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Some Strange Properties of Quadratic Stochastic Volterra Operators

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Abstract: One of the fascinating results in the one dimensional nonlinear dynamical system is that a mapping which maps a compact connected subset of the real line into itself is regular if and only if it does not have any order periodic points except fixed points. However, in general, this result does not hold true in the high dimensional case. In this paper, we provide a counter example for such kind of mappings among quadratic stochastic Volterra operators. Moreover, we showed an equivalence of notions of regularity, transitivity and Ergodic principle for quadratic stochastic Volterra operators acting on the finite dimensional simplex. Apart from these, we study the fixed point set of the composition of two quadratic stochastic Volterra operators.

Key words: Regularity · Transitivity · Ergodic princple · Quadratic stochastic operator · Volterra operator

INTRODUCTION

The theory of linear operators has been well studied since the last century. The simplest nonlinear operator is a quadratic operator. A quadratic operator is a primary source for investigations of dynamical properties of population genetics [1-2]. It was given a long self-contained exposition of the recent achievements and open problems in the theory of quadratic stochastic operators in [3].

Let $S^{m-1} = \left\{ x \in \mathbb{R}^{m-1} : \sum_{i=1}^{m} x_i = 1, x_i \geq 0, \forall i = \overline{1,m} \right\}$ be an $(m-1)$ dimensional simplex. A mapping $V : S^{m-1} \to S^{m-1}$ defined as follows

$$x'_k \equiv (Vx)_k := \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad \forall k = \overline{1,m} \tag{1}$$

is said to be *a quadratic stochastic operator* where

$$x = (x_1, x_2, ..., x_m) \in S^{(m-1)}$$

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^{m} P_{ij,k} = 1 \quad \forall i, j = \overline{1,m}$$

The main problem of the theory of the dynamical system is to classify states $x^{(0)}$ based on the behavior of the trajectory $\{x^{(n)}\}_{n=1}^{\infty}$. It is particularly interesting when the trajectory repeats. In this case we say that $x^{(0)}$ is *a periodic point*, i.e., $x^{(0)}$ is *a periodic point* if there is the smallest positive integer k such that $x^{(k)} = x^{(0)}$. The number k is called *a period* of the periodic point $x^{(0)}$. A *fixed point* is a periodic point of period-1, that is, a point $x^{(0)}$ such that $x^{(1)} = x^{(0)}$.

If the trajectory of every point converges to some fixed point (the limiting fixed point might be depended on the initial point) then a mapping is called *regular*. It is clear that if a mapping is regular then it does not have any order periodic point except fixed points. It turns out that, in one dimensional case, the converse statement holds true as well. More precisely, one of the fascinating results in one dimensional nonlinear dynamical system is that a mapping which maps a compact connected subset of the real line into itself is regular if and only if it does not have any order periodic points [4]. The most incredible result is that a mapping which maps a compact connected subset of the real line into itself is regular if and only if it does not have any period-2 points [4].

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It is natural to seek an analogy of these incredible results in the high dimensional case. However, in general, these results do not hold true in the high dimensional case. The regularity of the nonlinear operator acting on the high dimensional space could not be described in term of an absence of periodic points. In this paper, we provide a counter example for such kind of mappings among quadratic stochastic Volterra operators. Therefore, the study of the regularity is independent of interest. We showed an equivalence of notions of regularity, transitivity and Ergodic principle for quadratic stochastic Volterra operators acting on the finite dimensional simplex. Apart from these, we study the fixed point set of the composition of two quadratic stochastic Volterra operators.

Regularity, Transitivity and Ergodic Principle: The regularity problem was concerned for the operator (1) in [5]. Namely, it was studied the following problem: find the number $\alpha_m > 0$ such that $P_{ij,k} > \alpha_m, \forall i,j,k = \overline{1,m}$, implies the regularity of quadratic stochastic operators (1). It was shown [5] that if $\alpha_m = \frac{1}{2m}$ then the operator (1) is regular. The main problem is to find the smallest positive number among all α_m for fixed m (if any) such that any quadratic stochastic operator under the condition $P_{ij,k} > \alpha_m, \forall i,j,k = \overline{1,m}$ is regular. One can check that $\inf \alpha_2 = \frac{1}{2}\left(3-\sqrt{7}\right)$. If $m \geq 3$ the problem remains open.

However, the regularity problem was studied intensively for another class of quadratic stochastic operators [6-9] which could not be covered by previous cases.

Definition: *An operator (1) is called a quadratic stochastic Volterra operator if*

$$P_{ij,k} = 0, \ k \notin \{i,j\} \text{ for any } \forall i,j = \overline{1,m}$$

Any quadratic stochastic Volterra operator can be written in the following form

$$(Vx)_k \equiv x'_k := x_k\left(1+\sum_{i=1}^{m} a_{ki}x_i\right) \tag{2}$$

where $A_m = (a_{ki})_{k,i=1}^{m}$ is a skew-symmetric matrix with $a_{ki} \in [-1,1]$. A nonlinear stochastic Volterra operator was studied in [6-8].

Let us consider the following quadratic stochastic Volterra operator $V : S^2 \to S^2$

$$V: \begin{cases} x'_1 = x_1\left(1+ax_2-bx_3\right) \\ x'_2 = x_2\left(1-ax_1+cx_3\right) \\ x'_3 = x_3\left(1+bx_1-cx_2\right) \end{cases} \tag{3}$$

where, $0 < a, b, c < 1$. It is easy to check that fixed points of the operator (3) are the vertices of the simplex e_1 (1,0,0), e_2 (0,1,0), e_3 (0,0,1) and $C = \left(\frac{c}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)$. The operator (3) does not have any order periodic points. However, the trajectory of any interior point of the simplex except the fixed point C does not converge [6-8], [10]]. More interestingly, the first order arithmetic mean of the trajectory of the operator (3) starting from the interior point of the simplex except the fixed point C does not converge [11-12]. Surprisingly, we have the following incredible result.

Theorem [13]: *Any order arithmetic mean of the trajectory of the operator (3) starting from the interior point of the simplex except the fixed point* C *does not converge.*

There exists a one-to-one mapping between the set of all quadratic stochastic Volterra operators and the set of skew-symmetric matrices.

Definition: *A skew-symmetric matrix* A_m *is called transversal if all even order leading (principal) minors are nonzero. A quadratic stochastic Volterra operator (2) is called transversal if the corresponding skew-symmetric matrix* A_m *is transversal.*

Theorem [6-7]: *The set of all transversal quadratic stochastic Volterra operators is an open, everywhere-dense subset of the set of all quadratic stochastic Volterra operators.*

In the sequel, we will only consider transversal quadratic stochastic Volterra operators without mentioning the transversality. The main approach to study the dynamics of quadratic stochastic Volterra operator is to construct its fixed points' chart by means of tournaments [6-7].

A tournament is a complete directed graph. A tournament is called *transitive* if it does not contain a cycle of length 3. It is clear that if a skew-symmetric matrix A_m corresponding to the Volterra operator (2) is transversal then $a_{ij} \neq 0$, $\forall\ i \neq j$. Therefore, we can construct a tournament corresponding to a transversal Volterra operator (2) as follows [6-7]: a tournament T_m consists of m vertices and an edge directs form i to j if $a_{ij} < 0$ otherwise it directs form j to i. A Volterra operator (2) is called *transitive* if the corresponding tournament is transitive.

Theorem [6-7]: *If a quadratic stochastic Volterra operator (2) is transitive then it is regular*.

In this paper we want to prove the converse statement of this theorem. Moreover, we describe the regularity of quadratic stochastic Volterra operators in term of the Ergodic principle. In order to this end we introduce the notion of the Ergodic principle for quadratic stochastic operators.

Let us fix a norm $\|x\| = \sqrt{\sum_{i=1}^{m} x_i^2}$ in the Euclidean space $\mathbb{R}^m$. For every $x \in S^{m-1}$ we define its support as follows supp(x) = {i: $x_i \neq 0$}.

Definition: *We say that a quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ given by (1) satisfies the Ergodic principle on the simplex S^{m-1} if for any $x, y \in S^{m-1}$ with* supp(x) = supp(y) *one has* $\lim_{n\to\infty} \left\| V^n x - V^n y \right\| = 0$ *, where*

$$V^n(\bullet) = \underbrace{V\Big(V\big(...(V(\bullet))...\big)\Big)}_{n}$$

is n times compositions of V.

The following theorem is the main result.

Theorem 1: *Let $V : S^{m-1} \to S^{m-1}$ be a quadratic stochastic Volterra operator given by (2). The following statements are equivalent:*

- *V is regular;*
- *V is transitive;*
- *V satisfies the Ergodic principle;*
- *One has* $\lim_{n\to\infty} \left\| V^n x - V^{n+1} x \right\| = 0 \quad \forall x \in S^{m-1}$.

Compositions of Volterra Operators: The set of all ?xed points $Fix(V)$ of quadratic stochastic Volterra operator (2) is nonempty due to the Brower theorem.

Theorem [6-7]: *Let $V : S^{m-1} \to S^{m-1}$ be a quadratic stochastic transversal Volterra operator given by (2). Then the following inequality holds true*

$$\binom{m}{1} \le \left| Fix(V) \right| \le \binom{m}{1} + \binom{m}{3} + \binom{m}{5} + ... + \binom{m}{2\left[\frac{m-1}{2}\right]+1}$$

One can easily prove the following results.

Remark: *Prof. Ganikhodjaev R. has conjectured that the composition of two quadratic stochastic transversal Volterra operators has the finite number of fixed points*.

However, in general, this conjecture is not true. We are going to provide a counter example for such kinds of quadratic stochastic transversal Volterra operators.

Let $V_1, V_2 : S^2 \to S^2$ be a quadratic stochastic Volterra operators

$$V_1 : \begin{cases} x' = x(1 + a_1 y - b_1 z) \\ y' = y(1 - a_1 x + c_1 z) \\ z' = z(1 + b_1 x - c_1 y) \end{cases}$$

$$V_2 : \begin{cases} x'' = x(1 + a_2 y - b_2 z) \\ y'' = y(1 - a_2 x + c_2 z) \\ z'' = z(1 + b_2 x - c_2 y) \end{cases}$$

It is easy to check that V_1, V_2 are transversal if and only if $a_i, b_i, c_i \neq 0$ for i=1,2 and the vertices of the simplex S^2 are fixed points of operators V_1, V_2. Operators V_1, V_2 might have fixed points in the set S^2.

A priori, the vertices of the simplex S^2 are fixed points of the composition operator $V_2 \circ V_1$. Therefore, we have that $\left| Fix(V_2 \circ V_1) \cap \partial S^2 \right| \ge 3$.

Proposition: *Let $V_1, V_2 : S^2 \to S^2$ be a quadratic stochastic transversal Volterra operators. Then the set $Fix(V_2 \circ V_1) \cap \partial S^2$ is finite.*

Now, we study fixed points of the operator $V_2 \circ V_1$ in the set S^2.

Theorem 2: *Let $V_1, V_2 : S^2 \to S^2$ be a quadratic stochastic transversal Volterra operators with $Fix(V_2 \circ V_1) \cap \text{int } S^2 \neq \emptyset$. Then the operator $V_2 \circ V_1$ has a finite number of fixed points in the set S^2 if and only if*

$$(a_1 + b_1 + c_1)^2 + (a_2 + b_2 + c_2)^2 > 0$$

Theorem 3: *Let $V_1, V_2 : S^2 \to S^2$ be a quadratic stochastic transversal Volterra operators with $Fix(V_2 \circ V_1) \cap \text{int } S^2 \neq \emptyset$. The following statements hold true*

- *If* $(a_1+b_1+c_1)^2+(a_2+b_2+c_2)^2>0$ *then* $\left|Fix(V_2\circ V_1)\cap \operatorname{int} S^2\right|\geq 2$;
- *If* $(a_1+b_1+c_1)^2+(a_2+b_2+c_2)^2=0$ *then* $\left|Fix(V_2\circ V_1)\cap \operatorname{int} S^2\right|=\aleph_1$.

Remark: *In the case*
$$(a_1+b_1+c_1)^2+(a_2+b_2+c_2)^2=0$$
the set $Fix(V_2\circ V_1)\cap \operatorname{int} S^2$ *is a connected line segment.*
Let us provide an example of such kind operators.

Example: Let us consider the following quadratic stochastic transversal operators

$$V_1:\begin{cases} x'=x\left(1-\frac{3}{4}y+\frac{1}{12}z\right)\\ y'=y\left(1+\frac{3}{4}x+\frac{5}{6}z\right)\\ z'=z\left(1-\frac{1}{12}x-\frac{5}{6}y\right)\end{cases}$$

$$V_2:\begin{cases} x''=x\left(1+\frac{2}{3}y-\frac{1}{6}z\right)\\ y''=y\left(1-\frac{2}{3}x-\frac{5}{6}z\right)\\ z''=z\left(1+\frac{1}{6}x+\frac{5}{6}y\right)\end{cases}$$

One can easily check that, these operators do not have any fixed points in the simplex except the vertices of the simplex S^2. For any point $(x_0,y_0,z_0)\in \operatorname{int} S^2$, the trajectory of the first operator V_1 converges to the fixed point (0,1,0) and the trajectory of the second operator V_2 converges to the fixed point (0,0,1).
However, one can easily check that

$$Fix(V_2\circ V_1)\cap \operatorname{int} S^2=\left\{(4-10\alpha,\alpha,9\alpha-3):\frac{1}{3}<\alpha<\frac{4}{10}\right\}$$

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