FINITE AXISYMMETRIC DEFORMATIONS OF AN INITIALLY CYLINDRICAL MEMBRANE REINFORCED WITH INEXTENSIBLE CORDS

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[Received 1 December 1969]

SUMMARY

We consider the axisymmetric deformations of an initially cylindrical membrane composed of an elastic, homogeneous, isotropic and incompressible material possessing a strain energy function and reinforced by two families of perfectly flexible and inextensible helicoidal cords. The solutions obtained include a wide range of deformations and they are valid for the general strain energy function. Applications to specific boundary-value problems are also given.

1. Introduction

THE theory of large elastic deformations of reinforced bodies, and in particular of reinforced membranes, has been developed by J. E. Adkins and R. S. Rivlin who also obtained the solutions to a number of problems. Most of these solutions are given by Green and Adkins (**1**) who also give references to original sources. The closely related theory of deformation of networks of inextensible cords has been formulated by Rivlin (**2**) whose paper motivated the problem described here.

Apart from their intrinsic interest, the deformations of reinforced solids present a technological interest as well since reinforcement by inextensible cords forms a common feature of such manufactured articles as pneumatic tyres and hose-pipes in which it is necessary to give added strength to the composite body.

Because of the complexity of the equations involved, the number of exact solutions obtained so far to problems in the theory of finite deformations of elastic membranes is limited. Some further solutions can be derived if a particular simplified form is assumed for the strain energy function but, even with this assumption, the number of existing solutions remains small.

In this paper we consider axisymmetric deformations of a cylindrical membrane reinforced with two families of inextensible helicoidal cords. The general solution obtained in section 2 is valid for the general strain energy function and a wide range of deformations compatible with the not very restrictive geometrical conditions assumed. In section 3 we describe

how the general solution can be applied to three different classes of problems and also give specific examples.

The notation adopted is, in the main, that of Green and Adkins (**1**) who give a full account of the theory of finite deformations and, in particular, of membrane theory. The reader is referred to (**1**) for details of the notation and the derivations of the basic formulae.

2. General equations

We consider axisymmetric deformations of an initially cylindrical membrane of uniform thickness $2h_0$ composed of an elastic, homogeneous, isotropic and incompressible material possessing a strain energy function $W = W(I_1, I_2)$ and reinforced by two families of perfectly flexible and inextensible cords.

We assume that the cords of the two families form constant angles $\pm\alpha$ with the generators of the undeformed membrane. It is also assumed that the lengths of the intercepts on any one cord of the one family by two adjacent cords of the other family are independent of the position on the surface and that these lengths are small compared with the radii of curvature at any point of the deformed or undeformed membrane. Moreover, it is assumed that no two cords of the same family are brought into contact as a result of the deformation and that intersecting cords of the two families do not move relative to each other at their point of intersection.

As any cylindrical membrane can be deformed into a circular cylinder without extension on its surface, it can be assumed, without loss of generality, that the undeformed membrane is a circular cylinder.

We refer the deformation to cylindrical polar coordinates and denote by (ρ, ϑ, η), ρ = constant, the coordinates in the undeformed configuration of a point which has coordinates (r, ϑ, z) in the deformed state. Because the deformation is axially symmetric,

$$r = r(\eta), \qquad z = z(\eta). \tag{2.1}$$

The elements of length in the undeformed and deformed configurations will be denoted by dS and ds respectively. The angle of dS with the generator of the undeformed membrane will be denoted by a and the element of the deformed meridian by $d\xi$. Thus

$$\left(\frac{ds}{dS}\right)^2 = \left(\frac{d\xi}{dS}\right)^2 + r^2\left(\frac{d\vartheta}{dS}\right)^2 = \left(\frac{d\xi}{d\eta}\right)^2\cos^2 a + \left(\frac{r}{\rho}\right)^2\sin^2 a,$$

whence

$$\left(\frac{ds}{dS}\right)^2 = \lambda_1^2\cos^2 a + \lambda_2^2\sin^2 a, \tag{2.2}$$

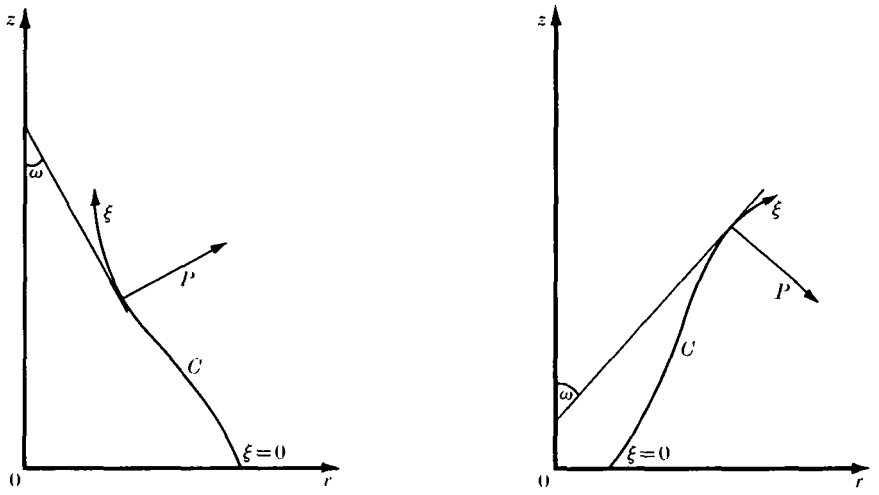

FIG. 1. Axial sections of the deformed membrane.

where we have used the notation

$$\lambda_1 = d\xi/d\eta, \qquad \lambda_2 = r/\rho. \tag{2.3}$$

From the symmetry of the system and the incompressibility of the material it follows that λ_1, λ_2 and $\lambda_3 = (\lambda_1\lambda_2)^{-1}$ are the principal extension ratios in terms of which the strain invariants I_α are given by

$$I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2, \qquad I_2 = \lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2}. \tag{2.4}$$

Since the cords are inextensible, for $a = \alpha$ we obtain from (2.2)

$$\lambda_1^2 \cos^2\alpha+\lambda_2^2 \sin^2\alpha = 1, \tag{2.5}$$

which gives λ_1 in terms of λ_2 and α.

Let $\pm\beta(\eta)$ be the angles of the two families of cords with the meridian of the deformed section. Then, if dS, ds are on the same cord,

$$r\rho^{-1} \sin\alpha\, dS = \sin\beta\, ds, \qquad \cos\beta\, ds = d\xi = \lambda_1 d\eta = \lambda_1 \cos\alpha\, dS.$$

From these relations we obtain

$$\sin\beta = \lambda_2 \sin\alpha, \qquad \cos\beta = \lambda_1 \cos\alpha. \tag{2.6}$$

Let Δ and δ denote the distances between adjacent cords in the undeformed and deformed membrane respectively. From

$$\Delta = \rho \cos\alpha\, d\vartheta, \qquad \delta = r \cos\beta\, d\vartheta$$

and (2.6) we obtain

$$\delta = \lambda_1\lambda_2\, \Delta. \tag{2.7}$$

We now introduce the further assumption that the meridian C of the deformed membrane does not intersect the z-axis and it has no finite element parallel or perpendicular to the z-axis. Moreover, and without loss of generality, it is assumed that the tangent to C is nowhere parallel or perpendicular to the z-axis, except perhaps at an end point of C; otherwise C may be divided into segments for which this condition does hold. With this assumption the equations of equilibrium

$$\frac{d}{d\xi}(rT_1) = T_2\frac{dr}{d\xi}, \qquad \kappa_1 T_1 + \kappa_2 T_2 = P,$$

which are valid for any axisymmetric deformation of a membrane, can be written in the form

$$\frac{d}{dr}(rT_1) = T_2, \qquad \kappa_1 T_1 + \kappa_2 T_2 = P, \tag{2.8}$$

where T_1, T_2 are the stress resultants, per unit length of the deformed membrane, in the directions of the meridian and the circle of latitude respectively, κ_α the principal curvatures, κ_1 being the curvature of the meridian, and P is the internal pressure. If $\omega < \pi/2$ denotes the angle formed by the tangent to the meridian of the deformed membrane and the axis of symmetry then

$$\kappa_1 = \frac{d}{dr}\cos\omega, \qquad \kappa_2 = r^{-1}\cos\omega. \tag{2.9}$$

By using (2.3) and (2.9) the equations (2.8) are reduced to

$$\frac{d}{d\lambda_2}(\lambda_2 T_1) = T_2, \qquad \frac{d}{d\lambda_2}(\lambda_2 T_1\cos\omega) = P\rho\lambda_2. \tag{2.10}$$

The stress resultants T_α can be resolved into two parts:

$$T_\alpha = T'_\alpha + T''_\alpha, \tag{2.11}$$

where T'_α is due to the deformation of the material and can be expressed in terms of the strain energy function W by

$$\left.\begin{aligned} T'_1 &= 4h_0\lambda_3(\lambda_1^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right) = \frac{2h_0}{\lambda_2}\frac{\partial W}{\partial \lambda_1}, \\ T'_2 &= 4h_0\lambda_3(\lambda_2^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_1^2\frac{\partial W}{\partial I_2}\right) = \frac{2h_0}{\lambda_1}\frac{\partial W}{\partial \lambda_2}, \end{aligned}\right\} \tag{2.12}$$

and T''_α is due to the tension in the cords. The expressions $(2.12)_{2,4}$ for the parts T'_α of the stress resultants were given by Pipkin (3).

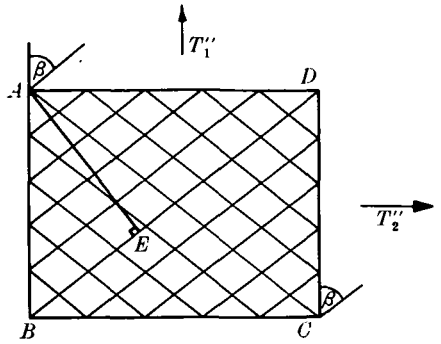

FIG. 2. Element of area of the deformed membrane.

By considering the element of area ABCD of the deformed configuration shown in Fig. 2 we obtain

$$T_1'' r\,d\vartheta = 2\left(\tau\frac{AE}{\delta}\right)\cos\beta = 2\tau\,\frac{r\cos\beta\,d\vartheta}{\lambda_1\lambda_2\Delta}\cos\beta = \frac{2\tau\lambda_1^2\cos^2\alpha}{\lambda_1\lambda_2\Delta}r\,d\vartheta = \frac{2\tau\lambda_1\cos^2\alpha}{\lambda_2\Delta}rd\vartheta,$$

$$T_2''\,d\xi = 2\left(\tau\,\frac{AE}{\delta}\right)\sin\beta = 2\tau\,\frac{\sin\beta\,d\xi}{\lambda_1\lambda_2\Delta}\,\sin\beta = \frac{2\tau\lambda_2^2\sin^2\alpha}{\lambda_1\lambda_2\Delta}\,d\xi = \frac{2\tau\lambda_2\sin^2\alpha}{\lambda_1\Delta}\,d\xi,$$

whence

$$T_1'' = \frac{2\cos^2\alpha}{\Delta}\frac{\lambda_1}{\lambda_2}\tau, \qquad T_2'' = \frac{2\sin^2\alpha}{\Delta}\frac{\lambda_2}{\lambda_1}\tau. \tag{2.13}$$

By using (2.11) to (2.13) the first equation of equilibrium $(2.10)_1$ is reduced to

$$\frac{d\tau}{d\lambda_2} - \frac{2\lambda_2\sin^2\alpha}{1-\lambda_2^2\sin^2\alpha}\tau = \frac{h_0\Delta}{1-\lambda_2^2\sin^2\alpha}\frac{d}{d\lambda_2}\left(W-\lambda_1\frac{\partial W}{\partial\lambda_1}\right) \tag{2.14}$$

with solution

$$\tau = \frac{\Delta h_0\{W-\lambda_1(\partial W/\partial\lambda_1)\}+A}{1-\lambda_2^2\sin^2\alpha}, \tag{2.15}$$

where A is an integration constant.

From (2.4), (2.5), (2.13) and (2.15) we obtain T_α'' as functions of λ_2. This result together with (2.4), (2.5), (2.11) and (2.12) gives T_α in terms of λ_2 and the integration constant A.

3. Applications

With the theory described in the previous section three different kinds of problems can now be solved.

A. *The meridian of the deformed membrane is specified*

This type of problem arises if, for example, the deformed membrane encloses a rigid axisymmetric body of a given shape and all the membrane is in contact with the enclosure.

In this case ω is given in terms of $\lambda_2 = r/\rho$ by the equation of the meridian of the enclosure and T_α has already been determined in terms of λ_2. Thus the internal pressure P applied to the membrane is easily obtained from $(2.10)_2$.

B. *The internal pressure P is given in terms of λ_2*

Then, by integrating $(2.10)_2$, we obtain

$$\omega = \cos^{-1}\left[\left(\tfrac{1}{2}\rho\int_{\Lambda_2}^{\lambda_2} P\lambda_2\, d\lambda_2 + B\right)\left(h_0\frac{\partial W}{\partial\lambda_1} + \frac{\cos^2\alpha}{\Delta}\lambda_1\tau\right)^{-1}\right], \tag{3.1}$$

where B is an integration constant and Λ_2 the value of λ_2 at $z = 0$. The formula (3.1) determines the equation of the deformed meridian in the form of a relation between $r = \lambda_2\rho$ and ω.

From Fig. 1 it is readily seen that

$$\frac{z}{\rho} = \pm\int_{\Lambda_2}^{\lambda_2} \cot\omega\, d\lambda_2, \qquad \frac{\xi}{\rho} = \pm\int_{\Lambda_2}^{\lambda_2} \frac{d\lambda_2}{\sin\omega}, \tag{3.2}$$

where the $+$ or $-$ sign is used according to whether λ_2 is an increasing or a decreasing function of ξ, and $\xi = 0$ for $z = 0$. We also obtain

$$\frac{\eta}{\rho} = \pm\int_{\Lambda_2}^{\lambda_2} \frac{d\lambda_2}{\lambda_1\sin\omega}, \tag{3.3}$$

where η is measured from the cross-section of the undeformed membrane which corresponds to the cross-section $z = 0$ of the deformed configuration. The formulae (3.2) and (3.3) are valid for the deformation of any membrane subject to the conditions stated in section 2.

The constant Λ_2 and the integration constants A and B involved in this solution will be determined by the appropriate boundary conditions of any specific problem.

Let us, for instance, consider the deformation under uniform internal pressure P_0 of an initially cylindrical membrane the edges of which are bonded on circular rings of radius ρ. This corresponds to the physical problem of the deformation under constant pressure of the flexible joint between two co-axial rigid tubes of radius ρ.

If $z = 0$ is the plane of symmetry of the deformed configuration which is perpendicular to its axis of symmetry and $2l_0$, $2l$ denote the axial lengths of the undeformed and deformed membrane respectively then, from $(3.2)_1$

and (3.3), we obtain

$$\frac{l_0}{\rho} = -\int_{\Lambda_2}^{1} \frac{d\lambda_2}{\lambda_1 \sin\omega}, \qquad \frac{l}{\rho} = -\int_{\Lambda_2}^{1} \cot\omega \, d\lambda_2. \tag{3.4}$$

Since $\omega = 0$ for $\lambda_2 = \Lambda_2$ and $P = P_0 = \text{constant}$, equation (3.1) reduces to

$$\omega = \cos^{-1}\left[\left\{\tfrac{1}{4}\rho P_0(\lambda_2^2 - \Lambda_2^2) + \left(h_0\frac{\partial W}{\partial \lambda_1} + \frac{\cos^2\alpha}{\Delta}\lambda_1\tau\right)_{\lambda_2=\Lambda_2}\right\}\left(h_0\frac{\partial W}{\partial \lambda_1} + \frac{\cos^2\alpha}{\Delta}\lambda_1\tau\right)^{-1}\right]. \tag{3.5}$$

From (2.5), (2.15) and (3.5) we derive ω in terms of λ_2 and the integration constant A, and by substituting in (3.4) we obtain l_0 and l in terms of Λ_2 and A. Thus, if the pressure P_0 and the axial lengths of the undeformed and deformed membrane are known, the constants Λ_2 and A can be obtained and the deformed configuration determined.

A solution to this problem for the case of a membrane fabric composed of two families of inextensible cords was given by Rivlin (**2**).

C. *A sector of the membrane deforms into a given shape while the remainder of the membrane is under a given internal pressure* $P = P(\lambda_2)$

This, for example, is the case of the deformation of a cylindrical membrane by an enclosed rigid axisymmetric solid of a given shape and a uniform internal pressure P_0 applied on that part of the membrane which is not in contact with the enclosure. A solution to this problem valid for an elastic non-reinforced membrane was given by Kydoniefs (**4**).

Mixed boundary-value problems of this type can be solved by a method similar to that used in (**4**). The sectors of the membrane which deform to a given shape or under a given pressure are considered separately in the way described in A and B respectively and the solutions obtained are matched. This matching is possible because the integration constant A involved in the solution (2.15) retains the same value over the deformed meridian and can be calculated at any of its points, and in particular at a point of the common boundary of the two sections considered.

The solutions given in this paper are exact, valid for the general strain energy function and include all possible axisymmetric deformations of the reinforced membrane, subject to the not very restrictive conditions stated at the beginning of the previous section. In order to solve specific boundary-value problems it is only necessary to determine the arbitrary constants and evaluate the integrals in (3.1) and (3.2). If a specified form, for example the Mooney (**5**) or the neo-Hookean form suggested by Rivlin (**6**) is assumed for the strain energy function, numerical results for the

problems described in B and C can be obtained, though this involves a fairly long computation.

Acknowledgement

I am very grateful to Professor A. J. M. Spencer for his advice and help. The results described in this paper were obtained in the course of research supported by a contract of the U.S. Department of Commerce, National Bureau of Standards, with Lehigh University.

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