AUTOMATIC STRUCTURES FOR TORUS LINK GROUPS

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Abstract. A general result of Epstein and Thurston implies that all link groups are automatic, but the proof provides no explicit automaton. Here we show that the groups of all torus links are groups of fractions of so-called Garside monoids, i.e., roughly speaking, monoids with a good theory of divisibility, which allows us to prove that those groups are automatic, but, in addition, gives a completely explicit description of the involved automata, thus partially answering a question of D. F. Holt.

Key words : knot and link groups, presentation of groups, automatic groups, Artin groups.

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Introduction

Epstein and Thurston showed in [20, Chapter XII] that the fundamental group of every Haken 3-manifold is biautomatic, except if it carries either nil geometry or sol geometry. As a corollary, every link group is biautomatic. However, the proof is not constructive in the sense that it provides no explicit automaton. Here, we propose a combinatorial group theory point of view to adjoint to this geometric group theory result. The aim of this paper is to show that all the torus link groups are the groups of fractions of monoids that admit a nice theory of divisibility, like spherical Artin or Birman-Ko-Lee monoids : all the torus link groups belong to the wide class of so-called Garside groups. Dehornoy stated in [15] that every Garside group is biautomatic by proving that each Garside monoid of which it is the group of fractions provides a simple explicit biautomatic structure. Therefore, our result enables us to construct an explicit finite state automaton computing normal forms for each torus link group.

This paper is organized as follows. In Section 1, we gather earlier results of [15] and [17] about Garside monoids and groups. In Section 2, we compute a first well-presented (monoid) presentation for every torus link group, then we complete the latter to obtain a so-called complemented presentation. In Section 3, using the previous results, we show that every torus link group is a Garside group, and, therefore, that it admits a biautomatic structure. Several examples are studied. Finally, in Section 4, we discuss the existence of other automatic structures for these groups.

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1. Background from Garside groups

In this section we list some basic properties of Garside monoids and groups, and summarize Dehornoy's results about biautomaticity of Garside groups. Finally, we recall an effective criterion for recognizing a Garside monoid from a given monoid presentation. For all the results quoted here, we refer the reader to [17], [15] and [14].

Main definitions and basic properties

Assume that M is a monoid. We say that M is *conical* if 1 is the only invertible element in M. For a, b in M, we say that b is a *left divisor* of a (or that a is a *right multiple* of b) if $a = bd$ holds for some d in M. An element c is a *right lower common multiple* (or a *right lcm*) of a and b if it is a right multiple of both a and b, and every common right multiple of a and b is a right multiple of c. Right divisor, left multiple, and left lcm are defined symmetrically. For a, b in M, we say that b *divides* a (or that b is a divisor of a) if $a = cbd$ holds for some c, d in M.

If c, c' are two right lcm's of a and b, necessarily c is a left divisor of c', and c' is a left divisor of c. If we assume M to be conical and cancellative, we have $c = c'$. In this case, the unique right lcm of a and b is denoted by $a \vee b$. If $a \vee b$ exists and M is left cancellative, then there exists a unique element c satisfying $ac = b\vee$. This element is denoted by $a \backslash b$. We define the left lcm $\wedge$ and the left operation $/$ symmetrically. In particular, we have

$$a \vee b = a(a \backslash b) = b(b \backslash a); \quad \text{and} \quad a \wedge b = (b/a)a = (a/b)b.$$

Cancellativity and conicity implies that left and right divisibility are order relations.

Definition. [15] A monoid M is said to be *Garside*[(*)] if M is conical and cancellative, every pair of elements in M admits a left lcm and a right lcm, and M admits a Garside element, defined to be an element whose left and right divisors coincide and generate M.

By [9], all spherical Artin monoids are Garside monoids. The braid monoids of the complex reflection groups $G_7; G_{11}; G_{12}; G_{13}; G_{15}; G_{19}$ and G_{22} given in [8] (see [17, 29]), Garside's hypercube monoids ([21, 29]), the dual braid monoids for spherical Artin groups ([6, 4, 29, 3, 31]) are also Garside monoids.

Example 1.1. Let us consider the monoid $M_\cdot$ with presentation

$$\langle x, y, z : xzxy = yzx^2,\ yzx^2z = zxyzx,\ zxyzx = xzxyz \rangle.$$

The monoid $M_\cdot$ is a typical example of a Garside monoid, and, in addition, $M_\cdot$ has the distinguishing feature to be not anti-automorphic, contrary to all spherical Artin monoids.

If M is a Garside monoid, then M satisfies Ore's condition [12], and it embeds in a group of right fractions and, symmetrically, in a group of left fractions. In this case, by construction, every right fraction ab^{-1} with a, b in M can be expressed as a left fraction $c^{-1}d$, and conversely. Therefore, the two groups coincide, and there is no ambiguity in speaking of the group of fractions of a Garside monoid.

(*) Garside monoids as defined above are called Garside monoids in [3, 4, 11, 15, 18, 30, 31], but they were called "small Gaussian" or "thin Gaussian" in previous papers [17, 27, 28, 29], where a more restricted notion of a Garside monoid was also considered.

Definition. A group G is a Garside group if there exists a Garside monoid of which G is the group of fractions.

Lemma 1.2. [15] Assume that M is a Garside monoid. Then the following identities holds in M:

$$(ab) \wedge (ac) = a(b \wedge c); \tag{1.1}$$

$$c^{-1}(ab) = (c^{-1}a)((a^{-1}c)^{-1}b); \qquad (ab)^{-1}c = b^{-1}(a^{-1}c); \tag{1.2}$$

$$(a \vee b)^{-1}c = (a^{-1}b)^{-1}(a^{-1}c) = (b^{-1}a)^{-1}(b^{-1}c); \qquad c^{-1}(a \vee b) = (c^{-1}a) \vee (c^{-1}b). \tag{1.3}$$

Lemma 1.3. [17] Assume that M is a Garside monoid. Then the following equivalent assertions hold:

(i) There exists a mapping δ from M into the integers satisfying $\delta(a) > 0$ for every $a \neq 1$ in M, and satisfying $\delta(ab) \geq \delta(a) + \delta(b)$ for every a, b in M;

(ii) For every set X that generates M and for every a in M, the lengths of the decompositions of a as products of elements in X have a finite upper bound.

Definition. [17] A monoid is said to be atomic if it satisfies the equivalent conditions of Lemma 1.3. An atom is defined to be a non trivial element a such that $a = bc$ implies $b = 1$ or $c = 1$. The norm function $|\cdot|$ of an atomic monoid M is defined such that, for every a in M, $|a|$ is the upper bound of the lengths of the decompositions of a as products of atoms.

Example 1.4. The monoid $M_\dagger$ defined by the presentation

$$\langle x, y : xyxyxyx = yy \rangle$$

is another example of a Garside monoid, which, as for it, admits no additive norm, i.e., no norm ν satisfying $\nu(ab) = \nu(a) + \nu(b)$.

By the previous lemma, every element in a Garside monoid has finitely many left divisors, only then, for every pair of elements (a, b), the common left divisors of a and b admit a right lcm, which is therefore the left gcd of a and b. This left gcd will be denoted by $a \wedge b$. We define the right gcd symmetrically.

Lemma 1.5. [15] Assume that M is a Garside monoid. Then it admits a finite generating subset that is closed under $\cdot, =, \vee, \wedge$ and $^{-1}$.

Every Garside monoid admits a finite set of atoms, and this set is the minimal generating set [17]. The hypothesis that there exists a finite generating subset that is closed under $\cdot$ implies that the closure of the atoms under $\cdot$ is finite; its elements are called right primitive elements. In particular, the closure of the atoms under $\cdot$ and $\vee$ is finite; its elements are called simple elements, and their right lcm is denoted by Δ. It turns out that the set of the simple elements is also the closure of atoms under $=$ and $^{-1}$. So, the element Δ is both the right and the left lcm of the simple elements, and it is called the Garside element of the monoid. If M is a Garside monoid and S is the set of simple elements in M, then $(S, \wedge, \vee, 1, \Delta)$ is a finite lattice which completely determines M.

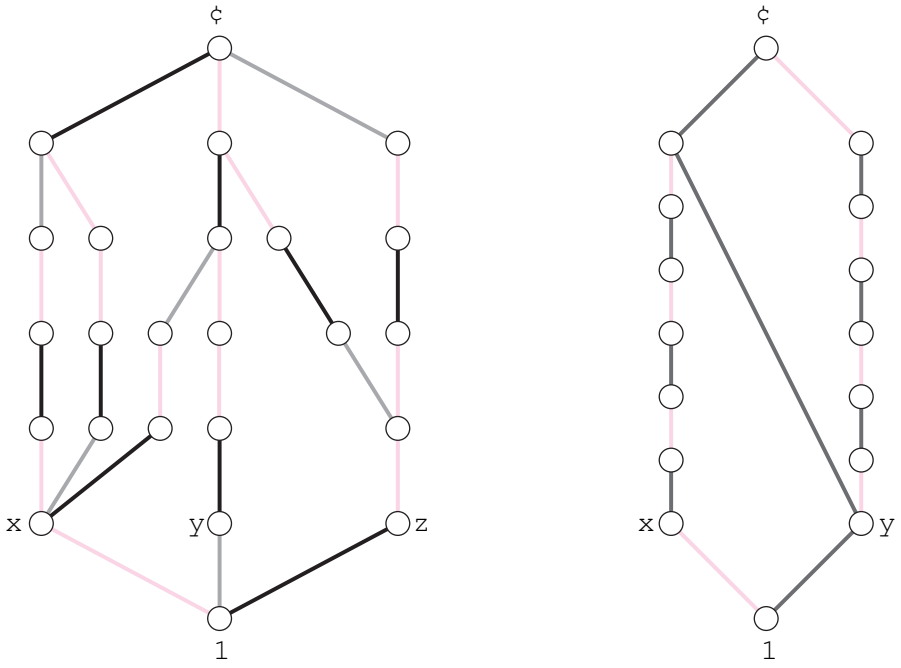


Figure 1. The lattice of simples in $M_{\cdot}$ (on the left) and in $M_{\dagger}$ (on the right).

Example 1.6. The lattice of simple elements in $M_{\cdot}$ (Example 1.1) and in $M_{\dagger}$ (Example 1.4) are displayed in Figure 1.

Proposition 1.7. [15] Assume that M is a Garside monoid, S is the set of its simple elements, and $¢$ is its Garside element.

(i) Let k be a non negative integer. Then, S^k is both the set of all left divisors of $¢^k$ and the set of all right divisors of $¢^k$.

(ii) The functions $a \mapsto (a¢)¢$ and $a \mapsto ¢(¢/a)$ from S into itself extend into automorphisms ℓ and e of M that map S^k into itself for every k, and the equalities

$$a¢ = ¢\,\ell(a), \quad \text{and} \quad ¢a = e(a)¢$$

hold for every a in M.

Normal forms and automatic structures

If S generates the monoid M, every element of M can be decomposed as a product of elements of S. The classical idea for obtaining a unique distinguished decomposition is to push the elements to the right (or to the left), i.e., to require the last element of the decomposition to be the maximal possible one. So a $\mathcal{n}$-closed family is what is needed for this process to lead to a unique decomposition (see [17, 15, 14, 27]).

Definition. Assume that M is a Garside monoid, and G is its group of fractions. For c in G, the (right fractional) normal form of c is defined to be the unique decomposition

$$a_p \cdot\cdot\cdot a_1 b_1^{-1} \cdot\cdot\cdot b_q^{-1},$$

where $a_p, \ldots, b_q$ are simple elements in M all distinct from 1, and we have $a_1 \wedge b_1 = 1$, $a_i \wedge (\Delta = a_{i-1}) = 1$ for $2 \leq i \leq p$, and $b_i \wedge (\Delta = b_{i-1}) = 1$ for $2 \leq i \leq q$.

The previous normal forms are associated with a bi-automatic structure. More than an existence result, simple automata that compute normal forms can be constructed:

Proposition 1.8. [15] Assume that M is a Garside monoid. Let A be the set of atoms, S be the set of simples, and Δ be the Garside element. Let define $T : S \times A \to S$ by $T(s, x) = (\Delta = (\Delta = x)) \vee (\Delta = s)) \vee \Delta$. Then, for every word u on A, we have $u \wedge \Delta = T(1, u)$, i.e., the result of reading u by the automaton $(T, 1)$ is the last term of the right normal form of u.

Corollary 1.9. [15] Every Garside group admits a bi-automatic structure.

Example 1.10. From the lattice of simples in M^+ (Figure 1), one can easily construct a transducer, i.e., a finite state automaton with output, computing the right normal form of every element in M^+: see Figure 2. The initial state is represented with a big black arrow : the state 1. Clear (resp. dark) arrows represent the reading of the letter x (resp. the letter y). During the reading of a word w, one concatenates the labels (eventually empty) of crossed arrows. At the end of the reading of w, the ambient state is the first simple of the right normal form $N(w)$ of w and the word w' obtained by concatenating the various labels is the word that remains to normalize: we have $N(w) = N(w')\Delta$.

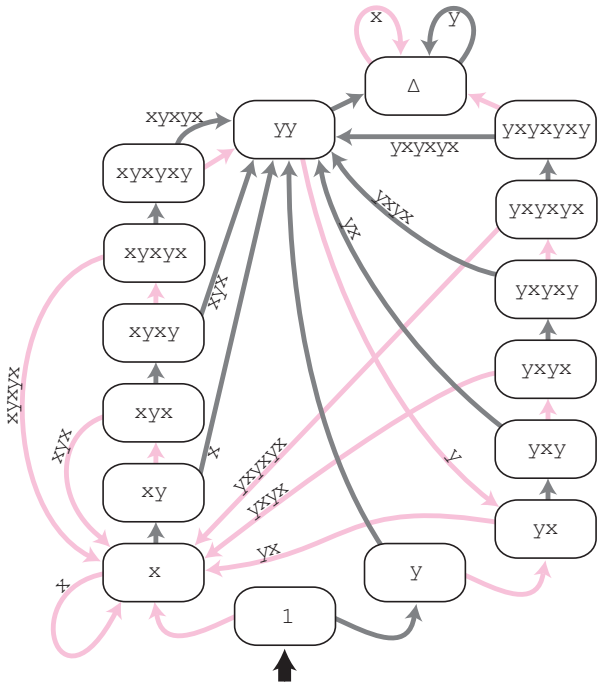


Figure 2. A transducer computing the right normal form in M^+.

Recognizing Garside monoids

We conclude this section by recalling how to effectively recognize Garside monoids from a presentation. In fact, a correspondence can be established between Garside monoids and presentations of a particular syntactic form. The connection relies on a combinatorial technique called word reversing (word reversing in [14]), and it leads to two fundamental applications. On the one hand, it allows to effectively decide whether a given presentation defines a Garside monoid. On the other hand, given a Garside monoid presentation, it allows one to compute the main operations of the monoid.

For a nonempty set A — usually called an alphabet in this context — we denote by A^* the free monoid with basis A. We use ε for the empty word.

Definition. [15] We say that a monoid presentation $\langle A : R \rangle$ is complemented iff, for every pair (x, y) in A^2, there exists exactly one relation $xu = yv$ in R; in this case, u and v are denoted by $x \backslash y$ and $y \backslash x$ respectively. The operation $\backslash$ on A can be extended into a unique partial operation $\backslash$ on A^* satisfying the rules of word reversing:

$$(uv) \backslash w = v \backslash (u \backslash w);$$
$$w \backslash (uv) = (w \backslash u) \backslash ((u \backslash w) \backslash v).$$

A complemented monoid presentation $\langle A : R \rangle$ is called coherent on $B \subseteq A^*$ iff, for every (u, v, w) in B^3, we have

$$((u \backslash v) \backslash (u \backslash w)) \backslash ((v \backslash u) \backslash (v \backslash w)) = \varepsilon;$$

and called normed if there exists a mapping ℓ of A^* to the integers such that, for all x, y in A, and all u, v in A^*, we have $\ell(xv) \geq \ell(v)$, $\ell(vx) \geq \ell(v)$ and $\ell(ux(xy)v) = \ell(uy(yx)v)$.

The Garsidity criterion we shall use in the sequel is:

Proposition 1.11. [14] Assume that M is a monoid with a finite coherent normed and complemented presentation. Assume that M is cancellative and admits a Garside element. Then M is a Garside monoid.

For alternate Garsidity criteria and details, we refer to [15], see also [14, 16, 17].

Remark. We know no such criterion allowing to prove that a given finite complemented presentation defines a (non Garside) Gaussian monoid, i.e., a Gaussian monoid without a Garside element. The coherence condition is certainly necessary, but not sufficient. Let us mention moreover that we know actually no example of finitely generated Gaussian monoid or group that is not Garside: the braid group B_1 is Gaussian and not Garside, but it is not finitely generated.

2. Monoid presentations for torus link groups

In this section we compute a first well-fitted (monoid) presentation for every torus link group, and then complete the latter to obtain a complemented presentation.

Every oriented link can be realized as a closed braid $\hat{b}$ for some braid b. Following Artin [2] (see also [5]), if b is an n-strand braid, then the link group of $\hat{b}$ | i.e., the fundamental group $\pi_1(S^3 \setminus \hat{b})$ of the complement of $\hat{b}$ | has a presentation of the form

$$\langle x_1, \ldots, x_n \,:\, x_i = x_i \cdot b \text{ for } 1 \leq i \leq n \rangle,$$

where $u \cdot b$ denotes the image of the element u under some automorphism of the free group based on $\{x_1, \ldots, x_n\}$.

Definition. A torus link is a link that can be drawn on the surface of a standard torus without self-intersections. Given integers p and q, the (p,q)-torus link wraps meridionally around the torus p times and wraps longitudinally around the torus q times.

Applying Artin's theorem to the particular case of torus links, we obtain that, for $2 \leq p \leq q$, the group of the (p,q)-torus link is

$$\langle x_1, \ldots, x_p \,:\, x_i = x_i \cdot (\sigma_1 \cdots \sigma_{p-1})^q \text{ for } 1 \leq i \leq p \rangle,$$

since, as shown in Figure 3, the (p,q)-torus link is the closure of the braid $(\sigma_1 \cdots \sigma_{p-1})^q$, where the σ_i's denote the Artin generators of B_n.

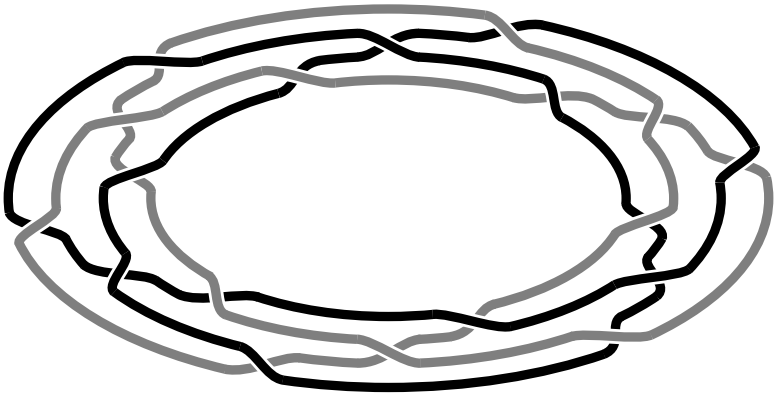


Figure 3. The (p,q)-torus link is the closure of the braid $(\sigma_1 \cdots \sigma_{p-1})^q$. (here, the (4,6)-torus link as the closure of $(\sigma_1 \sigma_2 \sigma_3)^6$).

Lemma 2.1. For $2 \leq p \leq q$, the (p,q)-torus link group is

$$\langle x_1, \ldots, x_p \,:\, x_i (x_1 \cdots x_p)^{\lfloor \frac{q-i}{p} \rfloor + 1} = (x_1 \cdots x_p)^{\lfloor \frac{q-i}{p} \rfloor + 1} x_{p-(i-q) \bmod p} \text{ for } 1 \leq i \leq p \rangle. \qquad (2.1)$$

Proof. We consider the action of the braid group B_p onto the free group on $\{x_1,\dots,x_p\}$ defined by

$$x_i \cdot \sigma_j = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{for } i = j \\ x_{i-1} & \text{for } i = j+1 \\ x_i & \text{otherwise.} \end{cases}$$

Let $[z]$ denote the integer part of z. We show by induction on $q \geq 2$ that, for every p with $2 \leq p \leq q$, we have

$$x_i \cdot (\sigma_1 \cdots \sigma_{p-1})^q = (x_1 \cdots x_p)^{[\frac{qi}{p}]+1} x_{p[\frac{qi}{p}]+p-q+i} (x_1 \cdots x_p)^{i-[\frac{qi}{p}]-1}$$

for every $1 \leq i \leq p$. For $p = q = 2$, we find

$$x_1 \cdot \sigma_1^2 = x_1 x_2 x_1^{i-1} \cdot \sigma_1 = (x_1 x_2 x_1^{i-1})(x_1) = (x_1 x_2 x_1^{i-1} x_1) = (x_1 x_2) x_1 (x_1 x_2)^{i-1};$$
$$x_2 \cdot \sigma_1^2 = x_1 \cdot \sigma_1 = x_1 x_2 x_1^{i-1} = (x_1 x_2) x_2 (x_1 x_2)^{i-1}.$$

Assume $q > 2$. By induction hypothesis we have

$$x_i \cdot (\sigma_1 \cdots \sigma_{p-1})^q = (x_1 \cdots x_p)^{[\frac{qi-i-1}{p}]+1} x_{p[\frac{qi-i-1}{p}]+p-q+i+1} (x_1 \cdots x_p)^{i-[\frac{qi-i-1}{p}]-1} \cdot (\sigma_1 \cdots \sigma_{p-1});$$

for $1 \leq i < p$. We easily compute $(x_1 \cdots x_p)^k \cdot \sigma_i = (x_1 \cdots x_p)^k$ for $1 \leq i < p$, hence

$$(x_1 \cdots x_p)^k \cdot (\sigma_1 \cdots \sigma_{p-1}) = (x_1 \cdots x_p)^k$$

for $k \neq 0$ (also a direct consequence of the fact that the considered action is an automorphism), and

$$x_i \cdot (\sigma_1 \cdots \sigma_{p-1}) = \begin{cases} (x_1 \cdots x_p) x_p (x_1 \cdots x_p)^{-1} & \text{for } i = 1 \\ x_{i-1} & \text{for } 1 < i \leq p. \end{cases}$$

Now, $p[\frac{qi-i-1}{p}] + p - q + i + 1 = 1$ is equivalent to $[\frac{qi-i-1}{p}] + 1 = \frac{qi-i}{p} = [\frac{qi-i}{p}]$. We deduce

$$x_1 \cdot (\sigma_1 \cdots \sigma_{p-1})^q = (x_1 \cdots x_p)^{[\frac{qi-i}{p}]} (x_1 \cdots x_p) x_p (x_1 \cdots x_p)^{-1} (x_1 \cdots x_p)^{i-[\frac{qi-i}{p}]},$$

and

$$x_i \cdot (\sigma_1 \cdots \sigma_{p-1})^q = (x_1 \cdots x_p)^{[\frac{qi-i}{p}]+1} x_{p[\frac{qi-i}{p}]+p-q+i} (x_1 \cdots x_p)^{i-[\frac{qi-i}{p}]-1},$$

for $1 < i < p$. Finally, for $i = p$, a direct computation gives the required result

$$x_p \cdot (\sigma_1 \cdots \sigma_{p-1})^q = (x_1 \cdots x_p)^{[\frac{q}{p}]} x_{p[\frac{q}{p}]+p-q} (x_1 \cdots x_p)^{-[\frac{q}{p}]},$$

which completes the induction. □

Now we have to distinguish two cases in order to complete the presentation obtained above into a new one, that will be eligible for applying the Garside-ity criterion of Proposition 1.11.

By Lemma 2.1, for $p \geq 2$ and $k \geq 1$, the (p,pk)-torus link group is

$$\langle x_1,\dots,x_p : x_i(x_1\cdots x_p)^k = (x_1\cdots x_p)^k x_i \text{ for } 1 \leq i \leq p-1\rangle,$$

i.e.,

$$\langle x_1,\dots,x_p : (x_1\cdots x_p)^k = (x_2\cdots x_p x_1)^k = \cdots = (x_p x_1\cdots x_{p-1})^k \rangle. \tag{2.2}$$

Let us fix $2 \leq p \leq q$ and assume that p does not divide q. Let φ,λ be the integer part and the remainder of q/p. Then we have $q = p\varphi+\lambda$ with $1 \leq \lambda \leq p-1$. The (p,q)-torus link group is then

$$\begin{aligned}
\langle x_1,\dots,x_p : (x_1\cdots x_p)^{\varphi+1} &= (x_2\cdots x_p)(x_1\cdots x_p)^{\varphi} x_{p-\lambda+1}\\
&= (x_3\cdots x_p)(x_1\cdots x_p)^{\varphi} x_{p-\lambda+1}x_{p-\lambda+2}\\
&= \cdots\\
&= (x_\lambda\cdots x_p)(x_1\cdots x_p)^{\varphi} x_{p-\lambda+1}\cdots x_{p-1}\\
(x_1\cdots x_p)^{\varphi} x_1 &= x_{\lambda+1}(x_1\cdots x_p)^{\varphi}\\
(x_1\cdots x_p)^{\varphi} x_2 &= x_{\lambda+2}(x_1\cdots x_p)^{\varphi}\\
&\vdots\\
(x_1\cdots x_p)^{\varphi} x_{p-\lambda} &= x_p(x_1\cdots x_p)^{\varphi}\rangle.
\end{aligned} \tag{2.3}$$

Definition. Let $2 \leq p \leq q$. We call (p,q)-torus link Artin monoid the monoid $A_{p,q}$ with presentation (2.2) for $p \mid q$ and with presentation (2.3) for $p \nmid q$.

By construction, for $2 \leq p \leq q$, the group admitting the presentation of $A_{p,q}$ is the (p,q)-torus link group. Now, we show that $A_{p,q}$ admits a complemented presentation. This new presentation (not unique a priori) will allow us in the following section to show that $A_{p,q}$ is a Garside monoid, which, in particular, will tell us that this complemented presentation is the right one, i.e., the minimal one.

Proposition 2.2. Let $2 \leq p \leq q$ with $p \nmid q$. The monoid $A_{p,q}$ admits the complemented presentation

$$\langle x_1,\dots,x_p : x_i(x_i \backslash x_j) = x_j(x_j \backslash x_i) \text{ for } 1 \leq i,j \leq p-1\rangle,$$

where $\backslash$ is defined by

$$x_{s\varphi+1} \backslash x_{t\varphi+1} = \begin{cases} w_{(t-s)0(\varphi-1)} & \text{for } t < s,\\ w_{(s-t)1(\varphi-1)}x_{(t-s-1)\varphi+1} & \text{for } t > s,\end{cases}$$

by

$$x_{s\varphi+\varepsilon} \backslash x_{t\varphi+1} = \begin{cases} w_{(t-s)0(\varphi-1)} & \text{for } t < s,\\ w_{(s-t)\varepsilon\varphi} & \text{for } s \leq t \leq s+1,\\ w_{(s-t)\varepsilon\varphi}w_{00(\varphi-2)}x_{(t-s-2)\varphi+1} & \text{for } s+1 < t,\end{cases}$$

and

$$x_{tβ+1}\backslash x_{sβ+e} = \begin{cases} w_{(tβ)1(β-1)}x_{(s-t-1)β+e} & \text{for } t < s, \\ w_{(tβ)1(sβ-tβ+β)} & \text{for } s \le t \le s+1, \\ w_{(sβ)0(2β-1)} & \text{for } s+1 < t, \end{cases}$$

for $e \neq 1$, and by

$$x_{sβ+e}\backslash x_{tβ+f} = \begin{cases} w_{(tβ)0(2β-1)} & \text{for } t < s-1, \\ w_{(sβ)eβ} & \text{for } t = s-1, \text{or for } s = t \text{ with } e \neq f, \\ w_{(sβ)eβ}w_{00(β-1)} & \text{for } t = s+1, \\ w_{(sβ)eβ}w_{00(β-2)}x_{(t-s-2)β+f} & \text{for } t > s+1, \end{cases}$$

for $e, f \neq 1$, with

$$w_{zeβ} = (x_1 \cdots x_p)^z x_{e+1} \cdots x_p (x_1 \cdots x_p)^β x_{1+p-β} \cdots x_{e-1+p-β};$$

implying $w_{z0β} = (x_1 \cdots x_p)^{z+1+β}$ and $w_{z1β} = (x_1 \cdots x_p)^z x_2 \cdots x_p (x_1 \cdots x_p)^β$.

Proof. First, the relations in (2.3a) are immediate consequences of those of the presentation above. Conversely, we have to find a relation of the form $x_i \cdots = x_j \cdots$ for $1 \le i, j \le p$ using only the relations of the presentation (2.3) for $A_{p,q}$, that is, using only the relations

$$x_e \cdots x_p (x_1 \cdots x_p)^β x_{p-β+1} \cdots x_{e-1+p-β} = x_f \cdots x_p (x_1 \cdots x_p)^β x_{p-β+1} \cdots x_{f-1+p-β}; \quad (1)$$

$$x_{sβ+e}(x_1 \cdots x_p)^{sβ} = (x_1 \cdots x_p)^{sβ} x_e; \quad (2)$$

for $1 \le e, f \le β$.

Case 1. Assume $i = sβ + e$ and $j = tβ + 1$ with $t < s$. Then we have

$$\begin{aligned} x_{sβ+e}(x_1 \cdots x_p)^{(t+1)β} &\overset{(2)}{=} (x_1 \cdots x_p)^{(t+1)β} x_{(s-t-1)β+e} \\ &= (x_1 \cdots x_p)^{tβ} x_1 x_2 \cdots x_p (x_1 \cdots x_p)^{β-1} x_{(s-t-1)β+e} \\ &\overset{(1)}{=} x_{tβ+1}(x_1 \cdots x_p)^{tβ} x_2 \cdots x_p (x_1 \cdots x_p)^{β-1} x_{(s-t-1)β+e}. \end{aligned}$$

This gives both the expression of $x_{sβ+1}\backslash x_{tβ+1}$ for $0 \le s \neq t$ and the expressions of $x_{sβ+e}\backslash x_{tβ+1}$ and $x_{tβ+1}\backslash x_{sβ+e}$ for $0 \le s < t$ and $e \neq 1$.

Case 2. Assume $i = sβ + e$ and $j = tβ + 1$ with $s \le t \le s+1$ and $e \neq 1$. Then we have

$$\begin{aligned} & x_{sβ+e}(x_1 \cdots x_p)^{sβ} x_{e+1} \cdots x_p (x_1 \cdots x_p)^β x_{1+p-β} \cdots x_{e-1+p-β} \\ &\overset{(2)}{=} (x_1 \cdots x_p)^{sβ} x_e x_{e+1} \cdots x_p (x_1 \cdots x_p)^β x_{1+p-β} \cdots x_{e-1+p-β} \\ &\overset{(1)}{=} (x_1 \cdots x_p)^{(s+1)β+1} = (x_1 \cdots x_p)^{tβ} x_1 x_2 \cdots x_p (x_1 \cdots x_p)^{(s-t+1)β} \\ &\overset{(2)}{=} x_{tβ+1}(x_1 \cdots x_p)^{tβ} x_2 \cdots x_p (x_1 \cdots x_p)^{(s-t+1)β}. \end{aligned}$$

This provides the expressions of $x_{sβ+e}\backslash x_{tβ+1}$ and $x_{tβ+1}\backslash x_{sβ+e}$ for $0 \le s \le t \le s+1$ and $e \neq 1$.

Case 3. Assume $i = sfl + e$ and $j = tfl + 1$ with $s + 1 < t$ and $e \neq 1$. Then we have

$$\begin{aligned} x_{sfl+e}(x_1 \cdots x_p)^{sfi} x_{e+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{e-1+p-fl} (x_1 \cdots x_p)^{fi-1} x_{(t-s-2)fl+1} \\ \overset{(2)}{=} (x_1 \cdots x_p)^{sfi} x_e x_{e+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{e-1+p-fl} (x_1 \cdots x_p)^{fi-1} x_{(t-s-2)fl+1} \\ \overset{(1)}{=} (x_1 \cdots x_p)^{(s+2)fi} x_{(t-s-2)fl+1} \overset{(2)}{=} x_{tfl+1} (x_1 \cdots x_p)^{(s+2)fi}. \end{aligned}$$

This gives the expression of $x_{sfl+e} \backslash x_{tfl+1}$ and $x_{tfl+1} \backslash x_{sfl+e}$ for $0 < s + 1 < t$ and $e \neq 1$.

Case 4. Assume $i = sfl + e$ and $j = tfl + f$ with $t < s - 1$ and $e, f \neq 1$. Then we have

$$\begin{aligned} x_{sfl+e}(x_1 \cdots x_p)^{(t+2)fi} \overset{(2)}{=} (x_1 \cdots x_p)^{(t+2)fi} x_{(s-t-2)fl+e} \\ \overset{(1)}{=} (x_1 \cdots x_p)^{tfi} x_f x_{f+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{f-1+p-fl} (x_1 \cdots x_p)^{fi-1} x_{(s-t-2)fl+e} \\ \overset{(2)}{=} x_{tfl+f} (x_1 \cdots x_p)^{tfi} x_{f+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{f-1+p-fl} (x_1 \cdots x_p)^{fi-1} x_{(s-t-2)fl+e}. \end{aligned}$$

This provides the expression of $x_{sfl+e} \backslash x_{tfl+f}$ for either $0 < t + 1 < s$ or $0 < s + 1 < t$ and $e, f \neq 1$.

Case 5. Assume $i = sfl + e$ and $j = tfl + f$ with $t = s - 1$ and $e, f \neq 1$. Then we have

$$\begin{aligned} x_{sfl+e}(x_1 \cdots x_p)^{sfi} x_{e+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{e-1+p-fl} \\ \overset{(2)}{=} (x_1 \cdots x_p)^{sfi} x_e x_{e+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{e-1+p-fl} \\ \overset{(1)}{=} (x_1 \cdots x_p)^{(s+1)fi+1} = (x_1 \cdots x_p)^{(t+2)fi+1} \\ \overset{(1)}{=} (x_1 \cdots x_p)^{tfi} x_f x_{f+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{f-1+p-fl} \\ \overset{(2)}{=} x_{tfl+f} (x_1 \cdots x_p)^{tfi} x_{f+1} \cdots x_p (x_1 \cdots x_p)^{fi} x_{1+p-fl} \cdots x_{f-1+p-fl}. \end{aligned}$$

This provides the expression of $x_{sfl+e} \backslash x_{tfl+f}$ for either $t = s - 1$ or $s = t - 1$ and $e, f \neq 1$.

Case 6. Assume $i = sfl + e$ and $j = sfl + f$ with $1 \neq e \neq f \neq 1$. Then the computations are similar with the previous ones. This completes the proof. □

3. Torus link groups are Garside groups

The main result of this section is that all torus link groups are groups of fractions of Garside monoids, which, as a consequence, provides explicit automatic structures for each of them.

Proposition 3.1. Let $2 \bullet p \bullet q$. The monoid $A_{p,q}$ is a Garside monoid with Garside element $\Delta = (x_1 \cdots x_p)^{\frac{q}{p \wedge q}}$.

Proof. For $p, 2$ and $k, 1$, $A_{p,pk}$ is a Garside monoid (see [17, Example 5] [29, Chapter VI]) with Garside element $(x_1 \cdots x_p)^k$.

From now on, we assume $2 \bullet p \bullet q$ and $p \nmid q$. We shall use the Garsidity criterion of [14] described in Section 1. First, Adjan's criterion [1] (see also [32, 23]) allows us to claim that, for $2 \bullet p \bullet q$ and $p \nmid q$, the monoid $A_{p,q}$ is cancellative; indeed, the graphs $LG(2.3)$ and $RG(2.3)$ are star graphs centered respectively in x_1 and in x_p, hence cycle-free. We define fi, fl to be respectively the integer part and the remainder of $\frac{q}{p}$. In particular, we have $q = pfi + fl$ with $1 \bullet fl \bullet p - 1$.

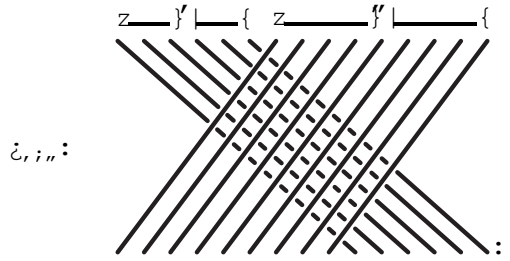
Next, in order to show that $\Delta = (x_1 \cdots x_p)^{\frac{q}{p \wedge q}}$ is a Garside element for $A_{p,q}$, we consider the relations of the presentation (2.1) for $A_{p,q}$, which can be rewritten as

$$\begin{aligned} x_i (x_1 \cdots x_p)^{\beta+1} &= (x_1 \cdots x_p)^{\beta+1} x_{i+p-\ell} && \text{for } 1 \leq i \leq \ell, \\ x_i (x_1 \cdots x_p)^{\beta} &= (x_1 \cdots x_p)^{\beta} x_{i-\ell} && \text{for } \ell+1 \leq i \leq p. \end{aligned}$$

Let $[z]$ denote the integer part of z. The map φ of $\{1,\dots,x_p\}$ defined by

$$\varphi(i) = p[\frac{q-i}{p}] + p - q + i = \begin{cases} i+p-\ell & \text{for } 1 \leq i \leq \ell, \\ i-\ell & \text{for } \ell+1 \leq i \leq p, \end{cases}$$

is a permutation of S_p with order $\frac{p}{p \wedge q}$. More precisely, let us consider the braid $\sigma_{\gamma,\delta}$ defined as the positive $\gamma+\varepsilon-$strand braid where the ε strands initially at positions $\gamma+1$ to $\gamma+\varepsilon$ cross over the first γ strands:



The minimal integer $k > 0$ for $\sigma_{\gamma,\delta}^k$ to be a pure braid is $k = \frac{\gamma+\varepsilon}{\gamma \wedge \varepsilon}$, and, moreover, in the pure braid $\sigma_{\gamma,\delta}^k$, every strand crosses over (resp. crosses under) the others $\frac{\varepsilon}{\gamma \wedge \varepsilon}$ times (resp. $\frac{\gamma}{\gamma \wedge \varepsilon}$ times). As φ is the permutation associated with $\sigma_{\ell,p-\ell}$, we deduce that the minimal integer $h > 0$ such that $x_i (x_1 \cdots x_p)^h = (x_1 \cdots x_p)^h x_i$ holds for every $1 \leq i \leq p$ is

$$h = \frac{\ell}{\ell \wedge (p-\ell)}(\beta+1) + \frac{p-\ell}{\ell \wedge (p-\ell)}\beta = \frac{q}{p \wedge q};$$

which proves that $\Delta = (x_1 \cdots x_p)^{\frac{q}{p \wedge q}}$ is a Garside element for $A_{p,q}$.

Finally, the presentation of Proposition 2.2 is normed, as the relations preserve the length. So, we are left with checking the local coherence, i.e., with checking that

$$(x_i \backslash x_j) \backslash (x_i \backslash x_k) = (x_j \backslash x_i) \backslash (x_j \backslash x_k) = \varepsilon$$

holds for $1 \leq i \neq j \neq k \leq p$. The verification is easy but somehow tedious: the decomposition into some 133 subcases and the systematic verification are given in Appendix. □

Gathering Proposition 3.9 and 3.1, we obtain:

Corollary 3.2. Every torus link group admits an explicit automatic structure given by the associated torus link Artin monoid.

Example 3.3. Figure 9 displays the lattice of simples in the monoid $A_{4,6}$.

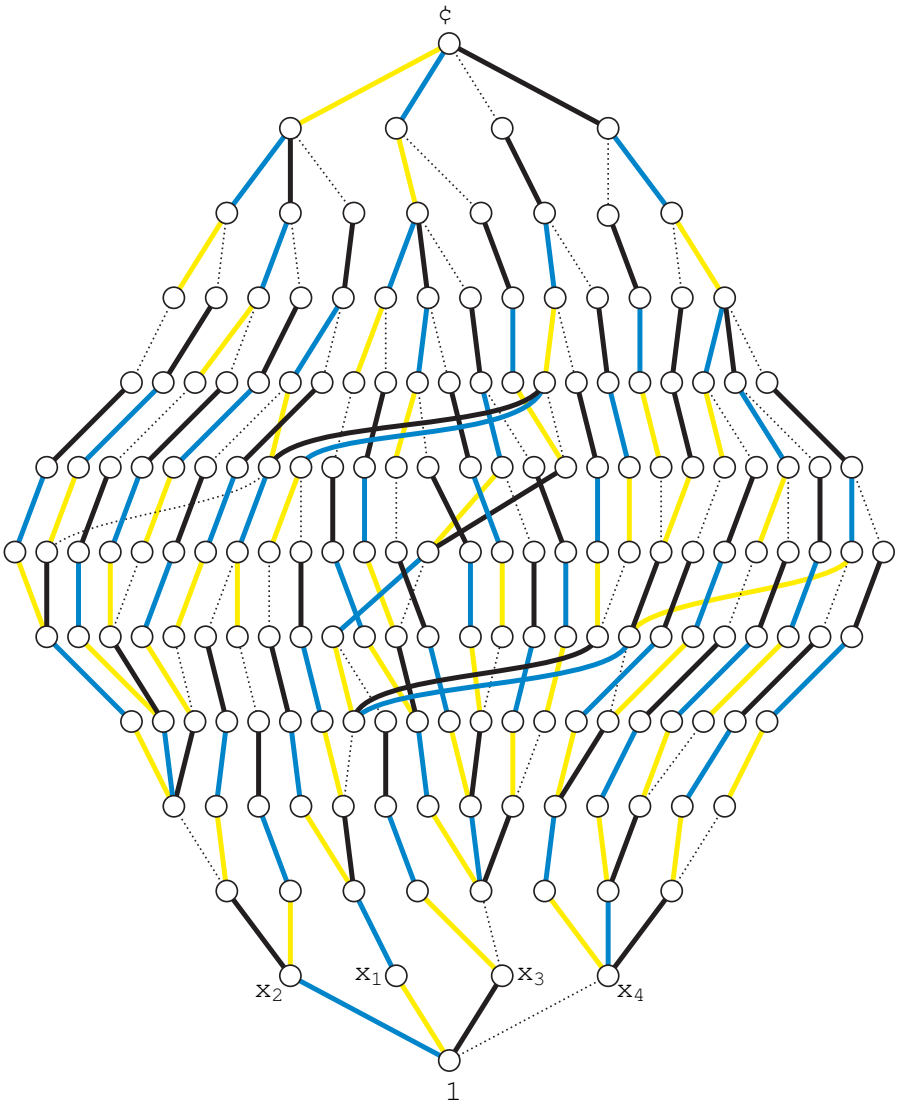


Figure 9. The lattice of simples in $A_{4,6}$.

Remark. A result of Burde-Zieschang [10, theorem 6.1] claims that a non-trivial knot whose group has a non-trivial center is a torus knot. By comparing this result with [28, Proposition 4.1], which claims that every Garside group is an iterated crossed product of Garside groups with a cyclic center generated by some power of its element ¢, we deduce

that, among all knots, only the torus knot $T(2,n)$ and the trivial knot have a group which is a Garside group.

4. Other monoids for torus link groups ?

Like every Garside group, a torus link group can be the group of fractions of several (Garside or not) monoids. For instance we know that, for $p, q > 1$ coprime, the (p,q)-torus knot group admits the (monoid) presentation $\langle x, y : x^p = y^q \rangle$, and, as already mentionned in [17], the monoid admitting this presentation is a Garside monoid. In this final section we consider other monoid presentations for torus link groups, inspired by the Wirtinger group presentations of a link group.

Definition. Let D be a regular projection of an oriented link L. We call *Wirtinger r-monoid* the monoid W_D admitting the presentation $\langle X : R \rangle$, where X is the set of Wirtinger generators, i.e. the set of all overcrossing arcs of D, and R is the set of positive relations associated with the crossings following the scheme of Figure 5.

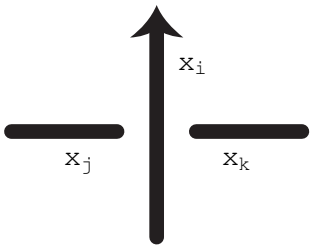


Figure 5. To this crossing corresponds the relation $x_i x_j = x_k x_i$.

Let $W_{p,q}$ denote the Wirtinger monoid associated with the diagram of the closure of the braid word $(\sigma_1 \cdots \sigma_{p-1})^q$, where the σ_i's are the Artin generators (see Section 2 and Figure 3).

It turns out that certain projections of torus links yield a Wirtinger r-monoid that is a Garside monoid, and, therefore, yield an explicit automatic structure for the associated torus link group. The point is that the automatic structure so obtained can be simpler that the general one described in the previous section.

For every $q > 1$ odd, the monoid $W_{2,q}$ is a Garside monoid, whose group of fractions is isomorphic to the type $I_2(q)$ Artin group. In particular, the group of the $(2,3)$-torus knot (i.e. the trefoil knot) is the braid group B_3.

In contradistinction, the monoids $W_{3,2}$ and $W_{4,2}$ are not Garside. The monoid $W_{3,2}$ admits the presentation

$$\langle \sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1} : \sigma_2^{-1}\sigma_1 = \sigma_1\sigma_2 = \sigma_2\sigma_2, \ \sigma_1\sigma_1^{-1} = \sigma_2\sigma_1 = \sigma_1^{-1}\sigma_2 \rangle,$$

and, in particular, the elements σ_1 and σ_2 do not admit a unique lcm (neither on the right, nor on the left). Nevertheless $W_{3,2}$ admits an element Δ whose left and right

divisors coincide and generate $W_{3,2}$, namely $\Delta = \sigma_2 \sigma_1^2$. This implies that any two elements in $W_{3,2}$ admit a right (and a left) common multiple. Moreover, adding the relations $\sigma_1 \sigma_2^2 = \sigma_2 \sigma_1^2 = \sigma_1^2 \sigma_1 = \sigma_2^2 \sigma_2$ to the previous presentation allows to obtain a complete presentation for the monoid $W_{3,2}$ which, according to a criterion of [16], implies that $W_{3,2}$ is cancellative. In particular, $W_{3,2}$ satisfies Ore's conditions, hence embeds into its group of fractions, namely B_3. Let us mention that this embedding result answers positively to a question of Han & Ko in [22] about this Sergiescu monoid for B_3. Figure 6 displays the Cayley graph of $W_{3,2}$ restricted to divisors of Δ.

The monoid $W_{4,2}$ does not embed into a group, for, in particular, it is not cancellative: indeed, $W_{4,2}$ is

$$\langle\, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \tau_1, \tau_2 : \sigma_1\sigma_2\tau_1 = \sigma_2\tau_1\sigma_2 = \sigma_2\sigma_1\tau_2,\ \sigma_1\sigma_2\sigma_1\tau_2 = \tau_1\sigma_2\sigma_1 = \sigma_2\sigma_1\tau_2,\ \sigma_2\sigma_1\sigma_2\tau_1 = \tau_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1\,\rangle,$$

and both $\sigma_1\sigma_2\sigma_1 = \tau_1\tau_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2 \neq \tau_2\tau_1\sigma_1$ hold in $W_{4,2}$.

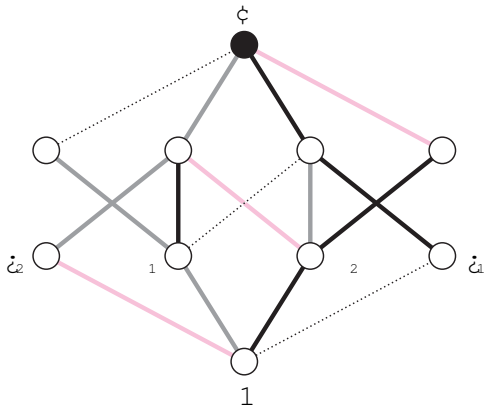


Figure 6. The monoid $W_{3,2}$ is not Garside, but it embeds into its group of fractions B_3.

The monoid $W_{3,4}$ is a Garside monoid; indeed, it is the monoid

$$\langle\, x_1, \ldots, x_8 : x_1x_7 = x_7x_2 = x_6x_1,\ x_2x_5 = x_5x_3 = x_3x_6,\ x_1x_4 = x_3x_1 = x_8x_3,\ x_4x_7 = x_5x_8 = x_7x_5 \,\rangle,$$

which admits the normed coherent complemented presentation

$$\begin{aligned}\langle\, x_1, \ldots, x_8 :\ & x_1x_4x_7 = x_2x_5x_1,\ x_1x_4 = x_3x_1,\ x_1x_4x_7 = x_4x_7x_3,\\ & x_1x_4x_7 = x_5x_1x_4,\ x_1x_7 = x_6x_1,\ x_1x_7 = x_7x_2,\ x_1x_4 = x_8x_3,\ x_2x_5 = x_3x_6,\\ & x_2x_5x_1 = x_4x_7x_3,\ x_2x_5 = x_5x_3,\ x_2x_5x_1 = x_6x_1x_5,\ x_2x_5x_1 = x_7x_2x_5,\\ & x_2x_5x_1 = x_8x_3x_7,\ x_3x_1x_7 = x_4x_7x_3,\ x_3x_6 = x_5x_3,\ x_3x_1x_7 = x_6x_1x_5,\\ & x_3x_1x_7 = x_7x_2x_5,\ x_3x_1 = x_8x_3,\ x_4x_7 = x_5x_8,\ x_4x_7x_3 = x_6x_1x_5,\\ & x_4x_7 = x_7x_5,\ x_4x_7x_3 = x_8x_3x_7,\ x_5x_1x_4 = x_6x_1x_5,\ x_5x_8 = x_7x_5,\\ & x_5x_1x_4 = x_8x_3x_7,\ x_6x_1 = x_7x_2,\ x_6x_1x_5 = x_8x_3x_7,\ x_7x_2x_5 = x_8x_3x_7 \,\rangle.\end{aligned}$$

A Garside element is $\Delta = x_1x_4x_7$, and $W_{3,4}$ is a Garside monoid, whose center is generated by the element Δ^4. The lattice of simple elements in $W_{3,4}$ is displayed on Figure 7. In contradistinction, the Artin–Tits monoid $W_{4,3}$ is not Garside.

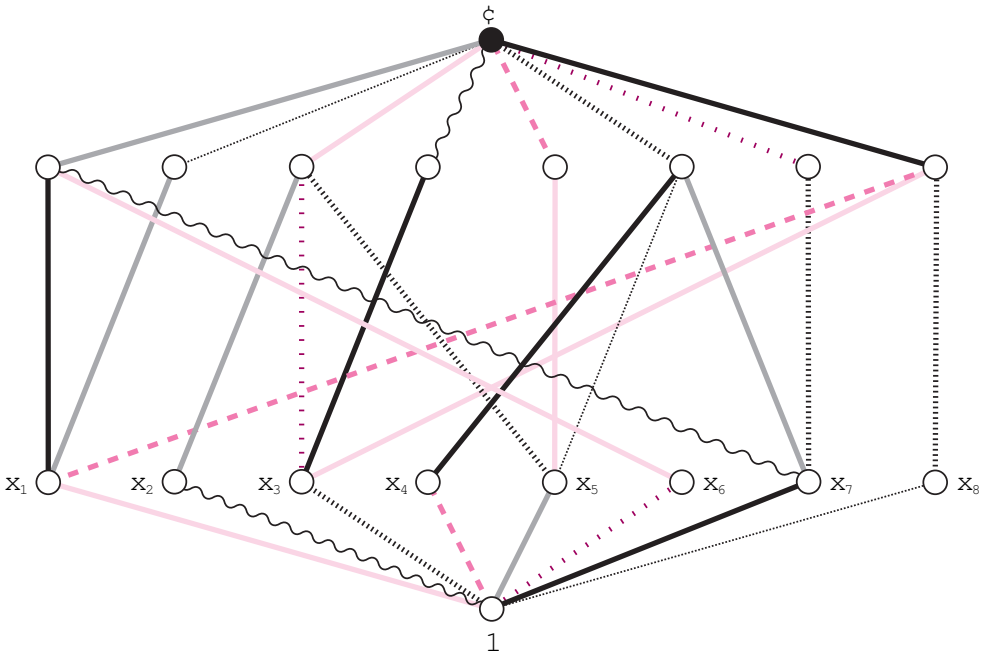


Figure 7. The lattice of simples in $W_{3,4}$.

Remark. There are many other examples of Garside monoids whose groups of fractions are isomorphic to a torus link group. For instance, the group of fractions of the monoid M_+ of Example 1.4 is isomorphic to the (3,4)-torus knot group. For every $k \geq 0$, the monoid $M_k^{(2,2)}$ presented by

$$\langle x, y : xy^2x\,¢\,(yx)^k y\,¢\,xy^2x = (yx)^k y\,¢\,xy^2x\,¢\,(yx)^k y \rangle$$

is a Garside monoid, whose Garside element is the square of the lcm of the atoms. The lattice of simples in $M_0^{(2,2)}$ is displayed in Figure 8. The groups of fractions of the monoids $M_k^{(2,2)}$ are all isomorphic to the (3,4)-torus knot group: take $z = xy^2(xy)^{k+1}$ and $t = y(xy)^{k+1}$, i.e., $x = (zt^{-1})^{k+2}t^{-1}$ and $y = t(tz^{-1})^{k+1}$.

Finally, $W_{4,6}$ is a Garside monoid: the lattice of the 68 simples in $W_{4,6}$ is displayed on Figure 9, and can be compared with the lattice of the 176 simples in $A_{4,6}$ on Figure 4.

We thus are naturally led to the two following problems:

Question 1. Among all possible Artin-Tits monoids of torus links, which ones are Garside?

Question 2. Among all possible Artin-Tits monoids of links, which ones embed into a group?

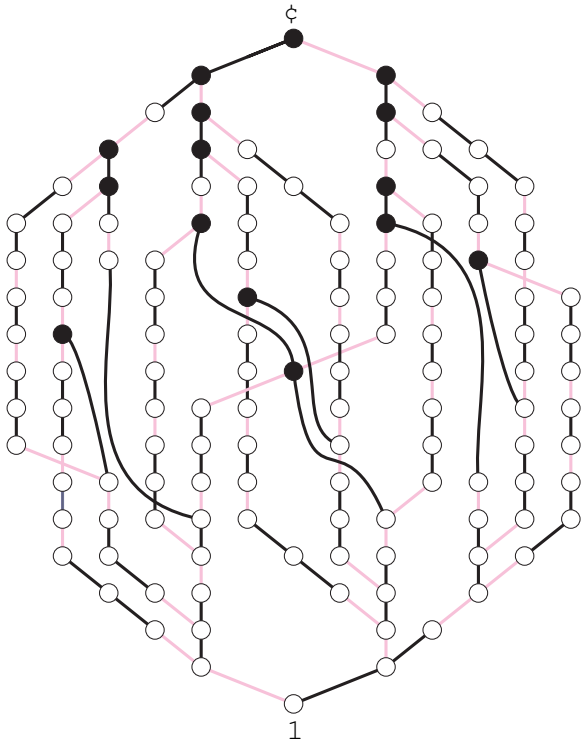


Figure 8. The lattice of simples in $\langle x, y : xyyxyxyyx = yxyyxy \rangle$.

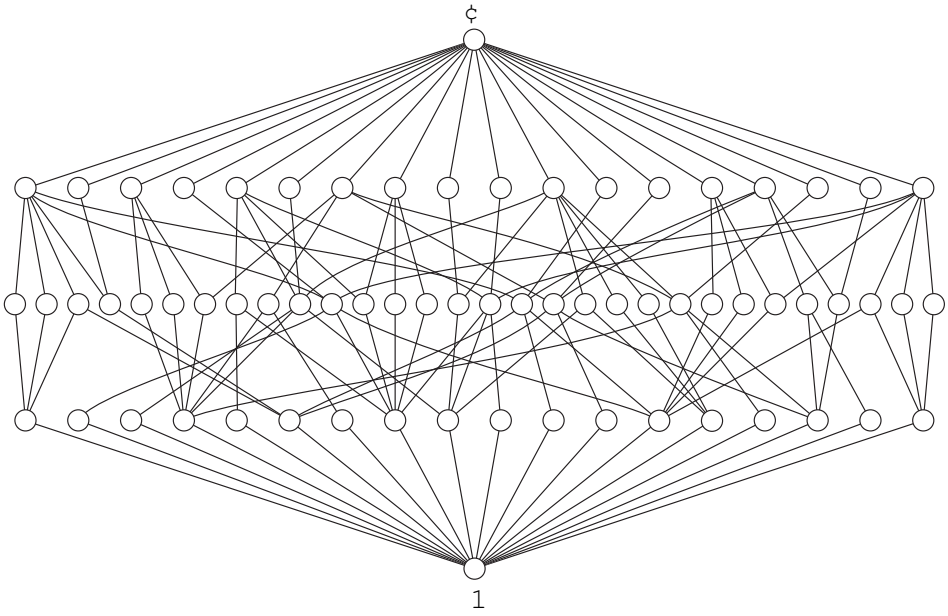


Figure 9. The (Hasse diagram of the) lattice of simples in $W_{4,6}$.

Appendix

Here is the full detail of verification of local coherence of the presentation of Proposition 2.4, completing the proof of Proposition 2.1. The notations here are the ones used in Proposition 2.2. Above all, let us observe that, from the relation

$$(x_1 \cdot\cdot\cdot x_p) \cdot (x_1 \cdot\cdot\cdot x_p)^{\varphi(i-1)} x_{(z-1)\varphi+1} = x_{z\varphi+1} \cdot (x_1 \cdot\cdot\cdot x_p)^{\varphi(i)}$$

of Presentation (2.3), we deduce

$$w_{000} \wedge x_{z\varphi+1} = w_{00(\varphi(i-2))} x_{(z-1)\varphi+1}$$
$$x_{z\varphi+1} \wedge w_{000} = w_{00(\varphi(i-1))}.$$

Let $i = s\varphi + e$, $j = t\varphi + f$, $k = z\varphi + g$ and $u_{ijk} = ((x_i \wedge x_j) \wedge (x_i \wedge x_k)) \wedge ((x_j \wedge x_i) \wedge (x_j \wedge x_k))$. By definition of $\wedge$, coherence always holds for triples (u,v,w) where two among u,v,w are equal: in the following, we shall assume $s = t) e \neq f, t = z) f \neq g$ and $z = s) g \neq e$.

Case 1: $e = 1, f = 1, g = 1$.
Subcase 1.1: $z < t < s$. We find

$$u_{ijk} = (w_{(t\varphi)0(\varphi(i-1))} \wedge w_{(z\varphi)0(\varphi(i-1))}) \wedge (w_{(t\varphi)1(\varphi(i-1))} x_{(s-t-1)\varphi+1} \wedge w_{(z\varphi)0(\varphi(i-1))}) = "\wedge" = ".$$

Subcase 1.2: $t < z < s$. We find

$$\begin{aligned} u_{ijk} &= (w_{(t\varphi)0(\varphi(i-1))} \wedge w_{(z\varphi)0(\varphi(i-1))}) \wedge (w_{(t\varphi)1(\varphi(i-1))} x_{(s-t-1)\varphi+1} \wedge w_{(t\varphi)1(\varphi(i-1))} x_{(z-t-1)\varphi+1}) \\ &= (x_1 \cdot\cdot\cdot x_p)^{(z-t)\varphi(i)} \wedge (x_{(s-t-1)\varphi+1} \wedge x_{(z-t-1)\varphi+1}) \\ &= (x_1 \cdot\cdot\cdot x_p)^{(z-t)\varphi(i)} \wedge w_{((z-t-1)\varphi)0(\varphi(i-1))} = ". \end{aligned}$$

Subcase 1.3: $t < s < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(t\varphi)0(\varphi(i-1))} \wedge w_{(s\varphi)1(\varphi(i-1))} x_{(z-s-1)\varphi+1}) \\ &\quad \wedge (w_{(t\varphi)1(\varphi(i-1))} x_{(s-t-1)\varphi+1} \wedge w_{(t\varphi)1(\varphi(i-1))} x_{(z-t-1)\varphi+1}) \\ &= (w_{((s-t-1)\varphi)1(\varphi(i-1))} x_{(z-s-1)\varphi+1}) \wedge (x_{(s-t-1)\varphi+1} \wedge x_{(z-t-1)\varphi+1}) = ". \end{aligned}$$

Subcase 1.4: $z < s < t$. We find

$$u_{ijk} = (w_{(s\varphi)1(\varphi(i-1))} x_{(t-s-1)\varphi+1} \wedge w_{(z\varphi)0(\varphi(i-1))}) \wedge (w_{(s\varphi)0(\varphi(i-1))} \wedge w_{(z\varphi)0(\varphi(i-1))}) = "\wedge" = ".$$

Subcase 1.5: $s < z < t$. We find

$$\begin{aligned} u_{ijk} &= (w_{(s\varphi)1(\varphi(i-1))} x_{(t-s-1)\varphi+1} \wedge w_{(s\varphi)1(\varphi(i-1))} x_{(z-s-1)\varphi+1}) \wedge (w_{(s\varphi)0(\varphi(i-1))} \wedge w_{(z\varphi)0(\varphi(i-1))}) \\ &= (x_{(t-s-1)\varphi+1} \wedge x_{(z-s-1)\varphi+1}) \wedge w_{((z-s-1)\varphi)0(\varphi(i-1))} = ". \end{aligned}$$

Subcase 1.6: $s < t < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(s\varphi)1(\varphi(i-1))} x_{(t-s-1)\varphi+1} \wedge w_{(s\varphi)1(\varphi(i-1))} x_{(z-s-1)\varphi+1}) \wedge (w_{(s\varphi)0(\varphi(i-1))} \\ &\quad \wedge w_{(t\varphi)1(\varphi(i-1))} x_{(z-t-1)\varphi+1}) \\ &= (x_{(t-s-1)\varphi+1} \wedge x_{(z-s-1)\varphi+1}) \wedge (w_{((t-s-1)\varphi)1(\varphi(i-1))} x_{(z-t-1)\varphi+1}) = ". \end{aligned}$$

Case 2: $e \neq 1, f = 1, g = 1$.

Subcase 2.1: $z < t < s$. We find

$$u_{ijk} = (w_{(tfi)0(fi-1)} \cdot w_{(zfi)0(fi-1)}) \cdot (w_{(tfi)1(fi-1)} x_{(s-t-1)fl+e} \cdot w_{(zfi)0(fi-1)}) = \varepsilon \cdot \varepsilon = \varepsilon.$$

Subcase 2.2: $t < z < s$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} \cdot w_{(zfi)0(fi-1)}) \cdot (w_{(tfi)1(fi-1)} x_{(s-t-1)fl+e} \cdot w_{(tfi)1(fi-1)} x_{(z-t-1)fl+1}) \\ &= w_{((z-t-1)fi)0(fi-1)} \cdot (x_{(s-t-1)fl+e} \cdot x_{(z-t-1)fl+1}) = \varepsilon. \end{aligned}$$

Subcase 2.3: $t < s \leq z \leq s+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} \cdot w_{(sfi)efi}) \cdot (w_{(tfi)1(fi-1)} x_{(s-t-1)fl+e} \cdot w_{(tfi)1(fi-1)} x_{(z-t-1)fl+1}) \\ &= w_{((s-t-1)fi)efi} \cdot (x_{(s-t-1)fl+e} \cdot x_{(z-t-1)fl+1}) = \varepsilon. \end{aligned}$$

Subcase 2.4: $t < s < z - 1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} \cdot w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(z-s-2)fl+1}) \\ &\quad \cdot (w_{(tfi)1(fi-1)} x_{(s-t-1)fl+e} \cdot w_{(tfi)1(fi-1)} x_{(z-t-1)fl+1}) \\ &= w_{((s-t-1)fi)efi} \cdot w_{00(fi-2)} x_{(z-s-2)fl+1} \cdot (x_{(z-s-2)fl+1} \cdot x_{(s-t-1)fl+e}) = \varepsilon. \end{aligned}$$

Subcase 2.5: $z < s \leq t \leq s+1$. We find

$$u_{ijk} = (w_{(sfi)efi} \cdot w_{(zfi)0(fi-1)}) \cdot (w_{(tfi)1(sfi-tfi+fi)} \cdot w_{(zfi)0(fi-1)}) = \varepsilon \cdot \varepsilon = \varepsilon.$$

Subcase 2.6: $t = s+1 = z+1$. We find

$$u_{ijk} = (w_{(sfi)efi} \cdot w_{(sfi)efi}) \cdot (w_{(tfi)1(sfi-tfi+fi)} \cdot w_{(zfi)0(fi-1)}) = \varepsilon \cdot \varepsilon = \varepsilon.$$

Subcase 2.7: $z = s+1 = t+1$. We find

$$u_{ijk} = (w_{(sfi)efi} \cdot w_{(sfi)efi}) \cdot (w_{(tfi)1(sfi-tfi+fi)} \cdot w_{(tfi)1(fi-1)} x_{(z-t-1)fl+1}) = \varepsilon \cdot (w_{000} \cdot x_1) = \varepsilon.$$

Subcase 2.8: $s \leq t \leq s+1 < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi} \cdot w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(z-s-2)fl+1}) \cdot (w_{(tfi)1(sfi-tfi+fi)} \cdot w_{(tfi)1(fi-1)} x_{(z-t-1)fl+1}) \\ &= \begin{cases} w_{00(fi-2)} x_{(z-s-2)fl+1} \cdot (w_{000} \cdot x_{(z-t-1)fl+1}) & \text{for } t = s \\ w_{00(fi-2)} x_{(z-s-2)fl+1} \cdot w_{00(fi-2)} x_{(z-t-1)fl+1} & \text{for } t = s+1 \end{cases} = \varepsilon. \end{aligned}$$

Subcase 2.9: $z+1 < s+1 < t$. We find

$$u_{ijk} = (w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(t-s-2)fl+1} \cdot w_{(zfi)0(fi-1)}) \cdot (w_{(sfi)0(2fi-1)} \cdot w_{(zfi)0(fi-1)}) = \varepsilon \cdot \varepsilon = \varepsilon.$$

Subcase 2.10: $s \leq z \leq s+1 < t$. We find

$$u_{ijk} = (w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(t-s-2)fl+1} \cdot w_{(sfi)efi}) \cdot (w_{(sfi)0(2fi-1)} \cdot w_{(zfi)0(fi-1)}) = \varepsilon \cdot \varepsilon = \varepsilon.$$

Subcase 2.11: $s+1 < z < t$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(t-s-2)fl+1} \cdot w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(z-s-2)fl+1}) \cdot (w_{(sfi)0(2fi-1)} \cdot w_{(zfi)0(fi-1)}) \\ &= (x_{(t-s-2)fl+1} \cdot x_{(z-s-2)fl+1}) \cdot \varepsilon = \varepsilon. \end{aligned}$$

Subcase 2.12: $s+1 < t < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(t-s-2)fl+1} \cdot w_{(sfi)efi} \cdot w_{00(fi-2)} x_{(z-s-2)fl+1}) \\ &\quad \cdot (w_{(sfi)0(2fi-1)} \cdot w_{(tfi)1(fi-1)} x_{(z-t-1)fl+1}) \\ &= (x_{(t-s-2)fl+1} \cdot x_{(z-s-2)fl+1}) \cdot w_{((t-s-2)fi)1(fi-1)} x_{(z-t-1)fl+1} = \varepsilon. \end{aligned}$$

Case 3: $e = 1, f \neq 1, g = 1$.

Subcase 3.1: $z < s < t$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(zfi)0(fi;1)}) n (w_{(sfi)0(fi;1)} nw_{(zfi)0(fi;1)}) = "n" = ":$$

Subcase 3.2: $s < z < t$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+1}) n (w_{(sfi)0(fi;1)} nw_{(zfi)0(fi;1)})$$
$$= (x_{(t;s;1)fl+f} nx_{(z;s;1)fl+1}) nw_{((z;s)fi)0(fi;1)} = ":$$

Subcase 3.3: $s < t \bullet z \bullet t+1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+1}) n (w_{(sfi)0(fi;1)} nw_{(tfi)ffi})$$
$$= (x_{(t;s;1)fl+f} nx_{(z;s;1)fl+1}) nw_{((t;s;1)fi)ffi} = ":$$

Subcase 3.4: $s < t < z-1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+1})$$
$$n (w_{(sfi)0(fi;1)} nw_{(tfi)ffi} w_{00(fi;2)} x_{(z;t;2)fl+1})$$
$$= (x_{(t;s;1)fl+f} nx_{(z;s;1)fl+1}) nw_{((t;s;1)fi)ffi} w_{00(fi;2)} x_{(z;t;2)fl+1} = ":$$

Subcase 3.5: Assume $z < t \bullet s \bullet t+1$. We find

$$u_{ijk} = (w_{(sfi)1(tfi;sfi+fi)} nw_{(zfi)0(fi;1)}) n (w_{(tfi)ffi} nw_{(zfi)0(fi;1)}) = "n" = ":$$

Subcase 3.6: Assume either $z = t+1 = s+1$ or $z = t = s-1$. We find

$$u_{ijk} = (w_{(sfi)1(tfi;sfi+fi)} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+1}) n (w_{(tfi)ffi} nw_{(tfi)ffi})$$
$$= (w_{00((t;s)fi)} nx_{(z;s;1)fl+1}) n" = ":$$

Subcase 3.7: $t \bullet s \bullet t+1 < z$. We find

$$u_{ijk} = \begin{cases} (w_{(sfi)1(tfi;sfi+fi)} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+1}) n (w_{(tfi)ffi} nw_{(tfi)ffi} w_{00(fi;2)} x_{(z;t;2)fl+1}) & \\ (x_1 \cdot\cdot\cdot x_p nx_{(z;s;1)fl+1}) nw_{00(fi;2)} x_{(z;t;2)fl+1} & \text{for } t = s \\ (w_{00(fi;2)} nx_{(z;s;1)fl+1}) nw_{00(fi;2)} x_{(z;t;2)fl+1} & \text{for } t+1 = s \end{cases} = ":$$

Subcase 3.8: $z < t < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi;1)} nw_{(zfi)0(fi;1)}) n (w_{(tfi)ffi} w_{00(fi;2)} x_{(s;t;2)fl+1} nw_{(zfi)0(fi;1)}) = "n" = ":$$

Subcase 3.9: $t \bullet z \bullet t+1 < s$. We find

$$u_{ijk} = (w_{(tfi)0(2fi;1)} nw_{(zfi)0(fi;1)}) n (w_{(tfi)ffi} w_{00(fi;2)} x_{(s;t;2)fl+1} nw_{(tfi)ffi}) = "n" = ":$$

Subcase 3.10: $t+1 < z < s$. We find

$$u_{ijk} = (w_{(tfi)0(2fi;1)} nw_{(zfi)0(fi;1)}) n (w_{(tfi)ffi} w_{00(fi;2)} x_{(s;t;2)fl+1} nw_{(tfi)ffi} w_{00(fi;2)} x_{(z;t;2)fl+1})$$
$$= w_{((z;t;2)fi)0(fi;1)} n (x_{(s;t;2)fl+1} nx_{(z;t;2)fl+1}) = ":$$

Subcase 3.11: $t+1 < s < z$. We find

$$u_{ijk} = (w_{(tfi)0(2fi;1)} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+1})$$
$$n (w_{(tfi)ffi} w_{00(fi;2)} x_{(s;t;2)fl+1} nw_{(tfi)ffi} w_{00(fi;2)} x_{(z;t;2)fl+1})$$
$$= w_{((s;t;2)fi)1(fi;1)} x_{(z;s;1)fl+1} n (x_{(s;t;2)fl+1} nx_{(z;t;2)fl+1}) = ":$$

Case 4: $e = 1, f = 1, g \neq 1$.

Subcase 4.1: $t < s < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} nw_{(sfi)1(fi-1)} x_{(z-s-1)fl+g}) \\ &\quad n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+1} nw_{(tfi)1(fi-1)} x_{(z-t-1)fl+g}) \\ &= (w_{((s-t-1)fi)1(fi-1)} x_{(z-s-1)fl+g}) n(x_{(s-t-1)fl+1} nx_{(z-t-1)fl+g}) = \varepsilon: \end{aligned}$$

Subcase 4.2: $t < z \leq s \leq z+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} nw_{(sfi)1(zfi-sfi+fi)}) n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+1} nw_{(tfi)1(fi-1)} x_{(z-t-1)fl+g}) \\ &= w_{((s-t-1)fi)1(zfi-sfi+fi)} n(x_{(s-t-1)fl+1} nx_{(z-t-1)fl+g}) = \varepsilon: \end{aligned}$$

Subcase 4.3: $t = z = s-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} nw_{(sfi)1(zfi-sfi+fi)}) n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+1} nw_{(tfi)1(zfi-tfi+fi)}) \\ &= w_{010} n(w_{(tfi)1(fi-1)} x_1 nw_{(tfi)1fi}) = w_{010} n(x_1 nx_1 \cdots x_p) = \varepsilon: \end{aligned}$$

Subcase 4.4: $t < z < s-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+1} nw_{(tfi)1(fi-1)} x_{(z-t-1)fl+g}) \\ &= w_{((z-t-1)fi)0(2fi-1)} n(x_{(s-t-1)fl+1} nx_{(z-t-1)fl+g}) = \varepsilon: \end{aligned}$$

Subcase 4.5: $z \leq t \leq z+1 < s$. We find

$$\begin{aligned} u_{ijk} &= \begin{cases} (w_{(tfi)0(fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+1} nw_{(tfi)1(zfi-tfi+fi)}) \\ \quad w_{00(fi-1)} n(x_{(s-t-1)fl+1} nw_{000}) \qquad \text{for } t = z \\ \quad w_{00(2fi-1)} n(w_{00(fi-2)} x_{(s-t-1)fl+1} n\varepsilon) \quad \text{for } t = z+1 \end{cases} \\ &= \varepsilon: \end{aligned}$$

Subcase 4.6: $z+1 < t < s$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+1} nw_{(zfi)0(2fi-1)}) \\ &= w_{((z-t-1)fi)0(2fi-1)} n\varepsilon = \varepsilon: \end{aligned}$$

Subcase 4.7: $s < t < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(fi-1)} x_{(t-s-1)fl+1} nw_{(sfi)1(fi-1)} x_{(z-s-1)fl+g}) n(w_{(sfi)0(fi-1)} nw_{(tfi)1(fi-1)} x_{(z-t-1)fl+g}) \\ &= (x_{(t-s-1)fl+1} nx_{(z-s-1)fl+g}) nw_{((t-s-1)fi)1(fi-1)} x_{(z-t-1)fl+g} = \varepsilon: \end{aligned}$$

Subcase 4.8: $s < z \leq t \leq z+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(fi-1)} x_{(t-s-1)fl+1} nw_{(sfi)1(fi-1)} x_{(z-s-1)fl+g}) n(w_{(sfi)0(fi-1)} nw_{(tfi)1(zfi-tfi+fi)}) \\ &= (x_{(t-s-1)fl+1} nx_{(z-s-1)fl+g}) nw_{((t-s-1)fi)1(zfi-tfi+fi)} = \varepsilon: \end{aligned}$$

Subcase4.9: $s+1<z+1<t$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+1}\,nw_{(sfi)1(fi;1)}x_{(z;s;1)fl+g})n(w_{(sfi)0(fi;1)}nw_{(zfi)0(2fi;1)})$$
$$= (x_{(t;s;1)fl+1}\,nx_{(z;s;1)fl+g})nw_{((z;s;1)fi)0(2fi;1)} = \text{"}:$$

Subcase4.10 $z=s=t;1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+1}\,nw_{(sfi)1(zfi;sfi+fi)})n(w_{(sfi)0(fi;1)}nw_{(tfi)1(zfi;tfi+fi)})$$
$$= (w_{(sfi)1(fi;1)}x_1 nw_{(sfi)1fi})nw_{010} = (x_1 nw_{000})nw_{010} = \text{"}:$$

Subcase4.11 $z=s<t;1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+1}\,nw_{(sfi)1(zfi;sfi+fi)})n(w_{(sfi)0(fi;1)}nw_{(zfi)0(2fi;1)})$$
$$= (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+1}\,nw_{(sfi)1fi})nw_{00(fi;1)} = (x_{(t;s;1)fl+1}\,nw_{000})nw_{00(fi;1)} = \text{"}:$$

Subcase4.12 $z=s+1<t;1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+1}\,nw_{(sfi)1(fi;1)}x_{(z;s;1)fl+g})n(w_{(sfi)0(fi;1)}nw_{(zfi)0(2fi;1)})$$
$$= (x_{(t;s;1)fl+1}\,nx_g)nw_{00(2fi;1)}) = \text{"}:$$

Subcase4.13 $z+1<s<t$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+1}\,nw_{(zfi)0(2fi;1)})n(w_{(sfi)0(fi;1)}nw_{(zfi)0(2fi;1)})$$
$$= w_{((s;z;2)fi)1(fi;1)}x_{(t;s;1)fl+1}\,n\text{"} = \text{"}:$$

Case5: $e=1, f\neq 1, g\neq 1$.
Subcase5.1: $s<z<t;1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+f}\,nw_{(sfi)1(fi;1)}x_{(z;s;1)fl+g})n(w_{(sfi)0(fi;1)}nw_{(zfi)0(2fi;1)})$$
$$= (x_{(t;s;1)fl+f}\,nx_{(z;s;1)fl+g})nw_{((z;s;1)fi)0(2fi;1)} = \text{"}:$$

Subcase5.2: $s<t\bullet z+1\bullet t+1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+f}\,nw_{(sfi)1(fi;1)}x_{(z;s;1)fl+g})n(w_{(sfi)0(fi;1)}nw_{(tfi)ffi})$$
$$= (x_{(t;s;1)fl+f}\,nx_{(z;s;1)fl+g})nw_{((t;s;1)fi)ffi} = \text{"}:$$

Subcase5.3: $s<t=z;1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+f}\,nw_{(sfi)1(fi;1)}x_{(z;s;1)fl+g})$$
$$n(w_{(sfi)0(fi;1)}nw_{(tfi)ffi}w_{00(fi;1)})$$
$$= (x_{(t;s;1)fl+f}\,nx_{(z;s;1)fl+g})nw_{((t;s;1)fi)ffi}w_{00(fi;1)} = \text{"}:$$

Subcase5.4: $s<t<z;1$. We find

$$u_{ijk} = (w_{(sfi)1(fi;1)}x_{(t;s;1)fl+f}\,nw_{(sfi)1(fi;1)}x_{(z;s;1)fl+g})$$
$$n(w_{(sfi)0(fi;1)}nw_{(tfi)ffi}w_{00(fi;2)}x_{(z;t;2)fl+g}$$
$$= (x_{(t;s;1)fl+f}\,nx_{(z;s;1)fl+g})nw_{((t;s;1)fi)ffi}w_{00(fi;2)}x_{(z;t;2)fl+g} = \text{"}:$$

Subcase 5.5: $z \bullet s \bullet z+1 < t$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(sfi)1(zfi;sfi+fi)}) n (w_{(sfi)0(fi;1)} nw_{(zfi)0(2fi;1)}) \\ &= (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(sfi)1(zfi;sfi+fi)}) n" = ": \end{aligned}$$

Subcase 5.6: $t;1 = s = z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(sfi)1(zfi;sfi+fi)}) n (w_{(sfi)0(fi;1)} nw_{(tfi)ffi}) \\ &= (x_f nw_{000}) nw_{0ffi} = ": \end{aligned}$$

Subcase 5.7: $z+1 < s < t$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(zfi)0(2fi;1)}) n (w_{(sfi)0(fi;1)} nw_{(zfi)0(2fi;1)}) \\ &= (w_{(sfi)1(fi;1)} x_{(t;s;1)fl+f} nw_{(zfi)0(2fi;1)}) n" = ": \end{aligned}$$

Subcase 5.8: $t \bullet s \bullet t+1 < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(tfi;sfi+fi)} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+g}) \\ &\quad n (w_{(tfi)ffi} nw_{(tfi)ffi} w_{00(fi;2)} x_{(z;t;2)fl+g}) \\ &= \begin{cases} w_{00(fi;2)} x_{(z;s;1)fl+g} nw_{00(fi;2)} x_{(z;t;2)fl+g} & \text{for } s = t+1 \\ (w_{000} nx_{(z;s;1)fl+g}) nw_{00(fi;2)} x_{(z;t;2)fl+g} & \text{for } s = t \end{cases} = ": \end{aligned}$$

Subcase 5.9: $s = t = z;1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(tfi;sfi+fi)} nw_{(sfi)1(fi;1)} x_{(z;s;1)fl+g}) n (w_{(tfi)ffi} nw_{(tfi)ffi} w_{00(fi;1)}) \\ &= (w_{000} nx_g) nw_{00(fi;1)} = ": \end{aligned}$$

Subcase 5.10 Assume either $s;1 \bullet t = z \bullet s$ or $s = t = z+1$. We find

$$u_{ijk} = (w_{(sfi)1(tfi;sfi+fi)} nw_{(sfi)1(zfi;sfi+fi)}) n (w_{(tfi)ffi} nw_{(tfi)ffi}) = ":$$

Subcase 5.11: $t+1 = z = s$. We find

$$u_{ijk} = (w_{(sfi)1(tfi;sfi+fi)} nw_{(sfi)1(zfi;sfi+fi)}) n (w_{(tfi)ffi} nw_{(tfi)ffi} w_{00(fi;1)}) = w_{00(fi;1)} nw_{00(fi;1)} = ":$$

Subcase 5.12 $z+1 = t = s;1$. We find

$$u_{ijk} = (w_{(sfi)1(tfi;sfi+fi)} nw_{(zfi)0(2fi;1)}) n (w_{(tfi)ffi} nw_{(tfi)ffi}) = ":$$

Subcase 5.13 $z+1 < t \bullet s \bullet t+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)1(tfi;sfi+fi)} nw_{(zfi)0(2fi;1)}) n (w_{(tfi)ffi} nw_{(zfi)0(2fi;1)}) \\ &= (w_{(sfi)1(tfi;sfi+fi)} nw_{(zfi)0(2fi;1)}) n" = ": \end{aligned}$$

Subcase 5.14: $t+1 < s < z$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(sfi)1(fi-1)} x_{(z-s-1)fl+g})$$
$$n(w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+1} nw_{(tfi)ffi} w_{00(fi-2)} x_{(z-t-2)fl+g})$$
$$= w_{((s-t-2)fi)1(fi-1)} x_{(z-s-1)fl+g} n(x_{(s-t-2)fl+1} nx_{(z-t-2)fl+g}) = ":$$

Subcase 5.15: $t+1 = z = s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(sfi)1(zfi-sfi+fi)}) n(w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+1} nw_{(tfi)ffi} w_{00(fi-1)})$$
$$= w_{010} n(x_1 nw_{000}) = ":$$

Subcase 5.16: $t+1 < z \le s \le z+1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(sfi)1(zfi-sfi+fi)}) n(w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+1} nw_{(tfi)ffi} w_{00(fi-2)} x_{(z-t-2)fl+g})$$
$$= w_{((s-t-2)fi)1(zfi-sfi+fi)} n(x_{(s-t-2)fl+1} nx_{(z-t-2)fl+g}) = ":$$

Subcase 5.17: $z+1 < t < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+1} nw_{(zfi)0(2fi-1)}) = "n" = ":$$

Subcase 5.18: $t-1 \le z \le t < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+1} nw_{(tfi)ffi}) = "n" = ":$$

Subcase 5.19: $t+1 = z < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+1} nw_{(tfi)ffi} w_{00(fi-1)})$$
$$= w_{00(fi-1)} n(x_{(s-t-2)fl+1} nw_{000}) = ":$$

Subcase 5.20: $t+1 < z < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+1} nw_{(tfi)ffi} w_{00(fi-2)} x_{(z-t-2)fl+g})$$
$$= w_{((z-t-2)fi)0(2fi-1)} n(x_{(s-t-2)fl+1} nx_{(z-t-2)fl+g}) = ":$$

Case 6: $e \neq 1, f = 1, g \neq 1$.

Subcase 6.1: $t < z < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+e} nw_{(tfi)1(fi-1)} x_{(z-t-1)fl+g})$$
$$= w_{((z-t-1)fi)0(2fi-1)} n(x_{(s-t-1)fl+e} nx_{(z-t-1)fl+g}) = ":$$

Subcase 6.2: $z \le t \le z+1 < s$. We find

$$u_{ijk} = \begin{cases} (w_{(tfi)0(fi-1)} nw_{(zfi)0(2fi-1)}) n(w_{(tfi)1(fi-1)} x_{(s-t-1)fl+e} nw_{(tfi)1(zfi-tfi+fi)}) & \\ w_{00(fi-1)} n(x_{(s-t-1)fl+e} nw_{000}) & \text{for } t = z \\ "n(w_{00(fi-2)} x_{(s-t-1)fl+e} n") & \text{for } t = z+1 \end{cases} = ":$$

Subcase 6.3: $z+1 < t < s$. We find

$$u_{ijk} = (w_{(tfi)0(fi;1)} nw_{(zfi)0(2fi;1)}) n (w_{(tfi)1(fi;1)} x_{(s;t;1)fl+e} nw_{(zfi)0(2fi;1)}) = "n" = ":$$

Subcase 6.4: $t < z \bullet s \bullet z+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi;1)} nw_{(sfi)efi}) n (w_{(tfi)1(fi;1)} x_{(s;t;1)fl+e} nw_{(tfi)1(fi;1)} x_{(z;t;1)fl+g}) \\ &= w_{((s;t;1)fi)efi} n (x_{(s;t;1)fl+e} nx_{(z;t;1)fl+g}) = ":" \end{aligned}$$

Subcase 6.5: $t < s = z-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi;1)} nw_{(sfi)efi} w_{00(fi;1)}) n (w_{(tfi)1(fi;1)} x_{(s;t;1)fl+e} nw_{(tfi)1(fi;1)} x_{(z;t;1)fl+g}) \\ &= w_{((s;t;1)fi)efi} w_{00(fi;1)} n (x_{(s;t;1)fl+e} nx_{(z;t;1)fl+g}) = ":" \end{aligned}$$

Subcase 6.6: $t < s < z-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(fi;1)} nw_{(sfi)efi} w_{00(fi;2)} x_{(z;s;2)fl+g}) \\ &\quad n (w_{(tfi)1(fi;1)} x_{(s;t;1)fl+e} nw_{(tfi)1(fi;1)} x_{(z;t;1)fl+g}) \\ &= w_{((s;t;1)fi)efi} w_{00(fi;2)} x_{(z;s;2)fl+g} n (x_{(s;t;1)fl+e} nx_{(z;t;1)fl+g}) = ":" \end{aligned}$$

Subcase 6.7: $z+1 < s \bullet t \bullet s+1$. We find

$$u_{ijk} = (w_{(sfi)efi} nw_{(zfi)0(2fi;1)}) n (w_{(tfi)1(sfi;tfi+fi)} nw_{(zfi)0(2fi;1)}) = "n" = ":"$$

Subcase 6.8: Assume either $z+1 = s = t$ or $z = s = t$ or $z = s = t-1$. We find

$$u_{ijk} = (w_{(sfi)efi} nw_{(sfi)efi}) n (w_{(tfi)1(sfi;tfi+fi)} nw_{(tfi)1(zfi;tfi+fi)}) = "n" = ":"$$

Subcase 6.9: $z+1 = s = t-1$. We find

$$u_{ijk} = (w_{(sfi)efi} nw_{(sfi)efi}) n (w_{(tfi)1(sfi;tfi+fi)} nw_{(zfi)0(2fi;1)}) = "n" = ":"$$

Subcase 6.10: $s = t = z-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi} nw_{(sfi)efi} w_{00(fi;1)}) n (w_{(tfi)1(sfi;tfi+fi)} nw_{(tfi)1(fi;1)} x_{(z;t;1)fl+g}) \\ &= w_{00(fi;1)} n (w_{000} nx_g) = ":" \end{aligned}$$

Subcase 6.11: $t = s+1 = z$. We find

$$u_{ijk} = (w_{(sfi)efi} nw_{(sfi)efi} w_{00(fi;1)}) n (w_{(tfi)1(sfi;tfi+fi)} nw_{(tfi)1(zfi;tfi+fi)}) = w_{00(fi;1)} nw_{00(fi;1)} = ":"$$

Subcase 6.12: $z+1 < s < t-1$. We find

$$u_{ijk} = (w_{(sfi)efi} w_{00(fi;2)} x_{(t;s;2)fl+1} nw_{(zfi)0(2fi;1)}) n (w_{(sfi)0(2fi;1)} nw_{(zfi)0(2fi;1)}) = "n" = ":"$$

Subcase 6.13: $s-1 \le z \le s < t-1$. We find

$$u_{ijk} = (w_{(sfi)efi}\, w_{00(fi-2)}\, x_{(t-s-2)fl+1}\, nw_{(sfi)efi})\, n\, (w_{(sfi)0(2fi-1)}\, nw_{(zfi)0(2fi-1)}) = \text{"n"} = \text{":}$$

Subcase 6.14: $s+1 = z = t-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi-2)}\, x_{(t-s-2)fl+1}\, nw_{(sfi)efi}\, w_{00(fi-1)})\, n\, (w_{(sfi)0(2fi-1)}\, nw_{(tfi)1(zfi-tfi+fi)}) \\ &= (x_1\, nw_{000})\, nw_{010} = \text{":} \end{aligned}$$

Subcase 6.15: $s+1 = z < t-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi-2)}\, x_{(t-s-2)fl+1}\, nw_{(sfi)efi}\, w_{00(fi-1)})\, n\, (w_{(sfi)0(2fi-1)}\, nw_{(zfi)0(2fi-1)}) \\ &= (x_{(t-s-2)fl+1}\, nw_{000})\, nw_{00(fi-1)} = \text{":} \end{aligned}$$

Subcase 6.16: $s+1 < t < z$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi-2)}\, x_{(t-s-2)fl+1}\, nw_{(sfi)efi}\, w_{00(fi-2)}\, x_{(z-s-2)fl+g}) \\ &\quad n\, (w_{(sfi)0(2fi-1)}\, nw_{(tfi)1(fi-1)}\, x_{(z-t-1)fl+g}) \\ &= (x_{(t-s-2)fl+1}\, nx_{(z-s-2)fl+g})\, nw_{((t-s-2)fi)1(fi-1)}\, x_{(z-t-1)fl+g} = \text{":} \end{aligned}$$

Subcase 6.17: $s+1 < z \le t \le z+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi-2)}\, x_{(t-s-2)fl+1}\, nw_{(sfi)efi}\, w_{00(fi-2)}\, x_{(z-s-2)fl+g})\, n\, (w_{(sfi)0(2fi-1)}\, nw_{(tfi)1(zfi-tfi+fi)}) \\ &= (x_{(t-s-2)fl+1}\, nx_{(z-s-2)fl+g})\, nw_{((t-s-2)fi)1(zfi-tfi+fi)} = \text{":} \end{aligned}$$

Subcase 6.18: $s+2 < z+1 < t$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi-2)}\, x_{(t-s-2)fl+1}\, nw_{(sfi)efi}\, w_{00(fi-2)}\, x_{(z-s-2)fl+g})\, n\, (w_{(sfi)0(2fi-1)}\, nw_{(zfi)0(2fi-1)}) \\ &= (x_{(t-s-2)fl+1}\, nx_{(z-s-2)fl+g})\, nw_{((z-s-2)fi)0(2fi-1)}) = \text{":} \end{aligned}$$

Case 7: $e \neq 1, f \neq 1, g = 1$.

Subcase 7.1: $z < t < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)}\, nw_{(zfi)0(fi-1)})\, n\, (w_{(tfi)ffi}\, w_{00(fi-2)}\, x_{(s-t-2)fl+e}\, nw_{(zfi)0(fi-1)}) = \text{"n"} = \text{":}$$

Subcase 7.2: $t < s-1 < z \le s+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(2fi-1)}\, nw_{(sfi)efi})\, n\, (w_{(tfi)ffi}\, w_{00(fi-2)}\, x_{(s-t-2)fl+e}\, nw_{(tfi)ffi}\, w_{00(fi-2)}\, x_{(z-t-2)fl+1}) \\ &= w_{((s-t-2)fi)efi}\, n\, (x_{(s-t-2)fl+e}\, nx_{(z-t-2)fl+1}) = \text{":} \end{aligned}$$

Subcase 7.3: $t+1 < s < z-1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(tfi)0(2fi-1)}\, nw_{(sfi)efi}\, w_{00(fi-2)}\, x_{(z-s-2)fl+1}) \\ &\quad n\, (w_{(tfi)ffi}\, w_{00(fi-2)}\, x_{(s-t-2)fl+e}\, nw_{(tfi)ffi}\, w_{00(fi-2)}\, x_{(z-t-2)fl+1}) \\ &= w_{((s-t-2)fi)efi}\, w_{00(fi-2)}\, x_{(z-s-2)fl+1}\, n\, (x_{(s-t-2)fl+e}\, nx_{(z-t-2)fl+1}) \end{aligned}$$

Subcase 7.4: $z < t = s - 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} nw_{(z,i)0(i-1)}) n (w_{(t,i)(f,i)} w_{00(i-1)} nw_{(z,i)0(i-1)}) = \text{"n"} = \text{":}$$

Subcase 7.5: $z = t = s - 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} nw_{(z,i)0(i-1)}) n (w_{(t,i)(f,i)} w_{00(i-1)} nw_{(t,i)(f,i)}) = \text{"n"} = \text{":}$$

Subcase 7.6: $t + 1 = s = z$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} nw_{(s,i)(e,i)}) n (w_{(t,i)(f,i)} w_{00(i-1)} nw_{(t,i)(f,i)}) = \text{"n"} = \text{":}$$

Subcase 7.7: $t + 1 = s = z - 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} nw_{(s,i)(e,i)}) n (w_{(t,i)(f,i)} w_{00(i-1)} nw_{(t,i)(f,i)} w_{00(i-2)} x_{(z-t-2)(l+1)}) = w_{000} nx_1 = \text{":}$$

Subcase 7.8: $t + 1 = s < z - 1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(s,i)(e,i)} nw_{(s,i)(e,i)} w_{00(i-2)} x_{(z-s-2)(l+1)}) n (w_{(t,i)(f,i)} w_{00(i-1)} nw_{(t,i)(f,i)} w_{00(i-2)} x_{(z-t-2)(l+1)}) \\ &= w_{00(i-2)} x_{(z-s-2)(l+1)} n (w_{000} nx_{(z-t-2)(l+1)}) = \text{":} \end{aligned}$$

Subcase 7.9: $z < t = s$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} nw_{(z,i)0(i-1)}) n (w_{(t,i)(f,i)} nw_{(z,i)0(i-1)}) = \text{"n"} = \text{":}$$

Subcase 7.10: $s = t \bullet z \bullet t + 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} nw_{(s,i)(e,i)}) n (w_{(t,i)(f,i)} nw_{(t,i)(f,i)}) = \text{"n"} = \text{":}$$

Subcase 7.11: $s = t < z - 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} nw_{(s,i)(e,i)} w_{00(i-2)} x_{(z-s-2)(l+1)}) n (w_{(t,i)(f,i)} nw_{(t,i)(f,i)} w_{00(i-2)} x_{(z-t-2)(l+1)}) = \text{":}$$

Subcase 7.12: $z < s = t - 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} w_{00(i-1)} nw_{(z,i)0(i-1)}) n (w_{(t,i)(f,i)} nw_{(z,i)0(i-1)}) = \text{"n"} = \text{":}$$

Subcase 7.13: $s = z = t - 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} w_{00(i-1)} nw_{(s,i)(e,i)}) n (w_{(t,i)(f,i)} nw_{(z,i)0(i-1)}) = \text{"n"} = \text{":}$$

Subcase 7.14: $s + 1 = t = z$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} w_{00(i-1)} nw_{(s,i)(e,i)}) n (w_{(t,i)(f,i)} nw_{(t,i)(f,i)}) = \text{"n"} = \text{":}$$

Subcase 7.15: $s + 1 = t = z - 1$. We find

$$u_{ijk} = (w_{(s,i)(e,i)} w_{00(i-1)} nw_{(s,i)(e,i)} w_{00(i-2)} x_{(z-s-2)(l+1)}) n (w_{(t,i)(f,i)} nw_{(t,i)(f,i)}) = w_{000} nx_1 = \text{":}$$

Subcase 7.16: $s+1 = t < z-1$. We find

$$u_{ijk} = (w_{(sfi)efi} w_{00(fi-1)} nw_{(sfi)efi} w_{00(fi-2)} x_{(z-s-2)fl+1}) n (w_{(tfi)ffi} nw_{(tfi)ffi} w_{00(fi-2)} x_{(z-t-2)fl+1})$$
$$= (w_{000} nx_{(z-s-2)fl+1}) nw_{00(fi-2)} x_{(z-t-2)fl+1} = "."$$

Subcase 7.17: $z < s < t-1$. We find

$$u_{ijk} = (w_{(sfi)efi} w_{00(fi-2)} x_{(t-s-2)fl+f} nw_{(zfi)0(fi-1)}) n (w_{(sfi)0(2fi-1)} nw_{(zfi)0(fi-1)}) = "n" = "."$$

Subcase 7.18: $s \bullet z \bullet s+1 < t$. We find

$$u_{ijk} = (w_{(sfi)efi} w_{00(fi-2)} x_{(t-s-2)fl+f} nw_{(sfi)efi}) n (w_{(sfi)0(2fi-1)} nw_{(zfi)0(fi-1)}) = "n" = "."$$

Subcase 7.19: $s+1 < z < t$. We find

$$u_{ijk} = (w_{(sfi)efi} w_{00(fi-2)} x_{(t-s-2)fl+f} nw_{(sfi)efi} w_{00(fi-2)} x_{(z-s-2)fl+1}) n (w_{(sfi)0(2fi-1)} nw_{(zfi)0(fi-1)})$$
$$= (x_{(t-s-2)fl+f} nx_{(z-s-2)fl+1}) nw_{((z-s-2)fi)0(fi-1)} = "."$$

Subcase 7.20: $s+1 < t \bullet z \bullet t+1$. We find

$$u_{ijk} = (w_{(sfi)efi} w_{00(fi-2)} x_{(t-s-2)fl+f} nw_{(sfi)efi} w_{00(fi-2)} x_{(z-s-2)fl+1}) n (w_{(sfi)0(2fi-1)} nw_{(tfi)ffi})$$
$$= (x_{(t-s-2)fl+f} nx_{(z-s-2)fl+1}) nw_{((t-s-2)fi)ffi} = "."$$

Subcase 7.21: $s+1 < t < z-1$. We find

$$u_{ijk} = (w_{(sfi)efi} w_{00(fi-2)} x_{(t-s-2)fl+f} nw_{(sfi)efi} w_{00(fi-2)} x_{(z-s-2)fl+1})$$
$$n (w_{(sfi)0(2fi-1)} nw_{(tfi)ffi} w_{00(fi-2)} x_{(z-t-2)fl+1})$$
$$= (x_{(t-s-2)fl+f} nx_{(z-s-2)fl+1}) nw_{((t-s-2)fi)ffi} w_{00(fi-2)} x_{(z-t-2)fl+1} = "."$$

Case 8: $e \neq 1$, $f \neq 1$, $g \neq 1$.
Subcase 8.1: $z+1 < t < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n (w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+e} nw_{(zfi)0(2fi-1)}) = "n" = "."$$

Subcase 8.2: $t-1 \bullet z \bullet t < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n (w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+e} nw_{(tfi)ffi}) = "n" = "."$$

Subcase 8.3: $z-1 = t < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n (w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+e} nw_{(tfi)ffi} w_{00(fi-1)})$$
$$= w_{00(fi-1)} n (x_{(s-t-2)fl+e} nw_{000}) = "."$$

Subcase 8.4: $t+1 < z < s-1$. We find

$$u_{ijk} = (w_{(tfi)0(2fi-1)} nw_{(zfi)0(2fi-1)}) n (w_{(tfi)ffi} w_{00(fi-2)} x_{(s-t-2)fl+e} nw_{(tfi)ffi} w_{00(fi-2)} x_{(z-t-2)fl+g})$$
$$= w_{((z-t-2)fi)0(2fi-1)} n (x_{(s-t-2)fl+e} nx_{(z-t-2)fl+g}) = "."$$

Subcase 8.5: $t+1 < z \le s \le z+1$. We find

$$u_{ijk} = (w_{(t)0(2)}nw_{(s)(e)})n(w_{(t)(f)}w_{00(i;2)}x_{(s;t;2)\{+e}nw_{(t)(f)}w_{00(i;2)}x_{(z;t;2)\{+g})$$
$$= w_{((s;t;2))(e)}n(x_{(s;t;2)\{+e}nx_{(z;t;2)\{+g}) = \varepsilon:$$

Subcase 8.6: $t+1 = z = s-1$. We find

$$u_{ijk} = (w_{(t)0(2)}nw_{(s)(e)})n(w_{(t)(f)}w_{00(i;2)}x_{(s;t;2)\{+e}nw_{(t)(f)}w_{00(i;1)})$$
$$= w_{0(e)}n(x_e nw_{000}) = \varepsilon:$$

Subcase 8.7: $t+1 < s = z-1$. We find

$$u_{ijk} = (w_{(t)0(2)}nw_{(s)(e)}w_{00(i;1)})$$
$$n(w_{(t)(f)}w_{00(i;2)}x_{(s;t;2)\{+e}nw_{(t)(f)}w_{00(i;2)}x_{(z;t;2)\{+g})$$
$$= w_{((s;t;2))(e)}w_{00(i;1)}n(x_{(s;t;2)\{+e}nx_{(z;t;2)\{+g}) = \varepsilon:$$

Subcase 8.8: $t+1 < s < z-1$. We find

$$u_{ijk} = (w_{(t)0(2)}nw_{(s)(e)}w_{00(i;2)}x_{(z;s;2)\{+g})$$
$$n(w_{(t)(f)}w_{00(i;2)}x_{(s;t;2)\{+e}nw_{(t)(f)}w_{00(i;2)}x_{(z;t;2)\{+g})$$
$$= w_{((s;t;2))(e)}w_{00(i;2)}x_{(z;s;2)\{+g}n(x_{(s;t;2)\{+e}nx_{(z;t;2)\{+g}) = \varepsilon:$$

Subcase 8.9: $z+1 < t = s-1$. We find

$$u_{ijk} = (w_{(s)(e)}nw_{(z)0(2)})n(w_{(t)(f)}w_{00(i;1)}nw_{(z)0(2)}) = \varepsilon n\varepsilon = \varepsilon:$$

Subcase 8.10: $z+1 = t = s-1$. We find

$$u_{ijk} = (w_{(s)(e)}nw_{(z)0(2)})n(w_{(t)(f)}w_{00(i;1)}nw_{(t)(f)}) = \varepsilon n\varepsilon = \varepsilon:$$

Subcase 8.11: $z = t = s-1$. We find

$$u_{ijk} = (w_{(s)(e)}nw_{(s)(e)})n(w_{(t)(f)}w_{00(i;1)}nw_{(t)(f)}) = \varepsilon n\varepsilon = \varepsilon:$$

Subcase 8.12: $z-1 = t = s-1$. We find

$$u_{ijk} = (w_{(s)(e)}nw_{(s)(e)})n(w_{(t)(f)}w_{00(i;1)}nw_{(t)(f)}w_{00(i;1)}) = \varepsilon n\varepsilon = \varepsilon:$$

Subcase 8.13: $t+1 = s = z-1$. We find

$$u_{ijk} = (w_{(s)(e)}nw_{(s)(e)}w_{00(i;1)})n(w_{(t)(f)}w_{00(i;1)}nw_{(t)(f)}w_{00(i;2)}x_{(z;t;2)\{+g})$$
$$= w_{00(i;1)}n(w_{000}nx_g) = \varepsilon:$$

Subcase 8.14: $t+1 = s < z-1$. We find

$$u_{ijk} = (w_{(s)(e)}nw_{(s)(e)}w_{00(i;2)}x_{(z;s;2)\{+g})n(w_{(t)(f)}w_{00(i;1)}nw_{(t)(f)}w_{00(i;2)}x_{(z;t;2)\{+g})$$
$$= w_{00(i;2)}x_{(z;s;2)\{+g}n(w_{000}nx_{(z;t;2)\{+g}) = \varepsilon:$$

Subcase 8.15 $z+1<s=t$. We find

$$u_{ijk}=(w_{(sfi)efi}\,nw_{(zfi)0(2fi-1)})n(w_{(tfi)ffi}\,nw_{(zfi)0(2fi-1)})="n"="".$$

Subcase 8.16 $z\bullet s=t\bullet z+1$. We find

$$u_{ijk}=(w_{(sfi)efi}\,nw_{(sfi)efi})n(w_{(tfi)ffi}\,nw_{(tfi)ffi})="n"="".$$

Subcase 8.17 $s+1=t+1=z$. We find

$$u_{ijk}=(w_{(sfi)efi}\,nw_{(sfi)efi}\,w_{00(fi-1)})n(w_{(tfi)ffi}\,nw_{(tfi)ffi}\,w_{00(fi-1)})=w_{00(fi-1)}nw_{00(fi-1)}="".$$

Subcase 8.18 $s+1=t+1<z$. We find

$$u_{ijk}=(w_{(sfi)efi}\,nw_{(sfi)efi}\,w_{00(fi-2)}x_{(z-s-2)fl+g})n(w_{(tfi)ffi}\,nw_{(tfi)ffi}\,w_{00(fi-2)}x_{(z-t-2)fl+g})="".$$

Subcase 8.19 $z+1<s=t-1$. We find

$$u_{ijk}=(w_{(sfi)efi}\,w_{00(fi-1)}nw_{(zfi)0(2fi-1)})n(w_{(tfi)ffi}\,nw_{(zfi)0(2fi-1)})="n"="".$$

Subcase 8.20 $z+1=s=t-1$. We find

$$u_{ijk}=(w_{(sfi)efi}\,w_{00(fi-1)}nw_{(sfi)efi})n(w_{(tfi)ffi}\,nw_{(zfi)0(2fi-1)})="n"="".$$

Subcase 8.21 $z=s=t-1$. We find

$$u_{ijk}=(w_{(sfi)efi}\,w_{00(fi-1)}nw_{(sfi)efi})n(w_{(tfi)ffi}\,nw_{(tfi)ffi})="n"="".$$

Subcase 8.22 $z=s+1=t$. We find

$$u_{ijk}=(w_{(sfi)efi}\,w_{00(fi-1)}nw_{(sfi)efi}\,w_{00(fi-1)})n(w_{(tfi)ffi}\,nw_{(tfi)ffi})="n"="".$$

Subcase 8.23 $s+1=t=z-1$. We find

$$\begin{aligned}u_{ijk}&=(w_{(sfi)efi}\,w_{00(fi-1)}nw_{(sfi)efi}\,w_{00(fi-2)}x_{(z-s-2)fl+g})n(w_{(tfi)ffi}\,nw_{(tfi)ffi}\,w_{00(fi-1)})\\&=(w_{000}\,nx_g)nw_{00(fi-1)}="".\end{aligned}$$

Subcase 8.24 $s+1=t<z-1$. We find

$$\begin{aligned}u_{ijk}&=(w_{(sfi)efi}\,w_{00(fi-1)}nw_{(sfi)efi}\,w_{00(fi-2)}x_{(z-s-2)fl+g})n(w_{(tfi)ffi}\,nw_{(tfi)ffi}\,w_{00(fi-2)}x_{(z-t-2)fl+g})\\&=(w_{000}\,nx_{(z-s-2)fl+g})nw_{00(fi-2)}x_{(z-t-2)fl+g}="".\end{aligned}$$

Subcase 8.25 $z+1<s<t-1$. We find

$$u_{ijk}=(w_{(sfi)efi}\,w_{00(fi-2)}x_{(t-s-2)fl+f}\,nw_{(zfi)0(2fi-1)})n(w_{(sfi)0(2fi-1)}nw_{(zfi)0(2fi-1)})="n".$$

Subcase 8.26: $s \bullet z+1 \bullet s+1 < t$. We find

$$u_{ijk} = (w_{(sfi)efi}\, w_{00(fi;2)}\, x_{(t;s;2)fl+f}\, nw_{(sfi)efi})n(w_{(sfi)0(2fi;1)}\, nw_{(zfi)0(2fi;1)}) = "n":$$

Subcase 8.27: $s+1 = z < t;\ 1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi;2)}\, x_{(t;s;2)fl+f}\, nw_{(sfi)efi}\, w_{00(fi;1)})n(w_{(sfi)0(2fi;1)}\, nw_{(zfi)0(2fi;1)}) \\ &= (x_{(t;s;2)fl+f}\, nw_{000})nw_{00(fi;1)} = ": \end{aligned}$$

Subcase 8.28: $s+1 = z = t;\ 1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi;2)}\, x_{(t;s;2)fl+f}\, nw_{(sfi)efi}\, w_{00(fi;1)})n(w_{(sfi)0(2fi;1)}\, nw_{(tfi)ffi}) \\ &= (x_f\, nw_{00(fi;1)})nw_{0ffi} = ": \end{aligned}$$

Subcase 8.29: $s+1 < z < t;\ 1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi;2)}\, x_{(t;s;2)fl+f}\, nw_{(sfi)efi}\, w_{00(fi;2)}\, x_{(z;s;2)fl+g})n(w_{(sfi)0(2fi;1)}\, nw_{(zfi)0(2fi;1)}) \\ &= (x_{(t;s;2)fl+f}\, nx_{(z;s;2)fl+g})nw_{((z;s;2)fi)0(2fi;1)} = ": \end{aligned}$$

Subcase 8.30: $s+1 < z \bullet t \bullet z+1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi;2)}\, x_{(t;s;2)fl+f}\, nw_{(sfi)efi}\, w_{00(fi;2)}\, x_{(z;s;2)fl+g})n(w_{(sfi)0(2fi;1)}\, nw_{(tfi)ffi}) \\ &= (x_{(t;s;2)fl+f}\, nx_{(z;s;2)fl+g})nw_{((t;s;2)fi)ffi} = ": \end{aligned}$$

Subcase 8.31: $s+1 < t = z;\ 1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi;2)}\, x_{(t;s;2)fl+f}\, nw_{(sfi)efi}\, w_{00(fi;2)}\, x_{(z;s;2)fl+g}) \\ &\quad n(w_{(sfi)0(2fi;1)}\, nw_{(tfi)ffi}\, w_{00(fi;1)}) \\ &= x_{(t;s;2)fl+f}\, nx_{(z;s;2)fl+g})nw_{((t;s;2)fi)ffi}\, w_{00(fi;1)} = ": \end{aligned}$$

Subcase 8.32: $s+1 < t < z;\ 1$. We find

$$\begin{aligned} u_{ijk} &= (w_{(sfi)efi}\, w_{00(fi;2)}\, x_{(t;s;2)fl+f}\, nw_{(sfi)efi}\, w_{00(fi;2)}\, x_{(z;s;2)fl+g}) \\ &\quad n(w_{(sfi)0(2fi;1)}\, nw_{(tfi)ffi}\, w_{00(fi;2)}\, x_{(z;t;2)fl+g}) \\ &= (x_{(t;s;2)fl+f}\, nx_{(z;s;2)fl+g})nw_{((t;s;2)fi)ffi}\, w_{00(fi;2)}\, x_{(z;t;2)fl+g} = ": \end{aligned}$$

This completes the proof of local coherence.

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