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Generating sets for uncleft algebras

Elsie Christine Muller
Iowa State University

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GENERATING SETS FOR UNCLEFT ALGEBRAS

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Elsie Christine Muller

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I. INTRODUCTION

A. Algebras Generated by Every Residue System (Gers Algebras)

A finite dimensional algebra with identity may possess a residue system modulo its radical N such that all of A is generated algebraically by this residue system. This property is generally not an invariant of the algebra. A class of algebras for which it is invariant is the object of study in this thesis.

Definition. If every residue system modulo the radical N generates A, then A will be called a gers algebra.

A simple example of such an algebra is the following commutative algebra, A, over a field, K, of characteristic 2. Let $\underline{a}$ denote the coset represented by a.

$(\underline{\alpha}_1, \underline{\alpha}_2)$ with $\alpha_1 = 1$ is the basis of A/N.
$(\underline{u}_1, \underline{u}_2)$ is the basis of N/N^2.

The multiplication table is displayed below.

1	α_2	u_1	u_2
1	α_2	u_1	u_2
α_2	$f + u_1$	u_2	
u_1	u_2		
u_2			

Here, $f \in K-K^p$.

B. Main Problems Investigated in this Thesis

First, a characterization is sought of a gers algebra in terms of linear bases of A/N and N/N^2. Such a characterization will focus attention on constructive possibilities for such algebras. Second, necessary conditions are sought for the existence of special classes of gers algebras which will be described in Chapter II. In general, we will concentrate on results obtainable from a coordinatization of the algebras, rather than from a coordinate-free approach such as has been used in the general analysis of maximally uncleft algebras by Vinograde and Weeg [4].

C. Narrowing the Set of Algebras that Need to be Considered

We first note that if an algebra is gers then it must have a base field of finite characteristic, otherwise there exists at least one residue system modulo N such that $A = S + N$, S is a semi-simple algebra. This follows from what is sometimes called Wedderburn's Principal Theorem (see p. 47 of [1]) which states, "Let A be an algebra with radical N such that $A - N$ is separable. Then $A = S + N$, where S is zero or an algebra, and hence S is separable and equivalent to $A - N$." We see that even if the characteristic is finite, the algebra may not be gers. For example, if K is an arbitrary field and A, B, C are total matrix algebras over K, then $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ is not gers.

Next, we note that we can restrict ourselves to algebras which are completely primary, (hence, A/N is a division algebra) because essentially A must be a total matrix algebra over a gers completely primary algebra (see [4]).

D. Main Results

The main results will be found in Theorem 2 of Chapter II and Theorem 3 of Chapter III. We will show in Theorem 2 that a necessary and sufficient condition for an algebra, A, to be gers is that every set $\{\underline{a}^1, \underline{a}^2, \underline{a}^3\}$ contains a linear basis for A/N^2. Theorem 3 will derive the rank criterion for an algebra to be gers.

II. COORDINATIZING A GERS ALGEBRA

A. Bases for A/N

Suppose A/N has linear dimension s over K, and let $\alpha_1, \cdots, \alpha_s$ be elements of A which are linearly independent mod N. We shall also consider the products $\alpha_i \alpha_j$ and $\alpha_i \alpha_j \alpha_k$. Let $\underline{\alpha}^1$, $\underline{\alpha}^2$, $\underline{\alpha}^3$ refer to the collection of these products ordered lexicographically.

We first note that in an algebra, a linear basis of A/N will yield a residue system for A/N, and conversely, every residue system can be so generated from some linearly independent system. For a linear basis of A/N is a set of cosets belonging to A/N such that the representatives of these cosets are linearly independent modulo N, and each element of A/N can be expressed as a linear combination of the elements of the linear basis of A/N. Conversely, for every linearly independent set of representatives modulo N, all possible linear combinations of these elements generate a residue system.

<u>Lemma 1</u>. Every residue system consisting of linear combinations of $\underline{\alpha}^1$ (that is, of the set $\alpha_1, \cdots, \alpha_s$) generates (algebraically) all of A (mod N^2) if, and only if, the same is true when α_1 is restricted to 1.

Proof: Suppose whenever $\alpha_1 = 1$ that A is generated by $\underline{\alpha}^1$ (mod N^2). We must prove that A is gers (mod N^2); that is, we can get $\alpha_1 \neq 1$ from an $\underline{\alpha}^1$ where $\alpha_1 = 1$. Now,

$$(1 + n)^2 = (1 + n) + n \qquad (\mathrm{mod}\ N^2)$$

Hence, if $\alpha_1 = 1 + n$, then $\underline{\alpha}^1$ generates n $(\alpha_1^2 - \alpha_1 \equiv n)$ and $\alpha_1 - n = 1$. We are using the fact that in an algebra, any residue system must contain a representative, say $1 + n$, of the coset $1 + N$. Hence, α_1 can be chosen as $1 + n$ in a linear basis mod N.

<u>Lemma 2</u>. N is generated (algebraically) by any linear basis for N (modulo N^2).

Proof: Let $\underline{u}_1, \cdots, \underline{u}_\delta$ be a linear basis of N/N^2. Then $u_1, \cdots, u_\delta \in N$, but $\notin N^2$, for otherwise $\underline{u}_1, \cdots, \underline{u}_\delta$ would not be linearly independent modulo N^2.

Since N is a nilpotent radical, there exists some λ such that $N^\lambda = \{0\}$. Therefore, the product of any λ elements of N will be zero. Now,

$$N \supset N^2 \supset N^3 \supset \cdots \supset \{0\}.$$

Suppose $n \in N^i$ for some i. Then

$$n = \sum n_{a_1} n_{a_2} \cdots n_{a_i} \tag{1}$$

since each element of N^i is the finite sum of finite products with at least i factors. Each factor is in N. Some of these factors may also be a finite sum of finite products. For example, if

$$n_{a_i} = \sum n_{b_1} \cdots n_{b_j} ,$$

substitute in (1) for n_{a_i} and expand by the distributive law. If a term contains λ factors, the product is zero because N is nilpotent. The number of times such substitutions can be made is finite. If this were not the case, n could no longer be written as a finite sum of finite products.

After each n_k can no longer be factored, then n will be the sum of a finite number of terms, and each term will contain less than λ factors. Thus,

$$n = \sum_{k=1}^{p} n_{k_1} n_{k_2} \cdots n_{k_s} , \quad s \leq \lambda - 1.$$

Now,

$$n_{k_1} n_{k_2} \cdots n_{k_{\lambda-1}} \in N^{\lambda-1} , \quad n_{k_1} n_{k_2} \cdots n_{k_{\lambda-2}} \in N^{\lambda-2} ,$$
$$\ldots, \quad n_{k_1} n_{k_2} \in N^2, \quad \text{but} \quad n_{k_1} \notin N^2$$

since n_{k_1} is not a sum of factorable terms. Therefore,

$$n_{k_i} \not\equiv 0 \pmod{N^2} , \quad \text{but} \quad n_{k_i} \in N.$$

Since

$$\underline{n}_i = \sum_{\sigma=1}^{\delta} f_{i\sigma} \underline{u}_\sigma ,$$

therefore

$$n_i = \sum_{\sigma=1}^{\delta} f_{i\sigma}u_\sigma + n_i^*, \quad n_i^* \in N^2,$$

and

$$n_1n_2 = (\sum f_{1\sigma}u_\sigma + n_1^*)(\sum f_{2\sigma}u_\sigma + n_2^*).$$

Consequently, n_1n_2 is a sum of products of the u_σ's, and similarly for other monomials, so the representatives of the basis of N/N^2 generate all of N.

Theorem 1. A is gers if, and only if, every linear basis $\underline{\alpha}^1$ with $\alpha_1 = 1$ of A (modulo N) generates a linear basis of N (modulo N^2).

Proof: If A is gers, then by definition the statement is correct. Conversely, if every $\underline{\alpha}^1$ generates such a linear basis of N (modulo N^2), then by Lemma 2, $\underline{\alpha}^1$ generates all of A. Hence, by Lemma 1 all residue systems will generate A.

B. Generation of Elements of N (Modulo N^2)

We now wish to indicate how $\underline{\alpha}^1$, $\underline{\alpha}^2$, $\underline{\alpha}^3$ generates the linear basis for N/N^2. More generally, we prove

Theorem 2. A is gers if, and only if, every set $\{\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3\}$ contains a linear basis for A/N^2.

Proof: If A is gers, $\underline{\alpha}^1 = \{\alpha_1, \cdots, \alpha_s\}$ generates A. For products two at a time we note that

$$\alpha_i \alpha_j = \sum f_\sigma^{ij} \alpha_\sigma + n_{ij} \pmod{N^2} ,$$
$$f_\sigma^{ij} \in K , \quad n_{ij} \in N. \tag{1}$$

Hence,

$$n_{ij} = \alpha_i \alpha_j - \sum f_\sigma^{ij} \alpha_\sigma \pmod{N^2}.$$

But these n_{ij}'s may not give all of N/N^2. The remainder of N/N^2 can be traced by considering

$$(\alpha_i \alpha_j)\alpha_k = (\sum_\sigma f_\sigma^{ij}\alpha_\sigma)\alpha_k + n_{ij}\alpha_k \pmod{N^2} ;$$

hence, $n_{ij}\alpha_k$ is a linear combination of $(\alpha^1, \alpha^2, \alpha^3) \pmod{N^2}$. Now, consider for instance $\alpha_i \alpha_j \alpha_k \alpha_\ell$.

$$(\alpha_i \alpha_j)(\alpha_k \alpha_\ell) = (\sum f_\sigma^{ij}\alpha_\sigma + n_{ij})(\sum f_\tau^{k\ell}\alpha_\tau + n_{k\ell}) \pmod{N^2}$$
$$= \sum_{\sigma,\tau} f_\sigma^{ij} f_\tau^{k\ell} \alpha_\sigma \alpha_\tau + \sum_\sigma f_\sigma^{ij}\alpha_\sigma n_{k\ell}$$
$$+ \sum_\tau f_\tau^{k\ell} n_{ij} \alpha_\tau \pmod{N^2}.$$

As shown above,

$$n_{ij}\alpha_\tau = (\alpha_i \alpha_j)\alpha_\tau - (\sum f_\sigma^{ij}\alpha_\sigma)\alpha_\tau \pmod{N^2}.$$

Similarly,

$$n_{k\ell}\alpha_\sigma = (\alpha_k \alpha_\ell)\alpha_\sigma - (\sum f_s^{k\ell}\alpha_s)\alpha_\sigma \pmod{N^2}.$$

Thus, $\alpha_1\alpha_j\alpha_k\alpha_\ell$ is a linear combination of $\left\{\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3\right\}$ (mod N^2).

Now by induction any monomial can be expressed in terms of $\left\{\alpha^1, \alpha^2, \alpha^3\right\}$ (mod N^2). Assume $\alpha_1\alpha_2\alpha_3\cdots\alpha_n$, $n = k$, is a linear combination of $\left\{\alpha^1, \alpha^2, \alpha^3\right\}$ (mod N^2). Therefore,

$$\alpha_1\alpha_2\alpha_3\cdots\alpha_k = \sum_{\sigma,\tau,\mu} f_\sigma f_\tau f_\mu \alpha_\sigma\alpha_\tau\alpha_\mu + \sum_{\sigma,\tau} f_\tau f_\sigma \alpha_\sigma\alpha_\tau$$

$$+ \sum_{\sigma} f_\sigma \alpha_\sigma + n_{ij} \quad (\text{mod } N^2)$$

Also,

$$(\alpha_1\alpha_2\alpha_3\cdots\alpha_k)\alpha_{k+1} = (\sum_{\sigma,\tau,\mu} f_\sigma f_\tau f_\mu \alpha_\sigma\alpha_\tau\alpha_\mu)\alpha_{k+1}$$

$$+ (\sum_{\sigma,\tau} f_\tau f_\sigma \alpha_\sigma\alpha_\tau)\alpha_{k+1} + (\sum_{\sigma} f_\sigma\alpha_\sigma)\alpha_{k+1}$$

$$+ n_{ij}\alpha_{k+1} \quad (\text{mod } N^2)$$

From (1)

$$n_{ij}\alpha_{k+1} = (\alpha_i\alpha_j)\alpha_{k+1} - (\sum f_\sigma^{ij}\alpha_\sigma)\alpha_{k+1} \quad (\text{mod } N^2)$$

Therefore, $\alpha_1\alpha_2\alpha_3\cdots\alpha_k\alpha_{k+1}$ is a linear combination of $\left\{\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3\right\}$ (mod N^2), and $\alpha_1\alpha_2\alpha_3\cdots\alpha_n$ is a linear combination of $\left\{\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3\right\}$ (mod N^2) for every n. Thus, $\left\{\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3\right\}$ contains a linear basis for A/N^2.

Conversely, if $\left\{\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3\right\}$ is a linear basis for A/N^2, it contains a linear basis for N/N^2 since N/N^2 is a subset of A/N^2. Therefore, $\{\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3\}$ contains a linear basis for N and A is gers.

C. Dimension of N/N^2

Since it is sufficient to assume A/N is a division algebra, we can limit the cases to be considered by applying the following lemma.

Lemma 3. The dimension δ of N/N^2 over K is a multiple of s, the dimension of A/N over K.

Proof: Since A/N is a division algebra of finite dimension over K, N/N^2 is a vector space of finite dimension over A/N, because an element, a, may be identified mod N when multiplication is defined by $a(n + N^2) = an + N^2$. Hence,

$$[N/N^2 : K] = [N/N^2 : A/N][A/N : K] ;$$

that is,

$$\delta = [N/N^2 : A/N]s ,$$

(see Proposition 1 on p. 157 of [2]).

III. THE RANK CRITERION FOR AN ALGEBRA TO BE GERS

A. Comparison of Two Bases

Let $\underline{u} = (u_1, \ldots, u_\delta)$, where $\underline{u}_1, \ldots, \underline{u}_\delta$ is a basis of N/N^2.

Lemma 4. If $\underline{\alpha}^1$ and $\underline{\beta}^1$ are any two linear bases for A/N^2, it is sufficient for the study of gers algebras to fix $\underline{\alpha}^1$ and set $\underline{\beta}^1 = \underline{\alpha}^1 + \underline{u}D$, where D is a δ x s matrix with coefficients in K.

Proof: In general, $\underline{\beta}^1 = \underline{\alpha}^1 + \underline{u}D$, where S is a non-singular, s x s matrix. We wish to show that S can be restricted to the identity matrix I. Suppose $\underline{\beta}^1 = \underline{\alpha}^1 S$ where $\underline{\alpha}^1$ generates A and S is non-singular. Then $\underline{\alpha}^1 = \underline{\beta}^1 S^{-1}$, and $\underline{\beta}^1$ generates A. Hence, there is no need to take $S \neq I$, but it is necessary to distinguish $\underline{\alpha}^1$ from $\underline{\alpha}^1 + \underline{u}D$.

Lemma 5. If A is gers, $\underline{\alpha}^1$ a linear basis of A (mod N), $\underline{u}$ a linear basis of N (mod N^2), and $\underline{\beta}^1$ is any other linear basis for A (mod N), then

$$(\underline{\alpha}^1, \underline{\alpha}^2, \underline{\alpha}^3) \equiv (\underline{\alpha}, \underline{u})C_\alpha \pmod{N^2}$$

$$(\underline{\beta}^1, \underline{\beta}^2, \underline{\beta}^3) \equiv (\underline{\alpha}, \underline{u})C_\beta \pmod{N^2}$$

where

$$C_\alpha = \begin{bmatrix} I & M_2 & M_3 \\ 0 & N_2 & N_3 \end{bmatrix} \quad \text{and} \quad C_\beta = \begin{bmatrix} I & S_2 & S_3 \\ D & D_2 & D_3 \end{bmatrix}$$

with D arbitrary.

$$C_\beta = \begin{bmatrix} I & M_2 & M_3 \\ D & N_2 + N^* & N_3 + N^{**} \end{bmatrix},$$

where N^* and N^{**} are linear functions of the coefficients of D,

$$N^* = \sum N^*_{\sigma\tau} d_{\sigma\tau} \quad \text{and} \quad N^{**} = \sum N^{**}_{\sigma\tau} d_{\sigma\tau} \; .$$

Proof:

$$\beta_i \beta_j \equiv (\alpha_i + \sum u_\sigma d_{\sigma i})(\alpha_j + \sum u_\sigma d_{\sigma j})$$

$$\equiv \alpha_i \alpha_j + \sum u_\sigma \alpha_j d_{\sigma i} + \sum \alpha_i u_\sigma d_{\sigma j} \pmod{N^2}.$$

Therefore,

$$\underline{\beta}^2 = \underline{\alpha}^2 + \cdots \pmod{N^2}$$

$$\underline{\alpha}^2 = \underline{\alpha} M_2 + \underline{u} N_2 \; .$$

Hence,

$$\underline{\beta}^2 \equiv (\underline{\alpha}\, M_2 + \underline{u}\, N_2) + \underline{u}\, N^*$$

$$\equiv \underline{\alpha}\, M_2 + \underline{u}(N_2 + N^*) \pmod{N^2}$$

with

$$N^* = \sum N^*_{\sigma\tau} d_{\sigma\tau} \ .$$

Similarly,

$$\beta_i \beta_j \beta_k \equiv \alpha_i \alpha_j \alpha_k + \alpha_i \alpha_j \sum u_\sigma d_{\sigma k}$$
$$+ \alpha_i \sum u_\sigma d_{\sigma j} \alpha_k + \sum u_\sigma d_{\sigma i} \alpha_j \alpha_k \qquad (\text{mod } N^2)$$

Therefore,

$$\underline{\beta}^3 = \underline{\alpha}^3 + \cdots \qquad (\text{mod } N^2)$$

$$\underline{\alpha}^3 = \underline{\alpha}\, M_3 + \underline{u}\, N_3$$

Hence,

$$\underline{\beta}^3 = \underline{\alpha}\, M_3 + \underline{u}(N_3 + N^{**}) \qquad (\text{mod } N^2)$$

with

$$N^{**} = \sum N^{**}_{\sigma\tau} d_{\sigma\tau} \ .$$

Consequently,

$$S_i = M_i \ , \quad D_2 = N_2 + N^* \ , \quad \text{and} \quad D_3 = N_3 + N^{**} \ .$$

B. A Necessary and Sufficient Condition for A to be Gers

Based on the above analysis of linear bases for A/N and N/N^2, we derive the following matrix criterion for A to be gers.

Theorem 3. A is gers if, and only if, the rank of $(D, N_2 + N^*,$

$N_3 + N^{**})$, or of $(0,\ N_2 + N^* - DM_2,\ N_3 + N^{**} - DM_3)$ is δ for arbitrary choice of D, where D is a $\delta \times s$ matrix whose first column is zero, $(\alpha_1 = 1)$, and therefore (N_2, N_3) must be of rank δ.

Proof: If A is gers, then

$$(\underline{\alpha}^1,\ \underline{\alpha}^2,\ \underline{\alpha}^3) = (\underline{\alpha},\ \underline{u})C_\alpha \qquad (\text{mod } N^2)$$

where C_α is the matrix, $\begin{bmatrix} I & M_2 & M_3 \\ 0 & N_2 & N_3 \end{bmatrix}$. Let $C_\alpha = (c_{ij})$.

Then

$$(\underline{\alpha},\ \underline{u})C_\alpha = \left[\cdots, \sum_{i=1}^{s} \alpha_i c_{ij} + \sum_{i=1}^{\delta} u_i c_{(s+i)j}, \cdots\right].$$

$(\underline{\alpha}^1,\ \underline{\alpha}^2,\ \underline{\alpha}^3)$ contains $s + \delta$ linearly independent elements. Therefore, $(\underline{\alpha},\ \underline{u})C_\alpha$ contains $s + \delta$ linearly independent elements. The rank of C_α must be $s + \delta$ for if this were not so, there would be a linear relation among any $s + \delta$ columns of C_α and $(\underline{\alpha},\ \underline{u})C_\alpha$ would not contain $s + \delta$ linearly independent elements. Since I has rank s, (N_2, N_3) is of rank δ.

If $\underline{\beta}^1$ is an arbitrary basis for A/N, then $(\underline{\beta}^1,\ \underline{\beta}^2,\ \underline{\beta}^3)$ is a basis for A/N^2. From Lemma 5,

$$(\underline{\beta}^1,\ \underline{\beta}^2,\ \underline{\beta}^3) \equiv (\underline{\alpha},\ \underline{u})C_\beta \qquad (\text{mod } N^2)$$

where

$$C_\beta = \begin{bmatrix} I & M_2 & M_3 \\ D & N_2 + N^* & N_3 + N^{**} \end{bmatrix}.$$

Since A is gers, $\underline{\beta}^1$, $\underline{\beta}^2$, $\underline{\beta}^3$ must contain $s + \delta$ linearly independent elements. By the same reasoning as for C_α above, C_β must have rank $s + \delta$. By elementary column and row transformation, C_β can be reduced to

$$C_\beta^* = \begin{bmatrix} I & 0 & 0 \\ 0 & N_2 + N^* - DM_2 & N_3 + N^{**} - DM_3 \end{bmatrix}.$$

Therefore, the rank of $(D, N_2 + N^*, N_3 + N^{**})$ or of $(0, N_2 + N^* - DM_2, N_3 + N^{**} - DM_3)$ is δ.

Conversely, since C_β^* has rank $s + \delta$ for all D, in particular it has rank $s + \delta$ for $D = 0$. Therefore, C_α has rank $s + \delta$, and A is gers.

C. A Special Condition

It is important to notice that A may not be gers even though some $\underline{\alpha}^1$ generates A. As an example consider the algebra,

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \right\}$$

with a, b ϵ integers mod 2 where $\underline{\alpha}^1 = (\alpha_1)$,

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \underline{u} = (u_1), \quad u_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now, if

$$\hat{a}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

then $\hat{a}_1^2 = \alpha_1$ and $\hat{a}_1 - \hat{a}_1^2 = u_1$. Therefore, $\hat{a}_1$ does generate A, although A is not gers.

In case s = 3 if A were not gers, then the rank of (N_2, N_3) could be δ for some choice of $\underline{a}^1$. For example, let A be the algebra with the multiplication table as given below over a field $K = P_3(t)$, $a^3 = t$ where P_3 denotes the integers mod 3 and t is an indeterminate.

1	α_2	α_3	u_1	u_2	u_3
α_2	$\alpha_3 - u_1$	$t\alpha_2 + u_2$	u_2	u_3	tu_1
α_3		$t\alpha_2 + 2u_3$	u_3	tu_1	tu_2
u_1			0	0	0
u_2				0	0
u_3					0

Therefore,

$$[1, \alpha_2, \alpha_3; \alpha_2^2, \alpha_2\alpha_3, \alpha_3^2; \cdots]$$

$$= [1, \alpha_2, \alpha_3, u_1, u_2, u_3] \left[\begin{array}{c|ccc|c} & 0 & 0 & 0 & \\ I & 0 & t & t & \cdots \\ & 1 & 0 & 0 & \\ \hline & -1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ & 0 & 0 & 2 & \end{array}\right]$$

Hence,

$$N_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is of rank $3 = \delta$.

Now set $\hat{\alpha}_2 = \alpha_2$, $\hat{\alpha}_3 = \alpha_3 - u_1$. Then

1	$\hat{\alpha}_2$	$\hat{\alpha}_3$	. . .
$\hat{\alpha}_2$	$\hat{\alpha}_3$	$t\hat{\alpha}_2$	
$\hat{\alpha}_3$	$t\hat{\alpha}_2$	$t\hat{\alpha}_2$	
.			
.			
.			

Consequently, for this basis $(\hat{N}_2, \hat{N}_3) = 0$.

But this rank stipulation is certainly necessary for a gers algebra. Hence, we have

<u>Lemma 6</u>. A sufficient condition for A to be gers is that (N_2, N_3) has rank δ and

$$N^* - DM_2 = N^{**} - DM_3 = 0.$$

Proof: As remarked above, (N_2, N_3) must have rank δ. Hence, the maintenance of this rank is assured under the additional conditions in the Lemma.

D. Commutativity

If A is commutative, then A/N is an algebraic field extension of K. In this case we would not want to consider A/N separable, since then separability coincides with separability in Wedderburn's Principal Theorem, and A would not be gers. Now, in general, $A/N \supset G \supset K$, where G is the maximal separable inbetween field.

Lemma 7. If A is commutative and gers, it is sufficient to suppose that A/N is purely inseparable.

Proof: Let G be the maximal separable inbetween field mentioned above. Then it is known (see [3]) that A may be considered an algebra over K' where $K' \cong G$ over K, and hence A/N is purely inseparable where A is thus regarded.

IV. DEDUCTIONS FROM THE RANK CRITERION

A. Identities

It will be noted in one of the examples which follows in Section C that $N^* - DM_2 = 0$ and $N^{**} - DM_3 = 0$. In this chapter we will note the necessary conditions for these identities to be true.

B. Computations

The proof of Lemma 5 indicates the procedure for computing N^* and N^{**}. From Lemma 5

$$\beta_i\beta_j \equiv \alpha_i\alpha_j + \sum_{\sigma} u_\sigma\alpha_j d_{\sigma i} + \sum_{\sigma} \alpha_i u_\sigma d_{\sigma j} \pmod{N^2} \qquad (1)$$

Also,

$$u_i\alpha_j = \sum_{\mu=1}^{\delta} k_\mu^{ij} u_\mu \pmod{N^2}\ , \quad k_\mu^{ij} \in K \qquad (2)$$

$$(j = 2, \cdots, s;\ i = 1, \cdots, \delta)$$

If $j = 1$, $\alpha_j = 1$. If the algebra is commutative, $u_i\alpha_j = \alpha_j u_i$. If the algebra is not commutative, then

$$\alpha_i u_j = \sum_{\mu=1}^{\delta} \ell_\mu^{ij} u_\mu \pmod{N^2}\ , \quad \ell_\mu^{ij} \in K \qquad (3)$$

$$(i = 2, \cdots, s;\ j = 1, \cdots, \delta)$$

From (1), (2) and (3)

$$\left(\sum_{\sigma} u_\sigma \alpha_j d_{\sigma i} + \sum_{\sigma} \alpha_i u_\sigma d_{\sigma j}\right)$$

$$= \left[\sum_{\sigma} \left(\sum_{\mu} k_\mu^{\sigma j} u_\mu d_{\sigma i} + \sum_{\mu} \ell_\mu^{i\sigma} u_\mu d_{\sigma j}\right)\right]$$

Factor out the u's as a row vector, and get the (ij) column of N*:

$$\begin{array}{cc} \vdots & \\ \sum_{\sigma} (k_\mu^{\sigma j} d_{\sigma i} + \ell_\mu^{i\sigma} d_{\sigma j}) & (\mu, \sigma = 1, \cdots, \delta; \\ \vdots & i, j = 2, \cdots, s) \end{array}$$

Again, from Lemma 5

$$\beta_i \beta_j \beta_k \equiv \alpha_i \alpha_j \alpha_k + \alpha_i \alpha_j \sum u_\sigma d_{\sigma k} + \alpha_i \sum u_\sigma d_{\sigma j} \alpha_k + \sum u_\sigma d_{\sigma i} \alpha_j \alpha_k \quad (\text{mod } N^2) \tag{4}$$

By applying (3) twice,

$$\alpha_i(\alpha_j u_k) = \alpha_i\left(\sum \ell_\mu^{jk} u_\mu\right) = \sum_{\mu,\tau=1}^{\delta} \ell_\mu^{jk} \ell_\tau^{i\mu} u_\tau \tag{5}$$

From the multiplication table

$$\alpha_i \alpha_j = \sum_{\sigma=1}^{s} c_\sigma^{ij} \alpha_\sigma + n_{ij}$$

Then

$$\alpha_i \alpha_j u_k \equiv (\sum_\sigma c_\sigma^{ij} \alpha_\sigma + n_{ij}) u_k$$
$$\equiv \sum_{\sigma,\tau} c_\sigma^{ij} \ell_\tau^{\sigma k} u_\tau \quad (\text{mod } N^2)$$

A useful notation is

$$\alpha_i \alpha_j u_k \equiv \sum_\tau \lambda_\tau^{ij(k)} u_\tau \quad (\text{mod } N^2)$$

If the algebra is not commutative, then by applying (3) and (2)

$$(\alpha_i u_j)\alpha_k \equiv (\sum \ell_\sigma^{ij} u_\sigma)\alpha_k \equiv \sum \ell_\sigma^{ij} k_\tau^{\sigma k} u_\tau$$
$$\equiv \sum_\tau \lambda_\tau^{i(j)k} u_\tau \quad (\text{mod } N^2) \tag{6}$$

By applying (2) twice

$$(u_i \alpha_j)\alpha_k \equiv (\sum k_\sigma^{ij} u_\sigma)\alpha_k \equiv \sum k_\sigma^{ij} k_\tau^{\sigma k} u_\tau$$
$$\equiv \sum_\tau \lambda_\tau^{(i)jk} u_\tau \quad (\text{mod } N^2) \tag{7}$$

From (4), (5), (6), and (7)

$$[\sum_\sigma \alpha_i \alpha_j u_\sigma d_{\sigma k} + \alpha_i(\sum_\sigma u_\sigma d_{\sigma j})\alpha_k + (\sum u_\sigma d_{\sigma i})\alpha_j \alpha_k]$$

$$= [\sum_\sigma (\sum_{\alpha,\tau} \ell_\alpha^{j\sigma} \ell_\tau^{i\alpha} u_\tau d_{\sigma k} + \sum_{\alpha,\tau} \ell_\alpha^{i\sigma} k_\tau^{\alpha k} u_\tau d_{\sigma j} + \sum_{\alpha,\tau} k_\alpha^{\sigma j} k_\tau^{\alpha k} u_\tau d_{\sigma i}]$$

Factor out the u's as a row vector, and get the (ijk) column of N^*:

$$\begin{matrix} \vdots \\ \sum_{\sigma,\alpha} (\ell_\alpha^{j\sigma} \ell_\mu^{i\alpha} d_{\sigma k} + \ell_\alpha^{i\sigma} k_\mu^{\alpha k} d_{\sigma j} + k_\alpha^{\sigma j} k_\mu^{\alpha k} d_{\sigma i}) \\ \vdots \end{matrix}$$

$$(\mu, \sigma, \alpha = 1, \cdots, \delta;\ i, j, k = 2, \cdots, s)$$

Now, we proceed to determine the consequences of the identity, $N^* - DM_2 = 0$. If the algebra is non-commutative, then $1 < i,\ j \leq s$. Hence, there will be no more than $(s - 1)^2$ columns in M_2. Arrange them lexicographically, and designate a general element of M_2 by $m_{\epsilon(ij)}$ where ϵ indicates the row and (ij) the column of $m_{\epsilon(ij)}$. Thus,

$$\sum_{\sigma=1}^{\delta} \left[k_\mu^{\sigma j} d_{\sigma i} + \ell_\mu^{i\sigma} d_{\sigma j} \right] = \sum_{\epsilon=2}^{s} d_{\mu\epsilon} m_{\epsilon(ij)}$$

First, we consider the case where $\delta = 3$ and $s = 3$. Let $\mu = 1$. Then we obtain the following equations:

$$\sum_{\sigma=1}^{3} [k_1^{\sigma 2} d_{\sigma 2} + \ell_1^{2\sigma} d_{\sigma 2}] = \sum_{\epsilon=2}^{3} d_{1\epsilon} m_{\epsilon(22)}$$

$$\sum_{\sigma=1}^{3} [k_1^{\sigma 3} d_{\sigma 2} + \ell_1^{2\sigma} d_{\sigma 3}] = \sum_{\epsilon=2}^{3} d_{1\epsilon} m_{\epsilon(23)}$$

$$\sum_{\sigma=1}^{3} \left[k_1^{\sigma 2} d_{\sigma 3} + \ell_1^{3\sigma} d_{\sigma 2} \right] = \sum_{\epsilon=2}^{3} d_{1\epsilon} m_{\epsilon(32)}$$

$$\sum_{\sigma=1}^{3} \left[k_1^{\sigma 3} d_{\sigma 3} + \ell_1^{3\sigma} d_{\sigma 3} \right] = \sum_{\epsilon=2}^{3} d_{1\epsilon} m_{\epsilon(33)}$$

By equating the coefficients of the d's, we get

$$K_1 = (k_1^{\sigma j}) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} \quad 2 & \qquad\quad 3 \end{array} \\ \begin{bmatrix} m_{3(32)} & m_{2(23)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$L_1 = (\ell_1^{i\sigma}) = \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 1 & \qquad 2 & \quad 3 \end{array} \\ \begin{bmatrix} m_{3(23)} & 0 & 0 \\ m_{2(32)} & 0 & 0 \end{bmatrix} \end{array}$$

$$k_1^{12} + \ell_1^{21} = m_{2(22)} \qquad\qquad m_{2(22)} = m_{3(32)} + m_{3(23)}$$

$$k_1^{13} + \ell_1^{31} = m_{3(33)} \qquad\qquad m_{3(33)} = m_{2(23)} + m_{2(32)}$$

If $\mu = 2$, then

$$\sum_{\sigma=1}^{3} \left[k_2^{\sigma 2} d_{\sigma 2} + \ell_2^{2\sigma} d_{\sigma 2} \right] = \sum_{\epsilon=2}^{3} d_{2\epsilon} m_{\epsilon(22)}$$

$$\sum_{\sigma=1}^{3} \left[k_2^{\sigma 3} d_{\sigma 2} + \ell_2^{2\sigma} d_{\sigma 3} \right] = \sum_{\epsilon=2}^{3} d_{2\epsilon} m_{\epsilon(23)}$$

$$\sum_{\sigma=1}^{3} [k_2^{\sigma 2} d_{\sigma 3} + \ell_2^{3\sigma} d_{\sigma 2}] = \sum_{\epsilon=2}^{3} d_{2\epsilon} m_{\epsilon(32)}$$

$$\sum_{\sigma=1}^{3} [k_2^{\sigma 3} d_{\sigma 3} + \ell_2^{3\sigma} d_{\sigma 3}] = \sum_{\epsilon=2}^{3} d_{2\epsilon} m_{\epsilon(33)}$$

Therefore,

$$K_2 = (k_2^{\sigma j}) = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ m_{3(32)} & m_{2(23)} \\ 0 & 0 \end{bmatrix} \end{array}$$

$$L_2 = (\ell_2^{i\sigma}) = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0 & m_{3(23)} & 0 \\ 0 & m_{2(32)} & 0 \end{bmatrix} \end{array}$$

Similarly, in the case where $\mu = 3$ we get

$$\sum_{\sigma=1}^{3} [k_3^{\sigma 2} d_{\sigma 2} + \ell_3^{2\sigma} d_{\sigma 2}] = \sum_{\epsilon=2}^{3} d_{3\epsilon} m_{\epsilon(22)}$$

$$\sum_{\sigma=1}^{3} [k_3^{\sigma 3} d_{\sigma 2} + \ell_3^{2\sigma} d_{\sigma 3}] = \sum_{\epsilon=2}^{3} d_{3\epsilon} m_{\epsilon(23)}$$

$$\sum_{\sigma=1}^{3} [k_3^{\sigma 2} d_{\sigma 3} + \ell_3^{3\sigma} d_{\sigma 2}] = \sum_{\epsilon=2}^{3} d_{3\epsilon} m_{\epsilon(32)}$$

$$\sum_{\sigma=1}^{3} \left[k_3^{\sigma 3} d_{\sigma 3} + \ell_3^{3\sigma} d_{\sigma 3} \right] = \sum_{\epsilon=2}^{3} d_{3\epsilon} m_{\epsilon(33)}$$

Hence,

$$K_3 = (k_3^{\sigma j}) = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ m_{3(32)} & m_{2(23)} \end{bmatrix} \end{array}$$

$$L_3 = (\ell_3^{i\sigma}) = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & m_{3(23)} \\ 0 & 0 & m_{2(32)} \end{bmatrix} \end{array}$$

$$M_2 = \begin{array}{c c} & \begin{array}{cccc} (22) & (23) & (32) & (33) \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} m_{1(22)} & m_{1(23)} & m_{1(32)} & m_{1(33)} \\ m_{3(23)} + m_{3(32)} & m_{2(23)} & m_{2(32)} & 0 \\ 0 & m_{3(23)} & m_{3(32)} & m_{2(23)} + m_{2(32)} \end{bmatrix} \end{array}$$

Secondly, we consider the case where $\delta = 6$, $s = 3$. If $\mu = 1$, then

$$K_1 = (k_1^{\sigma j}) = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} m_{3(32)} & m_{2(23)} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$L_1 = (\ell_1^{i\sigma}) = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 2 \\ 3 \end{matrix} & \begin{bmatrix} m_{3(23)} & 0 & 0 & 0 & 0 & 0 \\ m_{2(32)} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$k_1^{12} + \ell_1^{21} = m_{2(22)} = m_{3(32)} + m_{3(23)}$$

$$m_{3(22)} = 0$$

$$m_{2(33)} = 0$$

$$k_1^{13} + \ell_1^{31} = m_{3(33)} = m_{2(23)} + m_{2(32)}$$

If $\mu = 2$, then

$$K_2 = (k_2^{\sigma j}) = \begin{array}{cc} & \begin{matrix} 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 0 \\ m_{3(32)} & m_{2(23)} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$L_2 = (\ell_2^{i\sigma}) = \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & m_{3(23)} & 0 & 0 & 0 & 0 \\ 0 & m_{2(32)} & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

If $\mu = 3$,

$$K_3 = (k_3^{\sigma j}) = \begin{array}{cc} & \begin{matrix} 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ m_{3(32)} & m_{2(23)} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$L_3 = (\ell_3^{i\sigma}) = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & m_{3(23)} & 0 & 0 & 0 \\ 0 & 0 & m_{2(32)} & 0 & 0 & 0 \end{bmatrix} \end{array}$$

If $\mu = 4$,

$$K_4 = (k_4^{\sigma j}) = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ m_{3(32)} & m_{2(23)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$L_4 = (\ell_4^{i\sigma}) = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & m_{3(23)} & 0 & 0 \\ 0 & 0 & 0 & m_{2(32)} & 0 & 0 \end{bmatrix} \end{array}$$

If $\mu = 5$,

$$K_5 = (k_5^{\sigma j}) = \begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ m_{3(32)} & m_{2(23)} \\ 0 & 0 \end{bmatrix} \end{array}$$

$$L_5 = (\ell_5^{i\sigma}) = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & m_{3(23)} & 0 \\ 0 & 0 & 0 & 0 & m_{2(32)} & 0 \end{bmatrix} \end{array}$$

If $\mu = 6$,

$$K_6 = (k_6^{\sigma j}) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ m_{3(32)} & m_{2(23)} \end{bmatrix} \end{array}$$

$$L_6 = (\ell_6^{i\sigma}) = \begin{array}{c} \\ 2 \\ 3 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \quad 6 \quad \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & m_{3(23)} \\ 0 & 0 & 0 & 0 & 0 & m_{2(32)} \end{bmatrix} \end{array}$$

Thus, the matrix M_2 is identical to M_2 in the case $\delta = 3$, $s = 3$.

Finally, we consider the general case $\delta = 3n$, $s = 3$ where n is a positive integer. From the equation

$$\sum_{\sigma=1}^{3n} \left[k_\mu^{\sigma 2} d_{\sigma 2} + \ell_\mu^{2\sigma} d_{\sigma 2} \right] = \sum_{\epsilon=2}^{3} d_{\mu\epsilon} m_{\epsilon(22)}$$

$$\mu = 1, 2, \cdots, 3n$$

we get

$$m_{3(22)} = 0$$

$$k_\mu^{\mu 2} + \ell_\mu^{2\mu} = m_{2(22)}$$

$$k_\mu^{t2} + \ell_\mu^{2t} = 0 \qquad t = 1, 2, \cdots, \mu-1, \mu+1, \cdots, 3n$$

From

$$\sum_{\sigma=1}^{3n} \left[k_\mu^{\sigma 3} d_{\sigma 2} + \ell_\mu^{2\sigma} d_{\sigma 3} \right] = \sum_{\epsilon=2}^{3} d_{\mu\epsilon} m_{\epsilon(23)}$$

$$\mu = 1,\ 2,\ \cdots,\ 3n$$

come the results

$$k_\mu^{\mu 3} = m_{2(23)} \ , \qquad \ell_\mu^{2\mu} = m_{3(23)}$$

$$k_\mu^{t3} = 0 \quad \text{and} \quad \ell_\mu^{2t} = 0 \ , \qquad t = 1,\ 2,\ \cdots,\ \mu-1,\ \mu+1,\ \cdots,\ 3n$$

Furthermore,

$$\sum_{\sigma=1}^{3n} \left[k_\mu^{\sigma 2} d_{\sigma 3} + \ell_\mu^{3\sigma} d_{\sigma 2} \right] = \sum_{\epsilon=2}^{3} d_{\mu\epsilon} m_{\epsilon(32)}$$

$$\mu = 1,\ 2,\ \cdots,\ 3n$$

gives

$$k_\mu^{\mu 2} = m_{3(32)} \ , \qquad \ell_\mu^{3\mu} = m_{2(32)}$$

$$k_\mu^{t2} = 0 \quad \text{and} \quad \ell_\mu^{3t} = 0 \ , \qquad t = 1,\ 2,\ \cdots,\ \mu-1,\ \mu+1,\ \cdots,\ 3n$$

Finally,

$$\sum_{\sigma=1}^{3n} \left[k_\mu^{\sigma 3} d_{\sigma 3} + \ell_\mu^{3\sigma} d_{\sigma 3} \right] = \sum_{\epsilon=2}^{3} d_{\mu\epsilon} m_{\epsilon(33)}$$

$$\mu = 1,\ 2,\ \cdots,\ 3n$$

produces the equations

$$m_{2(33)} = 0$$

$$k_{\mu}^{\mu 3} + \ell_{\mu}^{3\mu} = m_{3(33)}$$

$$k_{\mu}^{t3} + \ell_{\mu}^{3t} = 0 , \qquad t = 1, 2, \cdots, \mu-1, \mu+1, \cdots, 3n.$$

Therefore, for each $\delta = 3n$, $s = 3$, M_2 is identical to M_2 in the case $\delta = 3$, $s = 3$. From the matrix M_2 we get

$$\alpha_2^2 \equiv m_{1(22)} + [m_{3(23)} + m_{3(32)}]\alpha_2 + 0 \cdot \alpha_3 \quad (\text{mod } N)$$

$$\alpha_3^2 \equiv m_{1(33)} + 0 \cdot \alpha_2 + [m_{2(23)} + m_{2(32)}]\alpha_3 \quad (\text{mod } N)$$

Suppose A is commutative. Since A/N is a division algebra, A/N is a field. Furthermore,

$$A/N = K[\alpha_2, \alpha_3] .$$

A/N must not be a separable extension for if A/N is separable, then A cannot be gers. Also,

$$\alpha_2^2 \equiv m_{1(22)} + 2m_{3(23)}\alpha_2 \quad (\text{mod } N)$$

$$\alpha_3^2 \equiv m_{1(33)} + 2m_{2(23)}\alpha_3 \quad (\text{mod } N)$$

If characteristic of K is not 2 and if $m_{3(23)}m_{2(23)} \neq 0$, then A/N is separable. This contradicts the fact that A is gers. If $m_{2(23)} = 0$, then $\alpha_2^2 = m_{1(22)}$. Again, we have a separable equation.

Therefore, assume K is of characteristic 2. Then

$$[A/N : K] = 2^2 .$$

This contradicts the fact that $s = 3$. Therefore, A cannot be commutative.

Now, let us consider the cases in which $s = 2$. First, take the specific example with $\delta = 2$, $s = 2$. If $\mu = 1$, then

$$\sum_{\sigma=1}^{2} \left[k_1^{\sigma j} d_{\sigma i} + \ell_1^{i\sigma} d_{\sigma j} \right] = d_{12} m_{2(ij)} .$$

As a result,

$$k_1^{12} + \ell_1^{21} = m_{2(22)} \qquad K_1 = \begin{array}{c|c} & 2 \\ 1 & \left[\begin{array}{c} m_{2(22)} - \ell_1^{21} \\ -\ell_1^{22} \end{array}\right. \\ 2 & \end{array}\Bigg]$$

$$k_1^{22} + \ell_1^{22} = 0$$

If $\mu = 2$, then

$$\sum_{\sigma=1}^{2} \left[k_2^{\sigma j} d_{\sigma i} + \ell_2^{i\sigma} d_{\sigma j} \right] = d_{22} m_{2(ij)} .$$

Therefore,

$$k_2^{12} + \ell_2^{21} = 0 \qquad K_2 = \begin{array}{c|c} & 2 \\ 1 & \left[\begin{array}{c} -\ell_2^{21} \\ m_{2(22)} - \ell_2^{22} \end{array}\right. \\ 2 & \end{array}\Bigg]$$

$$k_2^{22} + \ell_2^{22} = m_{2(22)}$$

Secondly, consider the case with $\delta = 4$, $s = 2$. If $\mu = 1$, then

$$k_1^{12} + \ell_1^{21} = m_{2(22)}$$

$$k_1^{22} + \ell_1^{22} = 0$$

$$k_1^{32} + \ell_1^{23} = 0$$

$$k_1^{42} + \ell_1^{24} = 0$$

$$K_1 = \begin{array}{c c} & 2 \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} m_{2(22)} - \ell_1^{21} \\ -\ell_1^{22} \\ -\ell_1^{23} \\ -\ell_1^{24} \end{bmatrix} \end{array}$$

If $\mu = 2$, then

$$k_2^{22} + \ell_2^{22} = m_{2(22)}$$

$$k_2^{12} + \ell_2^{21} = 0$$

$$k_2^{32} + \ell_2^{23} = 0$$

$$k_2^{42} + \ell_2^{24} = 0$$

$$K_2 = \begin{array}{c c} & 2 \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -\ell_2^{21} \\ m_{2(22)} - \ell_2^{22} \\ -\ell_2^{23} \\ -\ell_2^{24} \end{bmatrix} \end{array}$$

Furthermore, if $\mu = 3$, then

$$k_3^{12} + \ell_3^{21} = 0$$

$$k_3^{22} + \ell_3^{22} = 0$$

$$k_3^{32} + \ell_3^{23} = m_{2(22)}$$

$$k_3^{42} + \ell_3^{24} = 0$$

$$K_3 = \begin{array}{c c} & 2 \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -\ell_3^{21} \\ -\ell_3^{22} \\ m_{2(22)} - \ell_3^{23} \\ -\ell_3^{24} \end{bmatrix} \end{array}$$

Finally, if $\mu = 4$, then

$$k_4^{12} + \ell_4^{21} = 0$$

$$k_4^{22} + \ell_4^{22} = 0$$

$$k_4^{32} + \ell_4^{23} = 0$$

$$k_4^{42} + \ell_4^{24} = m_{2(22)}$$

$$K_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} 2 \\ \begin{bmatrix} -\ell_4^{21} \\ -\ell_4^{22} \\ -\ell_4^{23} \\ m_{2(22)} - \ell_4^{24} \end{bmatrix} \end{array}$$

This leads us to the more general situation where $\delta = 2n$, n is a positive integer, and $s = 2$. The equations,

$$\sum_{\sigma=1}^{2n} \left[k_\mu^{\sigma 2} d_{\sigma 2} + \ell_\mu^{2\sigma} d_{\sigma 2} \right] = d_{\mu 2} m_{2(22)}$$

$$\mu = 1, 2, \cdots, 2n$$

give us the following results:

$$k_\mu^{\mu 2} + \ell_\mu^{2\mu} = m_{2(22)}$$

$$k_\mu^{t2} + \ell_\mu^{2t} = 0 \,, \quad t = 1, 2, \cdots, \mu-1, \mu+1, \cdots, 2n \,.$$

If K is of characteristic 2 and A is commutative,

$$2k_1^{12} = m_{2(22)} \quad \text{or} \quad m_{2(22)} = 0 \,.$$

Since $M_2 = \begin{array}{c} 1 \\ 2 \end{array} \begin{bmatrix} m_{1(22)} \\ m_{2(22)} \end{bmatrix}$,

$$\alpha_2^2 = m_{1(22)} \quad (\text{mod } N)$$

with $A/N = K[\alpha_2]$. Hence, A/N is a purely inseparable extension of K.

If K is not of characteristic 2 and A is commutative, then

$$k_1^{12} = \frac{1}{2} m_{2(22)} ,$$

$$k_\mu^{12} = 0 , \qquad \mu = 2, 3, \cdots, 2n.$$

Therefore,

$$u_1\alpha_2 = k_1^{12}u_1 + k_2^{12}u_2 + \cdots + k_{2n}^{12}u_{2n}$$

$$= \frac{1}{2} m_{2(22)}u_1$$

and

$$u_1(\alpha_2 - \frac{1}{2} m_{2(22)}) = 0 .$$

Since $\alpha_2 \notin K$ and $\frac{1}{2} m_{2(22)} \in K$, $\alpha_2 - \frac{1}{2} m_{2(22)} \neq 0$. Therefore, $u_1 = 0$. But this is impossible. Hence, K must be of characteristic 2 if A is commutative whenever $\delta = 2n$ and $s = 2$.

In considering the consequences of the identity, $N^{**} - DM_3 = 0$, designate a general element of M_3 by $m_{\epsilon(ijk)}$ where ϵ indicates the row and (ijk) the column of M_2 in which $m_{\epsilon(ijk)}$ is located. In the first case let $\delta = 3$, $s = 3$, $\mu = 1$. From the identity we get the equation

$$\sum_{\sigma,\alpha=1}^{3} (\ell_\alpha^{j\sigma}\ell_1^{i\alpha}d_{\sigma k} + \ell_\alpha^{i\sigma}k_1^{\alpha k}d_{\sigma j} + k_\alpha^{\sigma j}k_1^{\alpha k}d_{\sigma i}) = \sum_{\epsilon=2}^{3} d_{1\epsilon}m_{\epsilon(ijk)}$$

The results obtained are tabulated below:

$(ijk) = (222)$

$$\ell_1^{21}\ell_1^{21} + \ell_1^{21}k_1^{12} + k_1^{12}k_1^{12} = m_{2(222)} \qquad m_{3(222)} = 0$$

$$m_{3(23)}^2 + m_{3(23)}m_{3(32)} + m_{3(32)}^2 = m_{2(222)}$$

$(ijk) = (223)$

$$\ell_1^{21}k_1^{13} + k_1^{12}k_1^{13} = m_{2(223)} \qquad \ell_1^{21}\ell_1^{21} = m_{3(223)}$$

$$m_{3(23)}m_{2(23)} + m_{3(32)}m_{2(23)} = m_{2(223)}$$

$(ijk) = (232)$

$$\ell_1^{31}\ell_1^{21} + k_1^{13}k_1^{12} = m_{2(232)} \qquad \ell_1^{21}k_1^{12} = m_{3(232)}$$

$$m_{2(32)}m_{3(23)} + m_{2(23)}m_{3(32)} = m_{2(232)}$$

$$m_{3(23)}m_{3(32)} = m_{3(232)}$$

$(ijk) = (233)$

$$k_1^{13}k_1^{13} = m_{2(233)} \qquad \ell_1^{31}\ell_1^{21} + \ell_1^{21}k_1^{13} = m_{3(233)}$$

$$m_{2(23)}^2 = m_{2(233)}$$

$$m_{2(32)}m_{3(23)} + m_{3(23)}m_{2(23)} = m_{3(233)}$$

$(ijk) = (322)$

$$\ell_1^{21}\ell_1^{31} + \ell_1^{31}k_1^{12} = m_{2(322)} \qquad k_1^{12}k_1^{12} = m_{3(322)}$$

$$m_{3(23)}m_{2(32)} + m_{2(32)}m_{3(32)} = m_{2(322)}$$

$$m_{3(32)}m_{3(32)} = m_{3(322)}$$

$(ijk) = (323)$

$$\ell_1^{31}k_1^{13} = m_{2(323)} \qquad \ell_1^{21}\ell_1^{31} + k_1^{12}k_1^{13} = m_{3(323)}$$

$$m_{2(32)}m_{2(23)} = m_{2(323)}$$

$$m_{3(23)} + m_{2(32)} + m_{3(32)}m_{2(23)} = m_{3(323)}$$

$(ijk) = (332)$

$$\ell_1^{31}\ell_1^{31} = m_{2(332)} \qquad \ell_1^{31}k_1^{12} + k_1^{13}k_1^{12} = m_{3(332)}$$

$$m_{2(32)}^2 = m_{2(332)}$$

$$m_{2(32)}m_{3(32)} + m_{2(23)}m_{3(32)} = m_{3(332)}$$

$(ijk) = (333)$

$$\ell_1^{31}\ell_1^{31} + \ell_1^{31}k_1^{13} + k_1^{13}k_1^{13} = m_{3(333)} \qquad m_{2(333)} = 0$$

$$m_{2(32)}^2 + m_{2(32)}m_{2(23)} + m_{2(23)}^2 = m_{3(333)}$$

From these results we get the matrix M_3.

$$M_3 = \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} (222) \\ \left[\begin{array}{c} m_{3(23)}^2 + m_{3(23)}m_{3(32)} + m_{3(32)}^2 \\ 0 \end{array} \right. \end{array}$$

$$
\begin{array}{cc}
(223) & (232) \\
m_{3(23)}m_{2(23)} + m_{3(32)}m_{2(23)} & m_{2(32)}m_{3(23)} + m_{2(23)}m_{3(32)} \\
m^2_{3(23)} & m_{3(23)}m_{3(32)} \\
 & \\
(233) & (322) \\
m^2_{2(23)} & m_{3(23)}m_{2(32)} + m_{2(32)}m_{3(32)} \\
m_{2(32)}m_{3(23)} + m_{3(23)}m_{2(23)} & m^2_{3(32)} \\
 & \\
(323) & (332) \\
m_{2(32)}m_{2(23)} & m^2_{2(32)} \\
m_{3(23)}m_{2(32)} + m_{3(32)}m_{2(23)} & m_{2(32)}m_{3(32)} + m_{2(23)}m_{3(32)} \\
 & \\
(333) & \\
0 & \\
m^2_{2(32)} + m_{2(32)}m_{2(23)} + m^2_{2(23)} \Bigg] &
\end{array}
$$

From $N^{**} - DM_3 = 0$, in the case where $\delta = 2$, $s = 2$, we get

$$\sum_{\sigma,\alpha=1}^{2} (\ell_\alpha^{2\sigma}\ell_\mu^{2\alpha}d_{\sigma 2} + \ell_\alpha^{2\sigma}k_\mu^{\alpha 2}d_{\sigma 2} + k_\alpha^{\sigma 2}k_\mu^{\alpha 2}d_{\sigma 2}) = d_{\mu 2}m_{2(222)}$$

If $\mu = 1$, then

$$(\ell_1^{21})^2 + \ell_1^{21}k_1^{12} + (k_1^{12})^2 + \ell_2^{21}\ell_1^{22} + \ell_2^{21}k_1^{22} + k_2^{12}k_1^{22} = m_{2(222)}$$

or

$$-m_{2(22)}\ell_1^{21} + m_{2(22)}^2 + (\ell_1^{21})^2 - \ell_2^{22}\ell_1^{22} = m_{2(222)}$$

If $\mu = 2$, then

$$\ell_1^{22}\ell_2^{21} + \ell_1^{22}k_2^{12} + k_1^{22}k_2^{12} + (\ell_2^{22})^2 + \ell_2^{22}k_2^{22} + (k_2^{22})^2 = m_{2(222)}$$

or

$$\ell_1^{22}\ell_2^{21} + m_{2(22)}^2 - m_{2(22)}\ell_2^{22} + (\ell_2^{22})^2 = m_{2(222)}$$

Now in the case where $\delta = 4$ and $s = 2$, the following relationship holds:

$$\sum_{\sigma,\alpha=1}^{4} (\ell_\alpha^{2\sigma}\ell_\mu^{2\alpha}d_{\sigma 2} + \ell_\alpha^{2\sigma}k_\mu^{\alpha 2}d_{\sigma 2} + k_\alpha^{\sigma 2}k_\mu^{\alpha 2}d_{\sigma 2}) = d_{\mu 2}m_{2(222)}$$

If $\mu = 1$, then

$$(\ell_1^{21})^2 + \ell_1^{21}k_1^{12} + (k_1^{12})^2 + \ell_2^{21}\ell_1^{22} + \ell_2^{21}k_1^{22} + k_2^{12}k_1^{22}$$
$$\ell_3^{21}\ell_1^{23} + \ell_3^{21}k_1^{32} + k_3^{12}k_1^{32} + \ell_4^{21}\ell_1^{24} + \ell_4^{21}k_1^{42} + k_4^{12}k_1^{42}$$
$$= m_{2(222)}$$

or

$$(\ell_1^{21})^2 - m_{2(22)}\ell_1^{21} + m_{2(22)}^2 + \ell_2^{21}\ell_1^{22} + \ell_3^{21}\ell_1^{23} + \ell_4^{21}\ell_1^{24}$$
$$= m_{2(222)}$$

If $\mu = 2$, then

$$\ell_1^{22}\ell_2^{21} + \ell_1^{22}k_2^{12} + k_1^{22}k_2^{12} + (\ell_2^{22})^2 + \ell_2^{22}k_2^{22} + (k_2^{22})^2 + \ell_3^{22}\ell_2^{23} +$$

$$\ell_3^{22}k_2^{32} + k_3^{22}k_2^{32} + \ell_4^{22}\ell_2^{24} + \ell_4^{22}k_2^{42} + k_4^{22}k_2^{42} = m_{2(222)}$$

or

$$-m_{2(22)}\ell_2^{21} + \ell_1^{21}\ell_2^{21} + m^2_{2(22)} - m_{2(22)}\ell_2^{22} + (\ell_2^{22})^2 + \ell_3^{22}\ell_2^{23}$$

$$+ \ell_4^{22}\ell_2^{24} = m_{2(222)}$$

If $\mu = 3$, then

$$\ell_1^{23}\ell_3^{21} + \ell_1^{23}k_3^{12} + k_1^{32}k_3^{12} + \ell_2^{23}\ell_3^{22} + \ell_2^{23}k_3^{22} + k_2^{32}k_3^{22} + (\ell_3^{23})^2 +$$

$$\ell_3^{23}k_3^{32} + (k_3^{32})^2 + \ell_4^{23}\ell_3^{24} + \ell_4^{23}k_3^{42} + k_4^{32}k_3^{42} = m_{2(222)}$$

or

$$\ell_1^{23}\ell_3^{21} + \ell_2^{23}\ell_3^{22} + m^2_{2(22)} - m_{2(22)}\ell_3^{23} + (\ell_3^{23})^2 + \ell_4^{23}\ell_3^{24}$$

$$= m_{2(222)}$$

If $\mu = 4$, then

$$\ell_1^{24}\ell_4^{21} + \ell_1^{24}k_4^{12} + k_1^{42}k_4^{12} + \ell_2^{24}\ell_4^{22} + \ell_2^{24}k_4^{22} + k_2^{42}k_4^{22} + \ell_3^{24}\ell_4^{23}$$

$$+ \ell_3^{24}k_4^{32} + k_3^{42}k_4^{34} + (\ell_4^{24})^2 + \ell_4^{24}k_4^{42} + (k_4^{42})^2 = m_{2(222)}$$

or

$$\ell_1^{24}\ell_4^{21} + \ell_2^{24}\ell_4^{22} - m_{2(22)}\ell_3^{24} + \ell_3^{24}\ell_4^{23} - m_{2(22)}\ell_4^{24} + m_{2(22)}^2$$

$$+ (\ell_4^{24})^2 = m_{2(222)}.$$

From the case $\delta = 2$, $s = 2$ we get the following solution:

$$m_{2(22)} = \frac{\ell_1^{22}\ell_2^{21} + (\ell_2^{22})^2 - (\ell_1^{21})^2 + \ell_2^{22}\ell_1^{22}}{\ell_2^{22} - \ell_1^{21}}$$

Similarly, in the case $\delta = 4$, $s = 2$

$$m_{2(22)} = \frac{\ell_1^{21}\ell_2^{21} + (\ell_2^{22})^2 + \ell_3^{22}\ell_2^{23} + \ell_4^{22}\ell_2^{24}}{\ell_2^{21} + \ell_2^{22} - \ell_1^{21}}$$

or

$$m_{2(22)} =$$

$$\frac{\ell_1^{24}\ell_4^{21} + \ell_2^{24}\ell_4^{22} + \ell_3^{24}\ell_4^{23} + (\ell_4^{24})^2 - \ell_1^{23}\ell_3^{21} - \ell_2^{23}\ell_3^{22} - (\ell_3^{23})^2 - \ell_4^{23}\ell_3^{24}}{\ell_3^{24} + \ell_4^{24} - \ell_3^{23}}$$

C. Examples

Let us investigate further the example of a commutative algebra which was given in Chapter I, Section A. We repeat the multiplication table.

1	α_2	u_1	u_2
α_2	$f + u_1$	u_2	
u_1	u_2		
u_2			

$f \in K - K^p$

Note that $\alpha_2^3 = f\alpha_2 + u_2$. Therefore,

$$(\alpha^1, \alpha^2, \alpha^3) \equiv (\alpha, u)\begin{bmatrix} 1 & 0 & f & 0 \\ 0 & 1 & 0 & f \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \pmod{N^2}$$

Since $C_\alpha = \begin{bmatrix} I & M_2 & M_3 \\ 0 & N_2 & N_3 \end{bmatrix}$, then $M_2 = \begin{bmatrix} f \\ 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 \\ f \end{bmatrix}$,

$N_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $N_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

If $\beta^1 = (\beta_1, \beta_2)$ is an arbitrary basis for A/N, then $\beta^1, \beta^2, \beta^3$ is a basis for A/N^2.

$$(\beta^1, \beta^2, \beta^3) \equiv (\alpha, u)\begin{bmatrix} I & \cdot & \cdot & \cdot \\ D & \cdot & \cdot & \cdot \end{bmatrix} \pmod{N^2}$$

where D is arbitrary and $C_\beta = \begin{bmatrix} I & \cdot & \cdot & \cdot \\ D & \cdot & \cdot & \cdot \end{bmatrix}$. Since $\alpha_1 = 1$,

$$D = \begin{bmatrix} 0 & d_{12} \\ 0 & d_{22} \end{bmatrix}.$$

From the multiplication table

$$\beta_2^2 \equiv (\alpha_2 + u_1 d_{12} + u_2 d_{22})^2$$

$$\equiv \alpha_2^2 + \alpha_2 u_1 d_{12} + \alpha_2 u_2 d_{22} + u_1 \alpha_2 d_{12} + u_2 \alpha_2 d_{22} \pmod{N^2}$$

$$\equiv f + u_1 \pmod{N^2}$$

Therefore,

$$N^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Again from the multiplication table

$$\beta_2^3 \equiv (f + u_1)(\alpha_2 + u_1 d_{12} + u_2 d_{22})$$

$$\equiv f\alpha_2 + fu_1 d_{12} + fu_2 d_{22} + u_1 \alpha_2 \pmod{N^2}$$

$$\equiv f\alpha_2 + u_2 + uf\begin{bmatrix} d_{12} \\ d_{22} \end{bmatrix} \pmod{N^2}$$

$$\equiv \alpha_2^3 + uf\begin{bmatrix} d_{12} \\ d_{22} \end{bmatrix} \pmod{N^2}$$

Thus,

$$N^{**} = f\begin{bmatrix} d_{12} \\ d_{22} \end{bmatrix}.$$

Furthermore,

$$C_\beta = \begin{bmatrix} 1 & 0 & f & 0 \\ 0 & 1 & 0 & f \\ 0 & d_{12} & 1 & fd_{12} \\ 0 & d_{22} & 0 & 1 + fd_{22} \end{bmatrix}$$

For all choice of d_{ij}, $\begin{bmatrix} 1 & fd_{12} \\ 0 & 1 + fd_{22} \end{bmatrix}$ has rank $2 = \delta$.

If $1 + fd_{22} = 0$, then $d_{22} = f^{-1} \neq 0$. In that case $\begin{bmatrix} d_{12} & 1 \\ d_{22} & 0 \end{bmatrix} \neq 0$.

$$N_2 + N^* - DM_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$N_3 + N^{**} - DM_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + f\begin{bmatrix} d_{12} \\ d_{22} \end{bmatrix} - f\begin{bmatrix} d_{12} \\ d_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, in this example $N^* - DM_2 = N^{**} - DM_3 = 0$.

Our second example is a gers algebra over a field, K, of characteristic 2 for which $N^* - DM_2 \neq 0$. $f_1, f_2 \in K$ and $s = 4$, $\delta = 4$. The multiplication table follows:

$\alpha_1=1$	α_2	α_3	α_4	u_1	u_2	u_3	u_4
α_2	$f_2 + u_1$	$\alpha_4 + u_1$	$f_2\alpha_3 + u_3 + u_2$	u_2	f_2u_1	u_4	f_2u_3
α_3	α_4	f_1	$f_1\alpha_2$	u_3	u_4	f_1u_1	f_1u_2
α_4	$f_2\alpha_3 + u_3$	$f_1\alpha_2 + u_3$	$f_1f_2 + f_1u_1 + u_4$	u_4	f_2u_3	f_1u_2	$f_1f_2u_1$
u_1	u_2	u_3	u_4	0	0	0	0
u_2	f_2u_1	u_4	f_2u_3	0	0	0	0
u_3	u_4	f_1u_1	f_1u_2	0	0	0	0
u_4	f_2u_3	f_1u_2	$f_1f_2u_1$	0	0	0	0

$$(\alpha^1,\ \alpha^2,\ \alpha^3) = (\alpha,\ u)C_\alpha \pmod{N^2}$$

where

$$C_\alpha = \begin{bmatrix} I & M_2 & M_3 \\ 0 & N_2 & N_3 \end{bmatrix}$$

$$M_2 = \begin{array}{c} \begin{matrix} (22) & (23) & (24) & (32) & (33) & (34) & (42) & (43) & (44) \end{matrix} \\ \begin{bmatrix} f_2 & 0 & 0 & 0 & f_1 & 0 & 0 & 0 & f_1f_2 \\ 0 & 0 & 0 & 0 & 0 & f_1 & 0 & f_1 & 0 \\ 0 & 0 & f_2 & 0 & 0 & 0 & f_2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$N_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_3 = \begin{array}{c} \begin{matrix} (222) & (223) & (224) & (232) & (233) & (234) & (242) & (243) & (244) \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & f_1f_2 & 0 & f_1f_2 & 0 \\ f_2 & 0 & 0 & 0 & f_1 & 0 & 0 & 0 & f_1f_2 \\ 0 & f_2 & 0 & f_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_2 & 0 & 0 & 0 & f_2 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{ccccccccc} (322) & (323) & (324) & (332) & (333) & (334) & (342) & (343) & (344) \\ 0 & 0 & f_1f_2 & 0 & 0 & 0 & f_1f_2 & 0 & 0 \\ 0 & f_1 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 \\ f_2 & 0 & 0 & 0 & f_1 & 0 & 0 & 0 & f_1f_2 \\ 0 & 0 & 0 & 0 & 0 & f_1 & 0 & f_1 & 0 \end{array}$$

$$
\begin{array}{ccccccccc}
(422) & (423) & (424) & (432) & (433) & (434) & (442) & (443) & (444)
\end{array}
$$

$$
\left.\begin{array}{ccccccccc}
0 & f_1f_2 & 0 & f_1f_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f_1f_2 & 0 & 0 & 0 & f_1f_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_1f_2 & 0 & f_1f_2 & 0 \\
f_2 & 0 & 0 & 0 & f_1 & 0 & 0 & 0 & f_1f_2
\end{array}\right]
$$

$$
\begin{array}{ccccccccc}
(222) & (223) & (224) & (232) & (233) & (234) & (242) & (243) & (244)
\end{array}
$$

$$
N_3 = \left[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & f_1 & f_2 & f_1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & f_1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & f_2 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}\right.
$$

$$
\begin{array}{ccccccccc}
(322) & (323) & (324) & (332) & (333) & (334) & (342) & (343) & (344) \\
0 & 0 & f_1 & 0 & 0 & 0 & f_1 & f_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & f_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
\begin{array}{ccccccccc}
(422) & (423) & (424) & (432) & (433) & (434) & (442) & (443) & (444)
\end{array}
$$

$$
\left.\begin{array}{ccccccccc}
0 & f_1 & 0 & f_1 & 0 & 0 & 0 & 0 & f_1f_2 \\
0 & 0 & f_1 & 0 & 0 & 0 & f_1 & f_1 & 0 \\
0 & 0 & 0 & 0 & 0 & f_1 & f_2 & f_1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & f_1
\end{array}\right]
$$

$$
C_\beta = \begin{bmatrix} I & M_2 & M_3 \\ D & N_2 + N^* & N_3 + N^{**} \end{bmatrix}
$$

where

$$D = \begin{bmatrix} 0 & d_{12} & d_{13} & d_{14} \\ 0 & d_{22} & d_{23} & d_{24} \\ 0 & d_{32} & d_{33} & d_{34} \\ 0 & d_{42} & d_{43} & d_{44} \end{bmatrix}$$

$$N^* = \begin{array}{cccccccccc} (22) & (23) & (24) & (32) & (33) & (34) & (42) & (43) & (44) \\ \end{array}$$

$$\left[\begin{array}{ccccccccc} 0 & f_2d_{23} + f_1d_{32} & f_2d_{24} + f_1f_2d_{42} & f_1d_{32} + f_2d_{23} & 0 & f_1d_{34} + f_1f_2d_{43} & f_1f_2d_{42} + f_2d_{24} & f_1f_2d_{43} + f_1d_{34} & 0 \\ 0 & d_{13} + f_1d_{42} & d_{14} + f_1d_{32} & f_1d_{42} + d_{13} & 0 & f_1d_{44} + f_1d_{33} & f_1d_{32} + d_{14} & f_1d_{33} + f_1d_{44} & 0 \\ 0 & f_2d_{43} + d_{12} & f_2d_{22} + f_2d_{44} & d_{12} + f_2d_{43} & 0 & d_{14} + f_2d_{23} & f_2d_{22} + f_2d_{44} & f_2d_{23} + d_{14} & 0 \\ 0 & d_{33} + d_{22} & d_{34} + d_{12} & d_{22} + d_{33} & 0 & d_{24} + d_{13} & d_{12} + d_{34} & d_{13} + d_{24} & 0 \end{array}\right]$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} (222) & (223) & (224) & (232) \end{array} \\
N^{**} = & \left[\begin{array}{cccc}
f_2 d_{12} & f_2 d_{13} & f_2 d_{14} & f_1 f_2 + f_2 d_{13} + f_1 f_2 d_{42} \\
f_2 d_{22} & f_2 d_{23} & f_2 d_{24} & f_2 d_{23} \\
f_2 d_{32} & f_2 d_{33} & f_2 d_{34} & f_2 d_{33} \\
f_2 d_{42} & f_2 d_{43} & f_2 d_{44} & f_2 d_{43}
\end{array}\right.
\end{array}
$$

$$
\begin{array}{ccc}
(233) & (234) & (242) \\
f_1 d_{12} & f_1 f_2 (d_{44} + d_{33} + d_{22}) & f_2 d_{14} \\
f_1 d_{22} & f_1 d_{34} + f_1 f_2 d_{43} + f_1 d_{12} & f_2 d_{24} \\
f_1 d_{32} & f_2 d_{24} + f_2 d_{13} + f_1 f_2 d_{42} & f_2 d_{34} \\
f_1 d_{42} & 1 + d_{14} + f_2 d_{23} + f_1 d_{32} & f_2 d_{44}
\end{array}
$$

$$
\begin{array}{cccc}
(243) & (244) & (322) & (323) \\
f_1 f_2 d_{33} + f_1 d_{14} + f_1^2 d_{32} & f_1 f_2 d_{12} & f_2 d_{13} & f_1 d_{12} \\
f_1 + f_1 d_{14} + f_1^2 d_{32} + f_1 f_2 d_{43} & f_1 f_2 d_{22} & f_2 d_{23} & f_1 d_{22} \\
f_2 d_{13} + f_2 d_{24} + f_1 f_2 d_{42} & f_1 f_2 d_{32} & f_2 d_{33} & f_1 d_{32} \\
f_2 d_{23} + d_{14} + f_1 d_{32} & f_1 f_2 d_{42} & f_2 d_{43} & f_1 d_{42}
\end{array}
$$

$$
\begin{array}{cccc}
(324) & (332) & (333) & (334) \\
f_1 f_2 (d_{22} + d_{33} + d_{44}) & f_1 d_{12} & f_1 d_{13} & f_1 d_{14} \\
f_1 d_{34} + f_1 d_{12} + f_1 f_2 d_{43} & f_1 d_{22} & f_1 d_{23} & f_1 d_{24} \\
f_2 d_{24} + f_1 f_2 d_{42} + f_2 d_{13} & f_1 d_{32} & f_1 d_{33} & f_1 d_{34} \\
d_{14} + f_1 d_{32} + f_2 d_{23} & f_1 d_{42} & f_1 d_{43} & f_1 d_{44}
\end{array}
$$

$$
\begin{array}{cccc}
(342) & (343) & (344) & (422) \\
f_1f_2(d_{22} + d_{33} + d_{44}) & f_1d_{14} & f_1f_2d_{13} & f_2d_{14} \\
f_1d_{12} + f_1d_{34} + f_1f_2d_{43} & f_1d_{24} & f_1f_2d_{23} & f_2d_{24} \\
f_1f_2d_{42} + f_2d_{24} + f_2d_{13} & f_1d_{34} & f_1f_2d_{33} & f_2d_{34} \\
f_1d_{32} + d_{14} + f_2d_{23} & f_1d_{44} & f_1f_2d_{43} & f_2d_{44}
\end{array}
$$

$$
\begin{array}{ccc}
(423) & (424) & (432) \\
f_1f_2(d_{22} + d_{33} + d_{44}) & f_1f_2d_{12} & f_1f_2(d_{22}+d_{33}+d_{44}) \\
f_1f_2d_{43} + f_1d_{12} + f_1d_{34} & f_1f_2d_{22} & f_1d_{12}+f_1f_2d_{43}+f_1d_{34} \\
f_2d_{13} + f_1f_2d_{42} + f_2d_{24} & f_1f_2d_{32} & f_1f_2d_{42}+f_2d_{13}+f_2d_{24} \\
f_2d_{23} + f_1d_{32} + d_{14} & f_1f_2d_{42} & f_1d_{32}+f_2d_{23}+d_{14}
\end{array}
$$

$$
\begin{array}{cccc}
(433) & (434) & (443) & (444) \\
\left.\begin{matrix}
f_1d_{14} & f_1f_2d_{12} & f_1f_2d_{13} & f_1f_2d_{14} \\
f_1d_{24} & f_1f_2d_{22} & f_1f_2d_{23} & f_1f_2d_{24} \\
f_1d_{34} & f_1f_2d_{32} & f_1f_2d_{33} & f_1f_2d_{34} \\
f_1d_{44} & f_1f_2d_{42} & f_1f_2d_{43} & f_1f_2d_{44}
\end{matrix}\right]
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccccccc}
(22) & (23) & (24) & (32) & (33) & (34) & (42) & (43) & (44)
\end{array} \\
DM_2 = \begin{bmatrix}
0 & d_{14} & f_2d_{13} & d_{14} & 0 & f_1d_{12} & f_2d_{13} & f_1d_{12} & 0 \\
0 & d_{24} & f_2d_{23} & d_{24} & 0 & f_1d_{22} & f_2d_{23} & f_1d_{22} & 0 \\
0 & d_{34} & f_2d_{33} & d_{34} & 0 & f_1d_{32} & f_1d_{33} & f_1d_{32} & 0 \\
0 & d_{44} & f_2d_{43} & d_{44} & 0 & f_1d_{42} & f_1d_{43} & f_1d_{42} & 0
\end{bmatrix}
\end{array}
$$

$N_2 + N^* - DM_2 =$

$$\begin{array}{c}
\begin{array}{cccc} (22) & & (24) & \end{array} \\
\left[\begin{array}{cccc}
1 & \cdot & f_2d_{24} + f_1f_2d_{42} - f_2d_{13} & \cdot \\
0 & \cdot & 1 + d_{14} + f_1d_{32} - f_2d_{23} & \cdot \\
0 & \cdot & 1 + f_2d_{22} + f_2d_{44} - f_2d_{33} & \cdot \\
0 & & d_{34} + d_{12} - f_2d_{43} &
\end{array}\right.
\end{array}$$

$$\begin{array}{cc}
(34) & (42) \\
f_1d_{34} + f_1f_2d_{43} - f_1d_{12} & f_1f_2d_{42} + f_2d_{24} - f_2d_{13} \\
f_1d_{44} + f_1d_{33} - f_1d_{22} & f_1d_{32} + d_{14} - f_2d_{23} \\
d_{14} + f_2d_{23} - f_1d_{32} & 1 + f_2d_{22} + f_2d_{44} - f_2d_{33} \\
d_{24} + d_{13} - f_1d_{42} & d_{12} + d_{34} - f_2d_{43}
\end{array}$$

$$\begin{array}{c}
\begin{array}{cc} (43) & (44) \end{array} \\
\left.\begin{array}{cc}
f_1f_2d_{43} + f_1d_{34} - f_1d_{12} & f_1 \\
f_1d_{33} + f_1d_{44} - f_1d_{22} & 0 \\
1 + f_2d_{23} + d_{14} - f_1d_{32} & 0 \\
d_{13} + d_{24} - f_1d_{42} & 1
\end{array}\right]
\end{array}$$

After elementary column transformations, subtracting column (42) from (24), and subtracting column (34) from (43) the matrix contains

$$\begin{array}{c}
\begin{array}{cccc} (22) & (24) & (43) & (44) \end{array} \\
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & * & * & 1
\end{vmatrix} = 1 .
\end{array}$$

Therefore, $N_2 + N^* - DM_2$ has rank 4.

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