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Characterizing Ordered Bi-Ideals in Ordered Γ-Semigroups

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Abstract. The notion of a Γ-semigroup was introduced by Sen [8] in 1981. We can see that any semigroup can be considered as a Γ-semigroup. The aim of this article is to study the concept of (0-)minimal and maximal ordered bi-ideals in ordered Γ-semigroups, and give some characterizations of (0-)minimal and maximal ordered bi-ideals in ordered Γ-semigroups analogous to the characterizations of (0-)minimal and maximal ordered bi-ideals in ordered semigroups considered by Iampan [5].

Keywords: (0-)minimal ordered bi-ideal; maximal ordered bi-ideal; ordered Γ-semigroup.

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1. Introduction and Prerequisites

Let S be a semigroup. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$. The notion of a bi-ideal was first introduced by Good and Hughes [3] as early as 1952, and it has been widely studied. In 2008, Iampan [5] characterized the (0-)minimal and maximal bi-ideals in semigroups, and gave some characterizations of (0-)minimal and maximal bi-ideals in semigroups. Let S be an ordered semigroup. A subsemigroup B of S is called an *ordered bi-ideal* of S if $BSB \subseteq B$, and for any $a \in S$ and $b \in B$, $a \leq b$ implies $a \in B$. In 2008, Iampan [5] characterized the (0-)minimal and maximal ordered bi-ideals in ordered semigroups, and gave some characterizations of (0-)minimal

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and maximal ordered bi-ideals in ordered semigroups. In 2009, Chinram and Tinpun [1] studied some properties of bi-ideals and minimal bi-ideals in ordered Γ-semigroups.

The concept of a (ordered) bi-ideal is a very interesting and important thing in (ordered) semigroups. Now we also characterize the (0-)minimal and maximal ordered bi-ideals in ordered Γ-semigroups, and give some characterizations of (0-)minimal and maximal ordered bi-ideals in ordered Γ-semigroups.

To present the main results we first recall some definitions which is important here.

Let M and Γ be any two nonempty sets. M is called a Γ-*semigroup* [8] if there exists a mappings $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. A nonempty subset K of M is called a *sub*-Γ-*semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. For nonempty subsets A, B of M, let $A\Gamma B := \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$. We also write $a\Gamma B, A\Gamma b$ and $a\Gamma b$ for $\{a\}\Gamma B, A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively.

Examples of Γ-semigroups can be seen in [6, 7] and [9] respectively.

The following example comes from Dixit and Dewan [2].

Example 1.1. *Let* $M = \{-i, 0, i\}$ *and* $\Gamma = M$*. Then* M *is a ordered* Γ-*semigroup under the multiplication over complex number while* M *is not a semigroup under complex number multiplication.*

A partially ordered Γ-semigroup M is called an *ordered* Γ-*semigroup* (some authors called po-Γ-semigroup) if for any $a, b, c \in M$ and $\gamma \in \Gamma$,

$$a \leq b \text{ implies } a\gamma c \leq b\gamma c \text{ and } c\gamma a \leq c\gamma b.$$

If $(M; \leq)$ is an ordered Γ-semigroup, and K is a sub-Γ-semigroup of M, then $(K; \leq)$ is an ordered Γ-semigroup. For an element a of an ordered Γ-semigroup M, define $(a] := \{t \in M \mid t \leq a\}$ and for a subset H of M, define $(H] = \bigcup_{h \in H} (h]$, that is, $(H] = \{t \in M \mid t \leq h \text{ for some } h \in H\}$, and $H \cup a := H \cup \{a\}$. We observe here that

1. $H \subseteq (H] = ((H]]$.
2. For any subsets A and B of M with $A \subseteq B$, we have $(A] \subseteq (B]$.
3. For any subsets A and B of M, we have $(A \cup B] = (A] \cup (B]$.
4. For any subsets A and B of M, we have $(A \cap B] \subseteq (A] \cap (B]$.
5. For any elements a and b of M with $a \leq b$, we have $(a\Gamma M] \subseteq (b\Gamma M]$ and $(M\Gamma a] \subseteq (M\Gamma b]$.

Examples of ordered Γ-semigroups can be seen in [6] and [10] respectively.

A nonempty subset I of an ordered Γ-semigroup M is called an *ordered ideal* of M if $M\Gamma I \subseteq I, I\Gamma M \subseteq I$, and for any $x \in I, (x] \subseteq I$. A sub-Γ-semigroup Q of an ordered Γ-semigroup M is called an *ordered quasi-ideal* of M if $(M\Gamma Q] \cap (Q\Gamma M] \subseteq Q$, and for any $x \in Q, (x] \subseteq Q$. A sub-Γ-semigroup B of an ordered Γ-semigroup M is called an *ordered bi-ideal* of M if $B\Gamma M\Gamma B \subseteq B$, and for any $x \in B, (x] \subseteq B$. Then the notion of an ordered quasi-ideal is a generalization of the notion of an ordered ideal, and the notion of an ordered bi-ideal is a generalization of the notion of an ordered quasi-ideal. The intersection of all ordered bi-ideals of a sub-Γ-semigroup K of an ordered Γ-semigroup M containing a nonempty subset A of K is called the *ordered bi-ideal of K generated by A*. For $A = \{a\}$, let $B_K(a)$ denote the ordered bi-ideal of K generated by $\{a\}$. If $K = M$, then we also write $B_M(a)$ as $B(a)$. An element a of an ordered Γ-semigroup M with at least two elements is called a *zero element* of M if $x\gamma a = a\gamma x = a$ for all $x \in M$ and $\gamma \in \Gamma$, and $a \leq x$ for all $x \in M$, and denote it by 0. If M is an ordered Γ-semigroup with zero, then every ordered bi-ideal of M containing a zero element. An ordered Γ-semigroup M without zero is called *B-simple* if it has no proper ordered bi-ideals. An ordered Γ-semigroup M with zero is called *0-B-simple* if it has no nonzero proper ordered bi-ideals and $M\Gamma M \neq \{0\}$. An ordered bi-ideal B of an ordered Γ-semigroup M without zero is called a *minimal ordered bi-ideal* of M if there is no ordered bi-ideal A of M such that $A \subset B$. Equivalently, if for any ordered bi-ideal A of M such that $A \subseteq B$, we have $A = B$. A nonzero ordered bi-ideal B of an ordered Γ-semigroup M with zero is called a *0-minimal ordered bi-ideal* of M if there is no nonzero ordered bi-ideal A of M such that $A \subset B$. Equivalently, if for any nonzero ordered bi-ideal A of M such that $A \subseteq B$, we have $A = B$. Equivalently, if for any ordered bi-ideal A of M such that $A \subset B$, we have $A = \{0\}$. A proper ordered bi-ideal B of an ordered Γ-semigroup M is called a *maximal ordered bi-ideal* of M if for any ordered bi-ideal A of M such that $B \subset A$, we have $A = M$. Equivalently, if for any proper ordered bi-ideal A of M such that $B \subseteq A$, we have $A = B$.

Our purpose in this article is fourfold.

1. To introduce the concept of a B-simple and 0-B-simple ordered Γ-semigroup.
2. To characterize the properties of ordered bi-ideals in ordered Γ-semigroups.
3. To characterize the relationship between (0-)minimal ordered bi-ideals and (0-)B-simple ordered Γ-semigroups.
4. To characterize the relationship between maximal ordered bi-ideals and (0-)B-simple ordered Γ-semigroups.

2. Lemmas

We shall assume throughout this article that M stands for an ordered Γ-semigroup. Before the characterizations of ordered bi-ideals for the main theorems, we give some auxiliary results which are necessary in what follows. The following three lemmas are also necessary for our considerations, and easy to verify.

Lemma 2.1. *For any nonempty subset A of M, $(A\Gamma M\Gamma A \cup A\Gamma A \cup A]$ is the smallest ordered bi-ideal of M containing A. Furthermore, for any $a \in M$,*

$$B(a) = (a\Gamma M\Gamma a \cup a\Gamma a \cup a].$$

Lemma 2.2. *For any $a \in M, (a\Gamma M\Gamma a]$ is an ordered bi-ideal of M.*

Lemma 2.3. *Let $\{B_i \mid i \in I\}$ be a family of ordered bi-ideals of M. Then $\bigcap_{i\in I} B_i$ is an ordered bi-ideal of M if $\bigcap_{i\in I} B_i \neq \emptyset$.*

Lemma 2.4. *If M has no zero element, then the following statements are equivalent:*

(i) *M is B-simple.*
(ii) *$(a\Gamma M\Gamma a] = M$ for all $a \in M$.*
(iii) *$B(a) = M$ for all $a \in M$.*

Proof. Since M is B-simple, it follows from Lemma (2.2) that $(a\Gamma M\Gamma a] = M$ for all $a \in M$. Therefore (i) implies (ii). By Lemma (2.1), $B(a) = (a\Gamma M\Gamma a \cup a\Gamma a \cup a] = (a\Gamma M\Gamma a] \cup (a\Gamma a \cup a] = M \cup (a\Gamma a \cup a] = M$ for all $a \in M$. Thus (ii) implies (iii). If B is an ordered bi-ideal of M, and let $a \in B$, then $M = B(a) \subseteq B \subseteq M$, so $B = M$. Hence M is B-simple, we have that (iii) implies (i). □

Lemma 2.5. *If M has a zero element, then the following statements hold:*

(i) *If M is 0-B-simple, then $B(a) = M$ for all $a \in M \setminus \{0\}$.*
(ii) *If $B(a) = M$ for all $a \in M \setminus \{0\}$, then either $M\Gamma M = \{0\}$ or M is 0-B-simple.*

Proof. (i) Assume that M is 0-B-simple. Then for all $a \in M \setminus \{0\}, B(a)$ is a nonzero ordered bi-ideal of M. Hence $B(a) = M$ for all $a \in M \setminus \{0\}$.

(ii) Assume that $B(a) = M$ for all $a \in M \setminus \{0\}$ and $M\Gamma M \neq \{0\}$. If B is a nonzero ordered bi-ideal of M, and let $a \in B \setminus \{0\}$, then $M = B(a) \subseteq B \subseteq M$, so $B = M$. Therefore M is 0-B-simple. □

Lemma 2.6. *If B is an ordered bi-ideal of M, and K is a sub-Γ-semigroup of M, then the following statements hold:*

(i) *If K is B-simple such that $K \cap B \neq \emptyset$, then $K \subseteq B$.*
(ii) *If K is 0-B-simple such that $K \setminus \{0\} \cap B \neq \emptyset$, then $K \subseteq B$.*

Proof. (i) Assume that K is B-simple such that $K \cap B \neq \emptyset$, and let $a \in K \cap B$. By Lemma (2.2), $(a\Gamma K\Gamma a]$ is an ordered bi-ideal of K. It follows that $(a\Gamma K\Gamma a] = K$. Hence $K = (a\Gamma K\Gamma a] \subseteq (B\Gamma M\Gamma B] \subseteq (B] = B$, so $K \subseteq B$.

(ii) Assume that K is 0-B-simple such that $K \setminus \{0\} \cap B \neq \emptyset$, and let $a \in K \setminus \{0\} \cap B$. By Lemma (2.1) and (2.5) (i), $K = B_K(a) = (a\Gamma K\Gamma a \cup a\Gamma a \cup a] \cap K \subseteq (a\Gamma K\Gamma a \cup a\Gamma a \cup a] \subseteq (a\Gamma M\Gamma a \cup a\Gamma a \cup a] = B(a) \subseteq B$. Hence $K \subseteq B$.

Hence the proof is completed. □

We now give the main theorem of this article as bellow.

3. Main Results

The aim of this section is to characterize the relationship between minimal ordered bi-ideals and B-simple ordered Γ-semigroups, and 0-minimal ordered bi-ideals and 0-B-simple ordered Γ-semigroups, and maximal ordered bi-ideals and the set $\mathcal{U}$ in ordered Γ-semigroups.

Theorem 3.1. *If M has no zero element, and B is an ordered bi-ideal of M, then the following statements hold:*

- (i) *B is a minimal ordered bi-ideal without zero of M if and only if B is B-simple.*
- (ii) *If B is a minimal ordered bi-ideal with zero of M, then either $B\Gamma B = \{0\}$ or B is 0-B-simple.*

Proof. (i) Assume that B is a minimal ordered bi-ideal without zero of M. Then B is a sub-Γ-semigroup of M. Now, let A be an ordered bi-ideal of B. Then $A\Gamma B\Gamma A \subseteq A$. Define $H := \{h \in A \mid h \leq a_1\gamma_1 b\gamma_2 a_2$ for some $a_1, a_2 \in A, b \in B$ and $\gamma_1, \gamma_2 \in \Gamma\}$. Then $\emptyset \neq H \subseteq A \subseteq B$. To show that H is an ordered bi-ideal of M, let $h_1, h_2 \in H, x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. Then $h_1 \leq a_1\alpha_1 b_1\alpha_1' a_1'$ and $h_2 \leq a_2\alpha_2 b_2\alpha_2' a_2'$ for some $a_1, a_1', a_2, a_2' \in A, b_1, b_2 \in B$ and $\alpha_1, \alpha_1', \alpha_2, \alpha_2' \in \Gamma$, so $h_1\gamma_1 h_2 \leq a_1\alpha_1 b_1\alpha_1' a_1'\gamma_1 a_2\alpha_2 b_2\alpha_2' a_2'$ and $h_1\gamma_1 x\gamma_2 h_2 \leq a_1\alpha_1 b_1\alpha_1' a_1'\gamma_1 x\gamma_2 a_2\alpha_2 b_2\alpha_2' a_2'$. Since $B\Gamma M\Gamma B \subseteq B, b_1\alpha_1' a_1'\gamma_1 a_2\alpha_2 b_2 \in B$ and $b_1\alpha_1' a_1'\gamma_1 x\gamma_2 a_2\alpha_2 b_2 \in B$. Since $h_1\gamma_1 h_2 \in H\Gamma H \subseteq A\Gamma A \subseteq A$, we get $h_1\gamma_1 h_2 \in H$. Thus H is a sub-Γ-semigroup of M. Since $A\Gamma B\Gamma A \subseteq A$, we have $a_1\alpha_1 b_1\alpha_1' a_1'\gamma_1 x\gamma_2 a_2\alpha_2 b_2\alpha_2' a_2' \in A$. Since $h_1\gamma_1 x\gamma_2 h_2 \in H\Gamma M\Gamma H \subseteq B\Gamma M\Gamma B \subseteq B, h_1\gamma_1 x\gamma_2 h_2 \leq a_1\alpha_1 b_1\alpha_1' a_1'\gamma_1 x\gamma_2 a_2\alpha_2 b_2\alpha_2' a_2'$, and A is an ordered bi-ideal of B, we get $h_1\gamma_1 x\gamma_2 h_2 \in A$. Hence $h_1\gamma_1 x\gamma_2 h_2 \in H$, so $H\Gamma M\Gamma H \subseteq H$. Next, let $x \in M$ and $h \in H$ be such that $x \leq h$. Then $x \leq h \leq a_1\gamma_1 b\gamma_2 a_2$ for some $a_1, a_2 \in A, b \in B$ and $\gamma_1, \gamma_2 \in \Gamma$. Since $a_1\gamma_1 b\gamma_2 a_2 \in A\Gamma B\Gamma A \subseteq A \subseteq B$, and B is an ordered bi-ideal of M, we have $x \in B$. Since A is an ordered bi-ideal of B, we have $x \in A$. Hence $x \in H$, so H is an ordered bi-ideal of M. Therefore H is an ordered bi-ideal of M. Since B is a minimal ordered bi-ideal of M, we get $H = B$. Hence $A = B$, so B is B-simple.

Conversely, assume that B is B-simple. If A is an ordered bi-ideal of M such that $A \subseteq B$, then $A \cap B \neq \emptyset$. It follows from Lemma (2.6) (i) that $B \subseteq A$. Hence $A = B$, so B is a minimal ordered bi-ideal of M.

(ii) Similar to the proof of necessary condition of statement (i).

Therefore we complete the proof of the theorem. □

Using the method of the proof of Theorem (3.1) (i) and Lemma (2.6) (ii), we have Theorem (3.2).

Theorem 3.2. *If M has a zero element, and B is a nonzero ordered bi-ideal of M, then the following statements hold:*

(i) *If B is a 0-minimal ordered bi-ideal of M, then either $A\Gamma B\Gamma A = \{0\}$ for some nonzero ordered bi-ideal A of B or B is 0-B-simple.*

(ii) *If B is 0-B-simple, then B is a 0-minimal ordered bi-ideal of M.*

Theorem 3.3. *If M has no zero element but it has a proper ordered bi-ideal, then every proper ordered bi-ideal of M is minimal if and only if the intersection of any two distinct proper ordered bi-ideals is empty.*

Proof. Assume B_1 and B_2 are two distinct proper ordered bi-ideals of M. Then B_1 and B_2 are minimal. If $B_1 \cap B_2 \neq \emptyset$, then $B_1 \cap B_2$ is an ordered bi-ideal of M by Lemma (2.3). Since B_1 and B_2 are minimal, we get $B_1 = B_2$. It is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

The converse is obvious. □

Using the method of the proof of Theorem (3.3), we have Theorem (3.4).

Theorem 3.4. *If M has a zero element and a nonzero proper ordered bi-ideal, then every nonzero proper ordered bi-ideal of M is 0-minimal if and only if the intersection of any two distinct nonzero proper ordered bi-ideals is $\{0\}$.*

The following two theorems give some characterizations of maximal ordered bi-ideals in ordered Γ-semigroups.

Theorem 3.5. *Let B be an ordered bi-ideal of M. If either $M \setminus B = \{a\}$ for some $a \in M$ or $M \setminus B \subseteq (b\Gamma M\Gamma b]$ for all $b \in M \setminus B$, then B is a maximal ordered bi-ideal of M.*

Proof. Let A be an ordered bi-ideal of M such that $B \subset A$. Then we consider the following two cases:

Case 1: $M \setminus B = \{a\}$ for some $a \in M$.

Then $M = B \cup \{a\}$. Since $B \subset A$, we have $\emptyset \neq A \setminus B \subseteq M \setminus B = \{a\}$. Thus $A \setminus B = \{a\}$ and $A = B \cup \{a\} = M$.

Case 2: $M \setminus B \subseteq (b\Gamma M\Gamma b]$ for all $b \in M \setminus B$.

If $b \in A \setminus B \subseteq M \setminus B$, then $M \setminus B \subseteq (b\Gamma M\Gamma b] \subseteq (A\Gamma M\Gamma A] \subseteq (A] = A$. Hence $M = B \cup M \setminus B \subseteq B \cup A = A \subseteq M$, so $A = M$.

Therefore we conclude that B is a maximal ordered bi-ideal of M. □

Theorem 3.6. *If B is a maximal ordered bi-ideal of M, and $B \cup B(a)$ is an ordered bi-ideal of M for all $a \in M \setminus B$, then either*

(i) *$M \setminus B \subseteq (a\Gamma a \cup a]$ and $a\Gamma a\Gamma a \subseteq B$ for some $a \in M \setminus B$, and $(b\Gamma M\Gamma b] \subseteq B$ for all $b \in M \setminus B$ or*

(ii) *$M \setminus B \subseteq B(a)$ for all $a \in M \setminus B$.*

Proof. Assume that B is a maximal ordered bi-ideal of M, and $B \cup B(a)$ is an ordered bi-ideal of M for all $a \in M \setminus B$. Then we have the following two cases:

Case 1: $(a\Gamma M\Gamma a] \subseteq B$ for some $a \in M \setminus B$.

Then $a\Gamma a\Gamma a \subseteq a\Gamma M\Gamma a \subseteq (a\Gamma M\Gamma a] \subseteq B$, so $a\Gamma a\Gamma a \subseteq B$. Since $B \cup (a\Gamma a \cup a] = B \cup (a\Gamma M\Gamma a] \cup (a\Gamma a \cup a] = B \cup (a\Gamma M\Gamma a \cup a\Gamma a \cup a] = B \cup B(a)$, so $B \cup (a\Gamma a \cup a]$ is an ordered bi-ideal of M. Since $a \in M \setminus B$, we have $B \subset B \cup (a\Gamma a \cup a]$. Thus $B \cup (a\Gamma a \cup a] = M$ because B is a maximal ordered bi-ideal of M, so $M \setminus B \subseteq (a\Gamma a \cup a]$. If $b \in M \setminus B$, then $b \in (a\Gamma a \cup a]$. If $b \leq a$, then $(b\Gamma M\Gamma b] \subseteq (a\Gamma M\Gamma a] \subseteq B$. If $b \leq a\gamma a$ for some $\gamma \in \Gamma$, then $(b\Gamma M\Gamma b] \subseteq (a\gamma a\Gamma M\Gamma a\gamma a] \subseteq (a\Gamma M\Gamma a] \subseteq B$. Hence $(b\Gamma M\Gamma b] \subseteq B$ for all $b \in M \setminus B$. In this case, the condition (i) is satisfied.

Case 2: $(a\Gamma M\Gamma a] \not\subseteq B$ for all $a \in M \setminus B$.

If $a \in M \setminus B$, then $B \subset B \cup (a\Gamma M\Gamma a] \subseteq B \cup B(a)$ by Lemma (2.1). Since $B \cup B(a)$ is an ordered bi-ideal of M, and B is a maximal ordered bi-ideal of M, we have $B \cup B(a) = M$. Hence $M \setminus B \subseteq B(a)$ for all $a \in M \setminus B$. In this case, the condition (ii) is satisfied.

Hence the proof is completed. □

For an ordered Γ-semigroup M, let $\mathcal{U}$ denote the union of all nonzero proper ordered bi-ideals of M if M has a zero element, and let $\mathcal{U}$ denote the union of all proper ordered bi-ideals of M if M has no zero element. Then it is easy to verify Lemma (3.7).

Lemma 3.7. *$M = \mathcal{U}$ if and only if $B(a) \neq M$ for all $a \in M$.*

As a consequence of Theorem (3.6) and Lemma (3.7), we obtain Theorem (3.8).

Theorem 3.8. *If M has no zero element, then one of the following four conditions is satisfied:*

(i) *$\mathcal{U}$ is not ordered bi-ideal of M.*

(ii) *$B(a) \neq M$ for all $a \in M$.*

(iii) *There exists* $a \in M$ *such that* $B(a) = M, (a\Gamma a \cup a] \nsubseteq (a\Gamma M\Gamma a]$ *and* $a\Gamma a\Gamma a \subseteq \mathcal{U}$, M *is not* B*-simple,* $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$, *and* $\mathcal{U}$ *is the unique maximal ordered bi-ideal of* M.

(iv) $M \setminus \mathcal{U} \subseteq B(a)$ *for all* $a \in M \setminus \mathcal{U}$, M *is not* B*-simple,* $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$, *and* $\mathcal{U}$ *is the unique maximal ordered bi-ideal of* M.

Proof. Assume that $\mathcal{U}$ is an ordered bi-ideal of M. Then $\mathcal{U} \neq \emptyset$. Now, we consider the following two cases:

Case 1: $\mathcal{U} = M$.

It follows from Lemma (3.7) that $B(a) \neq M$ for all $a \in M$. In this case, the condition (ii) is satisfied.

Case 2: $\mathcal{U} \neq M$.

Then M is not B-simple. To show that $\mathcal{U}$ is the unique maximal ordered bi-ideal of M, let A be an ordered bi-ideal of M such that $\mathcal{U} \subset A$. If $A \neq M$, then A is a proper ordered bi-ideal of M. Thus $A \subseteq \mathcal{U}$, so it is a contradiction. Hence $A = M$, so $\mathcal{U}$ is a maximal ordered bi-ideal of M. Next, assume that B is a maximal ordered bi-ideal of M. Then $B \subseteq \mathcal{U} \subset M$ because B is a proper ordered bi-ideal of M. Since B is a maximal ordered bi-ideal of M, we have $B = \mathcal{U}$. Hence $\mathcal{U}$ is the unique maximal ordered bi-ideal of M. Since $\mathcal{U} \neq M$, it follows from Lemma (3.7) that $B(a) = M$ for some $a \in M$. Clearly, $B(a) = M$ for all $a \in M \setminus \mathcal{U}$. Thus $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$, so $\mathcal{U} \cup B(x) = M$ is an ordered bi-ideal of M for all $x \in M \setminus \mathcal{U}$. By Theorem (3.6), we have the following two cases:

(i) $M \setminus \mathcal{U} \subseteq (a\Gamma a \cup a]$ and $a\Gamma a\Gamma a \subseteq \mathcal{U}$ for some $a \in M \setminus \mathcal{U}$, and $(b\Gamma M\Gamma b] \subseteq \mathcal{U}$ for all $b \in M \setminus \mathcal{U}$ or

(ii) $M \setminus \mathcal{U} \subseteq B(a)$ for all $a \in M \setminus \mathcal{U}$.

Assume $M \setminus \mathcal{U} \subseteq (a\Gamma a \cup a]$ and $a\Gamma a\Gamma a \subseteq \mathcal{U}$ for some $a \in M \setminus \mathcal{U}$, and $(b\Gamma M\Gamma b] \subseteq \mathcal{U}$ for all $b \in M \setminus \mathcal{U}$. If $(a\Gamma a \cup a] \subseteq (a\Gamma M\Gamma a]$, then $M = B(a) = (a\Gamma M\Gamma a \cup a\Gamma a \cup a] = (a\Gamma M\Gamma a]$ by Lemma (2.1). By hypothesis, $M = (a\Gamma M\Gamma a] \subseteq \mathcal{U}$ and so $\mathcal{U} = M$. This is a contradiction. Hence $(a\Gamma a \cup a] \nsubseteq (a\Gamma M\Gamma a]$. In this case, the condition (iii) is satisfied. Now, assume $M \setminus \mathcal{U} \subseteq B(a)$ for all $a \in M \setminus \mathcal{U}$. In this case, the condition (iv) is satisfied.

Hence the theorem is now completed. □

Using the method of the proof of Theorem (3.8), we have Theorem (3.9).

Theorem 3.9. *If* M *has a zero element and* $M\Gamma M \neq \{0\}$, *then one of the following five conditions is satisfied:*

(i) $\mathcal{U}$ *is not ordered bi-ideal of* M.

(ii) $B(a) \neq M$ *for all* $a \in M$.

(iii) $\mathcal{U} = \{0\}, M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$, *and* $\mathcal{U}$ *is the unique maximal ordered bi-ideal of* M.

(iv) *There exists* $a \in M$ *such that* $B(a) = M, (a\Gamma a \cup a] \nsubseteq (a\Gamma M\Gamma a]$ *and* $a\Gamma a\Gamma a \subseteq \mathcal{U}$, M *is not 0-B-simple,* $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$, *and* $\mathcal{U}$ *is the unique maximal ordered bi-ideal of* M.

(v) $M \setminus \mathcal{U} \subseteq B(a)$ *for all* $a \in M \setminus \mathcal{U}$, M *is not 0-B-simple,* $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$, *and* $\mathcal{U}$ *is the unique maximal ordered bi-ideal of* M.

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