A NOTE ON THE OSEEN APPROXIMATION FOR A PARABOLOID IN A UNIFORM STREAM PARALLEL TO ITS AXIS

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SUMMARY

It is shown in this note that the Oseen equations have a simple solution when the body is in the form of a paraboloid of elliptic section in a stream of viscous, incompressible fluid which at infinity has uniform velocity in the direction of the axis. Particular attention is given to the case in which the paraboloid degenerates into a flat plate with a parabolic leading edge, and it is suggested that the solution of the Oseen equations in this case may throw some light on the corresponding three-dimensional boundary-layer problem.

1. Introduction

Carrier and Lewis (1) first gave the solution of the Oseen equations for the half-plane $z = 0$, $x > 0$ in a uniform stream. At about the same time Stewartson, working independently, produced the same solution by a particularly simple method which also gave the solution in the case of a parabolic cylinder. Stewartson did not publish this result. A recent paper by Kaplun (2) contains a solution similar to Stewartson's.

For the parabolic cylinder the solution of the Oseen equations can be put in the form

$$u = \frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial\chi}{\partial x}-\chi, \qquad v = 0, \qquad w = \frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial\chi}{\partial z}, \tag{1}$$

where $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2} = 0, \qquad \frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial z^2} = 2k\frac{\partial\chi}{\partial x}, \qquad k = U/2\nu,$$

and u, $w \to 0$ at infinity and $u = -U$, $w = 0$ on the cylinder.

Stewartson expresses these equations in terms of parabolic cylinder coordinates ξ, η defined by $x = \xi^2-\eta^2$, $z = 2\xi\eta$ in which the cylinder is the coordinate line $\eta =$ constant. He then seeks, and finds, functions ϕ, χ which are functions of η only.

The question naturally arises: Can this method be extended from the parabolic cylinder in two dimensions to the paraboloid in three dimensions by the use of paraboloidal coordinates? It is shown below that this extension is possible for paraboloids of elliptic section with vertices facing into the uniform stream.

2. The equations

Paraboloidal coordinates (ξ, η, ζ) can be defined as the roots λ of the equation

$$\frac{y^2}{a-\lambda}+\frac{z^2}{b-\lambda} = 4(x-\lambda) \tag{2}$$

when x, y, z are supposed given, $a > b \geqslant 0$ and $\xi \geqslant a$, $a \geqslant \eta \geqslant b$, $b \geqslant \zeta$.

Then the surface $\zeta = \zeta_0$ is the paraboloid of elliptic section

$$\frac{y^2}{a-\zeta_0}+\frac{z^2}{b-\zeta_0} = 4(x-\zeta_0) \tag{3}$$

and by a suitable choice of coordinates in the first place any such paraboloid, except that with a circular section, can be expressed as $\zeta = \zeta_0$ if a, b and ζ_0 are correctly chosen.

The coordinates x, y, z are given in terms of ξ, η, ζ by

$$\begin{aligned} x &= \xi+\eta+\zeta-a-b, \\ y^2 &= \frac{4}{(a-b)}(\xi-a)(a-\eta)(a-\zeta), \\ z^2 &= \frac{4}{(a-b)}(\xi-b)(\eta-b)(b-\zeta). \end{aligned} \tag{4}$$

If the flow has velocity U in the direction of the x-axis of infinity, the Oseen equations are

$$\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \\ U\frac{\partial w}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), \\ 0 &= \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \end{aligned} \tag{5}$$

with $u = -U$, $v = w = 0$ on $\zeta = \zeta_0$ and $u, v, w \to 0$ at infinity.

In the general three-dimensional problem it is not possible to write the solution of the equations (5) in a form corresponding to equations (1), but in this case a solution is to be sought of the form

$$
\begin{aligned}
u &= \frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial\chi}{\partial x}-\chi, \\
v &= \frac{\partial\phi}{\partial y}+\frac{1}{2k}\frac{\partial\chi}{\partial y}, \\
w &= \frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial\chi}{\partial z}, \\
p &= -\rho U\frac{\partial\phi}{\partial x},
\end{aligned} \tag{6}
$$

where $$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0, \tag{7}$$

$$\nabla^2\chi \equiv \frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}+\frac{\partial^2\chi}{\partial z^2} = 2k\frac{\partial\chi}{\partial x}, \tag{8}$$

where $k = U/2\nu$ and in paraboloidal coordinates

$$
\begin{aligned}
\nabla^2 \equiv{}& \frac{\{(\xi-a)(\xi-b)\}^{\frac{1}{2}}}{(\xi-\zeta)(\xi-\eta)}\frac{\partial}{\partial\xi}\left[\{(\xi-a)(\xi-b)\}^{\frac{1}{2}}\frac{\partial}{\partial\xi}\right]+ \\
&+\frac{\{(a-\eta)(\eta-b)\}^{\frac{1}{2}}}{(\eta-\zeta)(\xi-\eta)}\frac{\partial}{\partial\eta}\left[\{(a-\eta)(\eta-b)\}^{\frac{1}{2}}\frac{\partial}{\partial\eta}\right]+ \\
&+\frac{\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}}{(\eta-\zeta)(\xi-\zeta)}\frac{\partial}{\partial\zeta}\left[\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{\partial}{\partial\zeta}\right]
\end{aligned}
$$

and $$\frac{\partial}{\partial x} \equiv \frac{(\xi-a)(\xi-b)}{(\xi-\zeta)(\xi-\eta)}\frac{\partial}{\partial\xi}+\frac{(a-\eta)(\eta-b)}{(\xi-\eta)(\eta-\zeta)}\frac{\partial}{\partial\eta}+\frac{(a-\zeta)(b-\zeta)}{(\eta-\zeta)(\xi-\zeta)}\frac{\partial}{\partial\zeta}.$$

The existence of such a set of solutions implies that the vorticity always lies in a plane perpendicular to the axis of the paraboloid. Moreover, the form of equations (7) and (8) in paraboloidal coordinates makes it possible to find solutions of these equations which are functions of ζ only.

In this case (7) and (8) become

$$\frac{d}{d\zeta}\left[\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{d\phi}{d\zeta}\right] = 0, \tag{9}$$

$$\frac{d}{d\zeta}\left[\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{d\chi}{d\zeta}\right] = 2k\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}\frac{d\chi}{d\zeta}. \tag{10}$$

Solutions of these equations may be written as

$$\phi = A\log[a/2+b/2-\zeta+\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}], \tag{11}$$

$$\chi = B\int\limits_{-\infty}^{\zeta}\frac{e^{2ks}}{\{(a-s)(b-s)\}^{\frac{1}{2}}}\,ds, \tag{12}$$

where A and B are constants.

Equations (6) can now be written as

$$
\begin{aligned}
u &= \frac{(a-\zeta)(b-\zeta)}{(\xi-\zeta)(\eta-\zeta)}\frac{d}{d\zeta}\left(\phi+\frac{1}{2k}\chi\right)-\chi \\
&= \frac{\{(a-\zeta)(b-\zeta)\}^{\frac{1}{2}}}{(\xi-\zeta)(\eta-\zeta)}\left\{A+\frac{Be^{2k\zeta}}{2k}\right\}-\chi, \\
v &= \frac{-(b-\zeta)\{(\xi-a)(a-\eta)(a-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\frac{d}{d\zeta}\left(\phi+\frac{1}{2k}\chi\right) \\
&= \frac{-\{(\xi-a)(a-\eta)(b-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\left\{A+\frac{Be^{2k\zeta}}{2k}\right\}, \\
w &= \frac{-(a-\zeta)\{(\xi-b)(\eta-b)(b-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\frac{d}{d\zeta}\left(\phi+\frac{1}{2k}\chi\right) \\
&= \frac{-\{(\xi-b)(\eta-b)(a-\zeta)\}^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}(\xi-\zeta)(\eta-\zeta)}\left\{A+\frac{Be^{2k\zeta}}{2k}\right\}.
\end{aligned} \tag{13}
$$

The conditions $v = w = 0$ for $\zeta = \zeta_0$ are satisfied by taking $A = -\dfrac{Be^{2k\zeta_0}}{2k}$ and the condition $u = -U$ is satisfied by taking

$$\chi(\zeta_0) = B\int\limits_{-\infty}^{\zeta_0}\frac{e^{2ks}}{\{(a-s)(b-s)\}^{\frac{1}{2}}}\,ds = U. \tag{14}$$

It is clear that u, v, $w \to 0$ as $\zeta \to -\infty$ which corresponds to moving off to infinite distance from the solid, and it is not difficult to show that the vortex lines are the plane ellipses

$$\zeta = \text{constant}, \qquad x = \text{constant}.$$

Equation (3) cannot represent a paraboloid of circular section unless $a = b$, when the coordinates become degenerate and the method fails. However, a result due to Howarth may be employed in this case. The coordinates ξ, η are introduced, where

$$x = \xi^2-\eta^2, \qquad y^2+z^2 = 4\xi^2\eta^2.$$

A solution of the type given in equations (6) can then be found with ϕ and χ functions of η only. The equations corresponding to (9) and (10) are the equations

$$\frac{d}{d\eta}\left(\eta\frac{d\phi}{d\eta}\right) = 0 \tag{15}$$

and

$$\frac{d}{d\eta}\left(\eta\frac{d\chi}{d\eta}\right) = -4k\eta^2\frac{d\chi}{d\eta}, \tag{16}$$

with solutions

$$\phi = A\log\eta, \qquad \chi = B\int\limits_{\eta}^{\infty}\frac{e^{-2ks^2}}{s}\,ds. \tag{17}$$

The boundary conditions can then be satisfied by taking $2kA = Be^{-2k\eta_0^2}$,

$$B\int\limits_{\eta_0}^{\infty}\frac{e^{-2ks^2}}{s}\,ds = U,$$

where the paraboloid of revolution is the surface $\eta = \eta_0$.

The solution of the Oseen equations specified in equations (11), (12), (13) and (14), will in particular apply to the degenerate paraboloid $\zeta = \zeta_0 = b$, which is a flat plate with a parabolic leading edge. In dealing with this case it is convenient to take $b = \zeta_0 = 0$, when the parabolic plate is $z = 0$, $4ax > y^2$.

Then $$B = U\left(\int\limits_{-\infty}^{0}\frac{e^{2ks}\,ds}{\{-s(a-s)\}^{\frac{1}{2}}}\right)^{-1} = \frac{Ue^{-ka}}{K_0(ka)} = \frac{Ue^{-R/4}}{K_0(R/4)},$$

where R is a Reynolds number defined as

$$R = \frac{2aU}{\nu} = 4ak,$$

$2a$ being the semi-latus rectum of the parabola.

The fluid velocity $\mathbf{u}$ given by the approximation may then be written

$$\mathbf{u} = \mathbf{U}+\mathbf{u}_1+\mathbf{u}_2,$$

where $\mathbf{U} = (U,0,0)$, $\mathbf{u}_1 = (-\chi,0,0)$, and $\mathbf{u}_2$ is a vector of magnitude $\dfrac{2aUe^{-R/4}}{RK_0(R/4)}\dfrac{(1-e^{-2k\zeta})}{\{(\xi-\zeta)(\eta-\zeta)\}^{\frac{1}{2}}}$ directed in the ζ-direction—i.e. along the tangent to the curve $\xi =$ constant, $\eta =$ constant in the direction of ζ increasing. The vector $\mathbf{u}_1$ influences only the x-component of the flow and is a pure boundary layer term. As R increases it is significant in a smaller and smaller region round the plate. The secondary flow—the three-dimensional effect induced by the presence of the plate—is given by the term $\mathbf{u}_2$. The magnitude of this term is zero on the plate and at infinity and decreases like $1/R^{\frac{1}{2}}$ as $R \to \infty$. The direction of $\mathbf{u}_2$ is always away from the plate and in the limit, as the plate is approached, is normal to the plate.

This secondary velocity $\mathbf{u}_2$ may be considered as due to a source distribution of density $\dfrac{Ue^{-R/4}e^{-2k\zeta}\{-\zeta(a-\zeta)\}^{\frac{1}{2}}}{K_0(R/4)\{(\xi-\zeta)(\eta-\zeta)\}^{\frac{1}{2}}}$ per unit volume; and this density is always positive.

Again, in this case it is a simple matter to show that on the plate

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{B}{\xi\eta}, \qquad \left(\frac{\partial v}{\partial z}\right)_{z=0} = 0$$

so that, although the flow is strictly three-dimensional, the skin friction is in the direction of the flow at infinity. Moreover, when $\zeta = 0$ equations

(4) can be used to show that $\xi\eta = \frac{1}{4}(4ax-y^2)$ so that

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{2B}{(4ax-y^2)^{\frac{1}{2}}} = \frac{Ue^{-R/4}}{a^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}K_0(R/4)}, \tag{18}$$

where $\bar{x} = (x-y^2/4a)$ = distance from edge of plate measured parallel to the x-axis.

It is interesting to compare this result with the values of $(\partial u/\partial z)_{z=0}$ obtained from other methods of approximation. If the boundary-layer equations are taken in the form

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= \nu\frac{\partial^2 u}{\partial z^2}, \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= \nu\frac{\partial^2 v}{\partial z^2}, \\ u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0, \end{aligned} \tag{19}$$

Moore (3) has shown that there will be a solution with $v = 0$ and on the plate

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{0{\cdot}332UR^{\frac{1}{2}}}{2^{\frac{1}{2}}a^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}}. \tag{20}$$

The ratio of this value to the value given by equation (18) is independent of position on the plate.

However, the simplest approximation to the skin friction on the plate is obtained by taking Rayleigh's value for the skin friction (4), starting the solution from the leading edge of the plate. The value of $(\partial u/\partial z)_{z=0}$ obtained is

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{UR^{\frac{1}{2}}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}a^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}}. \tag{21}$$

The ratio of this value to the value obtained from the Oseen equations in equation (18) is independent of position on the plate and is a function only of the Reynolds number R. The way the ratio varies with R is shown in Fig. 1. Beginning at zero for $R = 0$ it is greater than 0·9 when $R = 4$ and rapidly approaches its limiting value of unity. Thus, except for very small values of the Reynolds number, the Oseen equations provide almost no information about the skin friction that cannot be obtained far more easily from this simple Rayleigh approximation. Nor is this result altogether surprising, for the equations which determine the Rayleigh approximation are obtained by forming boundary-layer equations like equations (19) *from the Oseen equations* instead of the full equations of motion. What is of interest is the fact that the omission of the $\partial^2/\partial y^2$ terms from these equations should have so little effect at large distances from the axis of the parabola: for it would be expected that far from the axis, where the direction of the tangent to the leading edge approaches the direction of the axis, 'edge effects' would become important and the terms in $\partial^2/\partial y^2$ would not be negligible in comparison with the $\partial^2/\partial z^2$ terms.

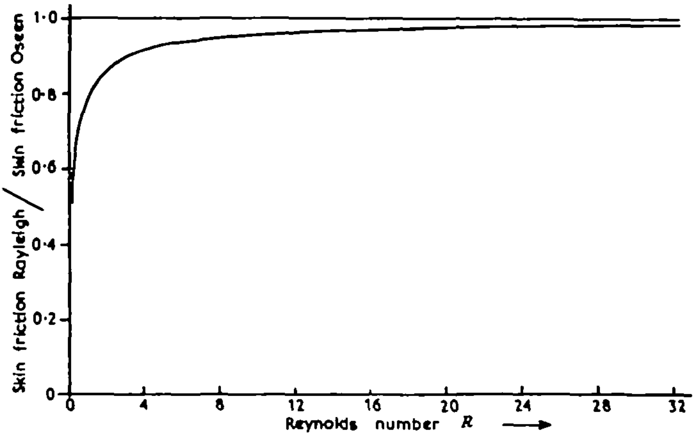

FIG. 1. Ratio of skin friction from Rayleigh approximation to skin friction from Oseen equations.

The fact that edge effects are of so little significance in this approximation suggests that equations (19) may give a valid approximation to the skin friction over the whole plate. Equations (19) ignore edge effects, but as they bear the same relationship to the full equations of motion as the equations determining the Rayleigh approximation do to the Oseen equations, it seems reasonable to suppose that edge effects will have very little effect on skin friction for high Reynolds numbers. If this is the case, equation (20) should give a much better approximation to the skin friction than equation (18) for high Reynolds numbers, for the limitations of the Oseen equations are well known.

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