MAGNETIC BLOCH FUNCTIONS AND VECTOR BUNDLES. TYPICAL DISPERSION LAWS AND THEIR QUANTUM NUMBERS

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I. In previous joint papers by the author and B. A. Dubrovin [1], [2] we computed completely the basic states of a two-dimensional, nonrelativistic electron with spin 1/2 in an external doubly periodic (in x and y) magnetic field $B(x, y)$ (directed along the z axis) and a zero electric field. The Hamiltonian in this case is the Pauli operator

$$H_0 = -\frac{1}{2}\left(\frac{\partial}{\partial x} - ieA_1\right)^2 - \frac{1}{2}\left(\frac{\partial}{\partial y} - ieA_2\right)^2 + e\sigma_3 B; \tag{1}$$

here $\hbar = m = c = 1$, $B = \partial_2 A_1 - \partial_1 A_2$, and $H_0\psi = \epsilon\psi$. Suppose that the lattice is rectangular, $z_{m,n} = mT_1 + inT_2$, and that the magnetic flux is integral and positive (generalization to a rational flux presents no difficulties):

$$\Phi = \iint_K B\,dx\,dy, \quad e\Phi = 2\pi N, \tag{2}$$

K is an elementary cell, $0 \le x \le T_1$, and $0 \le y \le T_2$.

Since $H_0\sigma_3 = \sigma_3 H_0$, we have a decomposition of the Hilbert space of square-summable, vector-valued functions ψ on the plane into a direct sum of two scalar spaces:

$$\mathcal{L}_2 = \mathcal{L}_2^{(+)} \oplus \mathcal{L}_2^{(-)}, \quad \sigma_3\psi = \pm\psi, \quad H_\pm \colon \mathcal{L}_2^{(\pm)} \to \mathcal{L}_2^{(\pm)}. \tag{3}$$

As was indicated in [3] for a localized field B, the basic states for $\Phi > 0$ are found in the periodic case only in the space $\mathcal{L}_2^{(+)}$ and have energy $\epsilon = 0$:

$$H_+\psi = 0. \tag{4}$$

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The formulas for the basic states are as follows (see [1] and [2]; for the properties of σ see [4]):

$$
\begin{aligned}
\psi = \psi_A &= \lambda \exp(-e\phi) \prod_{j=1}^{N} \sigma(z - a_j) \exp(az), \\
\phi &= \frac{1}{2\pi} \iint_K \ln|\sigma(z - z')|\, B(x', y')\, dx'\, dy', \quad z = x + iy, \\
\sigma(z) &= z \prod_{m^2+n^2\neq 0} (1 - z/z_{m,n}) \exp\left\{ z/z_{m,n} - \frac{1}{2} z^2/z_{m,n}^2 \right\}, \\
\operatorname{Re} a &= \operatorname{Re}\left\{ \frac{\eta_1}{T_1}\left[2\sum_{j=1}^{N} a_j - \frac{e}{\pi}\iint_k zB\, dx\, dy \right]\right\}, \\
\operatorname{Im} a &= \operatorname{Im}\left\{ \frac{\eta_2}{T_2}\left[2\sum_{j=1}^{N} a_j - \frac{e}{\pi}\iint_k zB\, dx\, dy \right]\right\}, \\
A &= (a_1, a_2, \ldots, a_N, \lambda),
\end{aligned} \tag{5}
$$

where λ is any number, $\eta_1 = \xi(T_{1/2})$, $i\eta_2 = \xi(iT_{2/2})$ and $\xi(z) = \sigma'/\sigma$.

The states (5) are "magnetic Bloch" states, i.e., they are the eigenstates for the operators of "magnetic translations" T_1^* and T_2^*, which commute with the Hamiltonian and have unimodular eigenvalues (this is a projective representation of the discrete group of translations):

$$
\begin{aligned}
T_1^* \psi_A &= \exp(ip_1 T_1)\psi_A = \psi_A(x + T_1, y) \exp\{-ie\eta_1 \Phi y/\pi\}, \\
T_2^* \psi_A &= \exp(ip_2 T_2)\psi_A = \psi_A(x, y + T_2) \exp\{-ie\eta \Phi y/\pi\}, \\
p_1 + ip_2 &= \frac{2\pi i}{T_1 T_2} \sum a_j + \text{const}, \quad T_1^* T_2^* = T_2^* T_1^* \exp(-e\Phi).
\end{aligned} \tag{6}
$$

The states (5) form a complete basis in $\mathcal{L}_2$ of solutions of the equation $H_0\psi = \epsilon\psi$ for $\epsilon = 0$ and generate a subspace $\mathcal{L}_2^0$ in $\mathcal{L}_2$ which is distinguished by the direct sum $\mathcal{L}_2 = \mathcal{L}_2^0 \oplus \mathcal{L}_2^1$ in a manner similar to the case of a discrete level; according to [2], it is possible to choose a discrete basis of localized "Wannier states" in $\mathcal{L}_2^0$ in place of the continuous magnetic Bloch basis (5).

We note a useful supplement to a result of [1] and [2].

Theorem 1. *For any integral or rational flux $e\Phi = 2\pi N M^{-1}$ the basic states* (5) *are separated from the remaining energy levels* (*eigenvalues of $H_0\psi = \epsilon\psi$*) *by a finite gap $\Delta\{B\}$.*

Conjecture. The gap $\Delta\{B\}$ varies continuously with the magnetic field $B(x, y)$ in the class of doubly periodic fields with arbitrary periods (which may vary and therefore pass through irrational fluxes).

The proof of Theorem 1 for integral fluxes follows easily from the following consideration: fixing the quasimomentum (p_1, p_2), we obtain an elliptic, selfadjoint operator $H_0(p_1, p_2)$ in a bundle over a compact manifold—the torus T^2, where the connectivity is defined by the field B. Therefore, the spectrum $\epsilon_j(p_1, p_2)$ is discrete, of finite multiplicity, and depends continuously on the parameters p_1, p_2. Following [1], [2], we know that a) $\epsilon_0 = \epsilon_{\min}(p_1, p_2) = 0$ for all (p_1, p_2); and b) the dimension of this eigensubspace is equal to N, and it varies continuously together with the quasimomenta (p_1, p_2) (without bifurcations) because of formulas (5). Further, the next eigenvalue $\epsilon_1(p_1, p_2)$ is positive and depends continuously on (p_1, p_2). The equality $\epsilon_1(p_1^0, p_2^0) = 0$ is impossible because of the absence of bifurcation of the eigensubspace with level $\epsilon_0 = 0$. Therefore, $\Delta = \min_{(p_1,p_2)} \epsilon_1(p_1, p_2) > 0$.

For rational fluxes the proof also reduces to the proof for integral fluxes.

Remark. The two-dimensional Pauli operator (1) with a zero electric potential on the subspace $\mathcal{L}_2^{(+)}$ reduces to the scalar Schrödinger operator (with spin 0)

$$H = H_+ = -\frac{1}{2}\left(\frac{\partial}{\partial x} - ieA_1\right)^2 - \frac{1}{2}\left(\frac{\partial}{\partial y} - ieA_2\right)^2 + eV(x, y) \tag{7}$$

with a nonzero but special electric potential V:

$$\partial_1 A_2 - \partial_2 A_1 = V(x, y) \tag{8}$$

(in the system of units $c = \hbar = m = 1$). Under the condition (8) we denote the operator H by H_0. Later we shall also consider the general Schrödinger operator (7) where the condition (8) is not satisfied.

For the Schrödinger operator (7) we have two integrable cases: a) $V \equiv 0$ and the field $B = \text{const}$ is homogeneous; b) condition (7) is satisfied, but the field B is arbitrary (only the lowest level $\epsilon = 0$ can be integrated). In both cases we denote the operator H by H_0.

II. An important property of the basic states (5) (which also occurs for the Landau levels in the homogeneous field $B = \text{const}$) is that the magnetic Bloch functions (5) for, integral number of quanta of the flux $N \neq 0$ form a topologically nontrivial vector bundle over the torus T^2. Under variation of any a_j over a lattice period $a_j \to a_j + T_1$ or $a_j \to a_j + iT_2$ the σ-function is multiplied by an exponential. This variation is compensated by the variation of the quantity $a(a_1, \ldots, a_N)$ in (5):

$$a(\ldots, a_j + T_1, \ldots) = a + 2\eta_1, \quad a(\ldots, a_j + iT_2, \ldots) = a + 2i\eta_2. \tag{9}$$

We thus obtain a "gluing law" for the complete space E of the vector bundle ξ with is defined by (9); from this it follows that

$$\begin{aligned}
&(\lambda, a_1, a_2, \ldots, a_N) \simeq (\lambda, a_{i_1}, a_{i_2}, \ldots, a_{i_N}),\\
&(\lambda, a_1, a_2, \ldots, a_N) \simeq (\lambda', a_1, a_2, \ldots, a_j + T_1, \ldots, a_N),\\
&(\lambda, a_1, a_2, \ldots, a_N) \simeq (\lambda'', a_1, a_2, \ldots, a_j + iT_2, \ldots, a_N),\\
&\lambda' = \lambda \exp\{2\eta_1 a_j + \eta_1 T_1 + i\pi\}, \quad \lambda'' = \lambda \exp\{2i\eta_2 a_j - \eta_2 T_2 + i\pi\}.
\end{aligned} \tag{10}$$

As indicated in [1] and [2], for a fixed quasimomentum we have a vector space $C^N(p_1, p_2)$ of functions ψ_A: they are all obtained from ψ_{A_0} by multiplication by a meromorphic, doubly periodic elliptic function with the same lattice, i.e. $\psi_A = \psi_{A_0}\chi$. The function $\chi(z)$ must have poles at some of the points a_j, so that the product again has no poles.

Lemma 1. *The mapping of quasimomentum*

$$p = p_1 + ip_2 \colon E \to \frac{2\pi i}{T_1 T_2} \sum_j a_j + \text{const}$$

transforms the manifold E of all magnetic Bloch functions (5) *of the basic state* ($\epsilon = 0$) *into a vector bundle ξ with fiber C^N over the torus T^2 obtained from the* *reciprocal lattice* (T_1^{-1}, T_2^{-1}). *This bundle is topologically nontrivial for all* $N > 0$ *and has nonzero first Chern class* $c_1(\xi) = 1 \neq 0$.

This lemma is derived from (10) in a topologically standard way, and we shall not prove it.

Remark. This lemma is also true for the magnetic Bloch functions of any Landau level in a homogeneous field $B = \text{const}$.[1]

III. Of course, the very fact of the occurrence of a situation of rank N (i.e., a bundle ξ with an N-dimensional fiber) for the magnetic Bloch functions over the torus T^2 implies very strong degeneracy for $N \geq 2$. This degeneracy should vanish under small perturbations. We shall consider small perturbations of the Hamiltonian by an electric, doubly periodic potential $W(x, y)$ with the same periods

$$H = H_0 + eW(x, y), \tag{11}$$

where the operator H_0 is any of those studied in §§I and II. For $N = 1$ a small perturbation (and therefore also a perturbation which is not small) leads only to the formation of a "dispersion law" $\epsilon(p_1, p_2)$ and spreading of any Landau level (or basic state for the operator (7), (8)) in a single magnetic zone due to the connectedness of the torus T^2. The topology of the family of Bloch functions itself—the "dispersion law"—does not change under small deformation of the operator for $N = 1$ and remains the same as in §II for the operators H_0. Thus, consideration of the single-quantum case $N = 1$ may lead to the illusion that the topology of all dispersion laws, although it is not trivial, is nevertheless completely determined by the flux of the external magnetic field B through an elementary cell—by the single integer N (this is actually the case for any small perturbations of the field $B = \text{const}$ for $N = 1$).

We consider the Hermitian form $\hat{W}(\psi_A)$ on the fibers of the bundle ξ which is defined by a perturbation $W(x, y)$ with the same periods (p_1 and p_2 are fixed):

$$\hat{W}(\psi_A) = \iint_K \psi_A W \bar{\psi}_A \, dx \, dy. \tag{12}$$

Here there arise the real eigenvalues

$$\epsilon_1(p_1, p_2) \geq \epsilon_2(p_1, p_2) \geq \cdots \geq \epsilon_N(p_1, p_2)$$

of the form $\hat{W}$ on the fibers $C^N(p_1, p_2)$.

Lemma 2. a) *In the class of doubly periodic, real functions* $W(x, y)$ *the condition of coalescence* $\epsilon_i = \epsilon_j$ *for fixed* (p_1, p_2) *is given by three independent conditions on the Fourier coefficients* (*this is also true in three-dimensional space*). *In particular, for functions in "general position"* $W(x, y)$ *the coalescence* $e_i(p_1^0, p_2^0) = \epsilon_j(p_1^0, p_2^0)$ *for at least one quasimomentum* p_1^0, p_2^0 *of the given dispersion law has codimension* 1 *in the function space* (*i.e., it is realized only at isolated points with respect to the parameter* τ *for "typical" one-parameter families of potentials* $W_\tau(x, y)$).

[1] For a homogeneous field,

$$e\phi = \frac{e\Phi}{2\pi}\left[\frac{\eta_1}{T_1}x^2 - \frac{\eta_2}{T_2}y^2 - \eta_1 x + \eta_2 y\right].$$

The operator A^n takes the functions (5) into the Bloch functions of the nth Landau level, where

$$A = -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + \frac{eB}{2}\left[\bar{z} + z\left(\frac{T_1\eta_2}{\pi} - \frac{1}{2}\right)\right] - \frac{e\Phi}{4\pi}(\eta_1 + i\eta_2).$$

b) *In the three-dimensional space of the parameters* (p_1, p_2, τ) *there may be stable singular points* (p_1^0, p_2^0, τ_0) *such that* $\epsilon_i = \epsilon_j$ (*for only one pair* i, j) *and the restrictions* ξ_i^δ *and* ξ_j^δ *of the one-dimensional bundles* ξ_i *and* ξ_j *to a small sphere* S_δ^2 *of radius* δ *surrounding the singular point are nontrivial* (*although their sum is trivial*), $\xi_1^\delta \oplus \xi_2^\delta \sim 0$ *on* S_δ^2,

$$q = c_1(\xi_i^\delta) = -c_1(\xi_j^\delta). \tag{13}$$

On passing through the value of the parameter $\tau = \tau_0$ *the dispersion laws "collide" and are changed by the quantum number* $q = \pm 1$*:*

$$\begin{aligned} c_1 = (\xi_j)_{\tau_0 - \delta} = c_1(\xi_j)_{\tau_0 + \delta} + q, \\ c_1 = (\xi_i)_{\tau_0 - \delta} = c_1(\xi_i)_{\tau_0 + \delta} - q. \end{aligned} \tag{14}$$

c) *For the Schrödinger operator in three-dimensional space the quasimomentum* $p_3 = \tau$ *plays the role of the parameter* τ; *therefore, the condition of coalescence for one quasimomentum* (p_1^0, p_2^0, p_3^0) *is stable, and the situation of part* b) *occurs.*

The following result is established using Lemma 2.

Theorem 2. a) *In the case of a small perturbing potential* $W(x, y)$ *in "general position" the eigenvalues of the form* $\hat{W}(\epsilon_1(p_,p_2) > \epsilon_2(p_1, p_2) > \cdots > \epsilon_N(p_1, p_2))$ *are distinct for any* (p_1, p_2) *and provide a decomposition of the family* (*bundle*) *of magnetic Bloch functions* ξ *of the unperturbed operator* H_0 *into a direct sum of one-dimensional* (*fiber* C^1) *complex bundles*

$$\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_N \tag{15}$$

with the single condition on the first Chern class

$$c_1(\xi) = 1 = \sum_{j=1}^{N} c_1(\xi_j). \tag{16}$$

b) *The "monodromy group" generated by permutations of the eigenvalues* ϵ_j *under basic circuits of the torus* T^2 *is, in general position, always trivial. Therefore, precisely* N *"decay" dispersion laws* $\epsilon_j(p_1, p_2)$ *are formed, with topological quantum numbers* $c_1(\xi_j) = m_j$ *which can be any integers* (*positive or negative*) *with the single relation* (16). *These dispersion laws have rank* 1 (*i.e., the fibers are one-dimensional*) *and are therefore stable under further deformation* (*which is not small*).

c) *In the three-dimensional case the potential* $W(x, y, z)$ *occasions the decay of the family of magnetic Bloch functions* (*the bundle* ξ) *into a sum of bundles* $\xi_1, \ldots, \xi_k$ (*which are not necessarily one-dimensional*), *where each of the* ξ_j *has fiber of dimension* k_j *and decomposes into a sum of one-dimensional bundles after removal of the singular points from the torus* T^3 *according to the dispersion laws* $(\epsilon_{j,1}, \ldots, \epsilon_{j,k_j})$*:*

$$\epsilon_j = \sum_{s=1}^{k_j} \xi_{j,s} \quad \text{on } T^2 \setminus (P_{j1} \cup \cdots \cup P_{jm}), \tag{17}$$

where the branches $\epsilon_{js} = \epsilon_{jt}$ *with topological invariants* $q_{j\alpha}$ *coalesce at the points* $P_{j\alpha}$.

Thus, by performing further large perturbations, we arrive at the following

Conclusion. For a “general” two-dimensional Schrödinger operator (7) in a stationary magnetic field which is periodic in (x, y) with an integral flux $N \geq 2$ and an electric field with a periodic potential there are a countable number of dispersion laws $\epsilon_j(p_1, p_2)$ for the magnetic Bloch functions. These dispersion laws (i.e., Bloch functions) form one-dimensional (fiber C^1) bundles over the torus T^2 of the reciprocal lattice and have “quantum numbers” $c_1(\xi_j) = m_j$ in no way connected with one another or with the flux N of the external magnetic field in the energy range where the perturbations of different Landau levels are “mixed” and cannot be separated from one another.[2] In a homogeneous magnetic field for sufficiently high energy levels a doubly periodic electric potential $W(x, y)$ produces only a small perturbation of the levels of the homogeneous field. Therefore, the perturbed dispersion laws which arise from them do not overlap; condition (16) is satisfied for the dispersion laws arising from each Landau level individually. For the general three-dimensional Schrödinger operator the “typical” dispersion laws do not form only one-dimensional bundles over the torus T^3, and the pairs of branches ϵ_j and ϵ_k coalesce for singular values of the quasimomentum.

Remark 1. Comparison with some results of the author and Kričever (see [5]) on integrable cases of rank greater than 1 shows that the conclusion regarding the occurrence of dispersion laws with completely random quantum numbers is probably also valid for $N = 0$ in periodic problems of dimension ≥ 2. This is probably also true for $N = 1$ if the perturbing potential is not small. However, here there is not an “integrable case” of even one dispersion law or of rank > 1 that might provide a proof from consideration of small perturbations.

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[2]The corresponding Bloch function $\psi_j(x, x_0, p)$, where $\psi_j = 1$ for $x = x_0$, has an algebraic number of zeros, equal to N for fixed $p\{x_{jk}(p)\}$ and equal to m_j for fixed $x\{p_{jl}(x)\}$. The poles are located at points $p_{jl}(x_0)$.