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A CLASS OF INTEGRAL INEQUALITY AND APPLICATION

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Abstract. We discuss a class of generalized retarded nonlinear integral inequalities, which not only include nonlinear compound function of unknown function but also include retarded items, and give upper bound estimation of the unknown function by integral inequality technique. This estimation can be used as tool in the study of differential equations with the initial conditions.

Introduction

Gronwall-Bellman inequalities [1, 2] and their various generalizations can be used important tools in the study of existence, uniqueness, boundedness, stability and other qualitative properties of solutions of differential equations, integral equations and integral-differential equations.

Abdeldaim and Yakout [3]) discussed the inequality

$$u(t)\le\left(u_0+\int_0^t f(s)u(s)ds\right)^2+\int_0^t f(s)\left(\int_0^s g(\lambda)\left(u(\lambda)+\int_0^\lambda f(\tau)u(\tau)d\tau\right)d\lambda\right)ds.$$

In this paper, we discuss a class of retarded nonlinear integral inequality and give upper bound estimation of the unknown function by integral inequality technique.

Main result

In this section, we discuss a class of retarded nonlinear integral inequality of Gronwall-Bellman type. Throughout this paper, let I = [0; ∞).

Theorem 1. Let $\varphi,\ \varphi',\alpha\in C(I,I)$ be increasing functions with $\varphi'(t)\le k,\alpha(t)\le t,\alpha(0)=0,\forall t\in I$; k,u_0 be positive constants with $2ku_0\ge 1$; we assume that $u(t)$ and $f(t)$ are nonnegative real-valued continuous functions defined on I and satisfy the inequality

$$u(t)\le\left(u_0+\int_0^{\alpha(t)} f(s)\varphi(u(s))ds\right)^2 +\int_0^{\alpha(t)} f(s)\left(\int_0^s g(\lambda)\left(\varphi(u(\lambda))+\int_0^\lambda f(\tau)\varphi(u(\tau))d\tau\right)d\lambda\right)ds\ , \tag{2.1}$$

For all $t\in I$, If $2\left(u_0+\varphi\left(u_o^{\,2}\right)\right)\int_0^{\alpha(t)} kf(s)\exp\left(\int_0^s\left(g(s)/(2u_o)+f(s)\right)d\tau\right)ds<1,$

then

$$u(t)\le u^2(0)+\int_0^t\alpha'(s)f(\alpha(s))\theta_1(s)ds,\forall t\in I, \tag{2.2}$$

Where

$$\theta_1(t)=\exp\left(2\int_0^t k\alpha'(s)f(\alpha(s))\theta_2(s)ds\right)$$
$$\left(2\varphi\left(u_0^{\,2}\right)u_0-\int_0^t\alpha'(s)g(\alpha(s))\theta_2(s)\exp\left(-2\int_0^s k\alpha'(\tau)f(\alpha(\tau))\theta_2(\tau)d\tau\right)ds\right), \tag{2.3}$$

for all $t\in I$,

$$\theta_2(t)=\frac{\left(u_0+\varphi\left(u_0^{\,2}\right)\right)\exp\left(\int_0^{\alpha(t)}\left(g(s)/(2u_0)+f(s)\right)ds\right)}{1-2\left(u_0+\varphi\left(u_0^{\,2}\right)\right)\int_0^{\alpha(t)}kf(s)\exp\left(\int_0^{s}\left(g(s)/\left(2u_{0,}\right)+f(s)\right)d\tau\right)ds},\forall t\in I. \tag{2.4}$$

Proof. Let $z(t)$ denote the function on the right-hand side of (2.1), which is a nonnegative and nondecreasing function on I with $z(0)=u_0^{\,2}$. Then (2.1) is equivalent to

$$u(t)\le z(t),u\left(\alpha(t)\right)\le z(\alpha(t))\le z(t),\forall t\in I. \tag{2.5}$$

Differentiating $z(t)$ with respect to t, using (2.5) we have

$$\begin{aligned}\frac{dz}{dt}&=2\alpha'(t)f\left(\alpha(t)\right)\varphi\left(u\left(\alpha(t)\right)\right)\left(u_0+\int_0^{\alpha(t)}f(s)\varphi\left(u(s)\right)ds\right)\\&+\alpha'(t)f\left(\alpha(t)\right)\int_0^{\alpha(t)}g(\lambda)\left(\varphi\left(u(\lambda)\right)+\int_0^{\lambda}f(\tau)\varphi\left(u(\tau)\right)d\tau\right)d\lambda\le\alpha'(t)f\left(\alpha(t)\right)\\&\left(2\varphi\left(z(t)\right)\left(u_0+\int_0^{\alpha(t)}f(s)\varphi\left(z(s)\right)ds\right)+\int_0^{\alpha(t)}g(\lambda)\left(\varphi\left(z(\lambda)\right)+\int_0^{\lambda}f(\tau)\varphi\left(z(\tau)\right)d\tau\right)d\lambda\right)\\&=\alpha'(t)f\left(\alpha(t)\right)Y_1(t),\forall t\in I,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}Y_1(t)&:=2\varphi\left(z(t)\right)\left(u_0+\int_0^{\alpha(t)}f(s)\varphi\left(z(s)\right)ds\right)\\&+\int_0^{\alpha(t)}g(\lambda)\left(\varphi\left(z(\lambda)\right)+\int_0^{\lambda}f(\tau)\varphi\left(z(\tau)\right)d\tau\right)d\lambda,\forall t\in I.\end{aligned} \tag{2.7}$$

Hence $Y_1(0)=2\varphi\left(u_0^{\,2}\right)u_0$, and $\varphi(z(\alpha(t)))\le\varphi\left(z(t)\right)\le Y_1(t)/(2u_0)$. Differentiating $Y_1(t)$ with respect to t, and using (2.6) and (2.7), we get

$$\begin{aligned}\frac{dY_1(t)}{dt}&\le2\varphi'\left(z(t)\right)\alpha'(t)f\left(\alpha(t)\right)Y_1(t)\left(u_0+\int_0^{\alpha(t)}f(s)\varphi\left(z(s)\right)ds\right)+2\alpha'(t)\varphi\left(z(t)\right)f\left(\alpha(t)\right)\\&*\varphi\left(z\left(\alpha(t)\right)\right)+\alpha'(t)g\left(\alpha(t)\right)\left(\varphi\left(z\left(\alpha(t)\right)\right)+\int_0^{\alpha(t)}f(\tau)\varphi\left(z(\tau)\right)d\tau\right)\\&\le2k\alpha'(t)f\left(\alpha(t)\right)Y_1(t)\left(u_0+\int_0^{\alpha(t)}\frac{f(s)Y_1(s)}{2u_0}ds\right)+\alpha'(t)f\left(\alpha(t)\right)\frac{Y^2{}_1(t)}{2u_0^{\,2}}\\&+\alpha'(t)g\left(\alpha(t)\right)\left(\frac{Y_1(t)}{2u_0}+\int_0^{\alpha(t)}\frac{f(s)Y_1(s)}{2u_0}ds\right)\\&\le\left(2k\alpha'(t)f\left(\alpha(t)\right)Y_1(t)+\alpha'(t)g\left(\alpha(t)\right)\right)\left(u_0+\frac{Y_1(t)}{2u_0}+\int_0^{\alpha(t)}\frac{f(s)Y_1(s)}{2u_0}ds\right).\end{aligned} \tag{2.8}$$

for all $t\in I$, where we use the conditions $\varphi'(t)\le k,2ku_0\ge1$. Let

$$Y_2(t)=u_0+\frac{Y_1(t)}{2u_0}+\int_0^{\alpha(t)}\frac{f(s)Y_1(s)}{2u_0}ds,$$

then $Y_2(0)=u_0+\varphi\left(u^2{}_0\right),Y_1(t)<2u_0Y_2(t)$. Differentiating $Y_2(t)$ with respect to t, and using (2.8), we have

$$\begin{aligned}\frac{dY_2(t)}{dt}&\le\frac{2k\alpha'(t)f\left(\alpha(t)\right)Y_1(t)+\alpha'(t)g\left(\alpha(t)\right)}{2u_0}Y_2(t)+\frac{\alpha'(t)f\left(\alpha(t)\right)Y_1\left(\alpha(t)\right)}{2u_0}\\&\le2k\alpha'(t)f\left(\alpha(t)\right)Y_2^{\,2}(t)+\left(\alpha'(t)g\left(\alpha(t)\right)/(2u_0)+\alpha'(t)f\left(\alpha(t)\right)\right)Y_2(t),\forall t\in I.\end{aligned} \tag{2.9}$$

Since $Y_2(t)>0$, from (2.9) we have

$$Y_2^{-2}(t)\frac{dY_2(t)}{dt} \le 2k\alpha'(t)f(\alpha(t)) + \left(\alpha'(t)g(\alpha(t))/(2u_0) + \alpha'(t)f(\alpha(t))\right)Y_2^{-1}(t), \forall t \in I. \quad (2.10)$$

Let $S_1(t) = Y_2^{-1}(t)$, then $S_1(0) = \left(u_0 + \varphi\left(u_0^2\right)\right)^{-1}$, from (2.10) we obtain

$$\frac{dS_1(t)}{dt} + \left(\alpha'(t)g(\alpha(t))/(2u_0) + \alpha'(t)f(\alpha(t))\right)S_1(t) \ge -2k\alpha'(t)f(\alpha(t)), \forall t \in I. \quad (2.11)$$

Consider the ordinary derential equation

$$\frac{dS_2(t)}{dt} + \left(\alpha'(t)g(\alpha(t))/(2u_0) + \alpha'(t)f(\alpha(t))\right)S_2(t) = -2k\alpha'(t)f(\alpha(t)), \forall t \in I, \quad (2.12)$$

$$S_2(0) = \left(u_0 + \varphi\left(u_0^2\right)\right)^{-1}, \quad (2.13)$$

The solution of Equation (2.12) and (2.13) is

$$S_2(t) = \left(u_0 + \varphi\left(u_0^2\right)\right)^{-1}\exp\left(-\int_0^t\left(\alpha'(s)g(\alpha(s))/(2u_0) + \alpha'(s)f(\alpha(s))\right)ds\right)$$

$$\left(1 - \left(u_0 + \varphi\left(u_0^2\right)\right)\int_0^t 2k\alpha'(s)f(\alpha(s))\exp\left(\int_0^s\left(\alpha'(\tau)g(\alpha(\tau))/(2u_0) + \alpha'(\tau)f(\alpha(\tau))\right)d\tau\right)ds\right)$$

$$= \left(u_0 + \varphi\left(u_0^2\right)\right)^{-1}\exp\left(-\int_0^{\alpha(t)}\left(g(s)/(2u_0) + f(s)\right)ds\right)$$

$$\left(1 - 2\left(u_0 + \varphi\left(u_0^2\right)\right)\int_0^{\alpha(t)} kf(s)\exp\left(\int_0^s\left(g(s)/(2u_0) + f(s)\right)d\tau\right)ds\right), \forall t \in I. \quad (2.14)$$

By (2.4), (2.11), (2.12),(2.13) and (2.14), we obtain

$$Y_2(t) = \frac{1}{S_1(t)} \le \frac{1}{S_2(t)} = \theta_2(t), \forall t \in I. \quad (2.15)$$

Where $\theta_2(t)$ as defined in (2.4). From (2.8) and (2.15), we have

$$\frac{dY_1(t)}{dt} \le 2k\alpha'(t)f(\alpha(t))\theta_2(t)Y_1(t) + \alpha'(t)g(\alpha(t))\theta_2(t), \forall t \in I.$$

The above inequality implies the following estimation for $Y_1(t)$:

$$Y_1(t) \le \theta_1(t), \forall t \in I. \quad (2.16)$$

Where $\theta_1(t)$ as defined in (2.3), thus from (2.6) and (2.16) we have

$$\frac{dz}{dt} \le \alpha'(t)f(\alpha(t))\theta_1(t), \forall t \in I. \quad (2.17)$$

By taking $t = s$ in (2.17) and integrating (2.17) from 0 to t, we get

$$u(t) \le z(t) \le u^2(0) + \int_0^{\alpha(t)}\alpha'(s)f(\alpha(s))\theta_1(s)ds, \forall t \in I.$$

The estimation (2.2) of unknown function in the inequality (2.1) is obtained.

Application

In this section we give an example for the application of our Theorem 1 to the following retarded differential-integral equation

$$\frac{dx(t)}{dt} = H\left(t, x(t), K\left(\alpha(t), x(\alpha(t))\right), K\left(t, x(t)\right)\right), t \in I_0 = [t_0, \infty), x(t_0) = x_0, \quad (3.1)$$

where $\alpha(t)$ satisfies $\alpha(t) \le t, 0 < \alpha'(t) \le 1$ and $\alpha(t_0) = t_0$. We assume that $K \in C(I_0, R)$, $H \in C(I_0, R \times R)$ and

$$H\left(t,x(t),K\left(\alpha(t),x\left(\alpha(t)\right)\right),K\left(t,x(t)\right)\right)=K\left(\alpha(t),x\left(\alpha(t)\right)\right)\left(2u_0+\int_{t_0}^{\alpha(t)}K\left(\tau,x(\tau)\right)d\tau\right)$$

$$+f\left(\alpha(t)\right)\left(\int_{t_0}^{\alpha(t)}g(\lambda)\left(\varphi\left(x(\lambda)\right)+\int_{t_0}^{\lambda}K\left(\tau,x(\tau)\right)d\tau\right)d\lambda\right). \tag{3.2}$$

$$\left|K\left(t,x(t)\right)\right|\le f(t)\varphi\left(x(t)\right). \tag{3.3}$$

where f,g are nonnegative real-valued continuous functions defned on I_0.

Corollary1. Consider nonlinear system (3.1) and suppose that $\varphi,\varphi'\in C^1\left(I_0,I_0\right)$ be increasing functions with $\varphi'(t)\le k,\forall t\in I;$ and K,H， satisfy the conditions (3.2) and (3.3), then

$$\left|x(t)\right|\le u_0^2+\int_0^t\alpha'(s)f\left(\alpha(s)\right)\theta_1(s)ds,\forall t\in I_0. \tag{3.4}$$

Where $\theta_1(t)$ is given by (2.3) in Theorem 1, $u_0^2=\left|x_0\right|$.

Proof. Applying Theorem 1 we get the estimation (3.16).

Summary

In the qualitative analysis of some classes of differential equations, it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. In current paper we study retarded differential inequalities and give corresponding estimations of the unknown functions, which provide explicit bounds used as tools in the study of boundedness, uniqueness, continuous dependence of the solution of initial value problem of a differential equation. We also give an example to illustrate the effectiveness of our results in estimation of solutions of some differential-integral equations.

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