Tight Weyl–Heisenberg Frames in $\ell^2(\boldsymbol{Z})$

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***Abstract*—Tight Weyl–Heisenberg frames in $\ell^2(Z)$ are the tool for short-time Fourier analysis in discrete time. They are closely related to paraunitary modulated filter banks and are studied here using techniques of the filter bank theory. Good resolution of short-time Fourier analysis in the joint time–frequency plane is not attainable unless some redundancy is introduced. That is the reason for considering overcomplete Weyl–Heisenberg expansions. The main result of this correspondence is a complete parameterization of finite length tight Weyl–Heisenberg frames in $\ell^2(Z)$ with arbitrary rational oversampling ratios. This parameterization follows from a factorization of polyphase matrices of paraunitary modulated filter banks, which is introduced first.**

I. Introduction

SHORT-TIME Fourier analysis, as originally proposed by Gabor [1], amounts to expanding a signal with respect to vectors in a Weyl–Heisenberg family Φ_{v,x_0,ω_0} that is generated from a single prototype window function v by translating it in time and frequency

$$\begin{aligned}\Phi_{v,x_0,\omega_0} &= \{v_{lm}: v_{lm}(x)\\ &= v(x-lx_0)e^{jm\omega_0 x}, l\in \boldsymbol{Z}, m\in \boldsymbol{Z}\}.\end{aligned} \tag{1}$$

The goal of such an expansion

$$f=\sum_{l,m} a_{lm}v_{lm}, \tag{2}$$

is to extract information on the spectral content of the signal without sacrificing information on its localization in time. The windows proposed by Gabor were Gaussians since they attain the lower bound on the localization in the time–frequency plane and therefore enable analysis with the best resolution in time and frequency jointly. Along with the development of Gabor's original scheme, it was observed that such expansions in $L^2(\boldsymbol{R})$ are stable only if the expansion vectors constitute an overcomplete family in this space [3], [4]. Another incarnation of the same phenomenon is expressed by the Balian–Low theorem [5], [6], which asserts that there are no orthogonal Weyl–Heisenberg bases that have good localization in both time and frequency. Even before these results on Weyl–Heisenberg expansions were established, it had been known in the signal processing community that redundant short-time Fourier transforms are advantageous over critically sampled transforms in terms of providing robustness, which is important in applications involving some processing in the Fourier domain [7]. In digital signal processing, we often encounter representations that are obtained as the inner products $\langle f, v_{lm}\rangle$ of a signal $f\in \ell^2(\boldsymbol{Z})$ with the vectors of a discrete-time Weyl–Heisenberg family $\Phi_{v,N,K}$ rather than expansions with respect to $\Phi_{v,N,K}$. The discrete-time family $\Phi_{v,N,K}$ is given by

$$\Phi_{v,N,K}=\{v_{lm}, l\in \boldsymbol{Z}_K m\in \boldsymbol{Z}\} \tag{3}$$

where $\boldsymbol{Z}_K=\{0,1,\cdots,k-1\}$, and

$$v_{lm}[n]=v[n-mN]e^{j(2\pi/K)ln}.$$

This kind of short-time Fourier transform can be implemented using modulated filter banks. An effect similar to that described by the Balian–Low theorem has been observed in [9], where it was shown that there are no critically sampled modulated filter banks with finite impulse responses that have good frequency selectivity and allow for perfect reconstruction using FIR filter banks.

These are the reasons for studying overcomlete Weyl–Heisenberg families of functions. Particularly interesting are tight frames of this kind. Dealing with tight frames automatically solves two problems associated with short-time Fourier representations that were a matter of debate in the past. Namely, the problem of finding expansion coefficients in (2) and the problem of synthesizing a signal from its inner products with vectors v_{lm}. If vectors v_{lm} constitute a tight frame in the considered Hilbert space ($\ell^2(\boldsymbol{Z})$ or $L^2(\boldsymbol{R})$), then the expansion and the inner product representation with respect to this family are equivalent, and any signal can be represented in a manner reminiscent of orthogonal expansions

$$f=\frac{1}{a}\sum_{l,m}\langle f, v_{lm}\rangle v_{lm} \tag{4}$$

where a is a constant factor. Furthermore, it was demonstrated by Daubechies that as soon as some redundancy is introduced, the situation with time–frequency localization changes drastically and that it is possible to attain tight Weyl–Heisenberg frames in $L^2(\boldsymbol{R})$ with good localization in both time and frequency [8].

In this paper, we consider tight Weyl–Heisenberg frames in $\ell^2(\boldsymbol{Z})$. These are equivalent to paraunitary modulated filter banks. We give a complete parameterization of FIR paraunitary modulated filter banks, i.e., tight Weyl–Heisenberg frames in $\ell^2(\boldsymbol{Z})$ with finite support in time and arbitrary rational

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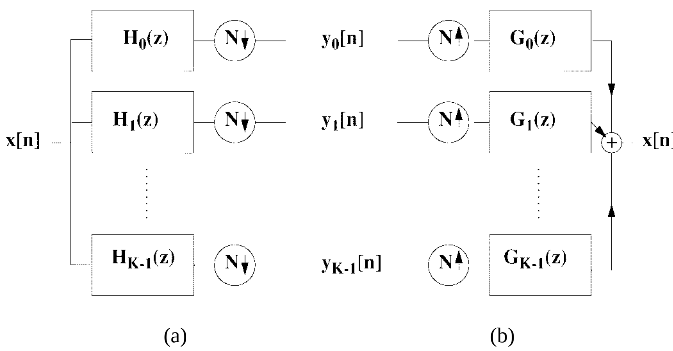

Fig. 1. Filter banks. (a) Analysis filter bank. (b) Synthesis filter bank.

oversampling ratios. A design example is also provided to demonstrate that the design flexibility attained with oversampled filter banks of this kind allows for linear-phase filters and significantly improved frequency selectivity over the critically sampled case.

Notation:

For a filter $H(z)$, $\tilde{H}(z)$ will denote the filter whose impulse response is the complex conjugate of the time-reversed version of the impulse response of $H(z)$. Similarly, when used with matrices of rational functions of the complex variable z, $\tilde{\boldsymbol{H}}(z)$ will denote the matrix obtained from $\boldsymbol{H}(z)$ by transposing it, conjugating all of the coefficients of the rational functions in $\boldsymbol{H}(z)$, and replacing z by z^{-1}. I_N denotes the $N \times N$ identity matrix.

II. The Polyphase Representation

Let $\Phi_{v,N,K}$, as given in (3), be a Weyl–Heisenberg family in $\ell^2(\boldsymbol{Z})$. We say that $\Phi_{v,N,K}$ is a tight frame in $\ell^2(\boldsymbol{Z})$ if for any $f \in \ell^2(\boldsymbol{Z})$

$$a||f||^2 = \sum_{l=0}^{K-1} \sum_{m=-\infty}^{\infty} |\langle f, v_{lm}\rangle|^2. \qquad (5)$$

If this is satisfied, any square summable sequence f can be expanded as given by (4). The inner products of a signal with the vectors v_{lm} can be obtained at the output of a K channel filter bank, as shown in Fig. 1(a), with filters that are time-reversed modulated versions of the window

$$H_i(z) = \tilde{V}(e^{j(2\pi i/K)}z)$$

where $V(z)$ denotes the z transform of v. On the other hand, the signal can be perfectly reconstructed from these inner products using the synthesis filter bank shown in Fig. 1(b), where the filters are modulated versions of the window

$$G_i(z) = V(e^{j(2\pi i/K)}z).$$

The parameterization of tight Weyl–Heisenberg frames will be given here as a parameterization of polyphase matrices corresponding to associated filter banks. Let M be the least common multiple of K and N, and let J and L be the two integers satisfying $JK = LN = M$. We consider the M-component polyphase representation of $V(z)$

$$V(z) = \sum_{j=0}^{M-1} z^{-j} V_j(z^M) \qquad (6)$$

where

$$V_j(z) = \sum_n v[j+nN]z^{-n}. \qquad (7)$$

The entries of the associated $K \times N$ polyphase matrix $\boldsymbol{V_p}(z)$ are polyphase components of the filters $V(e^{j(2\pi i/K)}z)$, and they can be represented in terms of the polyphase components of the window as

$$[\boldsymbol{V_p}(z)]_{m,n} = \sum_{l=0}^{L-1} e^{-j(2\pi/K)m(n+lN)} z^{-l} V_{n+lN}(z^L) \qquad (8)$$

where $L = M/N$. The polyphase matrix can be factored as

$$\boldsymbol{V_p}(z) = \boldsymbol{F}_K \boldsymbol{V}(z) \qquad (9)$$

where $\boldsymbol{F}_K$ is the $K \times K$ DFT matrix, and

$$\boldsymbol{V}(z) = [I_K \cdots I_K] \text{ diag } (V_0(z^L) \cdots V_{M-1}(z^L)) \cdot \begin{bmatrix} I_N \\ z^{-1}I_N \\ \cdots \\ z^{-(L-1)}I_N \end{bmatrix}. \qquad (10)$$

By inspection of the above factorization, we can see that the elements of $\boldsymbol{V}(z)$ are given by

$$[\boldsymbol{V}(z)]_{m,n} = z^{-q(m,n)} V_{m+p(m,n)K}(z^L) \qquad (11)$$

where $p(m,n)$ and $q(m,n)$ are integers satisfying

$$\begin{aligned} m + p(m,n)K &= n + q(m,n)N \\ p(m,n) \le J-1, &\quad q(m,n) \le L-1. \end{aligned} \qquad (12)$$

Note that these equations cannot be satisfied for every pair of integers m and n. In fact, for each n, there are exactly $J = M/K$ indices m that satisfy these constraints. Consequently, there are J nonzero elements in each row of $\boldsymbol{V}(z)$ and L nonzero elements in each column of $\boldsymbol{V}(z)$. The possible cases are illustrated by the following examples.

Example 1: K is a multiple of N. In this case, $J = 1$; therefore, there is a single nonzero element in each row of $\boldsymbol{V}(z)$. For $K = 6$ and $N = 3$, we have

$$\boldsymbol{V}(z) = \begin{bmatrix} V_0(z^2) & 0 & 0 \\ 0 & V_1(z^2) & 0 \\ 0 & 0 & V_2(z^2) \\ z^{-1}V_3(z^2) & 0 & 0 \\ 0 & z^{-1}V_4(z^2) & 0 \\ 0 & 0 & z^{-1}V_5(z^2) \end{bmatrix}.$$

□

Example 2: K and N are coprime. In this case, $J = N$; therefore, all elements of $\boldsymbol{V}(z)$ are nonzero. For $K = 3$ and $N = 2$, we obtain

$$\boldsymbol{V}(z) = \begin{bmatrix} V_0(z^3) & z^{-1}V_3(z^3) \\ z^{-2}V_4(z^3) & V_1(z^3) \\ z^{-1}V_2(z^3) & z^{-2}V_5(z^3) \end{bmatrix}.$$

□

Example 3: K and N have a common factor other than N. For $K = 6$ and $N = 4$, $\boldsymbol{V}(z)$ is equal to

$$\begin{bmatrix} V_0(z^3) & 0 & z^{-1}V_6(z^3) & 0 \\ 0 & V_1(z^3) & 0 & z^{-1}V_7(z^3) \\ z^{-2}V_8(z^3) & 0 & V_2(z^3) & 0 \\ 0 & z^{-2}V_9(z^3) & 0 & V_3(z^3) \\ z^{-1}V_4(z^3) & 0 & z^{-2}V_{10}(z^3) & 0 \\ 0 & z^{-1}V_5(z^3) & 0 & z^{-2}V_{11}(z^3) \end{bmatrix}.$$

□

III. PARAMETERIZATIONS

A necessary and sufficient condition on a Weyl–Heisenberg family $\Phi_{v,N,K}$ to be a tight frame in $\ell^2(\boldsymbol{Z})$ is given by the following proposition.

Proposition 1: A Weyl–Heisenberg family $\phi_{v,N,K}$ is a tight frame in $\ell^2(\boldsymbol{Z})$ if and only if the polyphase matrix $\boldsymbol{V_p}(z)$ is paraunitary $\tilde{\boldsymbol{V}}_{\boldsymbol{p}}(z)\boldsymbol{V_p}(z) = aI_N$. □

This proposition is proven for a more general case in [10], and it can be easily shown that the constant a is equal to redundancy of the frame, that is, $a = K/N$.

According to the factorization in (9), $\boldsymbol{V_p}(z)$ is paraunitary if and only if $\boldsymbol{V}(z)$ is paraunitary

$$\tilde{\boldsymbol{V}}(z)\boldsymbol{V}(z) = \frac{1}{N}I_N.$$

Noting that some of the elements of $\tilde{\boldsymbol{V}}(z)\boldsymbol{V}(z)$ are identically zero and taking symmetries into account, it can be observed that the paraunitariness condition imposes $N + (N(J-1)/2)$ different constraints. We now investigate the implications of these constraints for the three cases presented in the previous section.

Case 1) K is a multiple of N: $V(z)$ is constrained only by

$$\sum_{l=0}^{L-1} \tilde{V}_{i+lN}(z)V_{i+lN}(z) = \frac{1}{N}, \qquad i = 0, 1, \cdots, N-1. \quad (13)$$

Therefore, the polyphase components of the window of a tight frame $\Phi_{v,N,K}$ with an integer oversampling factor K/N are given as entries of any set of N $L \times 1$ paraunitary matrices. Note that the power complementarity condition in (13) means that $V_i(z), V_{i+N}(z), \cdots, V_{i+(L-1)N}(z)$ are polyphase components of a filter $F_i(z)$, which is orthogonal to its translates by multiples of L [2]. □

Case 2) K and N are coprime: In this case, the polyphase components of $V(z)$ are, up to a time delay, equal to the entries of a full $K \times N$ paraunitary matrix. In order to show this, we need to transform $\boldsymbol{V}(z)$. Let $\boldsymbol{D}_r(z)$ and $\boldsymbol{D}_l(z)$ be diagonal matrices of monomials selected so that the elements in the first row and the first column of $\boldsymbol{D}_l(z)\boldsymbol{V}(z)\boldsymbol{D}_r(z)$ are equal to the polyphase components of $\boldsymbol{V}(z)$. These matrices are given by $\boldsymbol{D}_l(z) = \text{diag}(1, z^{q(1,0)}, \cdots, z^{q(K-1,0)})$ and $\boldsymbol{D}_r(z) = \text{diag}(1, z^{q(0,1)}, \cdots, z^{q(0,N-1)})$. For instance, for the matrix $\boldsymbol{V}(z)$ in Example 2, $\boldsymbol{D}_l(z) = \text{diag}(1, z^2, z), \boldsymbol{D}_r(z) = \text{diag}(1, z)$, yielding

$$\boldsymbol{D}_l(z)\boldsymbol{V}(z)\boldsymbol{D}_r(z) = \boldsymbol{U}(z^L) = \begin{bmatrix} V_0(z^3) & V_3(z^3) \\ V_4(z^3) & z^3V_1(z^3) \\ V_2(z^3) & V_5(z^3) \end{bmatrix}.$$

In general, the entries of $\boldsymbol{D}_l(z)\boldsymbol{V}(z)\boldsymbol{D}_r(z)$ are equal to

$$[\boldsymbol{D}_l(z)\boldsymbol{V}(z)\boldsymbol{D}_r(z)]_{m,n} = z^{-q(m,n)+q(m,0)+q(0,n)}V_{m+p(m,n)K}(z^L). \quad (14)$$

It follows from (12) that

$$(-q(m,n) + q(m,0) + q(0,n))N = (-p(m,n) + p(m,0) + p(0,n))K \quad (15)$$

which implies that the delay factor $z^{-q(m,n)+q(m,0)+q(0,n)}$ in (14) can be only z^0 or z^L. Therefore, we can write

$$\boldsymbol{D}_l(z)\boldsymbol{V}(z)\boldsymbol{D}_r(z) = \boldsymbol{U}(z^L) \quad (16)$$

where $\boldsymbol{U}(z)$ is a matrix whose entries are, up to a delay, equal to the polyphase components of $V(z)$. Therefore, any paraunitary $K \times N$ matrix $\boldsymbol{U}(z)$ gives a window function $V(z)$ of a tight Weyl–Heisenberg frame. Conversely, any FIR tight Weyl–Heisenberg frame in $\ell^2(\boldsymbol{Z})$ with the oversampling ratio K/N can be obtained in this manner. □

Case 3) K and N have a common factor other than N: It can be easily observed that the paraunitariness of $\boldsymbol{V}(z)$ is equivalent to the paraunitariness of its submatrices $\boldsymbol{V}_i(z), i = 0, \cdots, N/J-1$, each of dimension $L \times J$. Note that L and J are coprime, and all submatrices $\boldsymbol{V}_i(z)$ have the same structure, in terms of the distribution of delay elements, as a $\boldsymbol{V}(z)$ matrix corresponding to a frame $\Phi_{v,J,L}$. For instance, in Example 3, $\boldsymbol{V}(z)$ is paraunitary if and only if its submatrices

$$\boldsymbol{V}_0(z) = \begin{bmatrix} V_0(z^3) & z^{-1}V_6(z^3) \\ z^{-2}V_8(z^3) & V_2(z^3) \\ z^{-1}V_4(z^3) & z^{-2}V_{10}(z^3) \end{bmatrix} \quad (17)$$

$$\boldsymbol{V}_1(z) = \begin{bmatrix} V_1(z^3) & z^{-1}V_7(z^3) \\ z^{-2}V_9(z^3) & V_3(z^3) \\ z^{-1}V_5(z^3) & z^{-2}V_{11}(z^3) \end{bmatrix} \quad (18)$$

are paraunitary [compare these submatrices with $\boldsymbol{V}(z)$ in Example 2]. According to the considerations in Case 2, it follows that paraunitariness of the submatrices $\boldsymbol{V}_i(z)$ is equivalent to paraunitariness of certain matrices $\boldsymbol{U}_i(z), i = 0, \cdots, N/J-1$, whose entries are up to a delay equal to the polyphase components of $V(z)$. For the matrix $\boldsymbol{V}(z)$ in Example 3, the corresponding matrices $\boldsymbol{U}_i(z)$ are given by

$$\boldsymbol{U}_0(z) = \begin{bmatrix} V_0(z) & V_6(z) \\ V_8(z) & zV_2(z) \\ V_4(z) & V_{10}(z) \end{bmatrix}$$

$$\boldsymbol{U}_1(z) = \begin{bmatrix} V_1(z) & V_7(z) \\ V_9(z) & zV_3(z) \\ V_5(z) & V_{11}(z) \end{bmatrix}. \quad (19)$$

Therefore, a prototype lowpass filter $\boldsymbol{V}(z)$ of any tight Weyl–Heisenberg frame with the oversampling ratio K/N can be obtained from N/J paraunitary matrices of size $L \times J$ by identifying their entries with the M polyphase components of $V(z)$. □

Further parameterizations of rectangular paraunitary matrices follow immediately from the parameterizations of square paraunitary matrices studied by Vaidayanathan [2] and from the fact that any $L \times J$ paraunitary matrix $(L > J)$ can be embedded into an $L \times L$ paraunitary matrix [10], [11].

IV. Design Example

The results presented in the previous section provide a complete parameterization of tight Weyl–Heisenberg frames of finite support in $\ell^2(\boldsymbol{Z})$ for any given rational oversampling ratio. Design of tight frames of this kind then amounts to an optimization procedure under these constraints.

The critically sampled case $N = K$ gives tight frames with no redundancy, that is, orthonormal bases. From the condition given in (13), we have that $\Phi_{v,K,K}$ is an orthonormal basis only if

$$\tilde{V}_i(z)V_i(z) = c, i = 0, 1, \cdots, K-1. \tag{20}$$

This result has two-fold implications. First, short-time Fourier analysis with critical sampling does not allow for overlaps between analysis windows. This creates blocking effects if some processing in the Fourier domain followed by synthesis is to be done. The other disadvantage of orthonormal Weyl–Heisenberg bases is that the only admissible window is the rectangular window of length K, which is not sufficient for good frequency resolution [9]. In the oversampled case $(N < K)$, no similar restriction is imposed on the window length. The acquired design freedom is illustrated by the following example.

Example 4: Consider the cases $K = 16$ and $N = 8$. With the additional requirement that the prototype filter $V(z)$ is symmetric, the design consists of finding a set of eight filters $F_i(z), i = 0, \cdots, 7$, each of which is orthogonal to its translates by integer multiples of 2. These filters can be represented in terms of their polyphase components as $F_i(z) = F_{i0}(z^2) + z^{-1}F_{i1}(z^2)$. Let

$$V(z) = \sum_{i=0}^{15} z^{-i} V_i(z^{16})$$

be the polyphase decomposition of $V(z)$. The design constraints are then satisfied by taking

$$V_i(z) = F_{i0}(z), V_{i+8}(z) = F_{i1}(z), \qquad i = 0, 1, 2, 3$$

and the remaining polyphase components $V_i(z)$ as the time-reversed versions of $V_{15-i}(z)$, respectively. The filter v obtained in this way is a symmetric window of a tight Weyl–Heisenberg frame $\Phi_{v,8,16}$. The magnitude response of an example window of this kind, with 64 taps, is shown in the bottom part of Fig. 2. The magnitude response is plotted in the frequency range from 0–0.25 of the sampling frequency. For comparison, the top part of Fig. 2 shows the magnitude response of the rectangular window of length 16, that is, the window of the orthonormal basis $\Phi_{v,16,16}$.

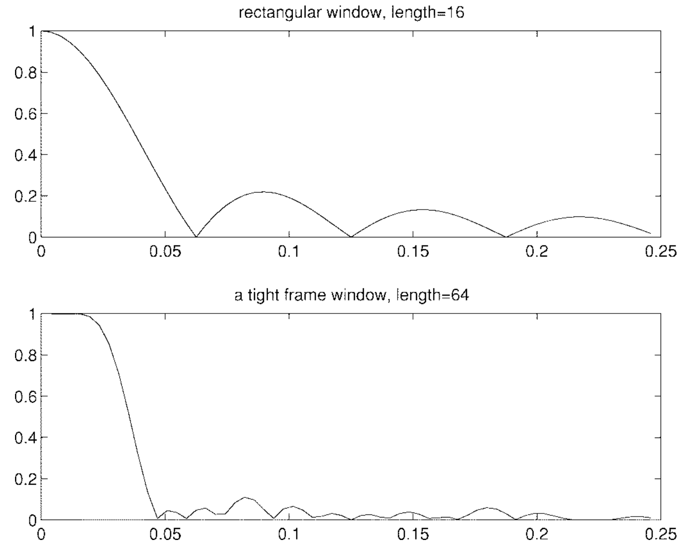

Fig. 2. Magnitude responses of the windows for tight Weyl-Heisenberg frames in $\ell^2(\boldsymbol{Z})$. (Top) Sixteen-tap rectangular window for the orthonormal basis $\Phi_{v,16,16}$. (Bottom) Sixty four-tap window for a $\Phi_{v,8,16}$ tight frame.

V. Conclusion

Tight Weyl–Heisenberg frames in $\ell^2(\boldsymbol{Z})$ were studied in this paper using filter bank tools. A complete parameterization of these frames with finite support in time was given. It was also demonstrated through a design example that redundant Weyl–Heisenberg families allow for windows with improved frequency selectivity over critically sampled ones.

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