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The relative variation of the vertical gravity gradient on the level ellipsoid

Franco Bocchio
Dipartimento di Georisorse e Territorio, Università di Udine, 33100 Udine, Italy

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SUMMARY

The relative variation of the vertical gravity gradient on an equipotential ellipsoid is given as a function of the geometrical flattening f of the ellipsoid and of the rotational parameter m up to the second order in the quantities f and m. General formulae are also obtained for the relative variation of vertical derivatives of the geopotential with differentiation order higher than two; it is seen that particularly simple first-order formulae are obtained. A possible application of the concept of 'flattening' of the vertical derivatives of the geopotential is also suggested.

Key words: gravity.

1 INTRODUCTION

If we consider the relative variation of the vertical derivatives—with differentiation order $n \geq 2$—of the geopotential W on the equipotential ellipsoid E with semi-axes $a, b,$ we can introduce the quantity

$$f_n = \frac{\left(\dfrac{\partial^n W}{\partial r^n}\right)_b - \left(\dfrac{\partial^n W}{\partial r^n}\right)_a}{\left(\dfrac{\partial^n W}{\partial r^n}\right)_a}, \tag{1.1}$$

which will be denoted, in analogy with the concept of gravity 'flattening' (Heiskanen & Mortiz 1967), as the 'flattening' of the nth vertical derivative of W. The subscripts a and b denote points at the equator and at the poles of E respectively. We observe that, since the radial derivatives coincide with the vertical derivatives at these points on the level ellipsoid, these points are also the extremal points for the value of the vertical derivative of the gravity potential. It will be shown that, if $f = (a - b)/a$ is the geometrical flattening of the ellipsoid and $m = \omega^2 R^3/GM$ is the rotational parameter, we have, to second order in the quantities f and m,

$$f_n = c_{n1} f + c_{n2} m + c_{n3} f^2 + c_{n4} fm + c_{n5} m^2, \tag{1.2}$$

where the coefficients $c_{n1}, c_{n2}, c_{n3}, c_{n4}$ and c_{n5} are constants which are shown to depend only on the differentiation order of the vertical derivative of the gravity potential. To the first order in f and m, it is thus seen that particularly simple formulae of the type

$$f_n = c_{m} f + c_{n2} m + O(2) \tag{1.3}$$

hold for the 'flattening' f_n of the vertical derivative of order n of the geopotential. In Section 2 the case $n = 2$, i.e. the vertical gravity gradient 'flattening' f_2, is considered in detail and the role of the gravitational part of the field is also examined. In Section 3, differentiation orders $n > 2$ of the vertical derivatives of the gravity potential are considered, and the general formulae for the coefficients of f and m in (1.3) are given.

2 THE RELATIVE VARIATION OF THE VERTICAL GRAVITY GRADIENT

We assume that the gravity potential is modelled by the function

$$W = V + \Phi = \frac{GM}{r}\left[1 - J_2\left(\frac{a}{r}\right)^2 P_2 - J_4\left(\frac{a}{r}\right)^4 P_4\right] + \frac{\omega^2 r^2}{3}(1 - P_2), \tag{2.1}$$

V and Φ being the gravitational and centrifugal potentials, respectively, P_2 and P_4 Legendre polynomials, ω the rotational speed, GM the geocentric constant, and a the semi-major axis of the equipotential ellipsoid.

In a second-order approach, the equation of E can be written

$$r = R(1 + \epsilon_2 P_2 + \epsilon_4 P_4), \tag{2.2}$$

where

$$R = a\left(1 - \frac{f}{3} - \frac{f^2}{5}\right) \tag{2.3}$$

is the ellipsoid's mean radius and

$$\epsilon_2 = -\tfrac{2}{3}(f + \tfrac{23}{42}f^2), \qquad \epsilon_4 = \tfrac{12}{35}f^2. \tag{2.4}$$

To the present degree of accuracy, we have

$$(1 + \epsilon_2 P_2 + \epsilon_4 P_4)^n = 1 + n\epsilon_2 P_2 + n\epsilon_4 P_4 + \frac{n(n-1)}{2}\epsilon_2^2 P_2^2. \tag{2.5}$$

It is known (Heiskanen & Moritz 1967) that the coefficients J_2 and J_4 in (2.1) are given by

$$\left.\begin{aligned} J_2 &= \frac{2}{3}f - \frac{m}{3} - \frac{f^2}{3} + \frac{2}{21}mf, \\ J_4 &= -\frac{4}{5}f^2 + \frac{4}{7}mf. \end{aligned}\right\} \tag{2.6}$$

We have from (2.1) that

$$\frac{\partial^2 W}{\partial r^2} = \frac{2GM}{r^3}\left[1 - 6\left(\frac{a}{r}\right)^2 J_2 P_2 - 15\left(\frac{a}{r}\right)^4 J_4 P_4\right] + \frac{2\omega^2}{3}(1 - P_2). \tag{2.7}$$

By substitution of (2.2) into (2.7) we see that, on the equipotential ellipsoid, the derivative (2.7) takes the values

$$\begin{aligned} \left(\frac{\partial^2 W}{\partial r^2}\right)_E = \frac{2GM}{R^3}&(1 - 3\epsilon_2 P_2 - 3\epsilon_4 P_4 + 6\epsilon_2^2 P_2^2 - 6J_2 P_2 \\ &+ 30J_2\epsilon_2 P_2^2 - 4fJ_2 P_2 - 15J_4 P_4) \\ &+ \frac{2\omega^2}{3}(1 - P_2), \end{aligned} \tag{2.8}$$

which at the poles and at the equator of E coincides with the vertical gravity gradient. Since at these points, referred to in what follows with the subscripts b and a, respectively, $P_2(1) = P_4(1) = 1$, $P_2(0) = -1/2$, and $P_4(0) = 3/8$, we have

$$\left.\begin{aligned} \left(\frac{\partial^2 W}{\partial r^2}\right)_a &= 2GMR^{-3}\Big(1 + \frac{3}{2}\epsilon_2 + \frac{m}{2} + 3J_2 - \frac{9}{8}\epsilon_4 + \frac{3}{2}\epsilon_2^2 \\ &\qquad + \frac{15}{2}\epsilon_2 J_2 + 2fJ_2 - \frac{45}{8}J_4\Big), \\ \left(\frac{\partial^2 W}{\partial r^2}\right)_b &= 2GMR^{-3}(1 - 3\epsilon_2 - 6J_2 - 3\epsilon_4 + 6\epsilon_2^2 \\ &\qquad + 30\epsilon_2 J_2 - 4J_2 f - 15J_4). \end{aligned}\right\} \tag{2.9}$$

On the level ellipsoid, the relative variation of the vertical gravity gradient, denoted as gravity gradient 'flattening', can thus be defined by

$$f_2 = \frac{\left(\frac{\partial^2 W}{\partial r^2}\right)_b - \left(\frac{\partial^2 W}{\partial r^2}\right)_a}{\left(\frac{\partial^2 W}{\partial r^2}\right)_a}. \tag{2.10}$$

We obtain

$$\begin{aligned} f_2 = \Big(&-\frac{9}{2}\epsilon_2 - 9J_2 - \frac{m}{2} - \frac{15}{8}\epsilon_4 + \frac{9}{2}\epsilon_2^2 + \frac{45}{2}\epsilon_2 J_2 \\ &- 6J_2 f - \frac{75}{8}J_4\Big)\Big(1 - \frac{3}{2}\epsilon_2 - \frac{m}{2} - 3J_2\Big). \end{aligned} \tag{2.11}$$

Considering (2.4) and (2.6), it follows that

$$f_2 = -3f + \tfrac{5}{2}m + \tfrac{5}{2}f^2 - \tfrac{45}{14}mf + \tfrac{5}{4}m^2. \tag{2.12}$$

If only the gravitational part of the field were considered, and the parameter

$$f_2' = \frac{\left(\frac{\partial^2 V}{\partial r^2}\right)_b - \left(\frac{\partial^2 V}{\partial r^2}\right)_a}{\left(\frac{\partial^2 V}{\partial r^2}\right)_a} \tag{2.13}$$

is introduced as the vertical gravitational gradient 'flattening', we see that, with

$$\left.\begin{aligned} \left(\frac{\partial^2 V}{\partial r^2}\right)_a &= 2GMR^{-3}(1 + \tfrac{3}{2}\epsilon_2 + 3J_2 - \tfrac{9}{8}J_4 + \tfrac{3}{2}\epsilon_2^2 \\ &\qquad + \tfrac{15}{2}\epsilon_2 J_2 + 2J_2 f - \tfrac{45}{8}J_4), \\ \left(\frac{\partial^2 V}{\partial r^2}\right)_b &= 2GMR^{-3}(1 - 3\epsilon_2 - 6J_2 - 3\epsilon_4 + 6\epsilon_2^2 \\ &\qquad + 30\epsilon_2 J_2 - 4J_2 f - 15J_4), \end{aligned}\right\} \tag{2.14}$$

we obtain

$$f_2' = -3f + 3m - \tfrac{15}{4}f^2 + \tfrac{11}{28}mf - 3m^2. \tag{2.15}$$

Notice that f_2' depends also on the rotational parameter m; this is due to the fact that it has been evaluated on E, which is equipotential for the gravity potential, not for the gravitational potential, and therefore its shape is conditioned by rotation.

We see that the first-order formulae

$$f_2 = -3f + \tfrac{5}{2}m + O(2) \tag{2.16}$$

and

$$f_2' = -3f + 3m + O(2), \tag{2.17}$$

from which we have also

$$f_2' - f_2 = \frac{m}{2} + O(2), \tag{2.18}$$

display a particular simple structure, closely resembling that of the first-order Clairaut formula; in the next section it will be seen that this is also true for the 'flattenings' of the vertical derivatives of the gravity potential of any differentiation order.

3 THE RELATIVE VARIATION OF THE *n*th ($n > 2$) VERTICAL DERIVATIVE OF THE GRAVITY POTENTIAL

It can be seen that we have

$$\begin{aligned} \frac{\partial^n W}{\partial r^n} = (-1)^n GM\Big[&n!\,r^{-(n+1)} - P_2 J_2 a^2 \frac{(n+2)!}{2} r^{-(n+3)} \\ &- P_4 J_4 a^4 \frac{(n+4)!}{24} r^{-(n+5)}\Big]. \end{aligned} \tag{3.1}$$

By substituting the radius r of the equipotential ellipsoid (2.2) into (3.1), we obtain the value of $\partial^n W/\partial r^n$ on E, i.e.

$$\left(\frac{\partial^n W}{\partial r^n}\right)_E = (-1)^n GMR^{-(n+1)}\Big[n! - n!(n+1)\epsilon_2 P_2 - \frac{(n+2)!}{2}J_2P_2 - n!(n+1)\epsilon_4 P_4 + \frac{n!(n+1)(n+2)}{2}\epsilon_2^2 P_2^2 + \frac{(n+2)!(n+3)}{2}\epsilon_2 J_2 P_2 - \frac{(n+2)!}{3} fJ_2P_2 - \frac{(n+4)!}{24}J_4P_4\Big]. \tag{3.2}$$

If we now proceed as in the previous section, we obtain

$$\left.\begin{aligned}\left(\frac{\partial^n W}{\partial r^n}\right)_a &= (-1)^n GMR^{-(n+1)}\Big[n! + \frac{n!(n+1)}{2}\epsilon_2 + \frac{(n+2)!}{4}J_2 - \frac{3n!(n+1)}{8}\epsilon_4 + \frac{n!(n+1)(n+2)}{8}\epsilon_2^2 + \frac{(n+2)!(n+3)}{8}\epsilon_2 J_2 + \frac{(n+2)!}{6}fJ_2 - \frac{(n+4)!}{64}J_4\Big],\\ \left(\frac{\partial^n W}{\partial r^n}\right)_b &= (-1)^n GMR^{-(n+1)}\Big[n! - n!(n+1)\epsilon_2 - \frac{(n+2)!}{2}J_2 - n!(n+1)\epsilon_4 + \frac{n!(n+1)(n+2)}{2}\epsilon_2^2 + \frac{(n+2)!(n+3)}{2}\epsilon_2 J_2 - \frac{(n+2)!}{3}fJ_2 - \frac{(n+4)!}{24}J_4\Big].\end{aligned}\right\} \tag{3.3}$$

If we now define the 'flattening' f_n of the vertical nth derivative of the gravity potential as in (1.1), we see that

$$f_n = -\frac{3(n+1)}{2}\epsilon_2 - \frac{3(n+2)!}{4n!}J_2 - \frac{5(n+1)}{8}\epsilon_4 + \frac{3(n+1)(3n+4)}{8}\epsilon_2^2 + \frac{9(n+2)!(n+2)}{n!}\epsilon_2 J_2 - \frac{(n+2)!}{2n!}fJ_2 - \frac{5(n+4)!}{192n!}J_4 + \frac{3}{16}\left[\frac{(n+2)!}{n!}\right]^2 J_2^2. \tag{3.4}$$

To the first order in f and m, it follows that

$$f_n = A(n)f + B(n)m + O(2), \tag{3.5}$$

where the coefficients

$$A(n) = -\frac{n(n+1)}{2}, \tag{3.6}$$

$$B(n) = \frac{(n+1)(n+2)}{4} \tag{3.7}$$

depend only on the differentiation order n of the vertical derivative of the gravity potential.

4 AN APPLICATION OF THE CONCEPT OF 'FLATTENING' OF THE VERTICAL DERIVATIVES OF THE GRAVITY POTENTIAL

Let us denote by f_1 the flattening of the first derivative of the gravity potential, i.e.

$$f_1 = \frac{\left(\frac{\partial W}{\partial r}\right)_b - \left(\frac{\partial W}{\partial r}\right)_a}{\left(\frac{\partial W}{\partial r}\right)_a}, \tag{4.1}$$

which according to the first-order Clairaut formula is given by

$$f_1 = -f + \tfrac{5}{2}m + O(2). \tag{4.2}$$

Considering the results given in Section 2, we can write

$$f = \frac{f_1 - f_2}{2} + O(2). \tag{4.3}$$

It is also seen that if eqs (3.5) to (3.7) are taken into account, for example up to $n = 4$, it follows that

$$f = f_2 + f_3 - f_4 + O(2). \tag{4.4}$$

This procedure can be generalized and we can conclude that the following relation holds:

$$f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n + O(2), \tag{4.5}$$

where $a_1, a_2, \ldots, a_n$ are suitable constants. We see, therefore, that the geometrical flattening f of the base ellipsoid can be obtained as a linear combination of 'flattenings' of the vertical derivatives of the geopotential.

REFERENCES

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