An Extension of the Cobham-Semënov Theorem

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Abstract

Let θ, θ' be two multiplicatively independent Pisot numbers, and let U, U' be two linear numeration systems whose characteristic polynomial is the minimal polynomial of θ and θ', respectively. For every $n \geq 1$, if $A \subseteq \mathbb{N}^n$ is $U-$ and $U'-$recognizable then A is definable in $\langle \mathbb{N}; + \rangle$.

1. Introduction

The Cobham-Semënov theorem [6, 19] states that for k, l multiplicatively independent integers, any set $X \subseteq \mathbb{N}^n$ which is $k-$ and $l-$recognizable is definable in $\langle \mathbb{N}; + \rangle$.

Alternative proofs of this result were proposed in [12] and [17]; Michaux and Villemaire presented in [16] a new proof involving the Büchi-Bruyère theorem, which provides a logical characterization of $k-$recognizable sets and enables to deal with problems on words by means of definability arguments. In [1] we use some results of [16] to solve related definability and decidability questions, from which we derived a new proof of the Cobham-Semënov theorem.

The study of non-classical numeration systems led to the notion of $U-$recognizable set, which naturally extends the one of $k-$recognizable set. Bruyère and Hansel have shown recently [3] that most of the computation models for $k-$recognizability can be generalized to the case of linear numeration systems whose characteristic polynomial is the minimal polynomial of a Pisot number. In particular the Büchi-Bruyère theorem can be extended to these numeration systems. Thanks to this result, we adapt here some ideas of [1] to prove that the Cobham-Semënov theorem still holds for two numeration systems satisfying the previous assumption:

Theorem 3.1. *Let θ, θ' be two multiplicatively independent Pisot numbers, and let U, U' be two linear numeration systems whose characteristic polynomial is the minimal polynomial of θ and θ', respectively. For every $n \geq 1$, if $A \subseteq \mathbb{N}^n$ is $U-$ and $U'-$recognizable then A is definable in $\langle \mathbb{N}; + \rangle$.*

This answers positively a conjecture by Michaux and Villemaire [15, p.377], and improves results by Fabre [8] and Point-Bruyre [18] where Theorem 3.1 is proved in case one of the Pisot numbers is an integer. In [15] it is shown that it is sufficient to prove Theorem 3.1 for the case where $n = 1$ and A is a subset of $\mathbb{N}$ which is expanding. This reduction step strongly relies on the results obtained by Michaux and Villemaire on (non)definability in Presburger Arithmetic (see [14, 15, 16]). Our proof involves this reduction step. Theorem 3.1 has been proved independently by Fagnot [9] and Hansel [13], both using Michaux-Villemaire's reduction step, but with quite different methods. Finally let us mention that Durand [7] proved a similar result for the case $n = 1$, with different assumptions.

In Section 2 we recall definitions and results related to finite automata, linear numeration systems and $U-$recognizability. Section 3 deals with the proof of the main result.

2. Preliminaries

This section provides the basic definitions, notations, results and tools that will be used in the paper.

2.1 Finite automata

Let Σ be a finite alphabet. We denote by Σ^* the set of finite words over Σ, including the empty word denoted by λ. For every $u \in \Sigma^*$ we denote by $\mu(u)$ the length of u. For all $u, v \in \Sigma^*$, we say that u is a *right factor* (resp. *left factor*) of v if there exists $w \in \Sigma^*$ such that $v = w \cdot u$ (resp. $v = u \cdot w$). We speak of *strict* factor in the case where $w \neq \lambda$.

We shall work with deterministic finite Σ-automata reading words from right to left. Our notation for such an automaton $\mathcal{A}$ will be $\mathcal{A} = (Q, q_0, \delta, Q')$, where Q is the finite set of states, $q_0 \in Q$ is the initial state, $Q' \subseteq Q$ is the set of accepting states, and δ is the transition function. We denote by δ^* the function from $Q \times \Sigma^*$ to Q which extends δ as follows:

$$\delta^*(q, \lambda) = q \text{ for every } q \in Q;$$
$$\delta^*(q, a) = \delta(q, a) \text{ for all } q \in Q, a \in \Sigma;$$
$$\delta^*(q, aw) = \delta(\delta^*(q, w), a) \text{ for all } a \in \Sigma,\ w \in \Sigma^*.$$

A word $w \in \Sigma^*$ is said to be *accepted* by the finite Σ-automaton $\mathcal{A}$ if $\delta^*(q_0, w) \in Q'$. We say that a state $q \in Q$ is *visited* during the computation of w by $\mathcal{A}$ if there exists a right factor v of w such that $\delta^*(q_0, v) = q$. A subset X of Σ^* is said to be Σ-*recognizable* if it is the set of accepted words of some finite Σ-automaton.

2.2 Linear numeration systems

We call *numeration system* any strictly increasing sequence of integers $U = (U_n)_{n \in \mathbb{N}}$ such that $U_0 = 1$ and $\{\frac{U_{n+1}}{U_n} : n \in \mathbb{N}\}$ is bounded. Every positive integer x can be represented as

$$x = a_n U_n + a_{n-1} U_{n-1} + \ldots + a_0 U_0$$

using the Euclidian algorithm: let n be such that $U_n \leq x < U_{n+1}$, and let $x_n = x$. For $i = n, n-1, \ldots, 1$ we compute the Euclidean division $x_i = a_i U_i + x_{i-1}$. Doing this we obtain a word $a_n a_{n-1} \ldots a_0$ over the *canonical alphabet* $\Sigma_U = \{0, 1, \ldots, c\}$, where c is the greatest integer less than $\sup\{\frac{U_{n+1}}{U_n} : n \in \mathbb{N}\}$. The word $a_n a_{n-1} \ldots a_0$ is called the *normalized U-representation* of x, and denoted by $\rho_U(x)$. We denote by $\mathcal{N}_U$ the set of normalized U-representations of integers:

$$\mathcal{N}_U = \{\rho_U(x) : x \in \mathbb{N}\}.$$

By convention 0 is represented by the empty word.

Conversely for every word $w = b_n b_{n-1} \ldots b_0$ over Σ_U we call *numerical value* of w the integer

$$\pi_U(w) = \sum_{i=0}^{m} b_i U_i.$$

Let $\prec$ denote the lexicographical ordering. The elements of $\mathcal{N}_U$ satisfy the following property:

Proposition 2.1. *For all $u, v \in \mathcal{N}_U$,*

$$\pi_U(u) < \pi_U(v) \Longleftrightarrow u \prec v.$$

A *linear numeration system* is a numeration system $U = (U_n)_{n \in \mathbb{N}}$ defined by a linear recurrence relation

$$U_n = d_{k-1} U_{n-1} + \ldots + d_0 U_{n-k}$$

for all $n \geq k$, with $d_i \in \mathbb{Z}$ for $i = 0, 1, \ldots, k-1$, and $d_0 \neq 0$. The polynomial

$$P_U(X) = X^k - d_{k-1} X^{k-1} - \ldots - d_1 X - d_0$$

is called the *characteristic polynomial* of the system U.

For generalities about linear numeration systems we refer the reader to [10]. In the sequel we will be concerned with linear numeration systems whose characteristic polynomial is the minimal polynomial of a Pisot number; they behave closely to classical numeration systems with respect to recognizability by finite automata (see e.g. [3, 11]). Recall that a *Pisot number* is an algebraic integer $\theta > 1$ such that the roots of its minimal polynomial, distinct from θ, have modulus less than 1.

For the rest of the paragraph $U = (U_n)_{n \in \mathbb{N}}$ will denote a linear numeration system whose characteristic polynomial P_U is the minimal polynomial of a Pisot number θ.

Under these assumptions, the roots $\theta_1 = \theta,\ \theta_2, \ldots,\ \theta_k$ of P_U are simple and

$$|\theta_j| < 1 \quad \text{whenever} \quad j \neq 1. \tag{1}$$

Moreover there exist complex constants $c_1, \ldots, c_k$ such that

$$\forall n \in \mathbb{N} \quad U_n = \sum_{i=1}^{k} c_i \theta_i^n \quad . \tag{2}$$

Let us define the function $\pi_\theta : \Sigma_U^* \to \mathbb{C}$ which maps every $w = a_n \cdots a_0 \in \Sigma_U^*$ to

$$\pi_\theta(w) = c_1 \sum_{i=0}^{n} a_i \theta^i.$$

From (1) and (2) one easily deduces the following:

Proposition 2.2. *There exists a constant e such that*

$$\forall w \in \Sigma_U^*, \quad |\pi_U(w) - \pi_\theta(w)| < e.$$

Now let u, v, w be words over Σ_U such that $uvw \in \mathcal{N}_U$ and $v \neq \lambda$. It is easily seen that $\pi_U(uv^n w) \to +\infty$ as $n \to +\infty$. From this fact and the above proposition we get:

Proposition 2.3. *Let $u, v, w \in \Sigma_U^*$ be such that $uvw \in \mathcal{N}_U$ and $v \neq \lambda$. Then*

$$\lim_{n \to +\infty} \frac{\pi_U(uv^n w)}{|\pi_\theta(uv^n w)|} = 1.$$

The next proposition follows easily from the previous one and the definition of π_θ.

Proposition 2.4. *Let u, v, w be words over Σ_U such that $uvw \in \mathcal{N}_U$ and $v \neq \lambda$. There exists $\kappa \in \mathbb{R}$ such that*

$$\lim_{n \to +\infty} \frac{\pi_U(uv^n w)}{\theta^{n\mu(v)} \kappa} = 1.$$

2.3 Logic and $U-$recognizable sets

The notion of $U-$*recognizable* set naturally extends the one of $p-$recognizable set, which concerns representations in an integer base $p \geq 2$, to arbitrary numeration systems.

Since we have to deal with subsets of $\mathbb{N}^n$ for an arbitrary integer $n \geq 1$, we shall extend our definition of ρ_U. Let n be a positive integer; for every $n-$tuple $x = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ we define $\rho_U(x)$ as the word (of $n-$tuples)

$$(0^{l-l_1}\rho_U(x_1), 0^{l-l_2}\rho_U(x_2), \ldots, 0^{l-l_n}\rho_U(x_n))$$

over Σ_U^n, where $l_i = \mu(\rho_U(x_i))$ and $l = \max\{l_1, \ldots, l_n\}$. Moreover we will denote by 0 the $n-$tuple $(0, 0, \ldots, 0)$.

Definition 2.5. Let n be a positive integer and U be a numeration system. A set $X \subseteq \mathbb{N}^n$ is said to be $U-$*recognizable* if the set $\rho_U(X)$ is Σ_U^n-recognizable.

For every integer $p \geq 2$, $p-$recognizability corresponds to $U-$recognizability for $U = (p^n)_{n \in \mathbb{N}}$.

The Büchi-Bruyère theorem ([2], see [4]) states that for every integer $k \geq 2$ a set $X \subseteq \mathbb{N}^n$ is $k-$recognizable if and only if X is definable[1] in the structure $\langle \mathbb{N}; +, V_k \rangle$ (where $V_k(x)$ denotes the greatest power of k which divides x). In [3], Bruyère and Hansel generalized this result to the case of linear numeration systems whose characteristic polynomial is the minimal polynomial of a Pisot number.

For any numeration system $U = (U_n)_{n \in \mathbb{N}}$, one defines the function $V_U : \mathbb{N} \to \mathbb{N}$ as follows: $V_U(0) = U_0 = 1$, and for every positive integer x, if $\rho_U(x) = a_n \ldots a_j 0^j$ with $a_j \neq 0$ then $V_U(x) = U_j$ (that is, $V_U(x)$ is the least U_i appearing in the normalized $U-$representation of x with a non-zero coefficient).

Theorem 2.6. (Bruyère,Hansel) *Let U be a linear numeration system whose characteristic polynomial is the minimal polynomial of a Pisot number. For every $n \geq 1$ a set $X \subseteq \mathbb{N}^n$ is $U-$recognizable if and only if X is definable in the structure $\langle \mathbb{N}; +, V_U \rangle$.*

2.4 The Cobham-Semënov theorem

Definition 2.7. Two reals $k, l > 1$ are said to be *multiplicatively dependent* if there exist two positive integers a, b such that $k^a = l^b$. Otherwise k,l are said to be *multiplicatively independent.*

Büchi proved [5] that for all multiplicatively dependent integers k,k', and every set $X \subseteq \mathbb{N}$, X is $k-$recognizable if and only if X is $k'-$recognizable. On the other hand, it is easily shown that any set $X \subseteq \mathbb{N}$ which is ultimately periodic (i.e. definable in $\langle \mathbb{N}; + \rangle$) is $k-$recognizable for every integer $k \geq 2$. The Cobham-Semënov theorem specifies the base-dependence of the notion of $k-$recognizable set.

Theorem 2.8. (Cobham,Semënov) *Let k, l be two multiplicatively independent integers. For every $n \geq 1$ and every set $X \subseteq \mathbb{N}^n$, if X is $k-$ and $l-$recognizable then X is definable in $\langle \mathbb{N}; + \rangle$.*

[1] by *definable* we will always mean *first-order definable*

The case $n = 1$ was proved by Cobham in [6]; Semënov extended the result to higher dimensions in [19].

3. The main result

In this section we prove the following theorem.

Theorem 3.1. *Let θ, θ' be two multiplicatively independent Pisot numbers, and let U, U' be two linear numeration systems whose characteristic polynomial is the minimal polynomial of θ and θ', respectively. For every $n \geq 1$, if $A \subseteq \mathbb{N}^n$ is $U-$ and $U'-$recognizable then A is definable in $\langle \mathbb{N}; + \rangle$.*

Remark: it follows from theorem 2.6 that if A is $U-$ and $U'-$recognizable then A is definable in $\langle \mathbb{N}; +, V_U \rangle$ and $\langle \mathbb{N}; +, V_{U'} \rangle$. Thus every relation definable in $\langle \mathbb{N}; +, A \rangle$ is definable in $\langle \mathbb{N}; +, V_U \rangle$ and $\langle \mathbb{N}; +, V_{U'} \rangle$ too, and therefore is $U-$ and $U'-$recognizable (by virtue of the same theorem).

We shall make use of the two following theorems, due to Michaux and Villemaire [16].

Definition 3.2. Let $(l_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of integers, and let $L = \{l_n : n \in \mathbb{N}\}$. We say that L is *expanding* if the set $\{l_{n+1} - l_n : n \in \mathbb{N}\}$ is not bounded.

Theorem 3.3. (Michaux,Villemaire) *Let $K \subseteq \mathbb{N}$. If K is not definable in $\langle \mathbb{N}; + \rangle$ then there exists an expanding set $L \subseteq \mathbb{N}$ which is definable in $\langle \mathbb{N}; +, K \rangle$.*

Theorem 3.4. (Michaux,Villemaire) *A set $A \subseteq \mathbb{N}^n$ is definable in $\langle \mathbb{N}; + \rangle$ if and only if every subset of $\mathbb{N}$ which is definable in $\langle \mathbb{N}; +, A \rangle$ is definable in $\langle \mathbb{N}; + \rangle$.*

The proof of theorem 3.1 is organized as follows: assuming for a contradiction that there exists $A \subseteq \mathbb{N}^n$ which is $U-$ and $U'-$recognizable and not definable in $\langle \mathbb{N}; + \rangle$, we use theorems 3.3 and 3.4 to define in $\langle \mathbb{N}; +, A \rangle$ a set $M = (m_n)_{n \in \mathbb{N}}$ such that $m_{n+1} - m_n \geq n$ for every $n \in \mathbb{N}$. From this property of $(m_n)_{n \in \mathbb{N}}$ we then deduce (lemmas 3.5 and 3.6) that $\rho_U(M)$ is a finite disjoint union of a finite set and of sets of the form $\{av^n b : n \in \mathbb{N}\}$. Now using the remark of the beginning of the section, M should be $U'-$recognizable too; we prove (by an application of the *pumping lemma*) that this contradicts the previous property on $\rho_U(M)$.

Proof of theorem 3.1.
Let θ and θ' be two multiplicatively independent Pisot numbers. Assume for a contradiction that there exists $A \subseteq \mathbb{N}^n$ which is $U-$ and $U'-$recognizable, and not definable in $\langle \mathbb{N}; + \rangle$, for U and U' two linear numeration systems whose characteristic polynomial is the minimal polynomial of θ and θ', respectively. By theorem 3.4 there exists a set $K \subseteq \mathbb{N}$ which is definable in $\langle \mathbb{N}; +, A \rangle$ and not definable in $\langle \mathbb{N}; + \rangle$. Then by theorem 3.3 we get an expanding set L which is definable in $\langle \mathbb{N}; +, K \rangle$. Let $(l_n)_{n \in \mathbb{N}}$ be the sequence of elements of L arranged in increasing order. Consider the function $f : L \to \mathbb{N}$ which maps every l_n to $(l_{n+1} - l_n)$. Now let M be the subset of L defined by

$$M = \{l_n : \forall i < n,\ f(l_i) < f(l_n)\}.$$

The set L is expanding, thus M is infinite. Let $(m_n)_{n \in \mathbb{N}}$ be the sequence of elements of M arranged in increasing order. From the definition of f and M one checks that

$$\forall n \in \mathbb{N}, \quad m_{n+1} - m_n \geq f(m_n) \geq m_n \geq n. \qquad (P)$$

Moreover M is definable in $\langle \mathbb{N}; +, L \rangle$ by the formula

$$M(x) \Longleftrightarrow \Big[L(x) \wedge\ \exists x' \{ L(x') \wedge x < x' \wedge \neg \exists z [x < z < x' \wedge L(z)] \wedge$$

$$\forall y \{ [y < x \wedge L(y)] \Longrightarrow \exists y' [L(y') \wedge x < y' \wedge \neg \exists z [y < z < y' \wedge L(z)] \wedge y' + x < x' + y] \} \} \Big]$$

(the relation $x < y$ is obviously definable in $\langle \mathbb{N}; + \rangle$).

Therefore M is definable in $\langle \mathbb{N}; +, A \rangle$, and thus U-recognizable. Hence $\rho_U(M)$ is Σ_U- recognizable. For the remainder of the proof let

$$\mathcal{A} = (Q, q_0, \delta, Q')$$

be a deterministic Σ_U-automata that recognizes $\rho_U(M)$.

The following lemma states an interesting consequence of property (P) for the set $\rho_U(M)$.

Lemma 3.5. *Let $u, v, w_1, w_2 \in \Sigma_U^*$, $v \neq \lambda$. If $\mu(w_1) = \mu(w_2)$, and furthermore if for every $n \in \mathbb{N}$, $uv^n w_1$ and $uv^n w_2$ belong to $\rho_U(M)$, then $w_1 = w_2$.*

Proof. Assume for a contradiction that $w_1 \neq w_2$. By our hypothesis uw_1 and uw_2 belong to $\rho_U(M)$, thus to $\mathcal{N}_U$; since $w_1 \neq w_2$ by proposition 2.1 we have $\pi_U(uw_1) \neq \pi_U(uw_2)$. Now $\mu(w_1) = \mu(w_2)$ thus $\pi_U(w_1) \neq \pi_U(w_2)$. Assume, for example, $\pi_U(w_2) > \pi_U(w_1)$. Let $a = \pi_U(w_2) - \pi_U(w_1)$, and let N be an integer such that $\pi_U(uv^N w_1) \geq m_{a+1}$ (such an integer exists since $v \neq \lambda$). There exist

$i_1, i_2 \geq a+1$ such that $\pi_U(uv^N w_1) = m_{i_1}$ and $\pi_U(uv^N w_2) = m_{i_2}$. Then if c denotes the length of w_1 and w_2,

$$\begin{aligned} m_{i_2} - m_{i_1} &= \pi_U(uv^N w_2) - \pi_U(uv^N w_1) \\ &= (\pi_U(uv^N 0^c) + \pi_U(w_2)) - (\pi_U(uv^N 0^c) + \pi_U(w_1)) \\ &= \pi_U(w_2) - \pi_U(w_1) \\ &= a \end{aligned}$$

Now

$$m_{i_2} - m_{i_1} \geq m_{i_1+1} - m_{i_1}$$

and it follows from (P) that

$$m_{i_1+1} - m_{i_1} \geq i_1 \geq a+1$$

which cannot be true. □

We now intend to show that that the set $\rho_U(M)$ is a finite disjoint union of a finite set and of sets of the form $\{av^n b : n \in \mathbb{N}\}$. To this end let us introduce some notations.

Let $S \subseteq \Sigma_U^*$ be the set of words w such that

(1) w is accepted by $\mathcal{A}$
(2) There do not exist two distinct right factors of w, say w_1 and w_2, such that

$$\delta^*(q_0, w_1) = \delta^*(q_0, w_2)$$

(that is, no state is visited several times during the computation of w by $\mathcal{A}$).

It is easily seen that S is nonempty, and finite. Now let $B \subseteq \Sigma_U^*$ be the set of words w such that

(1) w is a right factor of some element in S;
(2) there exists a word $u \neq \lambda$ such that $\delta^*(q_0, w) = \delta^*(q_0, uw)$;
(3) there is no strict right factor of w, say w', for which there exists a word $u' \neq \lambda$ such that

$$\delta^*(q_0, w') = \delta^*(q_0, u'w').$$

The set $\rho_U(M)$ is infinite, thus B is nonempty. We denote by $b_1, \ldots, b_\zeta$ the distinct elements of B. For $i = 1, \ldots, \zeta$, let v_i be a word of minimal length such that

$$\delta^*(q_0, b_i) = \delta^*(q_0, v_i b_i).$$

We denote by $\phi(i)$ the number of distincts words a such that $ab_i \in S$; these words will be denoted by $a_{i,1}, a_{i,2}, \ldots, a_{i,\phi(i)}$.

For all integers i, j such that $1 \leq i \leq \zeta$ and $1 \leq j \leq \phi(i)$, let us introduce the set

$$E_{i,j} = \{a_{i,j}{v_i}^n b_i \ : \ n \in \mathbb{N}\}.$$

We denote by $E_{0,0}$ the elements s of S for which there are no words w, u, with $u \neq \lambda$, such that w is a right factor of s and $\delta^*(q_0, w) = \delta^*(q_0, uw)$.

Finally let $\mathcal{I} = \{(i,j) \ : \ i = j = 0 \vee 1 \leq i \leq \zeta, 1 \leq j \leq \phi(i)\}$, and $t = \sum_{i=1}^{\zeta} \phi(i)$.

Lemma 3.6. *The family* $(E_{i,j})_{(i,j)\in\mathcal{I}}$ *is a partition of* $\rho_U(M)$.

Proof. We first proceed to show that for all distinct couples $(q,r), (q',r') \in \mathcal{I}$ the sets $E_{q,r}$ and $E_{q',r'}$ are disjoint. The result is obvious in the case where one of the couple is $(0,0)$. Now if q, q' are positive, assume that $E_{q,r} \cap E_{q',r'}$ is nonempty. In this case there exist two integers n, n' such that

$$a_{q,r}{v_q}^n b_q = a_{q',r'}{v_{q'}}^{n'} b_{q'}. \tag{3}$$

This equality implies that among the words b_q and $b_{q'}$, one is a right factor of the other; but it cannot be a strict right factor since it would contradict point (3) in the definition of B. Thus $b_q = b_{q'}$, that is $q = q'$. Assume now that $n \geq n'$; from the previous equality it follows that

$$a_{q,r}{v_q}^{n-n'} = a_{q,r'}; \tag{4}$$

The word $a_{q,r}{v_q}^{n-n'} b_q$ belongs to S, since the word $a_{q,r'} b_q$ does; it follows from the definition of S that we must have $n - n' = 0$. Therefore $n = n'$, and $a_{q,r} = a_{q,r'}$; thus $r = r'$, which cannot be.

Now there remains to prove that

$$\bigcup_{(i,j)\in\mathcal{I}} E_{i,j} = \rho_U(M).$$

If $w \in E_{0,0}$ then $w \in S$, hence $w \in \rho_U(M)$. Now suppose that w belongs to some $E_{q,r}$ with $q > 0$. There exists $n \in \mathbb{N}$ such that $w = a_{q,r}{v_q}^n b_q$. We have

$$\delta^*(q_0, a_{q,r}{v_q}^n b_q) = \delta^*(q_0, a_{q,r} b_q),$$

thus the fact that $a_{q,r} b_q$ belongs to S yields $a_{q,r}{v_q}^n b_q \in \rho_U(M)$. We have proved the inclusion $\bigcup_{(i,j)\in\mathcal{I}} E_{i,j} \subseteq \rho_U(M)$.

For the converse inclusion, let $u \in \rho_U(M)$. If there is no right factor u' of u for which there exists a word $v' \neq \lambda$ such that

$$\delta^*(q_0, v'u') = \delta^*(q_0, u'),$$

then $u \in E_{0,0}$. Otherwise let b be a right factor of u of minimal length such that there exists some word $v' \neq \lambda$ for which

$$\delta^*(q_0, v'b) = \delta^*(q_0, b).$$

The word u belongs to $\rho_U(M)$, thus b must be a right factor of some word of S; it follows that $b \in B$, that is $b = b_j$ for some positive integer $j \leq \zeta$. Now there exist $a \in \Sigma_U^*$ and $n \in \mathbb{N}$, such that $u = a{v_j}^n b_j$ and v_j is not a right factor of a. Let us show that ab_j belongs to S, which will ensure us that u belongs to some $E_{j,k}$. The word $a{v_j}^n b_j$ is accepted by $\mathcal{A}$, and

$$\delta^*(q_0, a{v_j}^n b_j) = \delta^*(q_0, ab_j);$$

therefore ab_j is accepted by $\mathcal{A}$. Thus there remains to show that no state is visited several times during the computation of ab_j by $\mathcal{A}$. Assume the contrary. Let b' be the smallest right factor of ab_j for which there exists w right factor of ab_j and distinct from b' such that

$$\delta^*(q_0, b') = \delta^*(q_0, w).$$

Then there exist $a', z \in \Sigma_U^*$, $z \neq \lambda$ such that $ab_j = a'zb'$ and

$$\delta^*(q_0, b') = \delta^*(q_0, zb').$$

The word b_j is a right factor of b', otherwise b' would be strict right factor of b_j, and the fact that

$$\delta^*(q_0, b') = \delta^*(q_0, zb')$$

would imply $b' \in B$, which contradicts the minimality of b_j. By setting $b' = a''b_j$ we then have $ab_j = a'za''b_j$ with

$$\delta^*(q_0, a''b_j) = \delta^*(q_0, za''b_j).$$

Moreover

$$\delta^*(q_0, b_j) = \delta^*(q_0, v_j b_j).$$

Thus for all $m_1, m_2 \in \mathbb{N}$, the word $a'z^{m_1}a''{v_j}^{m_2}b_j$ is accepted by $\mathcal{A}$. Let $r = \mu(v_j)$ and $s = \mu(z)$. For every $m \in \mathbb{N}$, the words $a'z^m z^r a'' b_j$ and $a'z^m a''{v_j}^s b_j$ are accepted by $\mathcal{A}$. But

$$\mu(z^r a'' b_j) = s\ r + \mu(a'') + \mu(b_j) = \mu(a''{v_j}^s b_j). \tag{5}$$

From lemma 3.5 we get $z^r a'' b_j = a''{v_j}^s b_j$, that is

$$z^r a'' = a''{v_j}^s \tag{6}$$

If $\mu(v_j) > \mu(za'')$ then from (6) the word za'' is a strict right factor of v_j; but

$$\delta^*(q_0, a''b_j) = \delta^*(q_0, za''b_j),$$

and this contradicts the minimality of v_j. It follows that $\mu(v_j) \leq \mu(za'')$ and by (6), v_j is a right factor of za'', *a fortiori* of $a = a'za''$, which contradicts the hypothesis on a. $\square$

The set A is $U'-$recognizable and M is definable in $\langle \mathbb{N}; +, A\rangle$, hence M is $U'-$recognizable too. We shall prove that this contradicts the property on $\rho_U(M)$ expressed by the previous lemma.

The set $\rho_{U'}(M)$ is infinite thus by the pumping lemma there exist $u, v, w \in \Sigma_{U'}^*$ such that $v \neq \lambda$ and for every $n \in \mathbb{N}$ the word uv^nw belongs to $\rho_{U'}(M)$. Set $X = \{\pi_{U'}(uv^nw) : n \in \mathbb{N}\}$. X is an infinite subset of M; moreover if $(x_n)_{n\in\mathbb{N}}$ denotes the strictly increasing sequence of elements of X then by Proposition 2.4 we get

$$\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = \theta'^e \tag{7}$$

where $e = \mu(v)$.

Let us recall that $\mathcal{I} = \{(i,j) : i = j = 0 \vee 1 \leq i \leq \zeta, 1 \leq j \leq \phi(i)\}$, and $t = \sum_{i=1}^{\zeta} \phi(i)$.

Set $\epsilon = \dfrac{\min\{|\theta'^i - \theta^j| : 1 \leq i \leq et\ ,\ j \geq 1\}}{2}$. We have $\epsilon > 0$ since θ and θ' are multiplicatively independent. Moreover let q be an integer such that $\theta^q > \theta'^{et} + 2\epsilon$.

From (7) we can deduce the existence of an integer N_0 such that

$$\forall n \geq N_0 \quad \forall d \leq t \quad \left| \frac{x_{n+d}}{x_n} - \theta'^{ed} \right| < \epsilon. \tag{8}$$

On the other hand by Proposition 2.4 we have

$$\forall (i,j) \in \mathcal{I} \setminus (0,0) \quad \forall k \leq q \qquad \lim_{n\to+\infty} \frac{\pi_U(a_{i,j}v_i^{n+k}b_i)}{\pi_U(a_{i,j}v_i^nb_i)} = \theta^{k\mu(v_i)}$$

thus there exists $N_1 \in \mathbb{N}$ such that

$$\forall n \geq N_1 \quad \forall (i,j) \in \mathcal{I} \setminus (0,0) \quad \forall k \leq q \qquad \left| \frac{\pi_U(a_{i,j}v_i^{n+k}b_i)}{\pi_U(a_{i,j}v_i^nb_i)} - \theta^{k\mu(v_i)} \right| < \epsilon. \tag{9}$$

Let $R = \max\{\pi_U(x) : x \in E_{0,0}\}$ and $R' = \max\{\pi_U(a_{i,j}v_i^nb_i) : (i,j) \in \mathcal{I} \setminus (0,0)\ ,\ n \leq N_1\}$. Fix an integer n such that $n > N_0$ and $x_n > \max(R, R')$. By lemma 3.6 and using the hypothesis $x_n > R$, every word among $\rho_U(x_n), \rho_U(x_{n+1}), \ldots, \rho_U(x_{n+t})$ belongs to a unique set among $(E_{i,j})_{(i,j)\in\mathcal{I}\setminus(0,0)}$; thus by the pigeon-hole principle there exist at least two distinct elements among $\rho_U(x_n), \rho_U(x_{n+1}), \ldots, \rho_U(x_{n+t})$, say $\rho_U(x_{n+l_1})$ and $\rho_U(x_{n+l_2})$ with $0 \leq l_1 < l_2 \leq t$, which belong to the same set $E_{i,j}$ for some $(i,j) \in \mathcal{I} \setminus (0,0)$. That is,

$$x_{n+l_1} = \pi_U(a_{i,j}v_i^{n'}b_i)$$

and

$$x_{n+l_2} = \pi_U(a_{i,j} v_i^{n'+k} b_i)$$

where $k \in \mathbb{N}$ and $n' > N_1$.

Now $n > N_0$ thus by (8)

$$\left| \frac{x_{n+l_2}}{x_{n+l_1}} - \theta'^{e(l_2-l_1)} \right| < \epsilon. \tag{10}$$

Let us now prove that $k \leq q$. Assume the contrary; we have

$$\frac{x_{n+l_2}}{x_{n+l_1}} = \frac{\pi_U(a_{i,j} v_i^{n'+k} b_i)}{\pi_U(a_{i,j} v_i^{n'} b_i)} > \frac{\pi_U(a_{i,j} v_i^{n'+q} b_i)}{\pi_U(a_{i,j} v_i^{n'} b_i)}.$$

Now $x_{n+l_1} > R'$ thus $n' > N_1$, hence by (9)

$$\left| \frac{\pi_U(a_{i,j} v_i^{n'+q} b_i)}{\pi_U(a_{i,j} v_i^{n'} b_i)} - \theta^{q\mu(v_i)} \right| < \epsilon$$

therefore

$$\frac{x_{n+l_2}}{x_{n+l_1}} > \theta^{q\mu(v_i)} - \epsilon \geq \theta^q - \epsilon > \theta'^{et} + \epsilon \geq \theta'^{e(l_2-l_1)} + \epsilon$$

which contradicts (10).

Thus $k \leq q$, so that we deduce from (9) and $n' > N_1$ that

$$\left| \frac{x_{n+l_2}}{x_{n+l_1}} - \theta^{k\mu(v_i)} \right| < \epsilon. \tag{11}$$

Finally (10) and (11) contradict the very definition of ϵ. □

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References

[1] A.Bès, *Undecidable extensions of Büchi Arithmetic and Cobham-Semënov Theorem*, this Journal, vol.62 no.4 (1997), 1280–1296.

[2] V.Bruyère, *Entiers et automates finis*, U.E. Mons, mémoire de licence, 1984-85.

[3] V.Bruyère, G.Hansel, *Bertrand numeration systems and recognizability*, Theoretical Computer Science 181 (1997), 17–43.

[4] V.Bruyère, G.Hansel, C.Michaux, R.Villemaire, *Logic and $p-$recognizable sets of integers*, Bull. Belg. Math. Soc. 1, 1994, 191–238.

[5] J.R.Büchi, *Weak second-order arithmetic and finite automata*, Zeitschrift für Math. Logic Grundlagen Math. 6 (1960), 66-92.

[6] A.Cobham, *On the base-dependance of sets of numbers recognizable by finite automata*, Math. Systems Theory 3 (1969), 186-192.

[7] F.Durand, *A generalization of Cobham's theorem*, Theory of Comp.Systems 31 (1998), 169–185.

[8] S.Fabre, *Une gnralisation du thorme de Cobham*, Acta Arithmetica 67 (1994), 197-208.

[9] I.Fagnot, *Cobham's Theorem and automaticity in non-standard bases*, preprint, Paris 7 University, 1998.

[10] A.S.Fraenkel, *Systems of numeration*, Am.Math.Monthly 92 (1985), 105–114.

[11] C.Frougny, B.Solomyak, *On representation of integers in linear numeration systems*, in Pollicott, Mark (ed.) et al., Ergodic theory of $\mathbb{Z}^d$ actions. Proceedings of the Warwick symposium, Warwick, UK, 1993-94. New York: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 228 (1996), 345–368.

[12] G.Hansel, *A propos d'un théorème de Cobham*, in: D.Perrin, ed., Actes de la Fête des Mots, Greco de Programmation, CNRS, Rouen, France, 1992.

[13] G.Hansel, *Systèmes de numération indépendants et syndéticité*, Theoretical Computer Science (to appear).

[14] C.Michaux, R.Villemaire, *Cobham's theorem seen through Büchi theorem*, Proc. Icalp'93, *Springer Lecture Notes in Comput. Sci.* **700** (1993) 325–334.

[15] C.Michaux, R.Villemaire, *Open questions around Büchi and Presburger arithmetics*, *in*: Logic: From foundations to applications, Proc. European Logic Colloquium'93, Oxford University Press (1996), 353–383.

[16] C.Michaux, R.Villemaire, *Presburger arithmetic and recognizability of sets of natural numbers by automata: new proofs of Cobham's and Semenov's theorems*, **Annals of Pure and Applied Logic** 77 (1996), 251–277.

[17] A.MUCHNIK, *Definable criterion for definability in Presburger Arithmetic and its application* (in Russian), preprint, Institute of new technologies, 1991.

[18] F.POINT, V.BRUYÈRE, *On the Cobham-Semenov theorem*, Theory of Computing Systems 30 (1997), 197–220.

[19] A.L.SEMENOV, *The Presburger nature of predicates that are regular in two number systems*, **Siberian Math. J.** 18 (1977), 289–299.

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