An Iterative Precoder Design for Successive Zero-Forcing Precoded Systems

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***Abstract*—Successive zero-forcing (SZF) is a precoding technique for broadcast channels that partially precancels the multiuser interference. Due to the residual interference, the optimal precoder design for SZF is difficult to find, and currently unknown. This letter proposes a suboptimal iterative precoder design for the sum rate maximization problem for SZF, using a local optimization method. In each iteration, the precoders are updated to maximize a first-order approximation of the sum rate, which can be easily solved since the approximation is concave with the precoders. By initializing the precoders with zero matrices, we analytically prove that the iterative design can achieve a larger sum rate than the block diagonalization (BD) approach. Moreover, numerical results demonstrate the superior sum-rate performance of the proposed design over the existing scheme, which is based on a heuristic method.**

***Index Terms*—MIMO, downlink channel, linear precoding.**

I. Introduction

Recent information theoretic studies have proved that dirty paper coding (DPC) is the capacity-achieving precoding technique for MIMO broadcast channels (MIMO BCs) [1]. However, DPC may be too complex for practical applications, mainly due to its nonlinear precoding nature. Thus, low-complexity linear precoding schemes are of great interest. In [2], a zero-forcing (ZF) method, called block diagonalization (BD), was introduced. BD completely eliminates the multiuser interference, and thus decomposes a MIMO BC into a set of parallel single-user MIMO channels. The performance of the BD approach largely depends on the orthogonality among users since the precoder of each user lies in the null space of the remaining users' channel matrices. When the users' channel matrices are highly correlated, BD offers a poor performance.

In [3], a linear precoding scheme, which is referred to as SZF, was proposed based on relaxing ZF constraints in BD. Specifically, for user k, only the interference caused by users 1 through $k-1$ is eliminated using the ZF technique. Consequently, user k is still subject to the interference from users $k+1$ through K, where K is the number of users. Currently, the optimal precoder design for SZF is still unknown. The main difficulty is due to the residual interference. Thus, in [3], a suboptimal precoder design was proposed using a heuristic method. First, the optimal covariance matrices for DPC are computed based on the algorithms in [4], [5]. Then these covariance matrices are projected onto corresponding null spaces, resulting from the ZF constraints, to find precoders for SZF. It was numerically shown in [3] that the average sum rate of the heuristic design is larger than that of BD for several system configurations. However, for some channel realizations, we observe that the heuristic method even yields a lower sum rate than the BD scheme.

Considering the sum-rate maximization (SRM) problem under a total power constraint, we propose an iterative precoder design for SZF, using a convex optimization-based approach. In fact, the SRM problem for SZF has no general convex programming equivalent, due to the residual interference. Hence, in each iteration of the proposed precoder design, we update the precoders to maximize a first-order approximation of the sum rate. This approximation turns out to be a concave function of the precoders, and thus the maximization can be efficiently solved using standard convex optimization tools, e.g. [6]. In other words, the precoders in each iteration are found to be a local minimizer of the SRM problem. We show that the iterative method surely converges to a local optimum from an arbitrary starting point. Moreover, by initializing the precoders with zero matrices, we prove that the achieved sum rate of the iterative design is larger than that of the BD scheme. Finally, we provide numerical results to illustrate the superior sum-rate performance of the proposed design over the heuristic method.

Notation: Standard notations are used in this letter. Bold lower and upper case letters represent vectors and matrices, respectively; $\boldsymbol{H}^H$ is the Hermitian transpose of $\boldsymbol{H}$; $\mathtt{tr}(\boldsymbol{H})$ and $|\boldsymbol{H}|$ are the trace and determinant of $\boldsymbol{H}$, respectively. $\boldsymbol{I}_M$ represents an $M \times M$ identity matrix. $\mathcal{N}(\boldsymbol{H})$ denotes the null space of $\boldsymbol{H}$.

II. Successive Zero-Forcing Precoding

We consider a downlink channel comprising of a base station (BS) equipped with N transmit antennas and K users, each with n_k receive antennas. We denote this downlink channel by the notation $N \times \{n_1, n_2, \dots, n_K\}$. For linearly precoded systems, the received signal at the kth user is given by

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{W}_k \boldsymbol{s}_k + \sum_{j \neq k} \boldsymbol{H}_k \boldsymbol{W}_j \boldsymbol{s}_j + \boldsymbol{n}_k, \tag{1}$$

where $\boldsymbol{H}_k \in \mathbb{C}^{n_k \times N}$ denotes the channel matrix, $\boldsymbol{s}_k \in \mathbb{C}^{L_k \times 1}$, $L_k \leq n_k$, is vector of modulated symbols, and $\boldsymbol{W}_k \in \mathbb{C}^{N \times L_k}$ is the precoder of the kth user, respectively. We assume that $E\{\boldsymbol{s}_k \boldsymbol{s}_k^H\} = \boldsymbol{I}_{n_k}$. The background noise $\boldsymbol{n}_k \in \mathbb{C}^{n_k \times 1}$ is assumed to be white complex-Gaussian with zero mean and covariance matrix $\boldsymbol{I}_{n_k}$. Generally, low-complexity precoding schemes are developed based on completely or partly eliminating the multiuser interference in (1). For example, the BD method completely cancels the multiuser interference by designing $\boldsymbol{W}_k$ such that $\boldsymbol{H}_j \boldsymbol{W}_k = \boldsymbol{0}$ for all $j \neq k$ [2]. In [3], the SZF was proposed by finding

Manuscript received September 8, 2011. The associate editor coordinating the review of this letter and approving it for publication was D. Michalopoulos.

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Digital Object Identifier 10.1109/LCOMM.2011.111611.111834

precoders such that $\boldsymbol{H}_j\boldsymbol{W}_k = \boldsymbol{0}$ for all $j < k$, i.e. only the interference caused by users 1 through $k-1$ is eliminated. Let $\bar{\boldsymbol{H}}_k = [\boldsymbol{H}_1^H \boldsymbol{H}_2^H \cdots \boldsymbol{H}_{k-1}^H]^H \in \mathbb{C}^{\sum_{i=1}^{k-1} n_i \times N}$. Then ZF constraints in SZF are equivalent to $\bar{\boldsymbol{H}}_k\boldsymbol{W}_k = \boldsymbol{0}$. The SRM problem for SZF under a total power constraint P is formulated as

$$\begin{array}{ll} \underset{\boldsymbol{W}_k}{\text{maximize}} & \sum_{k=1}^{K} \log \frac{|\boldsymbol{I}+\boldsymbol{H}_k(\sum_{j=1}^{k}\boldsymbol{W}_j\boldsymbol{W}_j^H)\boldsymbol{H}_k^H|}{|\boldsymbol{I}+\boldsymbol{H}_k(\sum_{j=1}^{k-1}\boldsymbol{W}_j\boldsymbol{W}_j^H)\boldsymbol{H}_k^H|} \\ \text{subject to} & \bar{\boldsymbol{H}}_k\boldsymbol{W}_k = \boldsymbol{0},\ \forall k \\ & \sum_{k=1}^{K} \mathtt{tr}(\boldsymbol{W}_k\boldsymbol{W}_k^H) \leq P. \end{array} \tag{2}$$

Clearly, the sum rate of the SZF scheme depends on the user ordering by which the interference cancellation is performed. Herein, we simply assume a natural user ordering and focus on the precoder design. We also assume that $N > \sum_{i=1}^{K-1} n_i$ so that the BS can support all users. Let $\boldsymbol{B}_k \in \mathbb{C}^{N\times\bar{n}_k}$, $\bar{n}_k = N - \sum_{i=1}^{k-1} n_i$, be an orthonormal basis of $\mathcal{N}(\bar{\boldsymbol{H}}_k)$, which can be efficiently computed using singular value decomposition or QR decomposition. Then, we can write $\boldsymbol{W}_k = \boldsymbol{B}_k\boldsymbol{D}_k$, where $\boldsymbol{D}_k \in \mathbb{C}^{\bar{n}_k\times L_k}$ is optimized under the power constraint. Since $\boldsymbol{B}_k^H\boldsymbol{B}_k = \boldsymbol{I}$, (2) reduces to

$$\begin{array}{ll} \underset{\boldsymbol{D}_k}{\text{maximize}} & \sum_{k=1}^{K} \log \frac{|\boldsymbol{I}+\sum_{j=1}^{k}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{D}_j\boldsymbol{D}_j^H\tilde{\boldsymbol{H}}_{k,j}^H|}{|\boldsymbol{I}+\sum_{j=1}^{k-1}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{D}_j\boldsymbol{D}_j^H\tilde{\boldsymbol{H}}_{k,j}^H|} \\ \text{subject to} & \sum_{k=1}^{K} \mathtt{tr}(\boldsymbol{D}_k\boldsymbol{D}_k^H) \leq P, \end{array} \tag{3}$$

where $\tilde{\boldsymbol{H}}_{k,j} = \boldsymbol{H}_k\boldsymbol{B}_j$. Let $\boldsymbol{S}_k \triangleq \boldsymbol{D}_k\boldsymbol{D}_k^H \in \mathbb{C}^{\bar{n}_k\times\bar{n}_k}$. Then (3) is equivalent to

$$\begin{array}{ll} \underset{\boldsymbol{S}_k\succeq\boldsymbol{0}}{\text{maximize}} & \sum_{k=1}^{K} R_k \\ \text{subject to} & \sum_{k=1}^{K} \mathtt{tr}(\boldsymbol{S}_k) \leq P, \end{array} \tag{4}$$

where $R_k = \log \frac{|\boldsymbol{I}_{n_k}+\sum_{j=1}^{k}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j\tilde{\boldsymbol{H}}_{k,j}^H|}{|\boldsymbol{I}_{n_k}+\sum_{j=1}^{k-1}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j\tilde{\boldsymbol{H}}_{k,j}^H|}$ is the data rate of the kth user, and we have dropped the rank constraints $\mathrm{rank}(\boldsymbol{S}_k) \leq \min(n_k, \bar{n}_k)$ in (4). Since the objective function in (4) is neither convex nor concave, finding its optimal solution is generally hard. Based on the observation that (2) is similar to the sum-capacity computation of the BC with added ZF constraints on the set of covariance matrices, [3] proposed a suboptimal design based on a heuristic method. Specifically, the heuristic method first solves (2) with ZF constraints being omitted, using the iterative water-filling algorithm proposed in [4]. Then the precoders for SZF are obtained by projecting the resulting optimal solutions into $\mathcal{N}(\bar{\boldsymbol{H}}_k)$ to satisfy the ZF constraints in (2). Due to the heuristic nature, this method only produces small gains over the BD scheme. In this letter, we propose a suboptimal design for SZF by finding a local maximum of (4)

III. Proposed Precoder Design

The proposed precoder design is based on an iterative method. Suppose after n iterations, precoders are denoted by $\{\boldsymbol{S}_k^{(n)}\}$. Due to the concavity of the $\log\det$ function, the data rate of the kth user is lower bounded by

$$\begin{aligned} R_k \geq\ & \log|\boldsymbol{I}_{n_k} + \textstyle\sum_{j=1}^{k}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j\tilde{\boldsymbol{H}}_{k,j}^H| \\ & - \textstyle\sum_{j=1}^{k-1}\mathtt{tr}(\boldsymbol{\Omega}_k^{-1}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j\tilde{\boldsymbol{H}}_{k,j}^H) - \log|\boldsymbol{\Omega}_k| \\ & + \textstyle\sum_{j=1}^{k-1}\mathtt{tr}(\boldsymbol{\Omega}_k^{-1}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j^{(n)}\tilde{\boldsymbol{H}}_{k,j}^H) \end{aligned} \tag{5}$$

Algorithm 1 The proposed iterative precoder design for SZF

Initialization: $\{\boldsymbol{S}_k^{(0)}\} = \boldsymbol{0}$; $n \leftarrow 1$.
repeat
  Solve (6) to find optimal solutions $\boldsymbol{S}_k^\star$.
  Update: $\boldsymbol{S}_k^{(n)} = \boldsymbol{S}_k^\star$; $n \leftarrow n+1$.
until maximum number of iterations.
Let $\boldsymbol{S}_k^{(n)} = \boldsymbol{D}_k\boldsymbol{D}_k^H$ be the Cholesky decomposition of $\boldsymbol{S}_k^{(n)}$.
Precoder $\boldsymbol{W}_k$ is found as $\boldsymbol{W}_k = \boldsymbol{B}_k\boldsymbol{D}_k$.

where $\boldsymbol{\Omega}_k = \boldsymbol{I}_{n_k} + \sum_{j=1}^{k-1}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j^{(n)}\tilde{\boldsymbol{H}}_{k,j}^H$. In fact, (5) is a first-order approximation of R_k at the operation points $\{\boldsymbol{S}_k^{(n)}\}$. This approximation of the $\log\det$ function is also used, e.g., in [7] for rank-minimization problem, and in [8] for general linear precoding. In the $(n+1)$th iteration, the precoders $\{\boldsymbol{S}_k^{(n+1)}\}$ are designed to be the optimal solution to the following optimization problem

$$\begin{array}{ll} \underset{\boldsymbol{S}_k\succeq\boldsymbol{0}}{\text{maximize}} & \sum_{k=1}^{K} f_k(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_k) \\ \text{subject to} & \sum_{k=1}^{K} \mathtt{tr}(\boldsymbol{S}_k) \leq P \end{array} \tag{6}$$

where $f_k(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_k) = \log|\boldsymbol{I}_{n_k} + \sum_{j=1}^{k}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j\tilde{\boldsymbol{H}}_{k,j}^H| - \sum_{j=1}^{k-1}\mathtt{tr}(\boldsymbol{\Omega}_k^{-1}\tilde{\boldsymbol{H}}_{k,j}\boldsymbol{S}_j\tilde{\boldsymbol{H}}_{k,j}^H)$. We have ignored the constants in (5) because they do not affect the maximization. We note that the objective function in (6) is concave with respect to $\{\boldsymbol{S}_k\}$. Thus, the optimal solution of (6) can be efficiently computed using standard convex optimization tools, e.g., CVX [6]. The proposed iterative precoder design is summarized in Algorithm 1. Initially, we set $\boldsymbol{S}_k^{(0)} = \boldsymbol{0}$ for all $k = 1, 2, \ldots, K$. In each iteration, we find optimal solutions $\boldsymbol{S}_k^\star$ of (6) to update the precoders. For $\boldsymbol{S}_k^\star$ to be a feasible precoder design, we must ensure that $\mathrm{rank}(\boldsymbol{S}_k^\star) \leq \min(n_k, \bar{n}_k)$. This can be proved as follows. First, we start with the optimal conditions (i.e. the KKT conditions) for (6), which are given by (7), shown at the top of the following page, where λ and $\boldsymbol{Z}_k \succeq \boldsymbol{0}$ are the dual variables associated with the power and positive semidefinite constraints $\boldsymbol{S}_k^\star \succeq \boldsymbol{0}$, respectively. In (7), we have used the fact that the gradient of $\log|\boldsymbol{I} + \tilde{\boldsymbol{H}}_k\boldsymbol{S}_k\tilde{\boldsymbol{H}}_k^H|$ with respect to $\boldsymbol{S}_k$ is $\tilde{\boldsymbol{H}}_k^H(\boldsymbol{I} + \tilde{\boldsymbol{H}}_k\boldsymbol{S}_k\tilde{\boldsymbol{H}}_k^H)^{-1}\tilde{\boldsymbol{H}}_k$. Using the slackness property $\boldsymbol{Z}_k\boldsymbol{S}_k^\star = \boldsymbol{0}$, we obtain

$$\tilde{\boldsymbol{H}}_{k,k}^H\boldsymbol{\Omega}_k^{-1}\tilde{\boldsymbol{H}}_{k,k}\boldsymbol{S}_k^\star = (\lambda\boldsymbol{I} + \boldsymbol{\Psi}_k)\boldsymbol{S}_k^\star \tag{8}$$

where $\boldsymbol{\Psi}_k = \sum_{j=k}^{K}\tilde{\boldsymbol{H}}_{j,k}^H\big[\boldsymbol{\Omega}_j^{-1} - \big(\boldsymbol{I}_{n_j} + \sum_{i=1}^{j}\tilde{\boldsymbol{H}}_{j,i}\boldsymbol{S}_i^\star\tilde{\boldsymbol{H}}_{j,i}^H\big)^{-1}\big]\tilde{\boldsymbol{H}}_{j,k}$. It is easy to see that $\lambda > 0$ since the power constraint in (6) is active at the optimum. Hence, $\lambda\boldsymbol{I} + \boldsymbol{\Psi}_k$ is invertible with probability one. It follows from (8) that $\mathrm{rank}(\boldsymbol{S}_k^\star) \leq \mathrm{rank}(\tilde{\boldsymbol{H}}_{k,k}) = \min(n_k, \bar{n}_k)$, which completes the proof.

Following the same arguments as those in [8], we can see that the proposed iterative design surely converges to a local optimum of (4) from any starting point. Moreover, we are able to prove that the proposed design, although suboptimal, achieves a larger sum rate than that of BD for any channel realization. This is a stronger result than that in [3], where the heuristic method is only shown to obtain a higher sum rate than BD on the average sense by Monte Carlo simulations. To see this, substituting $\boldsymbol{S}_k^{(0)} = \boldsymbol{0}$ into (5), we can equivalently

$$\sum_{j=k}^{K} \tilde{\boldsymbol{H}}_{j,k}^{H}\big[\big(\boldsymbol{I}_{n_j} + \sum_{i=1}^{j} \tilde{\boldsymbol{H}}_{j,i}\boldsymbol{S}_i^{\star}\tilde{\boldsymbol{H}}_{j,i}^{H}\big)^{-1} - \boldsymbol{\Omega}_j^{-1}\big]\tilde{\boldsymbol{H}}_{j,k} + \tilde{\boldsymbol{H}}_{k,k}^{H}\boldsymbol{\Omega}_k^{-1}\tilde{\boldsymbol{H}}_{k,k} + \boldsymbol{Z}_k - \lambda\boldsymbol{I} = \boldsymbol{0},\ \forall k = 1, 2 \ldots K. \tag{7}$$

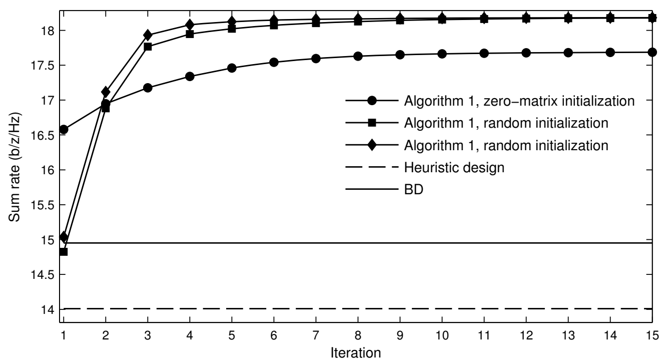

Fig. 1. Sum-rate versus number of iterations for a random channel realization of a $8 \times \{2, 2, 2, 2\}$ channel.

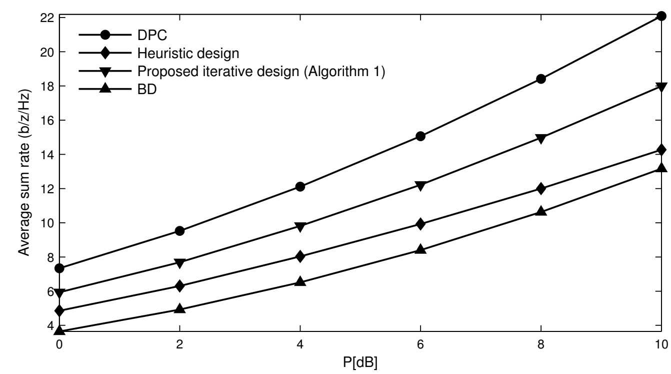

Fig. 2. Average sum-rate comparison of different precoding schemes for a $8 \times \{2, 2, 2, 2\}$ channel.

express the sum rate achieved after the first iteration as

$$\begin{array}{ll} \underset{\bar{\boldsymbol{S}}_k \succeq \boldsymbol{0}}{\text{maximize}} & \sum_{k=1}^{K}\Big\{\log|\boldsymbol{I}_{n_k} + \sum_{j=1}^{k}\boldsymbol{H}_k\bar{\boldsymbol{S}}_j\boldsymbol{H}_k^H| \\ & \quad - \sum_{j=1}^{k-1}\mathtt{tr}(\boldsymbol{H}_k\bar{\boldsymbol{S}}_j\boldsymbol{H}_k^H)\Big\} \\ \text{subject to} & \boldsymbol{H}_j\bar{\boldsymbol{S}}_k\boldsymbol{H}_j^H = \boldsymbol{0},\ \forall j < k \\ & \sum_{k=1}^{K}\mathtt{tr}(\bar{\boldsymbol{S}}_k) \le P. \end{array} \tag{9}$$

By forcing $\boldsymbol{H}_j\bar{\boldsymbol{S}}_k\boldsymbol{H}_j^H = \boldsymbol{0}$ for all $k \neq j$ as in the BD method, we can reduce (9) to

$$\begin{array}{ll} \underset{\bar{\boldsymbol{S}}_k \succeq \boldsymbol{0}}{\text{maximize}} & \sum_{k=1}^{K}\log|\boldsymbol{I}_{n_k} + \boldsymbol{H}_k\bar{\boldsymbol{S}}_k\boldsymbol{H}_k^H| \\ \text{subject to} & \boldsymbol{H}_j\bar{\boldsymbol{S}}_k\boldsymbol{H}_j^H = \boldsymbol{0},\ \forall j \neq k \\ & \sum_{k=1}^{K}\mathtt{tr}(\bar{\boldsymbol{S}}_k) \le P, \end{array} \tag{10}$$

The optimal value of (10) is actually the sum rate of the BD scheme [2]. Since (9) is a relaxation of (10), we conclude that the sum rate of the iterative design after the first iteration is larger than or equal to that of BD.

IV. Numerical Results

In this section we numerically compare the sum rate of the proposed design with the heuristic method in [3]. First, we demonstrate the convergence behavior of Algorithm 1 for a random channel realization of a downlink channel $8\times\{2, 2, 2, 2\}$ with three different choices for $\{\boldsymbol{S}_k^{(0)}\}$, one with zero matrices, the other two with random realizations. Since the proposed algorithm converges to a local minimum, the obtained sum rate strongly depends on $\{\boldsymbol{S}_k^{(0)}\}$. In Algorithm 1, we initially set $\{\boldsymbol{S}_k^{(0)}\} = \boldsymbol{0}$ to ensure that the iterative method provides a sum rate higher than BD. However, other choices for $\{\boldsymbol{S}_k^{(0)}\}$ can result in a better performance, as shown in Fig. 1. In this example, we can also see that the sum rate of the heuristic design is even lower than that of the BD scheme.

Fig. 2 plots the average sum rate versus the transmit power P for the same configuration as in Fig. 1. A Rayleigh quasi-static fading model is used in this simulation, where independent realizations of $\boldsymbol{H}_k$ are generated as zero mean and unit variance complex Gaussian random variables. Algorithm 1 is initialized by zero-matrix, and the maximum number of iterations is 5. The sum rate of DPC, which can be computed using the algorithms in [4], [5], is included as a benchmark. As can be seen in Fig. 2, the proposed iterative precoder design outperforms the heuristic method in [3], and the BD scheme.

V. Conclusions

We have presented an iterative precoder design for SZF based on a local minimization method. In each iteration, a convex optimization problem, which approximates the SRM problem, is solved to update the precoders. We prove that the iterative design for SZF converges to a local optimum, which achieves a higher sum rate than the BD approach. Numerical results are provided to demonstrate the superior sum-rate performance of the proposed design over the heuristic method and the BD scheme. The optimal precoder design for SZF is still an open problem, and left as a future work.

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