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Conditions for the validity of Faraday's law of induction and their experimental confirmation

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Abstract

This paper, as its main didactic objective, shows the conditions needed for the validity of Faraday's law of induction. Inadequate comprehension of these conditions has given rise to several paradoxes about the issue; some are analysed and solved in this paper in the light of the theoretical deduction of the induction law. Furthermore, an experimental set-up, in which such conditions are experimentally tested, is included. The experiment is not complicated and the method we use, and similar methods used elsewhere, is widely considered as suitable laboratory practice for students of first university courses in physics and engineering.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The non-fulfilment of the conditions used to derive, from Maxwell equations, Faraday's law of induction

$$\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{1}$$

(ε being the electromotive force in a circuit and Φ the coherent—i.e. according to the right-hand rule—magnetic flux through it) leads to situations in which equation (1) fails such as those described in [1, 2] (sliding cursor in a solenoid, winding wire around a rotating drum by means of sliding contacts and Hering's paradox) or, in contrast, seem to fail, as in [3] (Faraday's disc), but is indeed valid. We have carefully analysed the necessary theoretical conditions to derive Faraday's law of induction. An easy experimental set-up in which we proceed varying, in different ways, the magnetic flux in a circuit has been used to test those conditions.

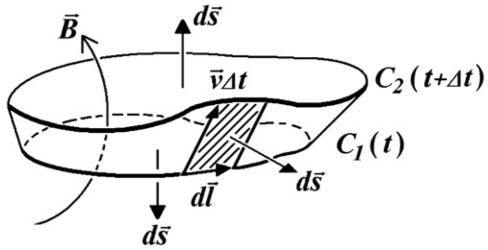

Figure 1. A suitable closed surface to deduce Faraday's law.

2. Deduction of Faraday's law

Using, as the starting point, some aspects of previous deductions [1, 4], let us begin by making a rigorous development of Faraday's law of induction to obtain its theoretical requirements.

In figure 1, a circuit moves and changes its shape in the presence of an external time-dependent magnetic field, $\vec{B}(t)$. $C_1(t)$ represents the shape and position of the circuit at the instant t, while $C_2(t + \Delta t)$ represents its shape and position at the moment $t + \Delta t$, Δt being very small. The circuit varies in such a way that all its positions along the time form the lateral surface of a volume *enclosed by this surface together with two arbitrary-shaped surfaces* limited only by the initial position of the circuit and its final position.

We can calculate the magnetic flux coming out from that volume and write at the instant $t + \Delta t$, taking into account the Maxwell equation, div $\vec{B} = 0$, and the Gauss theorem,

$$\iint_{S_2} \vec{B}(t + \Delta t) \cdot \mathrm{d}\vec{S} + \iint_{S_1} \vec{B}(t + \Delta t) \cdot \mathrm{d}\vec{S} + \iint_{S_\mathrm{L}} \vec{B}(t + \Delta t) \cdot \mathrm{d}\vec{S} = 0, \qquad (2)$$

where S_2 and S_1 are the surfaces limited only by the curves C_2 and C_1, respectively, while S_L is the lateral area formed by the successive positions of the circuit.

Let us choose a positive way along the circuit (for instance, that represented by $\mathrm{d}\vec{l}$ in figure 1). In such a case,

$$\iint_{S_2} \vec{B}(t + \Delta t) \cdot \mathrm{d}\vec{S} = \Phi(t + \Delta t) \qquad (3)$$

is the flux crossing the circuit at $t + \Delta t$, since $\mathrm{d}\vec{S}$ on S_2 is coherent with $\mathrm{d}\vec{l}$ (both vectors correspond to anticlockwise ways, as shown in figure 1).

On the other hand, if the external field varies continuously, we will have

$$\vec{B}(t + \Delta t) = \vec{B}(t) + \frac{\partial \vec{B}(t)}{\partial t}\Delta t + O(\Delta t^2). \qquad (4)$$

Then

$$\iint_{S_1} \vec{B}(t + \Delta t) \cdot \mathrm{d}\vec{S} = \iint_{S_1} \vec{B}(t) \cdot \mathrm{d}\vec{S} + \iint_{S_1} \frac{\partial \vec{B}(t)}{\partial t} \cdot \mathrm{d}\vec{S}\Delta t. \qquad (5)$$

Considering that the positive direction for $\mathrm{d}\vec{l}$ is that chosen in figure 1 and the fact that the fluxes appearing in equation (2) are outgoing fluxes,

$$\iint_{S_1} \vec{B}(t) \cdot \mathrm{d}\vec{S} = -\Phi(t), \qquad (6)$$

$\Phi(t)$ being the flux through the circuit at the instant t, when it was at the position C_1.

The last term in equation (5) is, taking into account the Maxwell equation, curl $\vec{E} = -\partial\vec{B}/\partial t$, and the Stokes theorem,

$$\iint_{S_1} \frac{\partial\vec{B}(t)}{\partial t}\cdot \mathrm{d}\vec{S}\Delta t = -\oint_{C_1} \vec{E}(t)\cdot(-\mathrm{d}\vec{l})\Delta t = \oint_{C_1} \vec{E}(t)\cdot \mathrm{d}\vec{l}\Delta t \tag{7}$$

because $\mathrm{d}\vec{l}$ has been chosen here opposite to that which would correspond to $\mathrm{d}\vec{S}$ in S_1 according to the right-hand rule.

Then, substituting equations (6) and (7) into equation (5), we get

$$\iint_{S_1} \vec{B}(t+\Delta t)\cdot \mathrm{d}\vec{S} = -\Phi(t) + \oint_{C_1} \vec{E}(t)\cdot \mathrm{d}\vec{l}\Delta t. \tag{8}$$

Finally, the last term on the left-hand side of equation (2), which corresponds to the outgoing flux through the lateral surface, can be written, *when* $\vec{B}(t)$ *varies in a continuous way* and $\vec{v}(t)$ is the velocity of the different points of the circuit, as

$$\begin{aligned}\iint_{S_L} \vec{B}(t+\Delta t)\cdot \mathrm{d}\vec{S} &= \oint_{C_1} [\vec{B}(t+\Delta t)\cdot \mathrm{d}\vec{l}\times\vec{v}(t)]\Delta t\\ &= \oint_{C_1} [\vec{B}(t)\cdot \mathrm{d}\vec{l}\times\vec{v}(t)]\Delta t + O(\Delta t^2),\end{aligned} \tag{9}$$

where the term corresponding to $\partial\vec{B}/\partial t$ has been included in the term of higher order $O(\Delta t^2)$.

Substituting equations (3), (8) and (9) into equation (2),

$$\Phi(t+\Delta t) - \Phi(t) + \oint_{C_1} \vec{E}(t)\cdot \mathrm{d}\vec{l}\Delta t + \oint_{C_1} [\vec{B}(t)\cdot \mathrm{d}\vec{l}\times\vec{v}(t)]\Delta t = 0 \tag{10}$$

or, using the cyclic property of the mixed product,

$$\oint_{C_1} \vec{E}(t)\cdot \mathrm{d}\vec{l} + \oint_{C_1} \vec{v}(t)\times\vec{B}(t)\cdot \mathrm{d}\vec{l} = -\frac{\Phi(t+\Delta t)-\Phi(t)}{\Delta t} \tag{11}$$

and taking limit when $\Delta t \to 0$,

$$\varepsilon = \oint_{C_1} [\vec{E}(t) + \vec{v}(t)\times\vec{B}(t)]\cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{12}$$

which is Faraday's law: the induced electromotive force (the work done by the force acting upon the positive unit charge, $\vec{E} + \vec{v}\times\vec{B}$, to complete one turn around the circuit) is equal to minus the time derivative of the coherent magnetic flux through the circuit.

Figure 2 shows a case in which Faraday's law would not be valid; since, at the moment $t + \Delta t/2$, a new branch in parallel is suddenly added to the circuit in such a way that, now, the sum of the three outgoing fluxes considered above is not zero because a new surface (that striped in figure 2) is needed to get a closed surface. Of course, a discontinuity in the circuit (like opening it to add a new branch in series) would also avoid the formation of the closed surface that is needed for the deduction.

The important conclusion of this section, which deserves to be highlighted, is that Faraday's law of induction is only valid if *the magnetic field varies in a continuous way* and if the circuit varies in such a way that all its positions along the time form the lateral surface of a volume *enclosed by this surface together with two arbitrary-shaped surfaces limited only by the initial position of the circuit and its final position*. In short, *a continuous variation along the time of the magnetic field and of the shape of the circuit* is required.

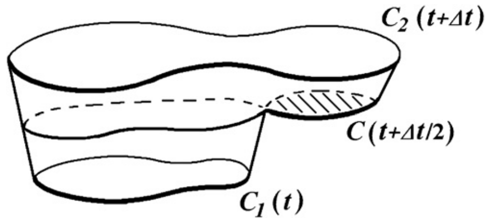

Figure 2. A non-suitable closed surface to deduce Faraday's law.

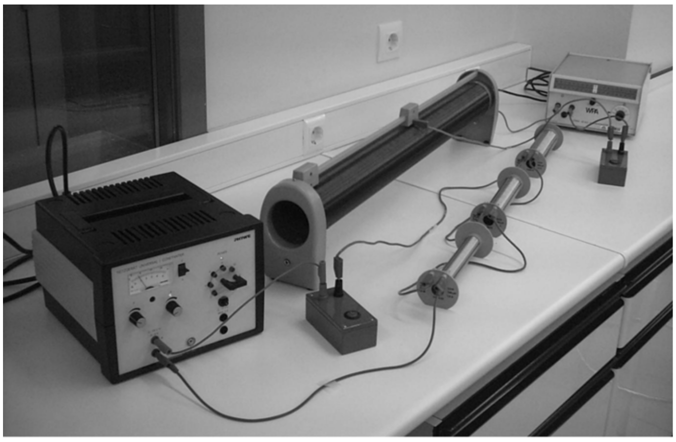

Figure 3. Experimental set-up to verify the conditions for Faraday's law.

3. Experimental details

We have made the experimental set-up as shown in the photograph of figure 3, including two circuits: the primary circuit has a dc source with several small solenoids in series, while the secondary includes a big solenoid and a ballistic galvanometer. The number of turns of the solenoid is changed with a slider wide enough to assure that there is always a contact with at least one of the turns at every moment. Interruptions in the circuit are so avoided when moving the slider, being produced, in contrast, in both circuits with switches. The small solenoids are placed inside the big one (although we have kept them outside it in the photograph for the sake of clarity). All the devices, except the galvanometer, have been supplied by PHYWE [5]. The ballistic galvanometer is a WPA Scientific Instrument.

Initially, a dc current of 1 A circulates by the inner solenoids in such a way that a constant magnetic field arises. It can be observed how the light signal of the ballistic galvanometer moves, showing that an induced current appears when the current intensity of the primary coils is roughly changed or the switch in the primary circuit is opened or closed. Although Faraday's law cannot be applied due to the theoretical discontinuity when opening or closing the switch, there is an induced electromotive force because the short-circuit of the primary happens during a short relaxation period. Nevertheless, it can be observed that the displacement of the light signal of the galvanometer, in spite of the very large flux change, is smaller than when the control of the intensity source is varied by hand from or to zero.

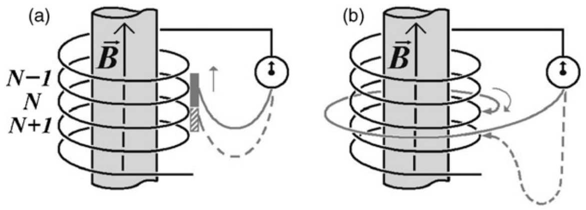

Figure 4. (a) The slider produces a discontinuous change in the number of turns of a circuit. (b) A non-suitable method to get a continuous change in the number of turns of a circuit.

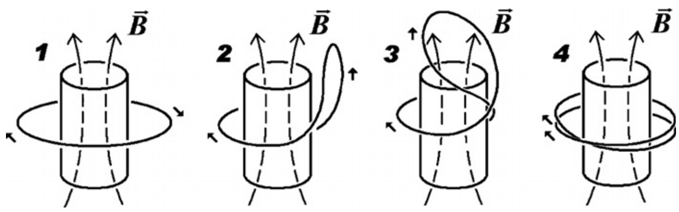

Figure 5. A suitable method to get a continuous change in the number of turns of a circuit.

In contrast, there is no induced current when the switch in the secondary circuit is connected/disconnected or when the sliding contact of the solenoid of the secondary circuit is roughly moved, although the number of turns of the big solenoid is drastically changed.

4. Discussion

In the last experiment of the previous section (that of moving the slider on the big solenoid) the electromotive force, ε, expressed in the middle part of equation (12) is null (there is no $\vec{E}$ because $\vec{B}$ is not time-dependent and $\vec{v} \neq \vec{0}$ only where $\vec{B} = \vec{0}$). As $\mathrm{d}\Phi/\mathrm{d}t \neq 0$ in the right-hand part of equation (12), Faraday's law does not hold.

When the slider in figure 4(a) is moved upwards, the contact with the ring $N + 1$ ceases and then starts the contact with the ring $N - 1$, keeping always the contact with the ring N. Thus, the circuit, which initially had two different paths (with $N + 1$ and N turns, respectively), changes into a single circuit (with N turns), being, then, again a circuit with two paths (with N and $N - 1$ turns). It seems clear that, in all this process, new branches in parallel are added and suppressed in a similar way to what happens in figure 2, and a closed surface as that required in figure 1 is not reached. This is the reason why equation (12) cannot be applied.

It is convenient to see what would happen if we try to change the number of turns of the secondary in a continuous way, sliding the contact along the helicoidal path determined by the wire around the solenoid in figure 4(b). Nevertheless, we can see that, in this way, for each turn of the solenoid being eliminated we would be forced to roll the wire coming from the galvanometer once around the solenoid, and so the real number of turns of the circuit keeps invariable and all the terms of equation (12) are null.

On the other hand, an example of how the wire can be rolled in such a way that the number of turns varies without discontinuities is shown in figure 5. It must be noted that, now, the wire has to cross the magnetic field lines so that each part of equation (12) leads to the same result and Faraday's law holds.

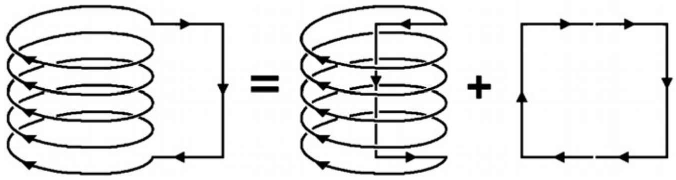

Figure 6. Surfaces to be used when applying Stokes law to a solenoid.

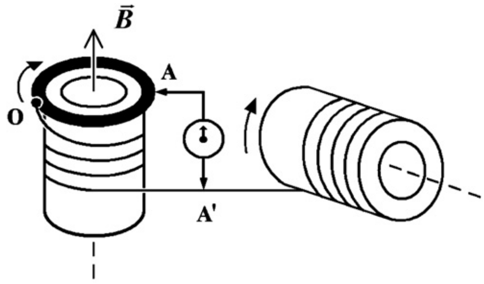

Figure 7. Winding wire around a rotating drum with sliding contacts.

In this case, Faraday's law can be written, as a function of the flux, ϕ, crossing each one of the variable number, N, of turns, in figure 5:

$$\varepsilon = -\frac{\mathrm{d}(N\phi)}{\mathrm{d}t} = -\frac{\mathrm{d}N}{\mathrm{d}t}\phi - N\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\mathrm{d}N}{\mathrm{d}t}\phi. \tag{13}$$

Therefore, it is not adequate to write Faraday's law as $\varepsilon = -N\,\mathrm{d}\phi/\mathrm{d}t$, as is done in some textbooks (see, for instance [6]). Moreover, to calculate the electromotive force, it is necessary to take, as the path for the simple integral of the left-hand part of equation (12), the line of the circuit and, due to Stokes theorem; the surface that the right-hand part of the double integral of equation (12) refers to, includes a spiral ramp that is crossed by the magnetic flux so many times as 'floors' (turns of the wire) it has [4]. This can be seen in figure 6.

A similar case to our experimental test with the slider is that of winding wire around a rotating drum described in [1] and shown in figure 7, for which again $\varepsilon = 0$ and $\mathrm{d}\Phi/\mathrm{d}t \neq 0$. Now, a path in parallel is suddenly added to the circuit (and the other is suppressed) each time the end of the wire O, placed on the upper black conductive ring in that figure, contacts the sliding cursor A. The two possible paths from O to A in the former conductive ring have a difference between them of one turn embracing the magnetic field and play the same role as the two rings contacting the slider of the solenoid in our experiment. Faraday's law cannot be applied here because the continuity condition regarding the change of the shape of the circuit is not fulfilled and no electromotive force is generated.

Another widely known case, in which a violation of Faraday's law seems to appear, is that of Faraday's disc (figure 8). Even some author [3] claims that the flux law is not applicable to this case.

Nevertheless, it is possible to apply the flux law if it is taken into account that the electrons displace through the material of the disc. If an electron goes into the disc by the point A at the border, it moves when going to the centre by a *path in the disc*, such as the curve marked between O and A in figure 8. But the contacting point A has displaced to A′, due to the rotation of the disc, during the movement of the electron and the path followed by the electron

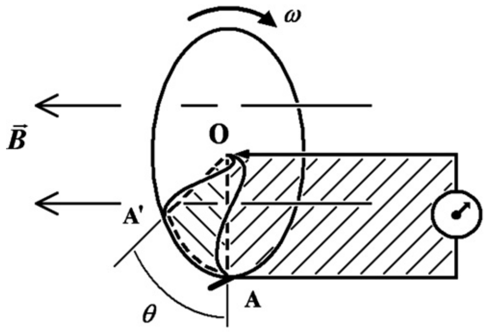

Figure 8. Faraday's disc.

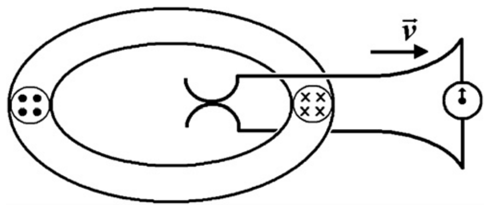

Figure 9. Hering's experiment: conductive magnetized toroid together with a circuit including a galvanometer.

has displaced to that position marked between O and A′ in figure 8. In this way, the sector OAA′ represents the change of area, and it must be noted that this change is independent of the particular path of each electron. The induced electromotive force, by each part of equation (12), is then

$$\varepsilon = \oint (\vec{E} + \vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{l} = \int_0^R \omega r B \, \mathrm{d}r = \frac{1}{2}\omega B R^2, \tag{14}$$

$$\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(-\pi R^2 B \frac{\theta}{2\pi}\right) = \frac{1}{2}\omega B R^2. \tag{15}$$

Assuming that the path of the electron is the *non-moving line* AO is to accept that the electrons displace *through the air* instead of through the material of the disc.

The so-called Hering's paradox (figure 9) gathers two of the physical situations analysed previously in this paper. In it, spring clips are able to slide over the surface of a magnetized toroid with a fairly high conductivity until the circuit with the clips herein stops to embrace the magnetic flux. While the contacts slide over the toroidal section (figure 10), a fixed path in such a section must be chosen (e.g. the marked path including the vertical diameter), in such a way that, when the contacts slide, there is no flux change and no electromotive force arises. When the spring clips establish a new contact with the toroid, at the beginning of the contact process (figure 10(a)) or they contact themselves again at the finish (figure 10(c)), new paths in parallel are added to the circuit and, just after, old paths are suppressed. There is no electromotive force either and Faraday's law does not hold now due to the cause mentioned before (discontinuities in the shape of the circuit).

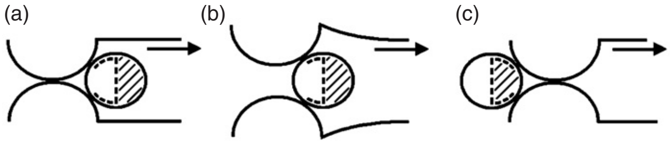

Figure 10. Hering's experiment: (a) two branches in parallel at the beginning of the contact process; (b) no flux variation arises when moving the spring clips; (c) two branches in parallel at the finish of the contact process.

Although in [1,2] it has been explained that Faraday's law cannot be applied to the same cases that have been analysed here (they do it either by using the direct result for the first member of equation (12) or by energetic considerations) the theoretical reason why the induction law does not hold (i.e. the conditions needed for the deduction) is not clarified.

5. Conclusions

With a simple experimental set-up and taking advantage of those available for other experiments in the laboratory, it is possible to show how the Maxwell equations are the fundamental laws of the electromagnetism and how Faraday's law of induction is only valid when it can be obtained from them (it is to say, when the external magnetic field varies in a continuous way and when it is possible to form a closed surface with the intermediate positions of the circuit, together with surfaces limited exclusively by its initial and final positions).

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