THE NODE-EDGE WEIGHTED 2-EDGE CONNECTED SUBGRAPH PROBLEM: LINEAR RELAXATION, FACETS AND SEPARATION

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ABSTRACT. Let $G = (V, E)$ be an undirected *k-edge connected* graph with weights c_e on edges and w_v on nodes. The minimum 2-edge connected subgraph problem, 2ECSP for short, is to find a 2-edge connected subgraph of G, of minimum total weight. The 2ECSP generalizes the well-known Steiner 2-edge connected subgraph problem. In this paper we study the convex hull of the incidence vectors corresponding to feasible solutions of 2ECSP. First, a natural integer programming formulation is given and it is shown that its linear relaxation is not sufficient to describe the polytope associated with 2ECSP even when G is series-parallel. Then, we introduce two families of new valid inequalities and we give sufficient conditions for them to be facet-defining. Later, we concentrate on the separation problem. We find polynomial time algorithms to solve the separation of important subclasses of the introduced inequalities, concluding that the separation of the new inequalities, when G is series-parallel, is polynomially solvable.

1. Introduction

Let $G = (V, E)$ be an undirected graph. G is said to be *k-edge* (resp. *k-node*) *connected* if for any pair of nodes $i, j \in V$ there exists at least k edge-disjoint (resp. node-disjoint) paths from i to j. Associate with each edge $e \in E$ a weight c_e and with each node $v \in V$ a weight w_v. The *node-edge weighted 2-edge connected subgraph problem*, denoted by 2ECSP, consists in finding a 2-edge connected subgraph of G (not necessarily spanning all the nodes in V), whose total weight of both nodes and edges is minimized. So the graphs considered in this paper are 2-edge connected. A related problem is to find a 2-node connected subgraph of G whose total weight of both nodes and edges is minimized. This problem is discussed in Section 4, where it is shown how the results obtained for 2ECSP may be applied.

To our knowledge this problem has never been considered in the literature, although some related problems have been studied. For instance, the case where the node weights are large negative numbers for some nodes $v \in T$ (terminals) and 0 for nodes $v \in V \setminus T$, the 2ECSP reduces to the well known Steiner 2-edge connected subgraph problem (STECSP) introduced by Monma, Munson and Pulleyblank in [9]. Given a graph and a set of *terminals* $T \subset V$, the problem is to find a minimum (edge) weight 2-edge connected subgraph of G spanning T. Polyhedral characterizations of the STECSP may be found in [1], [2] and in [8] and [3] when $T = V$. Closely related problems to

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the STECSP in network design were introduced by Grötschel and Monma in [6]. Stoer [10] surveys related works.

The Steiner 2-edge connected subgraph problem, where the only costs pertain to edges, arise in the design of reliable telecommunication networks: to link (to establish edges between) centers (nodes) that are already determined, at least total cost but that assures that all phone centers (a subset of special nodes) remain connected when one link fails. The 2ECSP is a direct generalization that recognizes that centers are built with costs too, so that a more realistic goal is to minimize the total costs of establishing nodes and links.

Let Z^* be the value of the optimal solution to 2ECSP. In what follows, we fix a node $r \in V$ called the *root*. Consider the problem of finding a 2-edge connected subgraph of G containing r whose total weight, of both nodes and edges, is minimized. We will refer to this problem as the *r-2-edge connected subgraph problem* (r-2ECSP). If Z_r^* denotes the value of the optimal solution of the r-2ECSP, then clearly $Z^* = \min_{r\in V}\{Z_r^*\}$. The idea of fixing a node r was introduced in [4]. It makes it easy to deal with the connectivity of the solutions and leads to a simple formulation of the r-2ECSP as an integer linear program.

We now give some standard definitions used throughout the paper. Consider $F \subseteq E$ and $U \subseteq V$, then $(x^F, y^U) \in \mathbb{R}^{|E|+|V|}$ denotes the *incidence vector* of the subgraph (U, F) of G, i.e., $x_e^F = 1$ if $e \in F$ and 0 otherwise, and $y_v^U = 1$ if $v \in U$ and 0 otherwise. As usual, for any subset of edges (resp. nodes) $F \subseteq E$ (resp. $U \subseteq V$), $x(F) = \sum_{e\in F} x_e$ (resp. $y(U) = \sum_{v\in U} y_v$). The set $E(W)$, for $W \subseteq V$, will denote the set of edges having both end-nodes in W and the set $\delta(W)$, called a *cut*, will denote the edges having one end-node in W and the other in $V \setminus W$. Also, by abuse of notation, $\delta(v) = \delta(\{v\})$ for $v \in V$. $G(W)$ will stand for the subgraph of G induced by W and $V(F)$ the set of nodes incident to the edge set F. If $W \subset S \subseteq V$, the set of edges having one end-node in W and the other in $S \setminus W$ is called an *S-cut* and denoted by $\delta_S(W)$ (i.e., $\delta_S(W)$ is the cut defined by W in the graph $G(S)$). Finally, for any set A, denote its complement by $\bar{A}$.

With the above definitions, the r-2ECSP can be formulated as an integer programming problem:

$$\text{minimize} \sum_{e\in E} w_e x_e + \sum_{v\in V} c_v y_v$$

subject to

$$\begin{array}{lll}
x(\delta(W)) - 2y_v \geq 0 & \text{for all } W \subset V, r \in W, v \notin W, & (1)\\
x_e \leq y_v, & \text{for all } v \in V,\ e \in \delta(v), & (2)\\
x_e \geq 0 & \text{for all } e \in E, & (3)\\
y_v \leq 1 & \text{for all } v \in V, & (4)\\
x_e, y_v \in \{0,1\} & \text{for all } e \in E, v \in V. & (5)
\end{array}$$

Let r-2ECSP(G)=conv$\{(x, y) \in \mathbb{R}^{|E|+|V|} : (x, y)$ satisfies $(1) - (5)\}$ be the polytope associated with the r-2ECSP.

Consider the polytope defined by inequalities (1)-(4), called the linear relaxation of r-2ECSP(G) and denoted by $P(G)$. The projection of $P(G)$ onto the edge variables is given by

$$\left.\begin{array}{ll}
0 \leq x_e \leq 1 & \text{for all } e \in E,\\
x(\delta(W)) \geq 2x_e & \text{for all } W \subseteq V, r \in W, e \notin E(W).
\end{array}\right\} \quad (6)$$

In [2], it was shown that the above polytope is integral when G is series-parallel. One may be tempted to claim that the same holds for $P(G)$; unfortunately, the following example shows the contrary. Let $H = (V, E)$ be the series-parallel graph defined in Figure 1, where $V = \{r, v_1, v_2, v_3\}$. Let $x_e^* = \frac{1}{2}$, for all $e \in E$, $y_r^* = y_{v_3}^* = 1$ and $y_{v_1}^* = y_{v_2}^* = \frac{1}{2}$: clearly $(x^*, y^*) \in P(H)$. Moreover, (x^*, y^*) is an extreme point of $P(H)$, but it violates the following valid constraint of r-2ECSP(H):

$$y_{v_1} + y_{v_2} - x_f \geq y_{v_3}.$$

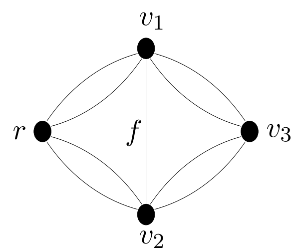

FIGURE 1. Example: the graph H.

In Section 2 we give a general form for this valid constraint. The above inequality defines, in fact, a facet of r-2ECSP(H), as will be shown in Theorem 7 (in a more general setting).

This paper studies the polytope r-2ECSP(G). First, in Section 2, we introduce a family of valid inequalities and give sufficient conditions for these inequalities to define facets of r-2ECSP(G). Section 3 shows that the separation problem associated with a subset of these inequalities is polynomially solvable. Using this result, we obtain a polynomial time algorithm for separating the inequalities in the case of series-parallel graphs. Concluding remarks are given in Section 4.

2. The polytope r-2ECSP(G)

We begin by discussing the dimension of r-2ECSP(G). Later, we introduce classes of valid inequalities and give conditions under which they define facets.

2.1. **The dimension.** Let $G = (V, E)$ be a 2-edge connected graph. Call a *2-cut* a cut containing exactly 2 edges, and let $E_{2c} = \{e \in E : e \text{ belongs to a 2-cut of } G\}$. We define the relation $\mathcal{R}$ between any two edges in E_{2c} as follows:

$$e\mathcal{R}f \iff \text{there exists a 2-cut defined by } e, f.$$

Clearly, $\mathcal{R}$ is an equivalence relation, and hence it induces a partition of $E_{2c} = E_{2c}^1 \cup E_{2c}^2 \cup \cdots \cup E_{2c}^l$ into disjoint equivalence classes. The removal of the edge set E_{2c}^i disconnects G into $|E_{2c}^i|$ 2-edge connected components, and induces a cycle when these components are seen as single nodes. Let R_i be the component containing r and let $V_{2c}^i \subseteq V \setminus R_i$ be such that the removal of any node in V_{2c}^i transforms the cycle induced by E_{2c}^i into a path (V_{2c}^i contains the endnodes of the edges in E_{2c}^i that do not belong to R_i and possibly other nodes; see Figure 2). Note that, since G is 2-edge connected: for all $i \neq j$, E_{2c}^j is included in one of the 2-edge connected components of $G = (V, E \setminus E_{2c}^i)$ and that $V_{2c}^i \cap V_{2c}^j = \emptyset$. Let $V_{2c} = \cup_{i=1}^l V_{2c}^i$.

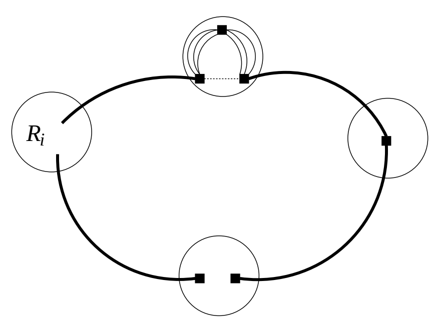

FIGURE 2. Edges of E_{2c}^i represented by bold edges. The squares are nodes in V_{2c}^i.

Lemma 1. *Given a graph $G = (V, E)$ and a fixed node $r \in V$. If $\sum_{e\in E} \alpha_e x_e + \sum_{v\in V} \beta_v y_v = \gamma$ is a valid equality of r-2ECSP(G) then:*

- $\gamma = 0$,
- $\alpha_e = 0$ *for all* $e \notin E_{2c}$, *and*
- $\beta_v = 0$ *for all* $v \notin V_{2c}$.

Proof. Since the zero vector is a feasible solution it follows that $\gamma = 0$. The equation $\beta_r = 0$ is obvious since r itself constitutes an r-2-edge connected subgraph.

Since G and $G \setminus \{f\}$, for all $e \notin E_{2c}$, are 2-edge connected it follows that $\alpha_e = 0$ for all $e \notin E_{2c}$. Hence $\sum_{e\in E_{2c}} \alpha_e + \sum_{v\in V} \beta_v = 0$.

Let $w \in V \setminus V_{2c}$, $w \neq r$. If G-w is 2−edge connected, then also $\sum_{e\in E_{2c}} \alpha_e + \sum_{v\in V\setminus\{w\}} \beta_v = 0$, implying $\beta_w = 0$. So suppose the contrary. Let S be a connected component of G-w containing r (S may consist of all the nodes of G-w). Remark that $|(S, \{w\})| \geq 3$, otherwise $w \in V_{2c}$. S may be partitioned into $S_1, S_2, \ldots, S_p$, where each $G(S_i)$ is a maximal 2-edge connected subgraph of $G(S)$; that is, if $G(W)$ is 2-edge connected for $W \subset S$, then $S_i \not\subset W$.

Let $r \in S_1$ and $T(S)$ be the graph obtained from $G(S)$ by shrinking the components S_i, $i = 1, \ldots, p$, and replacing them by nodes s_i. $T(S)$ is connected and by the maximality of each $G(S_i)$ it contains no cycles. So $T(S)$ is a tree with no edges in E^i_{2c}, $i = 1, \ldots, l$, otherwise the removal of w will transform the cycle induced by E^i_{2c} into a path. We conclude that $T(S)+w$ is 3-edge connected, hence there exists three edge-disjoint paths P_1, P_2 and P_3 in $T(S)+w$ from s_1 to w. These paths are also node-disjoint since $T(S)$ is a tree. Denote the nodes of each path P_i, by $\{s_1, s_{i_1}, \ldots, s_{i_{k_i}}, w\}$ and let $V_i = \{S_1, S_{i_1}, \ldots, S_{i_{k_i}}, w\}$ for $i = 1, \ldots, 3$. The following subgraphs of G are r-2-edge connected: $G(V_i \cup V_j)$, $i, j = 1, 2, 3$ and $i \neq j$. These graphs have in common only the nodes S_1 and w. This yields the equations:

$$\sum_{e\in E(V_i\cup V_j)\cap E_{2c}} \alpha_e + \sum_{v\in (V_i\cup V_j)} \beta_v = 0 \text{ for all } i, j = 1, 2, 3 \text{ and } i \neq j. \tag{7}$$

Also, $G(V_1 \cup V_2 \cup V_3)$ is r-2-edge connected, so

$$\sum_{e\in E(V_1\cup V_2\cup V_3)\cap E_{2c}} \alpha_e + \sum_{v\in (V_1\cup V_2\cup V_3)} \beta_v = 0. \tag{8}$$

The sum of the equations in (7) minus 2 times the equation (8) gives

$$\sum_{e\in E(S_1)\cap E_{2c}} \alpha_e + \sum_{v\in S_1} \beta_v + \beta_w = 0,$$

and since $G(S_1)$ is r-2-edge connected, we also have $\sum_{e\in E(S_1)\cap E_{2c}} \alpha_e + \sum_{v\in S_1} \beta_v = 0$, and therefore β_w=0. □

Theorem 2. *r-2ECSP(G) is of full dimension if and only if G is 3-edge connected.*

Proof. Necessity. Suppose that G is not 3-edge connected. If G is not connected or contains a bridge then it is clear that dim(r-2ECSP(G))< $|E| + |V|$. So suppose that G contains a 2-cut $\delta(W)$; that is, $\delta(W) = \{e_1, e_2\}$. Every r-2-edge connected subgraph of G verifies $x_{e_1} - x_{e_2} = 0$. Thus dim(r-2ECSP(G)) $\leq |E| + |V| - 1$.

Sufficiency. Let G be a 3-edge connected subgraph, and suppose that dim(r-2ECSP(G))< $|E|+|V|$. Then there must exist at least one valid equality of r-2ECSP(G), and from Lemma 1 this equality is the trivial equation $0 = 0$. □

Let $G = (V, E)$ be a 2-edge connected graph where the set E_{2c} contains at least one equivalence class, E^1_{2c}. Let $G_1 = (V_1, E_1)$ be the graph induced by R_1 with an additional edge, $\bar{e}$, joining the endnodes of the two edges in E^1_{2c} incident to R_i. Let $G_2 = (V_2, E_2)$ be the graph obtained from G by shrinking R_1, and let $\bar{r}$ be the resulting node.

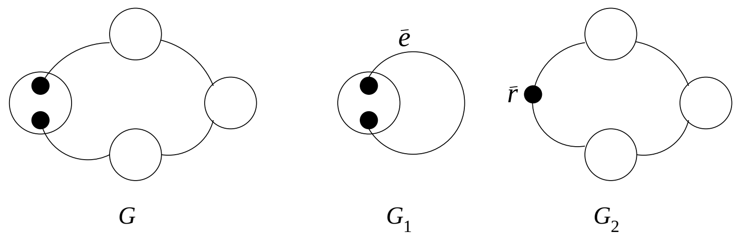

FIGURE 3. Decomposition of G into G_1 and G_2.

Lemma 3. $\dim(r\text{-}2ECSP(G)) = \dim(r\text{-}2ECSP(G_1)) + \dim(\bar{r}\text{-}2ECSP(G_2)) - 2.$

Proof. Let $\sum_{e\in E_1} \alpha^1_e x_e + \sum_{v\in V_1} \beta^1_v y_v = \gamma^1$ (resp. $\sum_{e\in E_2} \alpha^2_e x_e + \sum_{v\in V_2} \beta^2_v y_v = \gamma^2$) be an hyperplane containing r-2ECSP(G_1) (resp. $\bar{r}$-2ECSP(G_2)). Since $G(R_1)$ is 2-edge connected, then, $\sum_{e\in E_1\setminus\{\bar{e}\}} \alpha^1_e x_e + \sum_{v\in V_1} \beta^1_v y_v = \gamma^1$ and $\sum_{e\in E_2} \alpha^2_e x_e + \sum_{v\in V_2\setminus\{\bar{r}\}} \beta^2_v y_v = \gamma^2$ are hyperplanes containing r-2ECSP(G). Hence

$$\dim(r\text{-2ECSP}(G)) \le \dim(r\text{-2ECSP}(G_1)) + \dim(\bar{r}\text{-2ECSP}(G_2)) - 2.$$

Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be a collection of $\dim(r\text{-2ECSP}(G_1))$ (resp. $\dim(\bar{r}\text{-2ECSP}(G_2))$) linear independent r-2ECSP (resp. $\bar{r}$-2ECSP) subgraphs of G_1 (resp. G_2). Any graph H of $\mathcal{C}_1$ may be extended to an r-2ECSP subgraph of G by adjoining to H the edges E_2 and the nodes $V_2 \setminus \bar{r}$ if $\bar{e}$ is an edge of H, otherwise H itself is an r-2ECSP subgraph of G. Also if H is a graph of $\mathcal{C}_2$ then, one can replace $\bar{r}$ by $G(R_1)$ and obtains an r-2ECSP subgraph of G. Now it is easily seen that there exists at least $\dim(r\text{-2ECSP}(G_1)) + \dim(\bar{r}\text{-2ECSP}(G_2)) - 2$ linear independent r-2ECSP subgraphs of G. □

Theorem 4.

$$\begin{aligned}\dim(r\text{-}2ECSP(G)) &= |E| + |V| - \sum_{i=1}^{l} \left[|E^i_{2c}| + |V^i_{2c}| - 1\right] \\ &= |E \setminus E_{2c}| + |V \setminus V_{2c}| + l.\end{aligned}$$

Proof. We proceed by induction on the number of equivalence classes in E_{2c}. If $E_{2c} = \emptyset$, the result is shown by Theorem 2. Let us see the case of exactly one equivalence class (we call it E_{2c}).

It is easy to see that $\dim(r\text{-2ECSP}(G)) \le |E| + |V| - [|E_{2c}| + |V_{2c}| - 1]$. Indeed, the cycle defined by E_{2c} induces $|E_{2c}| + |V_{2c}| - 1$ linear independent hyperplanes. These are: $x(e_1) = x(e_2) = \cdots = x(e_k) = y(v_1) = y(v_2) = \cdots = y(v_l)$, where $E_{2c} = \{e_1, \ldots, e_k\}$ and $V_{2c} = \{v_1, \ldots, v_l\}$.

Let us see that $\dim(r\text{-2ECSP}(G)) \ge |E| + |V| - [|E_{2c}| + |V_{2c}| - 1]$. Let $\alpha x + \beta y = \gamma$, be an equality satisfied by all the incidence vectors of feasible solutions of r-2ECSP(G). Using Lemma 1, one may rewrite this equality as

$$\sum_{e\in E_{2c}} \alpha_e x_e + \sum_{v\in V_{2c}} \beta_v y_v = 0,$$

where $\sum_{e\in E_{2c}} \alpha_e + \sum_{v\in V_{2c}} \beta_v = 0$. Hence, the equation is implied by the hyperplanes described above.

Suppose the theorem is true for graphs with no more than m equivalence classes E_{2c} and suppose $G=(V,E)$ contains exactly $m+1$ equivalence classes of E_{2c}. W.l.o.g. let $E_{2c}^{m'+1}$ be an equivalence class such that $E(R_{m'+1})$ includes at least another equivalence class. Say that $E(R_{m'+1})$ includes m' equivalence classes. From $E_{2c}^{m'+1}$, construct the graphs G_1 and G_2 as in Lemma 3. Thus

$$\dim(r\text{-2ECSP}(G)) = \dim(r\text{-2ECSP}(G_1)) + \dim(\bar{r}\text{-2ECSP}(G_2)) - 2.$$

Notice that G_1 and G_2 contain, respectively, m' and m'' equivalence classes, such that $m'+m''=m+1$. By the induction hypothesis, we have

$$\dim(r\text{-2ECSP}(G_1)) = |E_1| + |V_1| - \sum_{i=1}^{m'} \left[|E_{2c}^i| + |V_{2c}^i| - 1\right],$$

and

$$\dim(\bar{r}\text{-2ECSP}(G_2)) = |E_2| + |V_2| - \sum_{i=m'+1}^{m+1} \left[|E_{2c}^i| + |V_{2c}^i| - 1\right],$$

where $E_{2c}^1, \dots, E_{2c}^{m+1}$ are the equivalence classes of E_{2c}.

The combination of the equalities above gives the claimed result

$$\begin{aligned}\dim(r\text{-2ECSP}(G)) &= |E_1| + |V_1| + |E_2| + |V_2| - \sum_{i=1}^{m+1} \left[|E_{2c}^i| + |V_{2c}^i| - 1\right] - 2 \\ &= |E| + |V| - \sum_{i=1}^{m+1} \left[|E_{2c}^i| + |V_{2c}^i| - 1\right].\end{aligned}$$

□

2.2. **Facet defining inequalities.** Given a graph $G=(V,E)$, a root vertex r and $r \in S \subseteq V$. If $G(\bar{S})$ is not connected, denote by $\bar{S}_1, \dots, \bar{S}_k$ the connected components of $G(\bar{S})$; with $\bar{S}_1 = \bar{S}$ when $G(\bar{S})$ is connected. Consider the following inequalities:

$$x(\delta_S(W)) + 2y(\bar{S}) - 2\sum_{i=1}^{k} x(T_i) \geq 2y_v, \tag{9}$$

$$x(\delta_S(W) \setminus \{e\}) + y(\bar{S}) - \sum_{i=1}^{k} x(T_i) \geq y_v, \tag{10}$$

where $T_i \subseteq E(\bar{S}_i)$ is a tree spanning $\bar{S}_i$, $i=1,\dots,k$, $W \subset S \subseteq V$ a proper subset of S, $v \in S \setminus W$ and $r \in W$. In inequalities (10), also add the condition that e is any edge in $\delta_S(W)$. Clearly inequalities (9) are a generalization of inequalities (1); they are the same when $S=V$.

Lemma 5. *Given a graph $G=(V,E)$ and a root vertex r then, for all $S \subseteq V$ with $r \in S$, inequalities (9) and (10) are valid for r-2ECSP(G).*

Proof. One can prove the validity of (9) and (10) by using the fact that the incidence vector of any r-2-edge connected subgraph of G satisfies $y(\bar{S}_i) - x(T_i) \geq \max_{v\in\bar{S}_i} y_v$, for all $i=1,\dots,k$, and the structure of the r-2-edge connected subgraph of G. Indeed, let us see this for inequalities (9). If $y_v=0$, the validity is trivial. Otherwise $y_v=1$, in this case, assume $x(\delta_S(W)) < 2$. From the 2-edge connectivity of the graph, this implies that at least one node, say u, in $\bar{S}$, satisfies $y_u=1$. Then $2y(\bar{S}) - 2\sum_{i=1}^{k} x(T_i) \geq 2\max_{v\in\bar{S}} y_v \geq 2y_u = 2$, and the inequality follows.

The validity of inequalities (10) is proved similarly. Nevertheless, interestingly, inequalities (10) can be derived by combining inequalities (1)-(4), as Chvátal-Gomory cuts of rank 1. For the sake of completeness we include this proof for the case where $G(\bar{S})$ is connected (the extension to general $G(\bar{S})$ is straightforward). That is, we show that

$$x(\delta_S(W)\setminus\{e\})+y(\bar{S})-x(T)\ge y_v,$$

is valid for $r\in W\subset S\subseteq V$, $v\in S\setminus W$ and T a spanning tree in $G(\bar{S})$.

If A and B are two node sets, (A,B) denotes the set of edges having one end-node in A and the other in B. Let $e=uw\in\delta_S(W)$ with $w\in S\setminus W$. From (1),

$$x(W,\bar{S})+x(\delta_S(W))=x(\delta(W))\ge 2y_v,$$

and

$$x(W,\bar{S})+x(\delta_S(W))=x(\delta(W))\ge 2y_w,$$

it follows that $x(W,\bar{S})+x(\delta_S(W))\ge y_v+y_w$, and by combining with $y_w\ge x_e$ we obtain $x(W,\bar{S})+x(\delta_S(W)\setminus\{e\})\ge y_v$. Also, inequalities (2) yield $x(W,\bar{S})\le\sum_{u\in\bar{S}}\mathrm{dg}_W(u)y_u$, where $\mathrm{dg}_W(u)$ denotes $|(\{u\},W)|$. Hence

$$x(\delta_S(W)\setminus\{e\})+\sum_{u\in\bar{S}}\mathrm{dg}_W(u)y_u\ge y_v \tag{11}$$

To complete the proof the following definitions are needed.

- Let v_0 be a special node of $\bar{S}$ and p be the length of the longest path in T having v_0 as an endnode.
- Define $L_0=\{v_0\}$ and $L_i=\{v\in\bar{S}:\exists\ u\in L_{i-1}\text{ with } e=uv\in T\}$, for $i=1,\dots,p$. Note that L_i is the i−th level of T when rooted at v_0.
- Let $v\in L_i$, $i\neq 0$, then the *father* of v, f_v, is the neighbor of v in L_{i-1} with $e=f_vv\in T$.
- Let $v\in L_i$, $i\neq p$, define $s^0(v)=\{v\}$ and $s^l(v)=\{w\in L_{i+l}:\exists\ u\in s^{l-1}(v)\text{ with } e=uw\in T\}$, for $l=1,\dots,p-i$. Let $\bar{S}_v=\cup_{l=0}^{p-i}s^l(v)$. $s^1(v)$ may be seen as the *sons* of v and $\bar{S}_v$ as the *progeny* of v.

The inequalities (2) imply

$$(|(W,\bar{S})|-|(W,\bar{S}_u)|)y_u \ \ge\ (|(W,\bar{S})|-|(W,\bar{S}_u)|)x_{f_uu}\ \text{ for all } u\in\bar{S}\setminus\{v_0\}, \tag{12}$$

$$|(W,\bar{S}_v)|y_u \ \ge\ |(W,\bar{S}_v)|x_{uv}\ \text{ for all } u\in\bar{S}\setminus L_p \text{ and } v\in s^1(u). \tag{13}$$

Summing, carefully, the inequalities (11), (12) and (13) yields

$$|(W,\bar{S})|\big(y(\bar{S})-x(T)\big)+x(\delta_S(W)\setminus\{e\})\ge y_v.$$

Hence, $|(W,\bar{S})|\big(y(\bar{S})-x(T)+x(\delta_S(W)\setminus\{e\})\big)\ge y_v$, and by dividing by $|(W,\bar{S})|$ and rounding up the result is obtained. □

For particular values of S, W, v, e and $F=\bigcup_{i=1}^{k}T_i$ (F is then a forest spanning $\bar{S}$), we will refer to (9) as (S,W,v,F) and to (10) as (S,W,v,e,F). When we write (S,W,v,T) or (S,W,v,e,T) we mean that $G(\bar{S})$ is connected and T is a spanning tree of $G(\bar{S})$. Note that when (i) $\delta_S(W)=\emptyset$ then inequalities (9) and (10) coincide, (ii) $\delta_S(W)=\{e\}$ then inequalities (9) are implied by (10) and $x_e\ge 0$, (iii) $S=V$ then inequalities (9) and (1) are the same and inequalities (10) are implied by (1) and (2).

Inequalities (9) are a generalization of the well-known *cut inequalities*. In [8], Mahjoub gives necessary and sufficient conditions for the cut inequalities to define facets for the polytope associated with STECSP when $T=V$. One can extend these results to get sufficient conditions for inequalities (9) to define facets of r-2ECSP(G). In the following we give sufficient statements under which inequalities (10) are facet-defining for r-2ECSP(G). These conditions may be weakened, but this would require more technical details and longer proofs. Our interest here is to show that inequalities (9) and (10) are necessary in a polyhedral description of r-2ECSP(G).

Let $G=(V,E)$ be a 3-edge connected graph. Let $(S,W,v',e',\cup_{i=1}^{k}T_i)$ be an inequality of type (10). Consider the graphs $G^1=(V^1,E^1)$ and $G^2=(V^2,E^2)$ induced by the nodes $S\cup\{\cup_{i=1}^{k-1}\bar{S}_i\}$ and $S\cup\bar{S}_k$, respectively. Note that G^1 and G^2 are 3-edge connected.

Lemma 6. *If $(S,W,v',e',\cup_{i=1}^{k-1}T_i)$ and (S,W,v',e',T_k) define facets of r-2ECSP(G^1) and r-2ECSP(G^2), respectively, and if $G(S)$ is not 3-edge connected then $(S,W,v',e',\cup_{i=1}^{k}T_i)$ defines a facet of r-2ECSP(G).*

Proof. Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be a collection of $|V^1|+|E^1|-1$ (resp. $|V^2|+|E^2|-1$) linearly independent r-2-edge connected subgraphs of G^1 (resp. G^2) verifying $(S,W,v',e',\cup_{i=1}^{k-1}T_i)$ (resp. (S,W,v',e',T_k)) as equation. Let $\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2$. Remark that the graphs of $\mathcal{C}$ are r-2-edge connected subgraphs of G and verify $(S,W,v',e',\cup_{i=1}^{k}T_i)$ as equation. The only graphs of $\mathcal{C}$ that may be linearly dependent are those having no node in $\bar{S}$, and since $G(S)$ is not 3-edge connected then by Theorem 2 each $\mathcal{C}_1$ and $\mathcal{C}_2$ may contain at most $|S|+|E(S)|-1$ linearly independent r-2-edge connected subgraphs . Hence the number of linearly dependent graphs in $\mathcal{C}$ is at most $|S|+|E(S)|-1$. Let dim be the maximal number of linearly independent graphs of $\mathcal{C}$; we have

$$\text{dim} \;\geq |V^1|+|E^1|+|V^2|+|E^2|-2-(|S|+|E(S)|-1)=|V|+|E|-1.$$

Since r-2ECSP(G) is of full dimension, this proves that $(S,W,v',e',\cup_{i=1}^{k}T_i)$ defines a facet of r-2ECSP(G). □

Remark that the above lemma also holds for inequalities (9).

For the next results some definitions are needed. Let $T\subset E$ be a tree of G spanning $V(T)\subset V$ and $v\notin V(T)$ a fixed node. A path $P=\{v_1,e_1,v_2,e_2,\dots,e_{k-1},v_k\}$ of T has the *2-edge connected property* with respect to v, if there exists graphs G_l, G^l, for all $l=1,\dots,k$, such that :

- G_l is an r-2-edge connected subgraph of G containing the subpath $\{v_l,e_l,\dots,e_{k-1},v_k\}$ and v, and none of the nodes in $V(T)\setminus\{v_l,\dots,v_k\}$.
- G^l is an r-2-edge connected subgraph of G containing the subpath $\{v_1,e_1,\dots,e_{l-1},v_l\}$ and v, and none of the nodes in $V(T)\setminus\{v_1,\dots,v_l\}$.

If there exists a collection of paths of T having the 2-edge connected property with respect to v, such that any edge of T is contained in at least one path of that collection, then T has *the 2-edge connected property with respect to v.*

Theorem 7. *Let $G=(V,E)$ be a 3-edge connected graph. Then an inequality (S,W,v',e',T), with $|\delta_S(W)|\leq 1$ defines a facet of r-2ECSP(G) if $G(\bar{S}\cup\{v'\})$ is 2-edge connected and T has the 2-edge connected property with respect to v'.*

Proof. Consider an inequality (S,W,v',e',T) verifying the hypotheses of the theorem. Note that the inequality becomes $y(\bar{S})-x(T)\geq y_{v'}$.

Consider the incidence vectors, (x,y), of an r-2-edge connected subgraph, satisfying $y(\bar{S})-x(T)=y_{v'}$. We shall prove that the only valid inequalities, satisfied at equality by all such incidence vectors, are equivalent to (S,W,v',e',T). Assume that $\alpha x+\beta y=\gamma$ for all $(x,y)\in r$-2ECSP(G) with $y(\bar{S})-x(T)=y_{v'}$. $(0,0)$ and the incidence vector of G verify $y(\bar{S})-x(T)=y_{v'}$, which implies, respectively, that $\gamma=0$, and

$$\sum_{e\in E}\alpha_e+\sum_{v\in V}\beta_v=0 \tag{14}$$

Also, since $G\setminus\{f\}$, for all $f\notin T$, are r-2-edge connected subgraphs and their incidence vectors verify $y(\bar{S})-x(T)=y_{v'}$, this implies that $\sum_{e\in E\setminus\{f\}}\alpha_e+\sum_{v\in V}\beta_v=0$, which combined with (14) yields

$$\alpha_f=0 \text{ for all } f\notin T.$$

Define G^* to be the graph obtained from G by shrinking $\bar{S}\cup\{v'\}$ and let v^* be the resulting node. Note that G^* is 3-edge connected. From above we know that all $(x,y)\in r$-2ECSP(G) with $y(\bar{S})-x(T)=y_{v'}$ verify

$$\sum_{v\in V}\beta_v y_v+\sum_{e\in T}\alpha_e x_e=0. \tag{15}$$

We claim that

$$\sum_{v\in V\setminus(\bar{S}\cup\{v'\})}\beta_v y^*_v+\beta_{v^*}y^*_{v^*}=0 \text{ for all } (x^*,y^*)\in r\text{-2ECSP}(G^*), \tag{16}$$

where $\beta_{v^*}=\sum_{v\in(\bar{S}\cup\{v'\})}\beta_v+\sum_{e\in T}\alpha_e$. In fact, suppose $(x^*,y^*)\in r$-2ECSP(G^*) does not satisfy (16). Define $y_v=y^*_v$ if $v\notin(\bar{S}\cup\{v'\})$; otherwise $y_v=y^*_{v^*}$, and $x_e=y^*_{v^*}$ if $e\in E(\bar{S}\cup\{v'\})$, otherwise $x_e=x^*_e$. Then (x,y) belongs to r-2ECSP(G) (since $G(\bar{S}\cup v')$ is 2-edge connected); moreover $y(\bar{S})-x(T)=y_{v'}$ and (x,y) does not verify (15), which is a contradiction. Now applying Lemma 1 to G^* and the equality (16) it follows that

$$\beta_v=0 \text{ for all } v\in V\setminus(\bar{S}\cup\{v'\}).$$

Next, we show that $\beta_v=-\alpha_e=-\beta_{v'}$ for all $v\in\bar{S}$, $e\in T$. Let $P=v_1,e_1,v_2,e_2,\dots,e_{k-1},v_k$ be a path of T having the 2-edge connected property with respect to v'. Then the incidence vectors of the graphs G_l, (resp. G^l) , for $l=1,\dots,k$, verify $y(\bar{S})-x(T)=y_{v'}$ and thus $\alpha x+\beta y=\gamma$, which implies $\beta_{v_i}+\alpha_{e_i}=0$ for $i=1,\dots,k-1$ and $\beta_{v_k}=-\beta_{v'}$ (resp. $\beta_{v_i}+\alpha_{e_{i-1}}=0$ for $i=2,\dots,k$, $\beta_{v_1}=-\beta_{v'}$). Combining these equalities we obtain $\beta_{v_i}=-\beta_{v'}$ for $i=1,\dots,k$ and $\alpha_{e_i}=\beta_{v'}$ for $i=1,\dots,k-1$. Moreover, since any edge of T is contained in a path of T having the 2-edge connected property with respect to v,

$$\beta_v=-\alpha_e=-\beta_{v'} \text{ for all } v\in\bar{S} \text{ and } e\in T.$$

We have shown that $\alpha x+\beta y=\gamma$ is $\beta_{v'}$ times $y(\bar{S})-x(T)=y_{v'}$. This means that (S,W,v',e',T) defines a facet of r-2ECSP(G). □

Corollary 8. *Suppose G is 3-edge connected. An inequality $(S,W,v',e',\cup_{i=1}^{k}T_i)$ with $|\delta_S(W)|\le 1$, defines a facet of r-2ECSP(G) if for $i=1,\dots,k$ the following hold: $G(\bar{S}_i\cup\{v'\})$ is 2-edge connected; T_i has the 2-edge connected property with respect to v'.*

Proof. Immediate from Lemma 6 and Theorem 7. □

Note that Theorem 7 and Corollary 8 may be used to generate a large class of graphs where inequalities (1)-(4) are not sufficient to describe r-2ECSP(G). The next lemma gives necessary conditions for inequalities (9) and (10) to define facets. The applicability of this result will become evident towards the end of the paper.

Lemma 9. *Let $G=(V,E)$ be a graph and r a fixed node. Inequalities (S,W,v,F) and (S,W,v,e,F) ($F=\cup_{i=1}^{k}T_i$) define facets of r-2ECSP(G) only if*

(i) *$G(W)$ is connected and,*

(ii) *every pendant node of T_i, for $i=1,\dots,k$, is connected to W and to $S\setminus W$; moreover, if $G(S\setminus W)$ is not connected then at least one of its connected components is connected to at least two pendant nodes of T_i, for all $i=1,\dots,k$.*

Proof. The proof is given for the inequality (S,W,v,F), i.e., an inequality (9). A similar proof holds for an inequality (10).

(i) If $G(W)$ is not connected, let W_1 be the connected component of $G(W)$ containing r, then the inequality (S,W,v,F) is implied by (S,W_1,v,F).

(ii) Let $v_l\in\bar{S}_l$ be a pendant node of T_l and e_l be the edge of T_l incident to v_l, for $1\le l\le k$. Suppose that v_l is not connected to W. Define $S'=S\cup\{v_l\}$; $\bar{S}'_i=\bar{S}_i$ for $i=1,\dots,k$, $i\neq l$;

$\bar{S}'_l = \bar{S}_l \setminus \{v_l\}$; $T'_i = T_i$, for $i = 1, \dots, k$, $i \neq l$; $T'_l = T_l \setminus \{e_l\}$. Note that T'_l is a tree spanning $\bar{S}'_l$, so that $y(\bar{S}'_l) - x(T'_l) \leq y(\bar{S}_l) - x(T_l)$ is valid. Hence the inequality $(S, W, v, \cup_{i=1}^k T_i)$ is implied by $(S', W, v, \cup_{i=1}^k T'_i)$. Thus it may be assumed that v_l is connected to W. Now if v_l is not connected to $S \setminus W$ then, by moving v_l to W we show that the inequality $(S, W, v, \cup_{i=1}^k T_i)$ is redundant.

Assume now that $G(S \setminus W)$ is not connected and that each of its connected components is connected to at most one pendant node of T_i, for $i = 1, \dots, k$. Then every pendant node is connected to $S \setminus W$. Consequently there is a connected component of $G(S \setminus W)$ which does not contain v, denoted by $\bar{W}_1$, that is connected to exactly one of the pendant nodes of T_i, for $i = 1, \dots, k$, (of course it may be connected to W and some nodes in $\bar{S}$). Let $v_l \in \bar{S}_l$ be such a node. Define $S' = S \cup \{v_l\}$, $W' = W \cup \bar{W}_1 \cup \{v_l\}$ and T'_i, $i = 1, \dots, k$, as defined above. As v_l is connected to W it follows that $G(W')$ is connected. Consequently, the inequality $(S, W, v, \cup_{i=1}^k T_i)$ is implied by $(S', W', v, \cup_{i=1}^k T'_i)$. □

Before beginning the next section two subclasses of inequalities (9) and (10) are given and, as shall be seen later, they can be separated in polynomial time.

Given a graph $G = (V, E)$, $S \subseteq V$, W a proper subset of S, $v \in S \setminus W$ and $r \in W$. Consider the following inequalities:

$$x(\delta_S(W)) + 2y(\bar{S}) \geq 2y_v, \tag{17}$$

$$x(\delta_S(W) \setminus \{e\}) + y(\bar{S}) \geq y_v. \tag{18}$$

(17) will be denoted by (S, W, v) and (18) by (S, W, v, e). Inequalities (17) (resp. (18)) are either included in inequalities (9) (resp. (10)) (when $\bar{S}$ is an independent set) or implied by (9) (resp. (10)).

3. Separation

The *separation problem* of a given set of inequalities is to determine whether a given vector satisfies this set of inequalities and, if not, to find an inequality in the set that is violated. It follows from the equivalence between separation and optimization [5] that if the separation problem is solvable in polynomial time then the optimization over this system of inequalities, is also polynomial.

The number of inequalities (2)-(4) is polynomial, thus their separation is straightforward. Also, the separation problem of inequalities (1) can be easily reduced to a min-cut problem and hence can be solved in polynomial time as well. From now on, we are given a point $(\bar{x}, \bar{y})$ satisfying inequalities (1)-(4). First, consider the separation of inequalities (9).

Let $G = (V, E)$ be a graph and $r \in V$ a root vertex. Let $(\bar{x}, \bar{y}) \in \mathbb{R}^{|E|+|V|}$ be a solution verifying inequalities (1)-(4). For $v \in V \setminus \{r\}$ and $S \subseteq V$, let $f^v(S)$ be the function defined as follows:

$$f^v(S) = \begin{cases} +\infty & \text{if } \{r, v\} \not\subset S \\ \min\limits_{r \in W \subset S \setminus \{v\}} \{\bar{x}(\delta_S(W))\} + 2\bar{y}(\bar{S}) - 2 \max\limits_{F \subseteq E(\bar{S})} \{\bar{x}(F) : F \text{ forest}\} & \text{otherwise.} \end{cases}$$

Note that given S and v, the value $f^v(S)$ can be computed in polynomial time by a single minimum r-v cut computation in $G(S)$, plus a maximum forest computation in $G(\bar{S})$.

Separating inequalities (9) reduces to the minimization of $f^v(S)$ among all subsets S of V and for every $v \in V$. If one finds $w \in V$ and $\hat{S}$ with $f^w(\hat{S}) < 2\bar{y}_w$ then $(\hat{S}, W, w, F)$ defines a violated inequality of type (9), where $\delta_{\hat{S}}(W)$ is a cut of minimum capacity (equal to $\bar{x}(\delta_{\hat{S}}(W))$) separating r and w, and $F = \cup_{i=1}^k T_i$ is a maximum forest (of weight $\bar{x}(F)$) spanning $\bar{\hat{S}}$. Otherwise, there exists no violated inequality of type (9). By the same manner one can define a function whose minimization solves the separation problem associated with inequalities (10). Unfortunately, $f^v(\cdot)$ is not a submodular function in general. However, there are some cases where the minimization of f^v can still be done in polynomial time.

In the following, the separation problem of inequalities (17) and (18) is discussed.

Construct a network $D_{(\bar{x},\bar{y})} = (N, A)$ from G and the vectors $\bar{x}$ and $\bar{y}$ as follows. Duplicate every node v of G into two nodes v', v''. Add two arcs (v', v'') with capacity $c(v', v'') = \bar{y}_v$, and (v'', v') with capacity $c(v'', v') = \infty$. Replace every edge $e = vw$ of G by two arcs (v'', w') and (w'', v') both having the capacity $c(v'', w') = c(w'', v') = \bar{x}_e$. An example is shown in Figure 3.

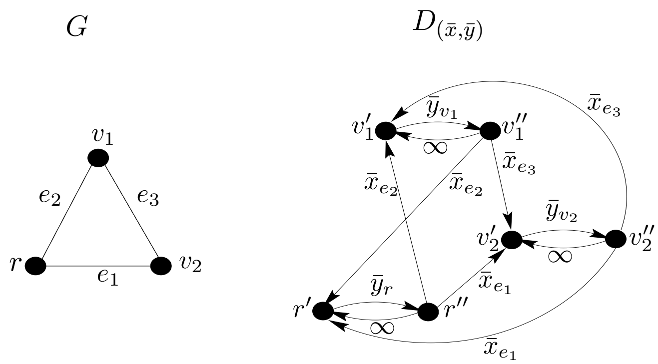

FIGURE 4. The values associated with the arcs of $D_{(\bar{x},\bar{y})}$ represent the capacities.

If U is a subset of N, $\delta^+(U) = \{(u, v) \in A : u \in U \text{ and } v \in N \setminus U\}$ is called a *directed-cut*. Let $V' = \{u_1, \dots, u_k\}$ be a node subset of V. $D^{V'}_{(\bar{x},\bar{y})}$ is the network obtained from $D_{(\bar{x},\bar{y})}$ by identifying u'_i with u''_i and the resulting node is u_i, for $i = 1, \dots, k$.

Separation of inequalities (17)

Define the function $g^v(\cdot)$:

$$g^v(S) = \begin{cases} +\infty & \text{if } \{r, v\} \not\subset S \\ \min\limits_{r\in W\subset S\setminus\{v\}} \{\bar{x}(\delta_S(W))\} + 2\bar{y}(\bar{S}) & \text{otherwise.} \end{cases}$$

Let $g^v(S^*) = \min_{S\subseteq V} g^v(S)$ for $v \in V$. If $g^v(S^*) \geq 2\bar{y}_v$ for all $v \in V$, then there is no violated inequality (17). Otherwise, we can show a violated inequality. It remains to see how to solve the minimization problem of $g^v(.)$. We show that this reduces to a min-cut problem in the network $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ defined from G and the vectors $\bar{x}$ and $2\bar{y}$.

Lemma 10. *For all $S \subseteq V$, $W \subset S$, $r \in W$ and $v \in S \setminus W$ there exists a directed-cut $\delta^+(U')$ of $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ separating r from v such that $c(\delta^+(U')) = \bar{x}(\delta_S(W)) + 2\bar{y}(\bar{S})$.*

Proof. Take $U' = \{r\} \cup (\bigcup_{v\in W}\{v', v''\}) \cup (\bigcup_{v\in\bar{S}}\{v'\})$. □

Lemma 11. *Let $\delta^+(U^*)$ be a minimum capacity directed-cut of $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ separating r from v. Then there exists $W \subset S' \subseteq V$, $r \in W$ and $v \in S' \setminus W$ with $\bar{x}(\delta_{S'}(W)) + 2\bar{y}(\bar{S}') = c(\delta^+(U^*))$.*

Proof. By Lemma 10, $c(\delta^+(U^*)) \neq \infty$. Hence $v'' \in U^*$ implies $v' \in U^*$. Define $\bar{S}'$ as the set of nodes v such that $v' \in U^*$ and $v'' \notin U^*$, and W as the set of nodes v such that $v', v'' \in U^*$. Add r to W. Now by the definition of $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ we have $\bar{x}(\delta_{S'}(W)) + 2\bar{y}(\bar{S}') = c(\delta^+(U^*))$. □

From the two lemmas above follows what had to be shown,

$$g^v(S^*) = c(\delta^+(U')) \geq c(\delta^+(U^*)) \geq g^v(S^*).$$

Separation of inequalities (18)

The separation of inequalities (18) is along the same lines as of inequalities (17).

To separate all inequalities (S, W, v, e) corresponding to a fixed v and e, consider $G' = (V, E \setminus \{e\})$ (i.e. G' is obtained from G by removing e). Then fix v and minimize the function

$$h^v(S) = \begin{cases} +\infty & \text{if } \{r, v\} \not\subset S \\ \min\limits_{r \in W \subset S \setminus \{v\}} \{\bar{x}(\delta_S(W))\} + \bar{y}(\bar{S}) & \text{otherwise,} \end{cases}$$

where $\delta_S(\cdot)$ is taken in G'. As for $g^v(.)$, this problem reduces to a minimum capacity directed-cut problem separating r from v in the network $D^{\{r,v\}}_{(\bar{x},\bar{y})}$ defined from G', the restriction of $\bar{x}$ on G' and the vector $\bar{y}$.

Let $h^v(S^*) = \min_{S \subseteq V} h^v(S)$ for $v \in V$. If $h^v(S^*) \geq \bar{y}_v$, then $\bar{x}(\delta_S(W) \setminus \{e\}) + \bar{y}(\bar{S}) \geq \bar{y}_v$ for all S. Hence there is no violated inequality (S, W, v, e) (for fixed v and e). If on the other hand $h^v(S^*) < \bar{y}_v$, we can exhibit a violated inequality (18). Repeating the procedure for every v and e, the separation problem for inequalities (18) is solved.

Remark. Say that an inequality (S, W, v, F) or (S, W, v, e, F) is of *class 1* if the nodes of $\bar{S}$ are pairwise non-adjacent. It is easy to see that inequalities (S, W, v) and (S, W, v, e) contain all inequalities of class 1. It follows that the separation problem for inequalities of class 1 is solvable in polynomial time.

Next, another family of inequalities (9) and (10) is introduced with the associated separation problem.

Consider inequalities (S, W, v, T) or (S, W, v, e, T) with $\delta_S(W) = \emptyset$, $G(\bar{S})$ connected and where T is a path spanning $\bar{S}$. Only the pendant nodes of T are connected with S. This subclass will be called inequalities of *class 2*.

If u and w represent the pendant nodes of T, then $G(V \setminus \{u, w\})$ contains at least two connected components, W_1 containing r and W_2 containing v. The separation problem reduces to finding a path $P = \{u = v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k = w\}$ in $G(V \setminus (W_1 \cup W_2))$ that minimizes

$$\sum_{i=1}^{k} \bar{y}_{v_i} - \sum_{i=1}^{k-1} \bar{x}_{e_i}. \tag{19}$$

If $\bar{y}_u + \bar{y}_w + \sum_{i=2}^{k-1} \bar{y}_{v_i} - \sum_{i=1}^{k-1} \bar{x}_{e_i} < \bar{y}_v$, then a violated inequality of class 2 is obtained, where $S = V \setminus \{u, v_2, \ldots, v_{k-1}, w\}$, $W = W_1$, $\bar{S} = \{u, v_2, \ldots, v_{k-1}, w\}$ and $T = \{e_1, \ldots, e_{k-1}\}$ is a path spanning $\bar{S}$. Otherwise, there is no violated inequality of class 2, where u and w are the pendant nodes of the path T spanning $\bar{S}$.

How is (19) to be solved? Given a triplet v, u and w such that $G \setminus \{u, w\}$ contains at least two connected components, W_1 containing r and W_2 containing v, construct the network $\bar{D}_{(\bar{x},\bar{y})}$ from the graph $G' = G(V \setminus (W_1 \cup W_2))$ as follows: replace each edge of G', $e = u_1u_2$, not incident to u nor to w, by two arcs (u_1, u_2) associated with a cost $c(u_1, u_2) = \bar{y}_{u_1} - \bar{x}_e$ and a reverse arc (u_2, u_1) with cost $c(u_2, u_1) = \bar{y}_{u_2} - \bar{x}_e$. If $e = uu_1$ (resp. $e = u_1w$) is an edge of G' incident to u (resp. w) then replace e by an arc (u, u_1) (resp. (u_1, w)) having a cost $c(u, u_1) = \bar{y}_u - \bar{x}_e$ (resp. $c(u_1, w) = \bar{y}_{u_1} + \bar{y}_w - \bar{x}_e$).

Problem (19) reduces to a min-cost path problem from u to w in $\bar{D}_{(\bar{x},\bar{y})}$. Since $(\bar{x}, \bar{y})$ verifies inequalities (2)-(4), it follows that the cost associated with each arc of $\bar{D}_{(\bar{x},\bar{y})}$ is nonnegative. One can apply, for example, Dijkstra's algorithm to find such a path.

Using the results above, it will be shown that separating inequalities (9) and (10) in series-parallel graphs may be done in polynomial time.

A *homeomorph* of K_4 (the complete graph on four nodes) is a graph obtained from K_4 when its edges are subdivided into paths by inserting new nodes of degree two. A graph is called *series-parallel* if it contains no homeomorph of K_4 as a subgraph.

Theorem 12. *If $G = (V, E)$ is a series-parallel graph, then inequalities (9) and (10) are either of class 1 or of class 2.*

Proof. Let $(S, W, v, \cup_{i=1}^{k} T_i)$ be an inequality (9). Suppose that there exists T_l, $1 \leq l \leq k$, which is not a path. Then T_l contains at least three pendant nodes. Suppose that $G(S \setminus W)$ is not connected. From Lemma 9 (ii), there exists a connected component $\bar{W}_1$ of $G(S \setminus W)$ connected to at least two pendant nodes of T_l, say v_1 and v_2. Let v_3 be a pendant node of T_l different from v_1 and v_2. By Lemma 9 (ii), the nodes v_1, v_2 and v_3 are connected to W, and by Lemma 9 (i) $G(W)$ is connected, so there is a K_4 that is a subgraph of G, see Figure 3. K_4 is obtained by shrinking

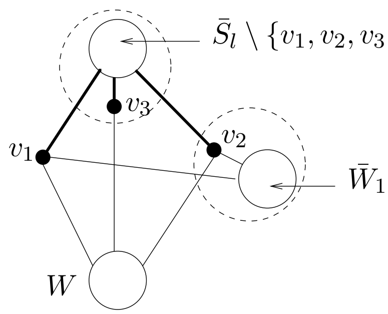

FIGURE 5. K_4 as subgraph. Bold edges belong to T_l.

the following connected components: W, $\bar{W}_1 \cup \{v_2\}$ and $\bar{S}_l \setminus \{v_1, v_2\}$. If $G(S \setminus W)$ is connected the same is obtained by replacing $\bar{W}_1$ by $S \setminus W$. Consequently, T_i is a path for all $i = 1, \dots, k$. Similarly one can show that in this case, $\delta_S(W) = \emptyset$ and $k = 1$ (we have only one path). It follows that $\bar{S}$ is either an independent set of G (class 1), or $k = 1$, T_1 a path such that only the endnodes are connected with S, and $\delta_S(W) = \emptyset$ (class 2). The same result holds for inequalities (10). □

4. Concluding remarks

Given a graph $G = (V, E)$, the node-edge weighted 2-edge connected subgraph problem has been introduced. This problem reduces to a sequence of $|V|$ r-edge connected subgraph problems (r-2ECSP). Inequalities (1)-(4) define a linear relaxation of the convex hull of the solutions of the r-2ECSP, r-2ECSP(G). These inequalities are based on a direct interpretation of the 2-edge connected property of the solutions. Unfortunately, this linear relaxation does not suffice to solve the problem even in particular classes of graphs (such as series-parallel graphs). Moreover, the graph given in Figure 1 is outer-planar, so it is more restricted than series-parallel graphs. Valid inequalities (9) and (10) of r-2ECSP(G) have been added in Section 2. We defined two classes among these inequalities, classes 1 and 2, and showed that their separation problem is polynomially solvable. This provides a new linear description, given by (1)-(4) plus inequalities of class 1 and 2, where the optimization can be performed in polynomial time. This linear relaxation provides better lower bounds on the value of the optimal solution of the problem. It has been shown that inequalities (9) and (10) are of class 1 and 2 when the underlying graph is series-parallel. An interesting question arises: *are inequalities (1)-(4),(9) and (10) sufficient to describe r-2ECSP(G) when G is series-parallel?* If the answer is positive, then there is a polynomial time algorithm to solve the node-edge weighted 2-edge connected subgraph problem in series-parallel graphs.

A consequence of the results of Section 2.1 regards the dimension of the Steiner 2-edge connected subgraph polytope discussed in the introduction. For a graph $G = (V, E)$ and a set of terminals T call STECSP(G, T) the convex hull of incidence vectors of 2-edge connected graphs spanning

T. Mahjoub [8] showed that when $T = V$, $\dim(\text{STECSP}(G, V)) = |E| - |E_{2c}|$. Following the ideas in Section 2.1 it is straightforward to extend this result to the general case. Indeed, let E^i_{2c} for $i = 1, \ldots, l$ be the partition of E_{2c} induced by the relation $\mathcal{R}$. Let E^i_{2c} for $i = 1 \ldots, l_1$ be the equivalence classes such that all nodes in T belong to the same connected component of $G^i = (V, E \setminus E^i_{2c})$; and let E^i_{2c} for $i = l_1 + 1 \ldots, l$ be the other equivalence classes. We have the following.

Lemma 13. $\dim(\text{STECSP}(G, T)) = |E| - |E_{2c}| + l_1$.

Note that if G is 3-edge connected, then STECSP(G, T) is full dimensional.

On a different matter, let us now look at a closely related problem to r-2ECSP: find a 2-node connected subgraph of G containing a fixed node r which minimizes the overall weight of both edges and nodes. Call this problem r-2NCSP and the associated polytope r-2NCSP(G). Each solution of r-2NCSP(G) is also a solution of r-2ECSP(G). Thus all valid inequalities (1)-(4) and (9)-(10) of r-2ECSP(G) are also valid for r-2NCSP(G). Consider the following valid inequalities for r-2NCSP(G) (which are not valid for r-2ECSP(G))

$$x(\delta_{V\setminus\{v\}}(W)) \geq y_w, \text{ for all } v \in V \setminus \{r\}, r \in W \subset V \setminus \{v\}, w \in (V \setminus \{v\}) \setminus W. \tag{20}$$

Note that (1)-(5) plus (20) give an integer linear formulation for r-2NCSP. The example of Figure 1 is a fractional extreme point of the linear relaxation of r-2NCSP(G) given by (1)-(4) and (20). But it violates inequalities of class 1 and 2. Thus inequalities (1)-(4), (20) and those of classes 1 and 2 provide a tighter linear relaxation for r-2NCSP(G). Note also that the separation problem of (20) reduces easily to a min-cut problem. The same question may be asked concerning the description of r-2NCSP(G) in series-parallel graphs.

The two linear relaxations associated with r-2ECSP(G) and r-2NCSP(G) may be used to solve the Steiner 2-edge and the Steiner 2-node connected subgraph problems.

We finish by noting that the separation problem of inequalities (9) and (10) is polynomially solvable in series-parallel graphs and that inequalities (17) and (18) can be separated in polynomial time for general graphs. What can be said about the separation of (9) and (10) in the general case?

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