A-Galois cover correponding to $H_1(X_o,\mathbb{Z})\twoheadrightarrow A$.

However, as Looijenga observes [Looijenga97, Section 2.1], the group $\mathbb{P}U(H_1(Y_o^A,\mathbb{Z})_C)$ is closely related to the similarly defined group $\mathbb{P}U(H_1(Y_o^C,\mathbb{Z})_C)$, where $p^C: Y_o^C\to X_o$ is the cover corresponding to $H_1(X_o,\mathbb{Z})\twoheadrightarrow A\twoheadrightarrow C$.

More precisely, if $A'=A/H$ is a quotient of A, then the Galois cover $p^{A'}: Y_o^{A'}\to X_0$ corresponding to $H_1(X_o,\mathbb{Z})\twoheadrightarrow A\twoheadrightarrow A'$, can be identified naturally with the quotient curve Y_o^A/H. The map $Y_o^A\to Y_o^{A'}$ induces an injective homomomorphism $H_1(Y_o^A,\mathbb{Z})\to H_1(Y_o^{A'},\mathbb{Z})$ with cokernel H. The homomorphism $H_1(Y_o^A,\mathbb{Z})\to H_1(Y_o^{A'},\mathbb{Z})$ is equivariant with respect to the surjection $A\twoheadrightarrow A'$ and so induces a $\mathbb{Z}[A']$-homomorphism

$$H_1(Y_o^A,\mathbb{Z})\otimes_{\mathbb{Z}[A]}\mathbb{Z}[A']\to H_1(Y_o^{A'},\mathbb{Z}), \tag{2.5}$$

which is injective and has cokernel H. Moreover the homomorphism (2.5) has degree $|H|$ in the sense that the natural $\mathbb{Z}[A']$ valued skew-Hermitian form on the left goes to $|H|$ times the natural $\mathbb{Z}[A']$ valued skew-Hermitian form on the right.

Applying this discussion to $A'=C$ for some $[A\twoheadrightarrow C]\in\mathfrak{C}(A)$, we conclude that

$$H_1(Y_o^A,\mathbb{Z})\otimes_{\mathbb{Z}[A]}\mathbb{Z}[C]\to H_1(Y_o^C,\mathbb{Z})$$

is an isogeny of skew-Hermitian modules defined over $\mathbb{Z}[C]$. Tensoring with R_C (over $\mathbb{Z}[C]$) and using the fact that $(H_1(Y_o^A,\mathbb{Z})\otimes_{\mathbb{Z}[A]}\mathbb{Z}[C])\otimes_{\mathbb{Z}[C]}R_C=H_1(Y_o^A,\mathbb{Z})\otimes_{\mathbb{Z}[A]}R_C=H_1(Y_o^A,\mathbb{Z})_C$ we get a well defined isogeny of skew-Hermitian R_C-modules:

$$H_1(Y_o^A,\mathbb{Z})_C\to H_1(Y_o^C,\mathbb{Z})_C.$$

In particular, the natural composition map

$$\mathrm{Sp}_A^\sharp(H_1(Y_o^A,\mathbb{Z}))\to\prod_{[A\twoheadrightarrow C]\in\mathfrak{C}(A)}U^\sharp(H_1(Y_o^A,\mathbb{Z})_C)\to\prod_{[A\twoheadrightarrow C]\in\mathfrak{C}(A)}U^\sharp(H_1(Y_o^C,\mathbb{Z})_C)$$

is an isogeny of arithmetic groups.

Therefore, in order for a group $\Gamma \subset \mathrm{Map}(X_o, A)$ to map to a subgroup of finite index in $\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$, it is necessary that Γ maps to a group of finite index in each $\mathbb{P}U(H_1(Y_o^C, \mathbb{Z})_C)$. Sufficient conditions for this to happen can be extracted from the proof of [Looijenga97, Theorem 2.4]:

Assume that $\Gamma \subset \mathrm{Map}(X_o, A)$ and that $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$ is fixed. Then necessarily $\Gamma \subset \mathrm{Map}(X_o, C)$. Assume in addition that Γ satisfies the following conditions

C(i) There exists a symplectic submodule $V \subset H_1(X_o, \mathbb{Z})$, such that

- $\mathrm{rank}(V) = 2g - 2$.
- $V \subset \ker[H_1(X_o, \mathbb{Z}) \twoheadrightarrow C]$.
- $Sp(V) \cap \mathrm{im}[\Gamma \to Sp(H_1(X_o, \mathbb{Z}))]$ is of finite index in $Sp(V)$.

C(ii) There exist embedded circles $E \neq E' \subset X_o$, such that

- E and E' are homologous (with the appropriate orientations).
- The class $[E] = [E'] \in H_1(X_o, \mathbb{Z})$ maps to a generator of C.
- $X - (E \cup E')$ splits into two connected components of genera 1 and $g - 1$.
- If t_E, $t_{E'}$ denote the right Dehn twists along E and E' respectively, then the mapping class $t_E^{E'} := t_E^{-1} \circ t_{E'} \in \mathrm{Map}(X_o)$ belongs to Γ.

Now [Looijenga97, Corollary 4.5], together with [Looijenga97, Lemma 4.6] and the proof of [Looijenga97, Theorem 2.4] imply that for every $\Gamma \subset \mathrm{Map}(X_o, A)$ satisfying **C(i)** and **C(ii)** the image

$$\mathrm{im}[\Gamma \to \mathbb{P}U(H_1(Y_o^C, \mathbb{Z}))]$$

will be a subgroup of finite index in $\mathbb{P}U(H_1(Y_o^C, \mathbb{Z}))$.

Next we turn to the case of a general abelian group A. Since we have an isogeny

$$\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z})) \to \prod_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} \mathbb{P}U(H_1(Y_o^C, \mathbb{Z})),$$

it suffices to check that to find conditions for a group $\Gamma \subset \mathrm{Map}(X_o, A)$ to map to a subgroup of finite index in the product $\prod_{C \in \mathfrak{C}(A)} \mathbb{P}U(H_1(Y_o^C, \mathbb{Z}))$.

The result [Looijenga97, Theorem 2.5] asserts that the image of the whole group

$$\bigcap_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} \mathrm{Map}(X_o, C) = \mathrm{Map}(X_o, A)$$

in $\prod_{C \in \mathfrak{C}(A)} \mathbb{P}U(H_1(Y_o^C, \mathbb{Z}))$ is a subgroup of finite index. Moreover, the proof of [Looijenga97, Lemma 5.2] implies that a subgroup $\Gamma \subset \mathrm{Map}(X_o, A)$ will map to a subgroup of finite index in the product, provided that the image of Γ is of finite index in each factor $\mathbb{P}U(H_1(Y_o^C, \mathbb{Z}))$ and that in addition Γ satisfies:

P(i) For every non-trivial $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$, there exists an embedded circle $E \subset X_o$, so that the Dehn twist t_E belongs to Γ, and so that the homology class $[E] \in H_1(X_o, \mathbb{Z})$ maps to a generator of C.

P(ii) For every pair $[A \twoheadrightarrow C_1] \in \mathfrak{C}(A)$ and $[A \twoheadrightarrow C_2] \in \mathfrak{C}(A)$ of *nontrivial* cyclic quotients of A, there exists an embedded circle $E \subset X_o$ so that:

- The cycle E is homologous to zero. That is $X_o - E$ splits into two connected components $X_o - E = X'_o \cup X''_o$.
- The homomorphisms

$$H_1(X'_o, \mathbb{Z}) \to H_1(X_o, \mathbb{Z}) \twoheadrightarrow C_1$$
$$H_1(X''_o, \mathbb{Z}) \to H_1(X_o, \mathbb{Z}) \twoheadrightarrow C_1$$

  are non-trivial.
- The homomorphism

$$H_1(X'_o, \mathbb{Z}) \to H_1(X_o, \mathbb{Z}) \twoheadrightarrow C_2$$

  is trivial.

To summarize - in order for a group $\Gamma \subset \mathrm{Map}(X_o, A)$ to map to a subgroup of finite index in $\mathbb{P}\,\mathrm{Sp}_A(H_1(Y_o^A, \mathbb{Z}))$ it is sufficient that: Γ satisfies conditions **C(i)** and **C(ii)** for all non-trivial cyclic quotients $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$; Γ satisfies the conditions **P(i)** and **P(ii)**; and Γ satisfies one extra condition corresponding to the trivial cyclic quotient of A, namely that Γ maps to a subgroup of finite index in $\mathrm{Sp}(H_1(X_o, \mathbb{Z}))$.

In the next section we show why these conditions hold under the hypothesis of Theorems A and B.

3 Proofs of the main theorems

In this section we derive the proofs of Theorems A and B.

3.1 Complex points and open orbits

Our first task is to show that the Zariski closure of the arithmetic group $\mathrm{Sp}^\sharp(H_1(Y_o^A, \mathbb{Z}))$ in $\mathrm{Sp}^\sharp(H_1(Y_o^A, \mathbb{C}))$ acts with an open orbit on $H_1(Y_o^A, \mathbb{C})$.

3.1.1. The complex points of $\mathrm{Sp}_A^\sharp$. Let $W_\mathbb{C}$ be a finite dimensional complex representation of A. For every such $W_\mathbb{C}$ we would like to build a direct sum decomposition in which the summands are labeled by the cyclic quotients $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$. Moreover if $W_\mathbb{C} = W \otimes_\mathbb{Z} \mathbb{C}$ for some free abelian group W equipped with an A action, then the decomposition in question should agree with (2.4).

We will describe such a decomposition by mimicking the construction of (2.4). Let $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$ and let $\chi \in \widehat{A}$ be a character which generates $\widehat{C} \subset \widehat{A}$. Let $\widetilde{\chi} : \mathbb{Q}[A] \to K_C \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ be as before and let $\widetilde{\chi}_{\mathbb{C}} := \chi \otimes \mathbb{C}$. Explicitly we have

$$\begin{array}{ccc} \mathbb{C}[A] & \xrightarrow{\widetilde{\chi}_{\mathbb{C}}} & K_C \otimes_{\mathbb{Q}} \mathbb{C} \\ \sum_{g \in A} c_g g & \longmapsto & \sum_{g \in A} c_g \chi(g). \end{array}$$

Put $R_{\mathbb{C},C} := R_C \otimes_{\mathbb{Z}} \mathbb{C} = K_C \otimes_{\mathbb{Q}} \mathbb{C}$. Note that by the Chinese reminder theorem $R_{\mathbb{C},C}$ is a $\mathbb{C}$-algebra isomorphic to the direct sum of $[K_C : \mathbb{Q}]$ copies of $\mathbb{C}$. As before $R_{\mathbb{C},C} = \mathrm{im}(\widetilde{\chi}_{\mathbb{C}}) \subset K_C \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$, and we set $I_{\mathbb{C},C} := \ker(\widetilde{\chi}_{\mathbb{C}})$. Define now

$$W_{\mathbb{C},C} := W_{\mathbb{C}} \otimes_{\mathbb{C}[A]} R_{\mathbb{C},C}\,.$$

The same reasoning as in Section 2.3.1 implies that the natural map

$$W_{\mathbb{C}} \to \bigoplus_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} W_{\mathbb{C},C}$$

is an isomorphism of complex vector spaces, and is equivariant with respect to the natural algebra isomorphism $\mathbb{C}[A] \cong \oplus_{C \in \mathfrak{C}(A)} R_{\mathbb{C},C}$. This yields the desired decomposition.

Let now W be a free abelian group of finite rank, which is equipped with a non-degenerate symplectic form and an action of A by symplectomorphisms. Set $W_{\mathbb{C}} := W \otimes_{\mathbb{Z}} \mathbb{C}$. We would like to understand the action of the arithmetic group $\mathrm{Sp}_A^\sharp(W)$ on the complex vector space $W_{\mathbb{C}}$.

Proposition 3.1 *The Zariski closure of* $\mathrm{Sp}_A^\sharp(W)$ *inside* $\mathrm{Sp}_A(W_{\mathbb{C}})$ *has a Zariski open orbit on* $W_{\mathbb{C}}$.

Proof. The previous discussion together with the analysis in Section 2.3.1 implies that

$$\mathrm{Sp}_A(W_{\mathbb{C}}) = \prod_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} G_C(\mathbb{C}),$$

where $G_C(\mathbb{C}) \subset GL(W_{\mathbb{C},C})$ is the group of complex points of a certain complex reductive group G_C, which we proceed to describe.

As we have seen in Section 2.3.1, the rational vector space $W_{\mathbb{Q},C} := W_C \otimes \mathbb{Q}$ has a natural structure of a K_C-vector space. Now, using the field extensions $\mathbb{Q} \subset K_C \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ we can write

$$\begin{aligned} W_{\mathbb{C},C} &= W_{\mathbb{Q},C} \otimes_{\mathbb{Q}} \mathbb{C} \\ &= (W_{\mathbb{Q},C} \otimes_{K_C} \mathbb{C}) \otimes_{\mathbb{C}} (K_C \otimes_{\mathbb{Q}} \mathbb{C}) \\ &= (W_{\mathbb{Q},C} \otimes_{K_C} \mathbb{C}) \otimes_{\mathbb{C}} R_{\mathbb{C},C}\,. \end{aligned}$$

Define a complex vector space $\Phi_C := W_{\mathbb{Q},C} \otimes_{K_C} \mathbb{C}$. With this notation we have an identification

$$W_{\mathbb{C},C} = \Phi_C \otimes_{\mathbb{C}} R_{\mathbb{C},C}\,, \tag{3.1}$$

which is clearly compatible with the $R_{\mathbb{C},C}$ module structures (here we consider the right hand side as a free $R_{\mathbb{C},C}$-module). Note that this identification is not canonical, since it depends on the choice of an embedding $K_C \subset \overline{\mathbb{Q}}$, which in turn depends on a choice of a generator $\chi \in \widehat{C}$.

The fact that $K_C \subset \mathbb{C}$ is preserved under complex conjugation allows us to define an anti-holomorphic involution '$\overline{\bullet}$' on Φ_C, whose real points are $W_{\mathbb{Q},C} \otimes_{K'_C} \mathbb{R} \subset \Phi_C$. In particular, the K_C-valued skew-Hermitian form $\langle \bullet, \bullet \rangle_C : W_{\mathbb{Q},C} \times W_{\mathbb{Q},C} \to K_C$ extends to a well-defined ordinary skew-Hermitian form

$$\varphi_C : \Phi_C \times \Phi_C \to \mathbb{C}.$$

In particular the natural $R_{\mathbb{C},C}$-valued skew-Hermitian form

$$\begin{aligned} \langle \bullet, \bullet \rangle_{\mathbb{C},C} : \qquad & W_{\mathbb{C},C} \times W_{\mathbb{C},C} \longrightarrow R_{\mathbb{C},C} \\ & (x,y) \longmapsto \textstyle\sum_{g \in A} (x, gy) g \end{aligned}$$

can be rewritten in terms of the identification (3.1) as

$$\langle a_1 \otimes r_1, a_2 \otimes r_2 \rangle_{\mathbb{C},C} = \varphi_C(a_1, a_2) r_1 \bar{r}_2, \qquad \text{for all } a_i \in \Phi_C, r_i \in R_{\mathbb{C},C}.$$

By definition the group G_C is the complexification of the real unitary group $U(W_{\mathbb{C},C})$, i.e. G_C is a complex reductive group, defined over the reals, such that $G_C(\mathbb{R}) = U(W_{\mathbb{C},C})$. The unitary group $U(W_{\mathbb{C},C})$ is defined in turn as the group of all $R_{\mathbb{C},C}$-linear automorphisms of $W_{\mathbb{C},C}$, which preserve $\langle \bullet, \bullet \rangle_{\mathbb{C},C}$.

On the other hand $R_{\mathbb{C},C} = K_C \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[t]/(g(t))$, where $g(t)$ is the cyclotomic polynomial

$$g(t) = \prod_{\substack{\mu \in \mu_d \\ \mu - \text{primitive}}} (t - \mu).$$

In other words, the algebra $R_{\mathbb{C},C}$ splits as a direct sum of $\phi(d) = [K_C : \mathbb{Q}] = |\operatorname{Gal}(K_C, \mathbb{Q})|$ copies of the algebra $\mathbb{C}$. Alternatively we can describe this algebra decomposition of $R_{\mathbb{C},C}$ by means of the isomorphism $R_{\mathbb{C},C} \cong \mathbb{C}[\operatorname{Gal}(K_C, \mathbb{Q})]$. Indeed, consider the character decomposition

$$\mathbb{C}[\operatorname{Gal}(K_C, \mathbb{Q})] = \bigoplus_{\alpha \in \widehat{\operatorname{Gal}(K_C, \mathbb{Q})}} R_\alpha,$$

where R_α is the one dimensional subspace of the complex vector space $\mathbb{C}[\operatorname{Gal}(K_C, \mathbb{Q})]$ on which each $\sigma \in \operatorname{Gal}(K_C, \mathbb{Q})$ acts by multiplication by $\alpha(\sigma)$. Note that with respect to the multiplicative structure on $R_{\mathbb{C},C} \cong \mathbb{C}[\operatorname{Gal}(K_C, \mathbb{Q})]$ we have $R_\alpha \cdot R_\beta = 0$ if $\alpha \neq \beta$, and $R_\alpha \cdot R_\alpha = R_\alpha$.

Next recall that under the inclusion $K_C \to \mathbb{C}$, the natural involution $\overline{\bullet}$ defined in Section 2.3.1 on K_C is compatible with the complex conjugation. Moreover it extends naturally

to an anti-linear involution $\overline{\bullet}$ on $R_{\mathbb{C},C} = K_C \otimes_{\mathbb{Q}} \mathbb{C}$ by the rule $\overline{a \otimes r} = \bar{a} \otimes \bar{r}$. In particular this involution on $R_{\mathbb{C},C}$ is compatible with the complex conjugation on $\mathbb{C} \subset R_{\mathbb{C},C}$. Finally, for an element $r \in R_\alpha$ and an element $g \in A$ we have $g \cdot \bar{r} = \overline{\bar{g} \cdot r} = \overline{g^{-1} \cdot r} = \overline{\alpha(g^{-1})} \bar{r}$, where $\overline{\alpha(g^{-1})}$ denotes the complex conjugate of the complex number $\alpha(g^{-1})$. But $\alpha(g^{-1}) = \alpha^{-1}(g) = \overline{\alpha(g)}$ and so $g \cdot \bar{r} = \alpha(g)\bar{r}$, i.e. $\bar{r} \in R_\alpha$.

This implies that the involution $\overline{\bullet}$ on $R_{\mathbb{C},C}$ preserves the algebra decomposition

$$R_{\mathbb{C},C} \cong \underbrace{\mathbb{C} \oplus \ldots \oplus \mathbb{C}}_{\phi(d) \text{ copies}},$$

and acts as the ordinary complex conjugation on each copy of $\mathbb{C}$.

This shows that $T \in GL(W_{\mathbb{C},C})$ will be $R_{\mathbb{C},C}$-linear if and only if under the decomposition

$$W_{\mathbb{C},C} = \underbrace{\Phi_C \oplus \ldots \oplus \Phi_C}_{\phi(d) \text{ copies}}$$

becomes block-diagonal, i.e. $T = M_1 \times \ldots \times M_{\phi(d)}$ with $M_i \in GL(\Phi_C)$. Moreover T will preserve the $R_{\mathbb{C},C}$-valued form $\langle \bullet, \bullet \rangle_{\mathbb{C},C}$ if and only if each M_i preserves the ordinary skew-Hermitian form φ_C on Φ_C. Altogether we have an isomorphism

$$U(W_{\mathbb{C},C}) = \underbrace{U(\Phi_C, \varphi_C) \times \ldots \times U(\Phi_C, \varphi_C)}_{\phi(d) \text{ copies}},$$

where $U(\Phi_C, \varphi_C)$ is the full unitary group of the complex vector space Φ_C, preserving the non-degenerate skew-Hermitian product φ_C, or equivalently, the non-degenerate Hermitian product $\sqrt{-1}\varphi_C$.

This shows that under the isomorphism $W_{\mathbb{C},C} \cong \Phi_C^{\oplus \phi(d)}$ the complexification $G_C(\mathbb{C})$ of $U(W_{\mathbb{C},C})$ can be identified with the group of all invertible block-diagonal matrices, i.e. $G_C(\mathbb{C}) = GL(\Phi_C)^{\times \phi(d)}$ and so

$$\mathrm{Sp}_A(W_{\mathbb{C}}) = \prod_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} GL(\Phi_C)^{\times \phi(d)}.$$

Finally, note that the group $U^\sharp(W_{\mathbb{C},C})$ consists of all elements in $U(W_{\mathbb{C},C})$ whose $R_{\mathbb{C},C}$-determinant is a square of a d-th root of unity in $R_{\mathbb{C},C}^\times$. Thus $U^\sharp(W_{\mathbb{C},C})$ consists of all $T = M_1 \times \ldots \times M_{\phi(d)}$ with $M_i \in U(\Phi_C, \varphi_C)$ and $\det(M_i) \in \boldsymbol{\mu}_d^{(2)} \subset \boldsymbol{\mu}_d \subset \mathbb{C}^\times$. Hence $U^\sharp(W_{\mathbb{C},C})$ contains $SU(\Phi_C, \varphi_C)^{\times \phi(d)}$ and so the Zariski closure of $\mathrm{Sp}_A^\sharp(W)$ in $\mathrm{Sp}_A(W_{\mathbb{C}})$ contains the group

$$\prod_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} SL(\Phi_C)^{\times \phi(d)}.$$

This shows that the Zariski closure of $\mathrm{Sp}_A^\sharp(W)$ acts with a Zariski open orbit on the complex vector space $W_{\mathbb{C}} = \oplus_{C \in \mathfrak{C}(A)} \Phi_C^{\oplus \phi(d)}$. The proposition is proven. $\square$

3.1.2. Geometric interpretation. In the special case when $W = H_1(Y_o^A, \mathbb{Z})$ it is instructive to identify the vector spaces $W_{\mathbb{C},C} = H_1(Y_o^A, \mathbb{C})_C$ and Φ_C in terms of the geometry of Y_o^A. This intepretation will not be needed in what follows but we include it here with the hope that it may clarify the exposition.

First we describe the direct summands $W_{\mathbb{C},C}$ explicitly as A-submodules in $W_{\mathbb{C}}$. For any cyclic quotient $[A \twoheadrightarrow C] \in \mathfrak{C}(A)$ denote by $\underline{W}_{\mathbb{C},C}$ the direct sum of all isotypic components of $W_{\mathbb{C}}$ corresponding to characters $\chi \in \widehat{C} \subset \widehat{A}$, such that χ generates the cyclic subgroup $\widehat{C} \subset \widehat{A}$. Then $W_{\mathbb{C},C}$ is a subrepresentation of W and we have a direct sum decomposition

$$W_{\mathbb{C}} = \bigoplus_{[A \twoheadrightarrow C] \in \mathfrak{C}(A)} \underline{W}_{\mathbb{C},C}. \tag{3.2}$$

Now we have the following easy

Lemma 3.2 *The composition map*

$$\underline{W}_{\mathbb{C},C} \hookrightarrow W_{\mathbb{C}} \to W_{\mathbb{C},C}$$

is an isomorphism of $R_{\mathbb{C},C}$-modules.

Proof. Recall that $\underline{W}_{\mathbb{C},C} = \oplus W_\chi$ where the direct sum runs over all generators $\chi \in \widehat{C}$, and W_χ is the isotypic component of the $\chi : A \to \mathbb{C}^\times$ in $W_{\mathbb{C}}$.

Consider the annihilating set of $I_{\mathbb{C},C}$:

$$\mathrm{ann}(I_{\mathbb{C},C}) := \{x \in W_{\mathbb{C}} | F \cdot x = 0, \text{ for all } F \in I_{\mathbb{C},C}\}.$$

Clearly $\mathrm{ann}(I_{\mathbb{C},C}) \subset W_{\mathbb{C}}$ is a vector subspace. Furthermore, since $I_{\mathbb{C},C} \triangleleft \mathbb{C}[A]$ is an ideal, it follows that $\mathrm{ann}(I_{\mathbb{C},C}) \subset W_{\mathbb{C}}$ is a $\mathbb{C}[A]$-submodule. But by definition $I_{\mathbb{C},C}$ acts trivially on $\mathrm{ann}(I_{\mathbb{C},C})$, and so the action of $\mathbb{C}[A]$ on $\mathrm{ann}(I_{\mathbb{C},C})$ factors through $\mathbb{C}[A] \to \mathbb{C}[A]/I_{\mathbb{C},C} \to R_{\mathbb{C},C}$.

Also if $x \in W_\chi$ for some generator $\chi \in \widehat{C}$, then for every $F = \sum_{g \in A} c_g g \in \mathbb{C}[A]$ we have $F \cdot x = \sum_{g \in A} c_g g \cdot x = (\sum_{g \in A} c_g \chi(g))x = \widetilde{\chi}_{\mathbb{C}}(F)x$. In other words $F \cdot x = 0$ iff $F \in \ker(\widetilde{\chi}_{\mathbb{C}})$, which shows that $\mathrm{ann}(I_{\mathbb{C},C}) = \underline{W}_{\mathbb{C},C}$ as $\mathbb{C}[A]$-modules (and hence as $R_{\mathbb{C},C}$-modules.

Finally, since $\mathbb{C}[A]$ is noetherian, the vector subspace $I_{\mathbb{C},C} \cdot W_{\mathbb{C}} \subset W_{\mathbb{C}}$ can be written as $I_{\mathbb{C},C} \cdot W_{\mathbb{C}} = F_1 \cdot W_{\mathbb{C}} + \ldots + F_k \cdot W_{\mathbb{C}}$, where the F_i's generate $I_{\mathbb{C},C}$. But by definition $F_i \in \mathrm{End}_{\mathbb{C}}(W_{\mathbb{C}})$ are commuting semisimple endomorphisms of $W_{\mathbb{C}}$ and therefore $W_{\mathbb{C}} = I_{\mathbb{C},C} \cdot W_{\mathbb{C}} \oplus (\cap_{i=1}^k \ker(F_i)) = I_{\mathbb{C},C} \cdot W_{\mathbb{C}} \oplus W_{\mathbb{C},C}$. By construction this is a decomposition as $\mathbb{C}[A]$-modules and so $W_{\mathbb{C},C} \subset W_{\mathbb{C}} \to W_{\mathbb{C}}/(I_{\mathbb{C},C} \cdot W_{\mathbb{C}}) = W_{\mathbb{C},C}$ is an isomorphism of $R_{\mathbb{C},C}$-modules. The lemma is proven. □

In view of this lemma we will not distinguish between $W_{\mathbb{C},C}$ and $\underline{W}_{\mathbb{C},C}$ from now on. Consider now $W_{\mathbb{C}} = H_1(Y_o^A, \mathbb{C})$. Using the finite map $p^A : Y_o^A \to X_o$ we get a canonical isomorphism

of $\mathbb{C}[A]$-modules:

$$H_1(Y_o^A, \mathbb{C}) = \bigoplus_{\chi \in \widehat{A}} H_1(X_o, \mathbb{L}_{\chi^{-1}}) \otimes_{\mathbb{C}} \mathbb{V}_\chi,$$

where $\mathbb{V}_\chi = (\mathbb{C}, \chi)$ is the one dimensional representation of A corresponding to the character χ, and $\mathbb{L}_{\chi^{-1}}$ is the rank one complex local system on X_o corresponding to the character $\pi_1(X_o) \to H_1(X_o, \mathbb{Z}) \to A \to \overline{\mathbb{Q}}^\times \subset \mathbb{C}^\times$. v In particular by Lemma 3.2 we conclude that for any $[A \twoheadrightarrow C] \in \mathbb{C}(A)$ we have:

$$H_1(Y_o^A, \mathbb{C})_C = \bigoplus_{\substack{\chi \in \widehat{C} \\ \chi - \text{generator}}} H_1(X_o, \mathbb{L}_{\chi^{-1}}) \otimes_{\mathbb{C}} \mathbb{V}_\chi$$

Now by the definition of Φ_C and the description of the decomposition of $R_{\mathbb{C},C}$ in the proof of Proposition 3.1 it follows that the choice of a generator $\chi \in \widehat{C}$ induces an isomorphism of complex vector spaces

$$H_1(X_o, \mathbb{L}_{\chi^{-1}}) \cong \Phi_C.$$

In terms of this identification, the Hermitian form $\sqrt{-1}\varphi_C$ on Φ_C can be identified with the cap product in homology. Indeed, the monodromy of the rank one local system $\mathbb{L}_{\chi^{-1}}$ is contained by definition in $\boldsymbol{\mu}_d \subset U(1) \subset \mathbb{C}$ and so the absolute value function on $\mathbb{C}$ induces a well defined Hermitian metric $h : \mathbb{L}_{\chi^{-1}} \otimes \overline{\mathbb{L}}_{\chi^{-1}} \to \mathbb{C}$. With this in mind $\sqrt{-1}\varphi_C$ can be identified with the composition

$$H_1(X_o, \mathbb{L}_{\chi^{-1}}) \times H_1(X_o, \mathbb{L}_{\chi^{-1}}) \stackrel{\bullet \cap \bullet}{\to} H_2(X_o, \mathbb{L}_{\chi^{-1}} \otimes \overline{\mathbb{L}}_{\chi^{-1}}) \stackrel{h}{\to} H_2(X_o, \mathbb{C}) \cong \mathbb{C},$$

i.e. $\sqrt{-1}\varphi_C$ is the natural polarization on the complex Hodge structure $H_1(X_o, \mathbb{L}_{\chi^{-1}})$.

3.2 Families with large monodromy

3.2.1. Proof of Theorem A. Let $f : X \to B$ be a smooth family of connected curves of genus $g \geq 3$ over a smooth quasi-projective base B. Let $o \in B$ be a base point and assume that $\mathrm{mon}(\pi_1(B, o) \subset \mathrm{Map}(X_o)$ is of finite index. We would like to show the existence of points $x_B \in M_B(X_o, n)$ and $x_{DR} \in M_{DR}(X_o, n)$ with Zariski dense monodromy orbits.

Let $A := \mathbb{Z}/n \times \widehat{\mathbb{Z}/n} \cong \mathbb{Z}/n \times \mathbb{Z}/n$ and let $1 \to \boldsymbol{\mu}_n \to H_n \to A \to 0$ be the corresponding finite Heisenberg group. Choose any surjective homomorphism $\pi_1(X_o) \twoheadrightarrow H_n$ and let $\rho : \pi_1(X_o) \to GL(V_n)$ be the representation of $\pi_1(X_o)$ obtained by composing this homomorphism with the Schrödinger representation $\phi_n : H_n \to GL(V_n)$ of H_n.

Due to the results of Sections 2.1 and 2.2, we know that points x_B and x_{DR} as above will exist if we can show that the pair $(f : X \to B, H_1(X_o, \mathbb{Z}) \twoheadrightarrow A)$ is good in the sense of definition 2.9. On the other hand the assumption that $\mathrm{mon}(\pi_1(B, o) \subset \mathrm{Map}(X_o)$ is of finite

index together with [Looijenga97, Collorary 2.6] imply that (in the notation of definition 2.9) the image

$$\mathrm{im}\left[\pi_1(B',o') \overset{\mathrm{mon}_A}{\to} \mathrm{LMap}(X_o,A) \to \mathrm{Sp}(H_1(Y_o^A,\mathbb{Z}))\right]$$

is a subgroup of finite index in $\mathrm{Sp}_A^\sharp(H_1(Y_o^A,\mathbb{Z}))$. Therefore by Proposition 3.1 it follows that the Zariski closure of this image in $\mathrm{Sp}(H_1(Y_o^A,\mathbb{C}))$ acts on $H_1(Y_o^A,\mathbb{C})$ with a Zariski open orbit. This shows that the pair $(f: X \to B, H_1(X_o,\mathbb{Z}) \twoheadrightarrow A)$ is good and concludes the proof of Theorem A. □

In fact, a close examination of the conditions from Looijenga's list (see Section 2.3.2) shows that a pair $(f: X \to B, H_1(X_o,\mathbb{Z}) \twoheadrightarrow A)$ can be good, even when the geometric monodromy of $f: X \to B$ is much smaller than the mapping class group $\mathrm{Map}(X_o)$. However one does need $\mathrm{mon}(\pi_1(X_o))$ to be relatively large - e.g. it can not be finite. In the next section we describe an explicit condition on the size of $\mathrm{mon}(\pi_1(X_o))$ which is sufficient for the existence of Zariski dense monodromy orbits on $M_B(X_o,n)$ and $M_{DR}(X_o,n)$.

3.2.2. Large monodromy. Let $f: X \to B$ a family of smooth connected curves of genus $g \geq 3$ over a smooth and quasi-projective B. Let $o \in B$ be a base point and assume for simplicity we have a holomorphic section $\sigma: B \to X$ of f. Let Λ denote the image $\mathrm{im}[\pi_1(B,o) \to \mathrm{Map}^1(X_o)]$ of the geometric monodormy map. Then we have:

Proposition 3.3 *Assume that* Γ *satisfies:*

(a) The image of Λ *in* $\mathrm{Sp}(H_1(X_o,\mathbb{Z}))$ *is a subgroup of finite index.*

(b) $\Lambda \supset \mathrm{Map}_2^1$*, where* Map_2^1 *is the mapping class group of some genus two handle of* X_o*.*

Then the monodromy groups $\mathrm{mon}_B^n(\pi_1(B,o))$ *and* $\mathrm{mon}_{DR}^n(\pi_1(B,o))$ *satisfy* **AGZD1** *and* $M_B(X/B,n) \to B$ *and* $M_{DR}(X/B,n) \to B$ *satisfy* **GZD** *when equipped with the non-abelian Gauss-Manin connection.*

Proof. The proof of this proposition is analogous to the proof of Theorem A above. The only new feature is that one has to exercise extra care in the choice of the surjection $\pi_1(X_o) \twoheadrightarrow H_n$. Indeed, as explained above, the only issue is whether we can choose the homomorphism $\pi_1(X_o) \twoheadrightarrow H_n$ so that the induced pair $(f: X \to B, H_1(X_o,\mathbb{Z}) \to A)$ is good.

Let $a_1,a_2,b_1,b_2 \subset X_o$ be free closed oriented loops, which form a standard basis in the homology of the genus two handle giving the subgroup $\mathrm{Map}_2^1 \subset \Lambda$. Consider the surjective homomorphism $H_1(X_o,\mathbb{Z}) \to A \cong \mathbb{Z}/n \times \mathbb{Z}/n$ given by $x \mapsto (x \cdot b_1 \bmod n, x \cdot b_2 \bmod n)$. Let $\pi_1(X_o) \twoheadrightarrow H_n$ be any lift of this homomorphism.

Now in view of Proposition 3.1 we only need to check the conditions **C(i)**, **C(ii)**, **P(i)** and **P(ii)** from Looijenga's list plus the extra condition that Λ maps to a subgroup of finite index in $\mathrm{Sp}(H_1(X_o,\mathbb{Z}))$.

The latter is just part (a) of the hypothesis ot the proposition. Moreover part (a) clearly implies the validity of **C(i)** for any non-trivial finite cyclic quotient of $H_1(X_o, \mathbb{Z})$ and in particular for the ones that factor through A.

Finally, by our choice of the homomorphism $\pi_1(X_o) \twoheadrightarrow H_n$ we see that any cyclic quotient $H_1(X_o, \mathbb{Z}) \to A \twoheadrightarrow \mathbb{Z}/d$ can be realized as $x \mapsto (x \cdot c) \bmod d)$, where $c \subset X_o$ is an oriented closed loop which is contained in the genus two handle from part (b) of the hypothesis of the proposition. But the conditions **C(i)**, **C(ii)**, **P(i)** and **P(ii)** obviously hold for the genus two handle (with $\Gamma = \mathrm{Map}_2^1$ and so must hold for the whole X_o and $\Gamma = \Lambda$. the proposition is proven. □

Thus from the viewpoint of the monodromy action on non-abelian cohomology, the monodromy $\Lambda = \mathrm{im}[\pi_1(B, o) \to \mathrm{Map}(X_o)]$ is *large* if it satisfies the conditions (a) and (b) from the hypothesis of Proposition 3.3.

3.3 Lefschetz pencils

We are now ready to prove Theorem B. Let Z be a smooth projective surface with $b_1(Z) = 0$. Let $\mathcal{O}_Z(1)$ be a very ample line bundle on Z and let $\mathbb{P}^1 \subset \mathbb{P}(H^0(Z, \mathcal{O}_Z(k)))$ be a generic line. Denote by $\varepsilon : \widehat{Z} \to Z$ the blow-up of Z at the base points of the pencil of curves $\{D_t\}_{t \in \mathbb{P}^1}$, and let $f : \widehat{Z} \to \mathbb{P}^1$ be the corresponding Lefschetz fibration. Let $p_1, \ldots, p_\mu \in \mathbb{P}^1$ be the critical points of f and let $B = \mathbb{P}^1 - \{p_1, \ldots, p_\mu\}$ and $X = f^1(B)$. Choose a base point $o \in B$. Let X_t, $t \in B$ denote the fiber of f over t, or equivalently, the strict transform of the divisor $D_t \subset Z$.

Fix an integer $n \geq 1$. We want to show that whenever k is bigger than some constant ℓ which depends only on Z and $\mathcal{O}_Z(1)$, the geometric monodromy of the smooth fibration $f : X \to B$ satisfies **AGZD1** on $M_B(X_o, n)$ and $M_{DR}(X_o, n)$.

For this it suffices to choose k so that the group $\Gamma := \mathrm{im}[\pi_1(B, o) \to \mathrm{Map}^1(X_o)]$ satisfies the hypothesis of Proposition 3.3. The fact that Γ must map to a subgroup of finite index in $\mathrm{Sp}(H_1(X_o, \mathbb{Z}))$ is a straightforward consequence of Deligne's strong irreducibility theorem [Deligne80, Section 4.4]. Indeed, it is well known (see [Deligne80, Proposition 4.3.3] or [Looijenga96, Lemma 3.4]) that the lattice of vanishing cycles in $H_1(X_o, \mathbb{Z})$ is a sublattice of finite index in the orthogonal complement to $\mathrm{im}[H_3(Z, \mathbb{Z}) \to H_1(X_o, \mathbb{Z})]$ with respect to the intersection form. Since by hypothesis $b_3(Z) = b_1(Z) = 0$, it follows that $H_3(Z, \mathbb{Z})$ is a finite abelian group and so the vanishing cycles of the Lefschetz pencil $f : \widehat{Z} \to \mathbb{P}^1$ span a sublattice of finite index in $H_1(X_o, \mathbb{Z})$. But for a Lefschetz pencil, all the Dehn twists along vanishing cycles are conjugate to each other in the monodromy group [Deligne80, Proposition 2.4.6], and so by [Deligne80, Theorem 4.4.2^a] and [Janssen83] the group Γ will be a subgroup of finite index in $\mathrm{Sp}(H_1(X_o, \mathbb{Z}))$.

Therefore it only remains to show that for a big enough k, one can satisfy part (b) of the hypothesis of Proposition 3.3. Consider the function $F : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, defined by $F(z_1, z_2) = z_1^6 + z_2^2$. The fiber of F over $0 \in \mathbb{C}$ has a singularity at $0 \in \mathbb{C}^2$ whose germ is an

isolated curve singularity of type $\boldsymbol{A}_5$. The miniversal deformation (see e.g. [Arnold *et al.*88, Chapter 1]) of this singularity is given by $G(z_1, z_2, u_1, \dots, u_5) := F(z_1, z_2) + \sum_{i=1}^{5} u_i z_1^{5-i}$. By choosing k big enough we can make $\mathcal{O}_Z(k)$ sufficiently ample, so that the linear system $|\mathcal{O}_Z(k)|$ contains a curve D with a single isolated singularity of type $\boldsymbol{A}_5$. Furthermore, by increasing k we may assume that inside $|\mathcal{O}_Z(k)|$ there exists a five dimensional projective space $\mathbb{P}^5 \subset |\mathcal{O}_Z(k)|$ which is linearly embedded, contains the point corresponding to D and in a neighborhood of D is isomorphic to the base of the miniversal unfolding $G(z, u) = 0$. Fix such a k. Let $U \subset |\mathcal{O}_Z(k)|$ denote the complement of the discriminant locus, and let $\mathcal{X} \to U$ be the family of all curves in the linear system $|\mathcal{O}_Z(k)|$ parameterized by U. By construction, the geometric monodromy group $\mathrm{im}[\pi_1(U, o) \to \mathrm{Map}^1(X_o)]$ corresponding to the family $\mathcal{X} \to U$, contains the monodromy image of the fundamental group of the complement of the bifurcation locus in the miniversal unfolding of an isolated curve singularity of type $\boldsymbol{A}_5$. It is well known however [Brieskorn71], [Arnold *et al.*88, Chapter 1] that the latter is isomorphic to the Artin braid group of a Dynkin graph of type A_5, and that the corresponding monodromy image is generated by the Dehn twists along five embedded circles $c_1, c_2, \dots, c_5 \subset X_o$ having the property that $c_i \cdot c_j = 0$ for $|i - j| \geq 2$ and $c_i \cdot c_{i+1} = 1$ for $i = 1, \dots, 4$. Since the dual graph of such a configuration of circles is a tree (namely the Dynkin graph A_5) it follows (see e.g. [Matsumoto00, Proposition 2.2]) that a tubular neighborhood of $\cup_{i=1}^{5} C_i$ is the ribbon graph of a genus two handle of X_o. Finally, it is known by the work Birman-Hilden [Birman-Hilden72] and Wajnryb [Wajnryb83], that the Dehn twists $\{t_{c_i}\}_{i=1}^{5}$ generate Map_2^1.

This shows part (b) of the hypothesis of Proposition 3.3 holds for Lefschetz pencils of sufficiently high degree and thus finishes the proof of Theorem B. □

In fact, the above proof of Theorem B gives the following slightly stronger statement, which is of independent interest:

Theorem 3.4 *Let Z be a smooth projective surface with $b_1(Z) = 0$. Let $\mathcal{O}_Z(1)$ be an ample line bundle on Z and let $n > 1$ be a fixed integer. Then there exists a positive integer ℓ (depending only on Z and $\mathcal{O}_Z(1)$), such that for every $k \geq \ell$ and for every Lefschetz fibration $f : \widehat{Z} \to \mathbb{P}^1$ in the linear system $|\mathcal{O}_Z(k)|$ there exists a smooth point $\rho \in M_B(X_o, n)$ which is fixed by a finite index subgroup $\Gamma \subset \pi_1(\mathbb{P}^1 - \{p_1, \dots, p_\mu\})$ and such that the Zariski closure of Γ in $GL(T_\rho M_B(X_o, n))$ acts with an open orbit on $T_\rho M_B(X_o, n)$.*

Remark 3.5 Note that in the proof above we could have used any other isolated curve singularity. In particular we could have used a singularity of type $\boldsymbol{E}_7$. Since the corresponding Artin braid group is known [Wajnryb83] to surject onto the group Map_3^1, we can work also with a genus three handle in X_o which will streamline the application of [Looijenga97, Theorem 2.5] even further and will allow one to skip the detailed analysis of the conditions in Looienga's list, needed for the proof of Proposition 3.3.

4 Further remarks

4.1 Topologically irreducible families

The requirement that our families have relatively large monodromy was used in an essential way in the proofs of Theorems A and B. However this requirement seems to be more an artifact of the method of proof rather than a real condition on the family $f : X \to B$ which is necessary for the density of the monodromy action. In this section we briefly examine some consequences of the density, which will allow us to probe the necessity of the 'large monodromy' condition.

Recall that a smooth family of curves $f : X \to B$ is called *topologically irreducible* if and only if there is no finite collection of disjoint embedded circles in X_o which is preserved by the geometric monodromy. We have the following simple

Lemma 4.1 *Let $f : X \to B$ be a family of smooth curves, such that the monodromy action of $\pi_1(B, o)$ has a Zariski dense orbit on the Betti moduli space $M_B(X_o, n)$ for some $n \geq 1$. Then $f : X \to B$ is topologically irreducible.*

Proof. Assume that one can find simple disjoint loops $a_1, \ldots, a_k \subset X_o$ such that the collection $\{a_1, \ldots, a_k\}$ of free homotopy classes on X_o is preserved by $\mathrm{mon}(\pi_1(B, o)) \subset \mathrm{Map}(X_o)$. Then for every $N \in \mathbb{Z}$ we have a well defined $\mathrm{mon}(\pi_1(B, o))$-invariant regular function $\psi_N : M_B(X_o, n) \to \mathbb{C}$ on $M_B(X_o, n)$, given by $\psi_N([\rho]) := \mathrm{Tr}(\prod_{i=1}^{k}(\rho(a_i))^N)$. But clearly for some N the function ψ_N will be non-constant and so $\mathrm{mon}(\pi_1(B, o))$ can not have a Zariski dense orbit on $M_B(X_o, n)$. The lemma is proven. □

In particular, all families of curves satisfying the hypothesis of Theorem B or more generally of Proposition 3.3 will be topologically irreducible. Recently C. McMullen has shown that topological irreducibility holds very generally: every non-isotrivial holomorphic family of curves is topologically irreducible [McMullen99, Proof of Theorem 3.1]. In particular, this corollary of our main results was already known.

When we started the current project we were hoping that topological irreducibility will allow one to distinguish symplectic Lefschetz pencils (whose topology tends to be much softer) from projective Lefschetz pencils. In the meantime however, Ivan Smith succeeded in showing [Smith00] that all symplectic Lefschetz fibrations over $\mathbb{P}^1$ are topologically irreducible. We still expect that the stronger property **GZD** (or the open orbit property from Theorem 3.4) will allow one to distinguish projective from symplectic Lefschetz pencils. We hope to return to examples of this type in a future paper.

McMullen's result (with the alternative symplectic proof by Smith when the base is $\mathbb{P}^1$) is the only evidence we have for the following conjectural generalization of Theorems A and B:

Conjecture 4.2 *For a family $f : X \to B$ assume that $\mathrm{mon}(\pi_1(B,o))$ is not a finite group. Then:*

(i) *There is no meromorphic function on $M_B(X_o,n)^{\mathrm{an}}$ which is invariant under the action of $\mathrm{mon}^n_B(\pi_1(B,o))$ (equivalently there is no meromorphic function on $M_{DR}(X_o,n)^{\mathrm{an}}$ which is $\mathrm{mon}^n_{DR}(\pi_1(B,o))$-invariant);*

(ii) *In the case of $M_B(X_o,n)$, considered with its natural structure of an affine algebraic variety, there exist a point $x_B \in M_B(X_o,n)$ so that the orbit*

$$\mathrm{mon}^n_B(\pi_1(B,o)) \cdot x_B \subset M_B(X_o,n)$$

is Zariski dense in $M_B(X_o,n)$.

Procesi's theorem [Procesi74] implies that the field of rational functions on $M_B(X_o,n)$ is generated by traces of evaluation maps for conjugacy classes of simple loops on X_o. One might hope (although we didn't find an argument) that the field of $\pi_1(B,o)$-invariant rational functions on $M_B(X_o,n)$ is similarly generated by the traces of evaluation maps at finite invariant collections of simple loops. If this were the case then McMullen's theorem would imply the validity of the variant of Conjecture 4.2 concerning algebraic meromorphic functions. The property **AGZD1** (i.e. the conjecture as it is stated using analytic meromorphic functions) would seem to remain more elusive.

4.2 Points Γ-near to a finite representation

We briefly describe here another variation on the basic result. Essentially, we have constructed a finite-image representation, the Schrödinger representation ρ, which corresponds to a smooth point in $M_B(X_o,n)$ and which turns out to be sufficient in order to get the Zariski-denseness property. In an attempt to better understand what is going on, we can explore a bit further the sense in which ρ is near the rest of $M_B(X_o,n)$.

Suppose a finitely presented group Γ acts on an affine variety M, and suppose $p \in M$ is a closed point in the smooth set of M, fixed by the action. We say that another point $q \in M$ is Γ-*near to* p if p lies in the closure of the orbit $\overline{\Gamma \cdot q}$. Let $N = \mathbf{Near}(M,p,\Gamma) \subset M$ denote the subset of points q which are Γ-near to p. It is Γ-invariant. Let T_pN denote its tangent cone at p, defined to be the set of limits of secants to M going from p to points $q \in N$ which approach p (the limits of secants may be taken in any real embedding of M). Note that $T_pN \subset T_pM$ is an invariant subset of the tangent space to M at p. An easy argument similar to that of Corollary 2.3 shows that if T_pN is Zariski-dense in T_pM then N is Zariski-dense in M.

Suppose $\gamma \in \Gamma$. Let $R \subset T_pM$ denote the span of the eigenvectors of γ whose eigenvalues have norm < 1. We claim that $R \subset T_pN$. Choose a smooth submanifold $V \subset M$ tangent to R. Let D be a ball neighborhood of p in M. Then the collection

$$\{V^k := \gamma^{-k}(W \cap \gamma^k D)\}$$

is a collection of manifolds with boundaries lying in the boundary of D, which are tangent to R at the origin, and whose curvature is bounded (the $V \cap \gamma^k D$ all lie in a sector preserved by γ^{-1} and in which γ^{-1} smooths things out). Thus these converge to a manifold V^∞ which is preserved by the action of γ and on which γ acts with all eigenvalues < 1. In particular, $V^\infty \subset N$ which shows that $R \subset T_p N$.

Lemma 4.3 *Suppose $\rho \in M_B(X_o, n)$ is the Schrödinger representation we have considered above. Suppose that a group $\Gamma = \pi_1(B, o)$ acts, satisfying one of the hypotheses of Theorem A or Theorem B. Let $N = \mathbf{Near}(M_B(X_o, n), \rho, \Gamma)$. Then $T_\rho N$ is Zariski-dense in $T_\rho M_B(X_o, n)$.*

Proof. Recall that

$$T_\rho M_B(X_o, n) = W_{\mathbb{C}},$$

and that the action of Γ factors through a surjection to a finite index subgroup of $\mathrm{Sp}^\sharp_A(W)$ where $W \subset W_{\mathbb{C}}$ is a $\mathbb{Z}$-submodule preserved by the finite group A. Furthermore recall the decomposition

$$W_{\mathbb{C}} = \bigoplus_{C \in \mathfrak{C}(A)} W_{\mathbb{C},C}$$

and (up to subgroups of finite index)

$$\mathrm{Sp}^\sharp_A(W) \sim \prod_{C \in \mathfrak{C}(A)} U^\sharp(W_C) \sim \prod_{C \in \mathfrak{C}(A)} SU(W_C).$$

Furthermore recall that $W_{\mathbb{C},C}$ is in fact a direct sum of $\phi(d)$ copies of Φ_C and that the real group $U(W_{\mathbb{C},C})$ (containing $U(W_C)$ as a lattice) decomposes as a direct sum of copies of $U(\Phi_C, \varphi_C)$. The hermitian forms φ_C are not definite (otherwise we would be treating a lattice in a compact group, which would have to be finite). Therefore there is an element $\gamma(\Phi_C) \in U(\Phi_C, \varphi_C)$ which has an eigenvalue of absolute value < 1. It follows since $U(W_C)$ is a lattice in $U(W_{\mathbb{C},C})$, that there is an element $\gamma_C \in U(W_C)$ such that all of the projections in the $\phi(d)$ factors of the form $U(\Phi_C, \varphi_C)$, have eigenvalues of norm < 1. Hence there is a vector $v_C \in W_{\mathbb{C},C}$ which is in the span of the eigenvectors of γ_C corresponding to eigenvalues of absolute value < 1, and which has a nonzero component in every one of the complex irreducible factors of the representation $W_{\mathbb{C},C}$ of Γ. Taking the product over C of the elements γ_C (then raising to a power if necessary to get an element of our group Γ) and taking the sum of the vectors v_C, we obtain an element $\gamma \in \Gamma$ and a vector $v \in W_{\mathbb{C}}$ such that v is in the span of the eigenvectors of γ corresponding to eigenvalues of absolute value < 1, and such that v has a nonzero component in all of the irreducible factors of $W_{\mathbb{C}}$. Recall from the end of the proof of Proposition 3.1 that the complex Zariski closure of Γ contains a product of special linear groups (corresponding to the decomposition into irreducible pieces of the representation $W_{\mathbb{C}}$). Therefore the orbit of the vector v under the action of Γ is Zariski-dense in $T_\rho M_B(X_o, n)$. On the other hand, from our discussion prior to the present lemma, v is in the Γ-invariant subset $T_\rho N$. Thus $T_\rho N$ is Zariski-dense. □

Corollary 4.4 *The moduli space $M_B(X_o, n)$ contains a smooth point, the Schrödinger representation ρ, such that the set of points* $\mathbf{Near}(M_B(X_o, n), \rho, \Gamma) \subset M_B(X_o, n)$ *which are Γ-near to ρ, is Zariski-dense in $M_B(X_o, n)$.*

It is clear that any Γ-invariant regular (i.e. holomorphic algebraic) function takes the same value at ρ as at every point of $\mathbf{Near}(M_B(X_o, n), \rho, \Gamma)$. In particular, this corollary implies the result that there are no Γ-invariant regular functions. This result is weaker than our main results about nonexistence of invariant meromorphic functions, but does provide a slightly different conceptual route to the topological irreducibility result refered to above.

Our motivation for introducing the notion of Γ-nearness is that the first and second authors asked some time ago whether there was any sense in which the finite-image representations could take up a big place in $M_B(X_o, n)$. The short answer to that question is that, by Jordan's theorem, the finite image representations occupy a rather small place in that there are only finitely many outside of representations which factor through a normalizer of a torus (and those which factor in this way lie in a closed subset of relatively high codimension). However, Corollary 4.4 provides the slightly more subtle answer that, in the presence of a large monodromy action, if you start out very near to a certain finite-image represenation (such as one of our Schrödinger representations) and then let the monodromy act, then you can get out to a significant part of $M_B(X_o, n)$.

4.3 Other groups

Finally, let us explicitly state that we expect that all of the results and conjectures of this paper to hold for Betti and deRham cohomology with coefficients in an arbitrary complex reductive group G. Specifically we make the following

Conjecture 4.5 *Let $f : X \to B$ be a smooth algebraic family of curves and G a complex reductive group. Then:*

(i) *Assume that f is not isotrivial. Then the families*

$$M_B(X/B, G) \to B \quad \textit{and} \quad M_{DR}(X/B, G) \to B$$

of relative Betti and de Rham cohomology with coefficients in G are **GZD** *when equipped with the non-abelian Gauss-Manin connection.*

(ii) *Assume that f comes from a projective Lefschetz pencil of sufficiently high degree. Then there exists a smooth point $\rho \in M_B(X_o, G)$ which is fixed by a finite index subgroup $\Gamma \subset \pi_1(B, o)$ and for which the Zariski closure of Γ in $GL(T_\rho M_B(X_o, G))$ acts with an open orbit on $T_\rho M_B(X_o, G)$.*

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