A Separation Principle for Non-UCO Systems

Manfredi Maggiore
Dept. of Electrical and Computer Eng.
University of Toronto
10 King's College Rd, Toronto, ON
M5S3G4

Kevin Passino
Dept. of Electrical Eng.,
The Ohio State University
2015 Neil Avenue, Columbus, OH
43210-1272

Systems Control Group Report No. 0203
02.08.08

Abstract

This paper introduces a new approach to output feedback stabilization of SISO systems which, unlike other techniques found in the literature, does not use quasi-linear high-gain observers and control input saturation to achieve separation between the state feedback and observer designs. Rather, we show that by using nonlinear observers working in state coordinates, together with a dynamic projection algorithm, the same kind of separation principle is achieved for the larger class of non-UCO systems, i.e., systems which are not *uniformly completely observable*. By working in state coordinates, this new approach avoids using knowledge of the inverse of the observability mapping to estimate the state of the plant, which is otherwise needed when using high-gain observers to estimate the output time derivatives.

1 Introduction

The area of nonlinear output feedback control has received much attention after the publication of the work [1], in which the authors developed a systematic strategy for the output feedback control of input-output linearizable systems with full relative degree, which employed two basic tools: an high-gain observer to estimate the derivatives of the outputs (and hence the system states in transformed coordinates), and control input saturation to isolate the peaking phenomenon of the observer from the system states. Essentially the same approach has later been applied in a number of papers by various researchers (see, e.g., [2, 3, 4, 5, 6, 7]) to solve different problems in output feedback control. In most of the papers found in the literature, (see, e.g., [1, 2, 3, 5, 4, 6]) the authors consider input-output feedback linearizable systems with either full relative degree or minimum phase zero dynamics. The work in [8] showed that for nonminimum phase systems the problem can be solved by extending the system dynamics with a chain of integrators at the input side. However, the results contained there are local. In [9], by putting together this idea with the approach found in [1], the authors were able to show how to solve the output feedback stabilization problem for smoothly stabilizable and uniformly completely observable (UCO) systems. The work in [7] unifies these approaches to prove a separation principle for a rather general class of nonlinear systems. The recent work in [10] relaxes the uniformity requirement of the UCO assumption by assuming the existence of one control input for which the system is observable. On the other hand, however, [10] requires the observability property to be complete, i.e., to hold on the entire state space. Another feature of that work is the relaxation of the smooth stabilizability assumption, replaced by the notion of asymptotic controllability (which allows for possibly non-smooth stabilizers).

A common feature of the papers mentioned above is their *input-output variable approach*, which entails using the vector $[y, \dot{y}, \ldots, y^{(n_y)}, u, \dot{u}, \ldots, u^{(n_u)}]^\top$ as feedback, for some integers n_y, n_u, where y and u denote the system output and input, respectively. This in particular implies that, when dealing with systems which are not input-output feedback linearizable, such approach requires the explicit knowledge of the inverse of the observability mapping, which in some cases may not be available.

This paper develops a different methodology for output feedback stabilization which achieves a separation principle for systems which are not UCO, i.e., systems that are observable on some regions of the state space and input space, rather than everywhere. For the implementation of our controller, the inverse of the observability mapping is not needed. These features are achieved by means of a *state-variable approach* employing two new tools: a nonlinear observer working in state coordinates (which is proved to be equivalent to the standard high-gain observer in output coordinates), and a dynamic projection operating on the observer dynamics which *eliminates* the peaking phenomenon in the observer states, thus avoiding the need to use control input saturation.

It is proved that, provided the observable region satisfies suitable topological properties, the proposed methodology yields closed-loop stability. In the particular case when the plant is globally stabilizable and UCO, this approach yields semiglobal stability, as in [9], *provided* a convexity requirement is satisfied.

2 Problem Formulation and Assumptions

Consider the following dynamical system,

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x,u)\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u, y \in \mathbb{R}$, f and h are known smooth functions, and $f(0,0) = 0$. Our control objective is to construct a stabilizing controller for (1) without the availability of the system states x. In order to do so, we need an observability assumption. Define the observability mapping

$$y_e \triangleq \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{H}\left(x, u, \ldots, u^{(n_u-1)}\right) \triangleq \begin{bmatrix} h(x,u) \\ \varphi_1(x,u,u^{(1)}) \\ \vdots \\ \varphi_{n-1}\left(x,u,\ldots,u^{(n_u-1)}\right) \end{bmatrix} \tag{2}$$

($y^{(n-1)}$ is the $n-1$-th derivative) where

$$\begin{aligned}\varphi_1(x,u,u^{(1)}) &= \frac{\partial h}{\partial x} f(x,u) + \frac{\partial h}{\partial u} u^{(1)}\\ \varphi_2(x,u,u^{(1)},u^{(2)}) &= \frac{\partial \varphi_1}{\partial x} f(x,u) + \frac{\partial \varphi_1}{\partial u} u^{(1)} + \frac{\partial \varphi_1}{\partial u^{(1)}} u^{(2)}\\ &\vdots\\ \varphi_{n-1}\left(x,u,\ldots,u^{(n_u-1)}\right) &= \frac{\partial \varphi_{n-2}}{\partial x} f(x,u) + \frac{\partial \varphi_{n-2}}{\partial u} u^{(1)} + \ldots + \frac{\partial \varphi_{n-2}}{\partial u^{(n_u-2)}} u^{(n_u-1)}\end{aligned} \tag{3}$$

where $0 \leq n_u \leq n$ ($n_u = 0$ indicates that there is no dependence on u). In the most general case, $\varphi_i = \varphi_i(x,u,\ldots,u^{(i)})$, $i = 1,2,\ldots,n-1$. In some cases, however, we may have that $\varphi_i = \varphi_i(x,u)$ for all $i = 1,\ldots,r-1$ and some integer $r > 1$. This happens in particular when system (1) has a well-defined relative degree r. Here, we do not require the system to be input-output feedback linearizable, and hence to possess a well-defined relative degree. In the case of systems with well-defined relative degree, $n_u = 0$ corresponds to having $r \geq n$, while $n_u = n$ corresponds to having $r = 0$. Next, augment the system dynamics with n_u integrators on the input side, which corresponds to using a compensator of order n_u. System (1) can be rewritten as follows,

$$\dot{x} = f(x,z_1),\ \dot{z}_1 = z_2,\ \ldots,\ \dot{z}_{n_u} = v \tag{4}$$

Define the extended state variable $X = [x^\top, z^\top]^\top \in \mathbb{R}^{n+n_u}$, and the associated *extended system*

$$\begin{aligned}\dot{X} &= f_e(X) + g_e v\\ y &= h_e(X)\end{aligned} \tag{5}$$

where $f_e(X) = [f^\top(x,z_1),\ z_2, \ldots, z_{n_u}, 0]^\top$, $g_e = [0, \ldots, 1]^\top$, and $h_e(X) = h(x,z_1)$. Now, we are ready to state our first assumption.

Assumption A1(Observability): System (1) is observable over an open set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ containing the origin, i.e., the mapping $\mathcal{F} : \mathcal{O} \to \mathcal{Y}$ (where $\mathcal{Y} = \mathcal{F}(\mathcal{O})$) defined by

$$Y = \begin{bmatrix} y_e \\ \dots \\ z \end{bmatrix} = \mathcal{F}(X) = \begin{bmatrix} \mathcal{H}(x, z) \\ z \end{bmatrix} \tag{6}$$

has a smooth inverse $\mathcal{F}^{-1} : \mathcal{Y} \to \mathcal{O}$,

$$\mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}(y_e, z) = \begin{bmatrix} \mathcal{H}^{-1}(y_e, z) \\ z \end{bmatrix}. \tag{7}$$

Remark 1: In the existing literature, an assumption similar to A1 can be found in [8] and [9]. It is worth stressing, however, that in both works the authors adopt a global observability assumption, i.e., the set $\mathcal{O}$ is taken to be $\mathbb{R}^n \times \mathbb{R}^{n_u}$. In many practical applications the system under consideration may be observable in some subset of $\mathbb{R}^n \times \mathbb{R}^{n_u}$ only, thus preventing the use of most of the output feedback techniques found in the literature, including the ones found in [8], [9], and [7]. On the other hand in A1 the mapping $\mathcal{H}$, viewed as a mapping acting on x parameterized by z, is assumed to be square (i.e., it maps spaces of equal dimension), thus implying that x can be expressed as a function of y, its $n-1$ time derivatives and z, i.e., $x = \mathcal{H}^{-1}([y, \dot{y}, \dots, y^{(n-1)}]^\top, z)$. In the works [9, 10], x is allowed to be a function depending on a possibly higher number of derivatives of y, rather than just $n-1$. In our setting, this is equivalent to assuming that $\mathcal{F}$ in A1, rather than being invertible, is just left-invertible. We are currently working on relaxing A1 and replace it by the weaker left-invertibility of $\mathcal{F}$.

Assumption A2(Stabilizability): The origin of (1) is locally stabilizable (or stabilizable) by a static function of x, i.e., there exists a smooth function $\bar{u}(x)$ such that the origin is an asymptotically stable (or globally asymptotically stable) equilibrium point of $\dot{x} = f(x, \bar{u}(x))$.

Remark 2: Assumption A2 implies that the origin of the extended system (5) is locally stabilizable (or stabilizable) by a function of X as well. A proof of the local stabilizability property for (5) may be found, e.g., in [11], while its global counterpart is a well known consequence of the integrator backstepping lemma (see, e.g., Theorem 9.2.3 in [12] or Corollary 2.10 in [13]). Therefore we conclude that for the extended system (5) there exists a smooth control $\bar{v}(X)$ such that its origin is asymptotically stable under closed-loop control. Let $\mathcal{D}$ be the domain of attraction of the origin of (5), and notice that, when A2 holds globally, $\mathcal{D} = \mathbb{R}^n \times \mathbb{R}^{n_u}$.

Remark 3: In [8] the authors consider affine systems and use a feedback linearizability assumption in place of our A2. Here, we consider the more general class of non-affine systems for which the origin is locally stabilizable (stabilizable). In this respect, our assumption A2 relaxes also the stabilizability assumption found in [9], while it is equivalent to Assumption 2 in [7].

3 Nonlinear Observer: Its Need and Stability Analysis

Assumption A2 allows us to design a stabilizing state feedback control $v = \phi(x, z)$. In order to perform output feedback control x should be replaced by its estimate. Many researchers in the past adopted an input-output feedback linearizability assumption ([1], [2], [5], [4], [6]) and transformed the system into normal form

$$\begin{aligned}
\dot{\pi}_i &= \pi_{i+1}, \qquad 1 \le i \le r-1 \\
\dot{\pi}_r &= \bar{f}(\pi, \Pi) + \bar{g}(\pi, \Pi)\, u \\
\dot{\Pi} &= \Phi(\pi, \Pi), \qquad \Pi \in \mathbb{R}^{n-r} \\
y &= \pi_1
\end{aligned} \tag{8}$$

In this framework the problem of output feedback control finds a very natural formulation, as the first r derivatives of y are equal to the states of the π-subsystem (i.e., the linear subsystem). The works [1, 2, 4] solve the output feedback control problem for systems with no zero dynamics (i.e., $r = n$), so that the first $n-1$ derivatives of y provide the entire state of the system. In the presence of zero dynamics (Π-subsystem), the use of input-output feedback linearization to put the system into normal form (8) forces the use of a minimum phase assumption (e.g, [5]) since the states of the Π-subsystem cannot be estimated from the derivatives of the output and, hence, cannot be controlled by output feedback. For this reason the output feedback control of nonminimum phase systems has been regarded in the past as a particularly challenging problem. Researchers who have addressed this problem (e.g., [8], [9]) rely on the explicit knowledge of $\mathcal{H}^{-1}$ in (7), $x = \mathcal{H}^{-1}(y_e, z_1, \dots, z_{n_u})$, so that estimation of the first $n-1$ derivatives of y (the vector y_e) provides an estimate of x, $\hat{x} = \mathcal{H}^{-1}(\hat{y}_e, z_1, \dots, z_{n_u})$, since the vector z, being the state of the controller, is known. Next, to estimate the derivatives of y, they employ an high-gain observer. Both the works [8] and [9] (the latter dealing with the larger class of stabilizable systems) rely on the knowledge of $\mathcal{H}^{-1}$ to prove closed-loop stability. In addition to this, the recent work [7] proves that a separation principle holds for a quite general class of nonlinear systems which includes (1) provided that $\mathcal{H}^{-1}$ is explicitly known and that the system is uniformly completely observable. Sometimes however, even if it exists, $\mathcal{H}^{-1}$ cannot be explicitly calculated (see, e.g., the example in Section 5) thus limiting the applicability of existing approaches. Hence, rather than designing an high-gain observer to estimate y_e and using $\mathcal{H}^{-1}(\cdot, \cdot)$ to get x, the approach adopted here is to estimate x directly using a nonlinear observer for system (1) and using the fact that the z-states are known. The observer has the form[1]

$$\begin{aligned}
\dot{\hat{x}} &= \hat{f}(\hat{x}, z, y) = f(\hat{x}, z_1) + \left[\frac{\partial \mathcal{H}(\hat{x}, z)}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\,[y(t) - \hat{y}(t)] \\
\hat{y}(t) &= h(\hat{x}, z_1)
\end{aligned} \tag{9}$$

where L is a $n \times 1$ vector, $\mathcal{E} = \mathrm{diag}\left[\rho, \rho^2, \dots, \rho^n\right]$, and $\rho \in (0, 1]$ is a fixed design constant.

[1] Throughout this section we assume A1 to hold globally, since we are interested in the ideal convergence properties of the state estimates. In the next section we will show how to modify the observer equation in order to achieve the same convergence properties when A1 holds over the set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$.

Notice that (9) does not require any knowledge of $\mathcal{H}^{-1}$ and has the advantage of operating in x-coordinates. The observability assumption A1 implies that the Jacobian of the mapping $\mathcal{H}$ with respect to x is invertible, and hence the inverse of $\partial\mathcal{H}(\hat{x}, z)/\partial\hat{x}$ in (9) is well defined. In the work [14], the authors used an observer structurally identical to (9) for the more restrictive class of input-output feedback linearizable systems with full relative degree. Here, by modifying the definition of the mapping $\mathcal{H}$ and by introducing a dynamic projection, we considerably relax these conditions by just requiring the general observability assumption A1 to hold. Furthermore, we propose a different proof than the one found in [14] which clarifies the relationship among (9) and the high-gain observers commonly found in the literature.

Theorem 1 *Consider system (4) and assume that A1 is satisfied for $\mathcal{O} = \mathbb{R}^{n+n_u}$, the state X belongs to a positively invariant, compact set Ω, and that the following time signal is bounded as follows*

$$|\alpha(\hat{y}_e(t), z(t)) + \beta(\hat{y}_e(t), z(t))v(t) - \alpha(y_e(t), z(t)) - \beta(y_e(t), z(t))v(t)| \leq \gamma\|\hat{y}_e(t) - y_e(t)\|, \tag{10}$$

for some $\gamma > 0$, for all $t \geq 0$, and for all $\hat{y}_e(0) \in \hat{\Omega}$ (a compact set), $y_e(t) \in \Omega$, where α and β are defined in (12), and $\hat{y}_e(t) = \mathcal{H}(\hat{x}(t), z(t))$. Choose L such that $A_c - LC_c$, where (A_c, B_c, C_c) is the controllable/observable canonical realization, is Hurwitz and choose $\hat{y}_e(0) \in \hat{\Omega}$.

Under these conditions and using observer (9), the following two properties hold

- *(i) Asymptotic stability of the estimation error: There exists $\bar{\rho}$, $0 < \bar{\rho} \leq 1$, such that for all $\rho \in (0, \bar{\rho})$, $\hat{x} \to x$ as $t \to +\infty$.*
- *(ii) Arbitrarily fast rate of convergence: For each positive T, ϵ, there exists ρ^*, $0 < \rho^* \leq 1$, such that for all $\rho \in (0, \rho^*]$, $\|\hat{x} - x\| \leq \epsilon \; \forall t \geq T$.*

Before proving this theorem, let us clarify the role of requirement (10). It is asked that the function $\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v$ satisfies a Lipschitz inequality at any time instant with a fixed Lipschitz constant γ. Notice that the boundedness of the control input $v(t)$ and the local Lipschitz continuity of the functions α and β are, in general, not sufficient to fulfill requirement (10) since, while $X(t)$, and hence $y_e(t) = \mathcal{H}(X(t))$, is assumed to be bounded for all $t \geq 0$, nothing can be said about the behavior of $\hat{y}_e(t)$ (this point is made clearer in the proof to follow). If α and β are globally Lipschitz functions and $v(t)$ is a bounded function of time, then requirement (10) is automatically satisfied. We will see in Section 4 that, without requiring α and β to be globally Lipschitz or any other additional assumption, (10) is always fulfilled by applying to the observer a suitable dynamic projection onto a fixed compact set.

Proof. Consider the filtered transformation

$$y_e = \mathcal{H}(x, z) = \begin{bmatrix} h(x, z_1) \\ \varphi_1(x, z_1, z_2) \\ \vdots \\ \varphi_{n-1}(x, z_1, \ldots, z_{n_u}) \end{bmatrix}. \tag{11}$$

A1 guarantees that $x = \mathcal{H}^{-1}(y_e, z)$ is well-defined, unique, and smooth. Let us express system (4) in new coordinates. By definition, $y_e = [y,\, \dot{y},\, \dots,\, y^{(n-1)}]^\top$ and, with φ_{n-1} defined in (3),

$$y^{(n)} = \left[\frac{\partial \varphi_{n-1}}{\partial x} f\left(\mathcal{H}^{-1}(y_e, z), z\right) + \sum_{k=1}^{n_u-1} \frac{\partial \varphi_{n-1}}{\partial z_k}\left(\mathcal{H}^{-1}(y_e, z), z\right)\, z_{k+1}\right] + \left[\frac{\partial \varphi_{n-1}}{\partial z_{n_u}}\left(\mathcal{H}^{-1}(y_e, z), z\right)\right] v$$
$$\triangleq \alpha(y_e, z) + \beta(y_e, z) v. \tag{12}$$

Hence, in the new coordinates (4) becomes

$$\dot{y}_e = A_c y_e + B_c \left[\alpha(y_e, z) + \beta(y_e, z) v\right]. \tag{13}$$

Next, transform the observer (9) to new coordinates $\hat{y}_e = \left[\hat{y}, \dot{\hat{y}}, \dots, \hat{y}^{(n-1)}\right]^\top = \mathcal{H}(\hat{x}, z)$ so that

$$\begin{aligned}\dot{\hat{y}}_{e,1} &= \frac{\partial h}{\partial \hat{x}} f(\hat{x}, z_1) + \frac{\partial h}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left[y - h(\hat{x}, z_1)\right] + \frac{\partial h}{\partial z_1}\dot{z}_1 \\ &= \hat{y}_{e,2} + \frac{\partial h}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left[y - h(\hat{x}, z_1)\right].\end{aligned} \tag{14}$$

Similarly, for $i = 2, \dots, n-1$

$$\dot{\hat{y}}_{e,i} = \frac{\partial \varphi_{i-1}}{\partial \hat{x}}(\hat{x}, z_1, \dots, z_i) \left\{ f(\hat{x}, z_1) + \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left(y - h(\hat{x}, z_1)\right) \right\} + \sum_{k=1}^{i} \frac{\partial \varphi_{i-1}}{\partial z_k} z_{k+1}. \tag{15}$$

By definition,

$$\hat{y}_{e,i+1} = \varphi_i(\hat{x}, z_1, \dots, z_{i+1}) = \frac{\partial \varphi_{i-1}}{\partial \hat{x}} f(\hat{x}, z_1) + \sum_{k=1}^{i} \frac{\partial \varphi_{i-1}}{\partial z_k} z_{k+1}.$$

Hence, we conclude that

$$\dot{\hat{y}}_{e,i} = \hat{y}_{e,i+1} + \frac{\partial \varphi_{i-1}}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left(y - h(\hat{x}, z_1)\right), \qquad i = 2, \dots, n-1 \tag{16}$$

Finally,

$$\dot{\hat{y}}_{e,n} = \alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z) v + \frac{\partial \varphi_{n-1}}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left[y - h(\hat{x}, z_1)\right]. \tag{17}$$

By using (14), (16), and (17) we can write, in compact form,

$$\begin{aligned}\dot{\hat{y}}_e &= A_c \hat{y}_e + B_c[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z) v] + \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right] \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left[y - h(\hat{x}, z_1)\right] \\ &= A_c \hat{y}_e + B_c[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z) v] + \mathcal{E}^{-1} L\left[y_{e,1} - \hat{y}_{e,1}\right].\end{aligned} \tag{18}$$

Define the observer error in the new coordinates, $\tilde{y}_e = \hat{y}_e - y_e$. Then, the observer error dynamics are given by

$$\dot{\tilde{y}}_e = (A_c - \mathcal{E}^{-1} L\, C_c)\tilde{y}_e + B_c\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z) v - \alpha(y_e, z) - \beta(y_e, z) v\right] \tag{19}$$

Next, define the coordinate transformation

$$\tilde{\nu} = \mathcal{E}'\tilde{y}_e, \qquad \mathcal{E}' \triangleq \operatorname{diag}\left[\frac{1}{\rho^{n-1}}, \frac{1}{\rho^{n-2}}, \dots, 1\right]. \tag{20}$$

In the new domain the observer error equation becomes

$$\dot{\tilde{\nu}} = \frac{1}{\rho}(A_c - LC_c)\tilde{\nu} + B_c\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v\right] \tag{21}$$

where, by assumption, $A_c - LC_c$ is Hurwitz. Let P be the solution to the Lyapunov equation

$$P(A_c - LC_c) + (A_c - LC_c)^\top P = -I, \tag{22}$$

and consider the Lyapunov function candidate $V_o(\tilde{\nu}) = \tilde{\nu}^\top P\tilde{\nu}$. Calculate the time derivative of V_o along the $\tilde{\nu}$ trajectories:

$$\dot{V}_o = -\frac{\tilde{\nu}^\top\tilde{\nu}}{\rho} + 2\tilde{\nu}^\top PB_c\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v\right]. \tag{23}$$

By virtue of (10), if $\hat{y}_e(0) \in \hat{\Omega}$ and $y_e(t) \in \Omega$, there exists a fixed scalar $\gamma > 0$, independent of ρ, such that the bracketed term in (23) can be bounded as follows

$$\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v\right] \leq \gamma\|\hat{y}_e - y_e\|, \tag{24}$$

and thus the time derivative of V_0 can be bounded as

$$\dot{V}_o \leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P\|\gamma\|\tilde{y}_e\|\|\tilde{\nu}\| \leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P\|\gamma\|\tilde{\nu}\|^2. \tag{25}$$

As we mentioned earlier, the local Lipschitz continuity of α and β and the boundedness of $v(t)$ for all $t \geq 0$ are not sufficient to guarantee that a bound of the type (24) hold for the bracketed term in (23). This is seen by noticing that the level sets of the Lyapunov function V_o, expressed in $\tilde{y}_e$ coordinates,

$$\Lambda_c = \{\tilde{y}_e \in \mathbb{R}^n \mid V_o(\mathcal{E}'\tilde{y}_e) \leq c\},$$

are parameterized by ρ and become larger as ρ is decreased. Thus, letting $\tilde{y}_e$ range over Λ_c, a straightforward application of Lipschitz inequality would result in a bound like (24) where γ, rather than being constant, is a function of ρ. In the next section we show that by introducing a dynamic projection one guarantees that (10) is automatically satisfied.

Defining $\bar{\rho} = \min\{1/(2\|P\|\gamma), 1\}$, we conclude that for all $\rho < \bar{\rho}$ the $\tilde{y}_e$ trajectories starting in $\hat{\Omega}$ will converge asymptotically to the origin, and hence part (i) of the theorem is proved.

As for part (ii), note that $\lambda_{min}(\mathcal{E}'P\mathcal{E}') \geq \lambda_{min}(\mathcal{E}')^2\lambda_{min}(P) = \lambda_{min}(P)$, since $\lambda_{min}(\mathcal{E}') = 1$. Next, $\lambda_{max}(\mathcal{E}'P\mathcal{E}') \leq \lambda_{max}(\mathcal{E}')^2\lambda_{max}(P) = 1/(\rho^{2(n-1)})\lambda_{max}(P)$, since $\lambda_{max}(\mathcal{E}') = 1/\rho^{(n-1)}$. Therefore

$$\lambda_{min}(P)\|\tilde{y}_e\|^2 \leq V_o = \tilde{y}_e^\top\mathcal{E}'P\mathcal{E}'\tilde{y}_e \leq \frac{1}{\rho^{2(n-1)}}\lambda_{max}(P)\|\tilde{y}_e\|^2. \tag{26}$$

Define $\bar{\epsilon}$ so that $\|\tilde{y}_e\| \leq \bar{\epsilon}$ implies that $\|\hat{x} - x\| \leq \epsilon$ (the smoothness of $\mathcal{H}^{-1}$ guarantees that $\bar{\epsilon}$ is well defined). By inequality (26) we have that $V_o \leq \bar{\epsilon}^2 \lambda_{min}(P)$ implies that $\|\tilde{y}_e\| \leq \bar{\epsilon}$, and $V_o(0) \triangleq V_o(\tilde{\nu}(0)) \leq (1/\rho^{2(n-1)})\lambda_{max}(P)\|\tilde{y}_e(0)\|^2$. Moreover, from (25)

$$\dot{V}_o(t) \leq -\left(\frac{1}{\rho} - 2\|P\|\gamma\right)\|\tilde{\nu}\|^2 \leq -\frac{1}{\lambda_{max}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)V_o(t). \tag{27}$$

Therefore, by the Comparison Lemma (see, e.g., [15]), $V_o(t)$ satisfies the following inequality

$$\begin{aligned} V_o(t) &\leq V_o(0)\exp\left\{-\frac{1}{\lambda_{max}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)t\right\} \\ &\leq \frac{1}{\rho^{2(n-1)}}\lambda_{max}(P)\|\tilde{y}_e(0)\|^2 \exp\left\{-\frac{1}{\lambda_{max}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)t\right\} \end{aligned} \tag{28}$$

which, for sufficiently small ρ, can be written as

$$V_o(t) \leq \frac{a_1}{\rho^{2n}}\exp\left\{-\frac{a_2}{\rho}t\right\}, \qquad a_1, a_2 > 0$$

An upper estimate of the time T such that $\|\hat{y}_e(t) - y_e(t)\| \leq \bar{\epsilon}$ (and thus $\|\hat{x}(t) - x(t)\| \leq \epsilon$) for all $t \geq T$, is calculated as follows

$$\frac{a_1}{\rho^{2n}}\exp\left\{-\frac{a_2}{\rho}t\right\} \leq \bar{\epsilon}^2\lambda_{min}(P) \text{ for all } t \geq T = \frac{\rho}{a_2}\ln\left(\frac{a_1}{\bar{\epsilon}^2\rho^{2n}\lambda_{min}(P)}\right).$$

Noticing that $T \to 0$ as $\rho \to 0$, we conclude that T can be made arbitrarily small by choosing a sufficiently small ρ^*, thus concluding the proof of part (ii).

■

Remark 4: Part (ii) of Theorem 1 implies that the observer convergence rate can be made arbitrarily fast. This property is essential for closed-loop stability.

Remark 5: Using inequality (28), we find the upper bound for the estimation error in y_e-coordinates

$$\|\tilde{y}_e\| \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\frac{1}{\rho^{n-1}}\|\tilde{y}_e(0)\|\exp\left\{-\frac{1}{2\lambda_{max}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)t\right\}. \tag{29}$$

Hence, during the initial transient, $\tilde{y}_e(t)$ may exhibit peaking, and the size of the peak grows larger as ρ decreases and the convergence rate is made faster. This phenomenon and its implications on output feedback control has been studied in the seminal work [1]. The analysis in that paper shows that a way to *isolate* the peaking of the observer estimates from the system states is to saturate the control input outside of the compact set of interest. The same idea has then been adopted in several other works in the output feedback control literature (see, e.g., [1, 2, 9, 3, 5, 4, 6, 7]). Rather than following this approach, in the next section we will present a new technique to *eliminate* the peaking phenomenon (rather than just isolate) which allows for the use of the weaker Assumption A1.

Remark 6: It is interesting to note that in y_e-coordinates (see (18)) the nonlinear observer (9) is almost identical to the high-gain observer found in [9]. The two observers would be identical (in y_e-coordinates) if the variable $\hat{y}_e$ in α and β in (18) was saturated outside a suitable compact set. In [9] such a saturation, together with a saturation of the control input v, guarantees that requirement (10) is always satisfied. In this paper the same result is obtained by replacing the saturation by a dynamic projection (see Section 4). Our observer has the advantage of avoiding the knowledge of the inverse of the mapping $\mathcal{H}$, as well as directly working in x coordinates.

4 Output Feedback Stabilizing Control

Consider system (5), by using assumption A2 and Remark 2 we conclude that there exists a smooth stabilizing control $v = \phi(x, z) = \phi(X)$ which makes the origin of (5) an asymptotically stable equilibrium point with domain of attraction $\mathcal{D}$. By the converse Lyapunov theorem found in [16], there exists a continuously differentiable function V defined on $\mathcal{D}$ satisfying, for all $X \in \mathcal{D}$,

$$\alpha_1(\|X\|) \le V(X) \le \alpha_2(\|X\|) \tag{30}$$

$$\lim_{X \to \partial\mathcal{D}} \alpha_1(\|X\|) = \infty \tag{31}$$

$$\frac{\partial V}{\partial X}\left(f_e(X) + g_e\, v\right) \le -\alpha_3(\|X\|) \tag{32}$$

where $\alpha_i,\ i = 1, 2, 3$ are class $\mathcal{K}$ functions (see [17] for a definition), and $\partial\mathcal{D}$ stands for the boundary of the set $\mathcal{D}$. Given any scalar $c > 0$, define

$$\Omega_c \stackrel{\triangle}{=} \{X \in \mathbb{R}^{n+n_u} \,|\, V \le c\}.$$

Clearly, $\Omega_c \subset \mathcal{D}$ for all $c > 0$ and, from (31), Ω_c becomes arbitrarily close to $\mathcal{D}$ as $c \to \infty$. Next, the following assumption is needed.

Assumption A3(Topology of $\mathcal{O}$): Assume that there exists a constant $c_2 > 0$ and a set $\mathcal{C}$ such that

$$\mathcal{F}\left(\Omega_{c_2}\right) \subset \mathcal{C} \subset \mathcal{Y}\ \ (= \mathcal{F}(\mathcal{O}))\,, \tag{33}$$

where $\mathcal{C}$ has the following properties

(i) The boundary of $\mathcal{C}$, $\partial\mathcal{C}$, is class C^1, i.e., there exists a C^1 function $g : \mathcal{C} \to \mathbb{R}$ such that $\partial\mathcal{C} = \{Y \in \mathcal{C} \,|\, g(Y) = 0\}$, and $(\partial g/\partial Y)^\top \ne 0$ on $\partial\mathcal{C}$.

(ii) $\mathcal{C}^{\bar{z}} = \{y_e \in \mathbb{R}^n \,|\, [{y_e}^\top, \bar{z}^\top]^\top \in \mathcal{C}\}$ is convex for all $\bar{z} \in \mathbb{R}^{n_u}$.

(iii) 0 is a regular value of $g(\cdot, \bar{z})$ for each fixed $\bar{z} \in \mathbb{R}^{n_u}$.

(iv) $\bigcup_{\bar{z} \in \mathbb{R}^{n_u}} \mathcal{C}^{\bar{z}}$ is compact.

Remark 7: See Figure 1 for a pictorial representation of condition (33). This assumption requires *in primis* that there exists a set $\mathcal{C}$ in Y coordinates which contains the image under $\mathcal{F}$ of a level set Ω_{c_2} of the Lyapunov function V and is contained in the image under $\mathcal{F}$ of the observable set $\mathcal{O}$. This guarantees in particular that $\Omega_{c_2} \subset \mathcal{O}$, and thus the state feedback trajectories starting in Ω_{c_2} never exit the observable

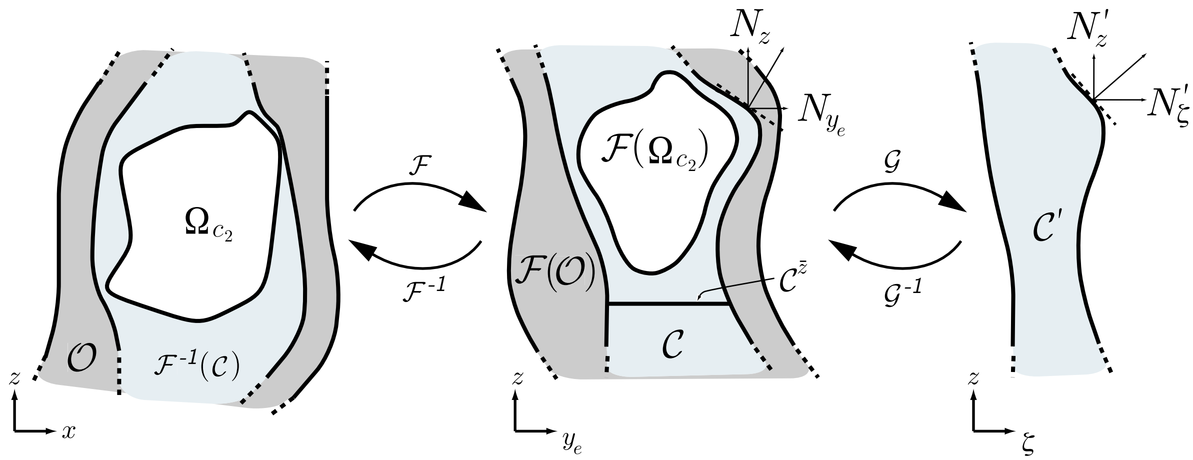


Figure 1: The mechanism behind the observer estimates projection.

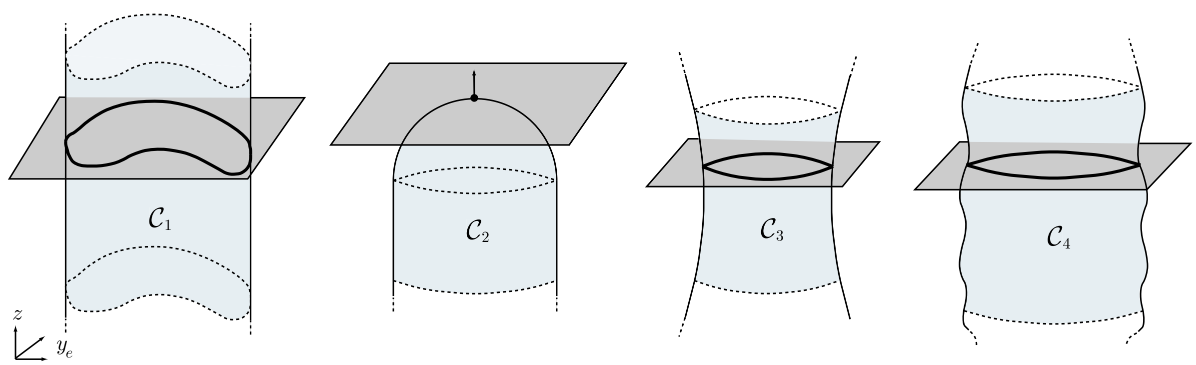


Figure 2: The domains $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ violate some of the requirements (i)-(iv) in A3, while $\mathcal{C}_4$ does not.

region $\mathcal{O}$. Furthermore, it is required that $\mathcal{C}$ possesses some basic topological properties: It is asked that the boundary of $\mathcal{C}$ be continuously differentiable (part (i)), every slice of $\mathcal{C}$ obtained by holding z constant at $\bar{z}$, $\mathcal{C}^{\bar{z}}$, is convex (part (ii)), the normal vector to each slice $\mathcal{C}^{\bar{z}}$ (which is given by $[\partial g/\partial y_e(y_e, \bar{z})]^\top$) does not vanish anywhere on the slice (part (iii)) and, finally, it is asked that the set $\mathcal{C}$ is compact in the y_e direction (part (iv)). Part (iv) can be replaced by the slightly weaker requirement that $\bigcup_{\bar{z}\in\Omega^z}\mathcal{C}^{\bar{z}}$ is compact (Ω^z is defined in the next section), without any change in the analysis to follow.

To further clarify the topology of the domains under consideration, consider the sets $\mathcal{C}_1$ to $\mathcal{C}_4$ in Figure 2, corresponding to the case $y_e \in \mathbb{R}^2$, $z \in \mathbb{R}$. While they all satisfy part (i), $\mathcal{C}_1$ does not satisfy part (ii) since its slices along z are not convex. $\mathcal{C}_2$ satisfies part (ii) but violates requirement (iii) because the normal vector to one of the slices has no components in the y_e direction. $\mathcal{C}_3$ satisfies parts (i)-(iii) but violates (iv) since the area of its slices grows unbounded as $z \to \infty$. $\mathcal{C}_4$ satisfies all the requirements above.

Note that, when the plant is UCO (and thus $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^{n_u}$) and $\mathcal{F}(\mathbb{R}^{n+n_u}) = \mathbb{R}^{n+n_u}$, A3 is always satisfied by a sufficiently large set $\mathcal{C}$ and any $c_2 > 0$. In order to see that, pick *any* c_2 and choose $\mathcal{C}$ to be any cylinder $\{Y \in \mathbb{R}^{n+n_u} \,|\, \|y_e\| \leq M\}$, where $M > 0$, containing $\mathcal{F}(\Omega_{c_2})$. The existence of $\mathcal{C}$ is guaranteed by the fact that the set $\mathcal{F}(\Omega_{c_2})$ is bounded. More generally, the same holds when $\mathcal{F}(\mathbb{R}^{n+n_u})$ is not all of $\mathbb{R}^{n+n_u}$ and $\mathcal{Y}^{\bar{z}} \triangleq \{y_e \in \mathbb{R}^n \,|\, \mathcal{F}(x, \bar{z})\}$ is convex for all $\bar{z} \in \mathbb{R}^{n_u}$.

Finally, notice that when the plant in *not* UCO but $\mathcal{O} = \mathcal{X} \times \mathbb{R}^{n_u}$, where $\mathcal{X}$ is an open set which is not all of $\mathbb{R}^n$, and the origin is globally stabilizable (i.e., A2 holds globally), one can choose $\mathcal{C} = D \times \mathbb{R}^{n_u}$, where $D \subset \mathbb{R}^n$ is *any* convex compact set with smooth boundary contained in $\mathcal{X}$ and containing the point $\mathcal{H}(0,0)$ (i.e., the origin in y_e coordinates). The scalar c_2 is then the largest scalar such that $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C}$ (notice, however, that c_2 does not need to be known for design purposes). Consequently, in the particular case when $n_u = 0$ (and hence the control input does not affect the mapping $\mathcal{H}$) and A2 holds globally, one can choose $\mathcal{C} = D$, where D is defined above.

4.1 Observer Estimates Projection

As we already pointed out in Remark 4, in order to isolate the peaking phenomenon from the system states, the approach generally adopted in several papers is to saturate the control input to prevent it from growing above a given threshold. This technique, however, does not eliminate the peak in the observer estimate and, hence, cannot be used to control general systems like the ones satisfying assumption A1, since even when the system state lies in the observable region $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, the observer estimates may enter the unobservable domain where (9) is not well defined. It appears that in order to deal with systems that are not completely observable, one has to eliminate the peaking from the observer by guaranteeing its estimates to be confined in a prespecified observable compact set.

A common procedure used in the adaptive control literature (see [18]) to confine vectors of parameter estimates within a desired convex set is gradient projection. This idea cannot be directly applied to our problem, mainly because $\dot{\hat{x}}$ is not proportional to the gradient of the observer Lyapunov function and, thus, the projection cannot be guaranteed to preserve the convergence properties of the estimate. Inspired by this idea, however, we propose a way to modify the $\dot{\hat{x}}$ equation which confines $\hat{x}$ to within a prespecified compact set while preserving its convergence properties.

Recall the coordinate transformation defined in (6) and let

$$\hat{y}_e^P = \mathcal{H}(\hat{x}^P, z), \qquad \tilde{y}_e^P = \hat{y}_e^P - y_e, \qquad \hat{Y}^P = [\hat{y}_e^{P\top}, z^\top]^\top, \tag{34}$$

where $\hat{x}^P$ is the state of the *projected* observer defined as[2]

$$\dot{\hat{x}}^P = \begin{cases} \left[\dfrac{\partial \mathcal{H}}{\partial \hat{x}^P}\right]^{-1} \left\{ \dot{\hat{y}}_e|_{\hat{x}^P} - \Gamma \dfrac{N_{y_e}(\hat{Y}^P)\left(N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z}\right)}{N_{y_e}(\hat{Y}^P)^\top \Gamma N_{y_e}(\hat{Y}^P)} - \dfrac{\partial \mathcal{H}}{\partial z}\dot{z} \right\} & \\ \qquad \text{if } N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z} \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} & \\ \hat{f}(\hat{x}^P, z, y) \qquad \text{otherwise} & \end{cases} \tag{35}$$

where $\dot{\hat{y}}_e|_{\hat{x}^P}$ denotes the time derivative of $y_e^P = \mathcal{H}(\hat{x}^P, z)$ when $\hat{x}^P$ evolves according to the observer dynamics (9), i.e.,

$$\dot{\hat{y}}_e|_{\hat{x}^P} = \frac{\partial \mathcal{H}}{\partial \hat{x}^P}\hat{f}(\hat{x}^P, z, y) + \frac{\partial \mathcal{H}}{\partial z}\dot{z},$$

[2]The projection defined in (35) is discontinuous in the variable $\hat{y}_e$, therefore raising the issue of the existence and uniqueness of its solutions. We refer the reader to Remark 8, were this issue is addressed and a solution is proposed.

$\Gamma = (S\mathcal{E}')^{-1}(S\mathcal{E}')^{-\top}$, $S = S^\top$ denotes the matrix square root of P (defined in (22)) and

$$N_{y_e}(\hat{Y}^P) = \left[\frac{\partial g}{\partial \hat{y}_e^P}(\hat{y}_e^P, z)\right]^\top, \; N_z(\hat{Y}^P) = \left[\frac{\partial g}{\partial z}(\hat{y}_e^P, z)\right]^\top$$

are the y_e and z components of the normal vector $N(\hat{Y}^P)$ to the boundary of $\mathcal{C}$ at $\hat{Y}^P$, i.e., $N(\hat{Y}^P) = [N_{y_e}(\hat{Y}^P)^\top, N_z(\hat{Y}^P)^\top]^\top$ (the function g is defined in A3). Notice that the dynamic projection (35) is well-defined since A3, part (iii), guarantees that N_{y_e} does not vanish (see also Remark 7). The following lemma shows that (35) guarantees boundedness and preserves convergence for $\hat{x}^P$.

Lemma 1 *: If A3 holds and (35) is used:*

(i) Boundedness: if $\hat{x}^P(0) \in \mathcal{H}^{-1}(\mathcal{C})$, then $\hat{x}^P(t) \in \mathcal{H}^{-1}(\mathcal{C})$ for all t.

If, in addition, $[x(t)^\top, z(t)^\top]^\top \in \Omega_{c_2}$ for all $t \geq 0$ and the assumptions of Theorem 1 are satisfied, then the following properties hold for the flow of the projected observer dynamics (35)

(ii) Requirement (10) in Theorem 1 is satisfied provided $\sup_{t\geq 0} v(t) < \infty$.

(iii) Preservation of original convergence characteristics: properties (i) and (ii) established by Theorem 1 remain valid for $\hat{x}^P$.

Proof. We start by introducing another coordinate transformation, $\zeta = S\mathcal{E}' y_e$, (similarly, let $\hat{\zeta}^P = S\mathcal{E}'\hat{y}_e^P$, $\tilde{\zeta}^P = S\mathcal{E}'\tilde{y}_e^P$), letting $\mathcal{G} = diag\,[S\mathcal{E}', I_{n_u \times n_u}]$, and letting $\mathcal{C}'$ be the image of the set $\mathcal{C}$ under the linear map $\mathcal{G}$, i.e., $\mathcal{C}' \triangleq \left\{[\zeta^\top, z^\top]^\top \in \mathbb{R}^{n+n_u} \,|\, \mathcal{G}^{-1}[\zeta^\top, z^\top]^\top \in \mathcal{C}\right\}$. Let $N'_\zeta(\zeta, z)$, $N'_z(\zeta, z)$ be the ζ and z components of the normal vector to the boundary of $\mathcal{C}'$. The reader may refer to Figure 1 for a pictorial representation of the sets under consideration. In order to prove part (i) of the Lemma, it is sufficient to show that the projection (35) renders $\mathcal{C}$ a positively invariant set for the $\hat{Y}^P$ trajectories which in turn guarantees that $\hat{x}^P(t) = \mathcal{H}^{-1}(\hat{Y}^P)$ is contained in the observable set $\mathcal{H}^{-1}(\mathcal{C})$ for all $t \geq 0$. After coordinate transformation (34) we have that

$$\dot{\hat{y}}_e^P = \frac{d}{dt}\left\{\mathcal{H}(\hat{x}^P, z)\right\} = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}^P}\dot{\hat{x}}^P + \frac{\partial \mathcal{H}}{\partial z}\dot{z}\right] = \begin{cases} \dot{\hat{y}}_e|_{\hat{x}^P} - \Gamma \dfrac{N_{y_e}(\hat{Y}^P)\left(N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z}\right)}{N_{y_e}(\hat{Y}^P)^\top \Gamma N_{y_e}(\hat{Y}^P)} \\ \qquad \text{if } N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z} \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ \dot{\hat{y}}_e|_{\hat{x}^P} \qquad \text{otherwise} \end{cases} \tag{36}$$

In order to relate $N'_\zeta(\hat{\zeta}^P, z)$, $N'_z(\hat{\zeta}^P, z)$ to $N_{y_e}(\hat{y}_e^P, z)$, $N'_z(\hat{\zeta}^P, z)$, recall from A3 that the boundary of $\mathcal{C}$ is expressed as the set $\partial\mathcal{C} = \{Y \in \mathbb{R}^{n+n_u} \,|\, g(Y) = 0\}$ and hence the boundary of $\mathcal{C}'$ is the set $\partial\mathcal{C}' = \{\zeta \in \mathbb{R}^n \,|\, g((S\mathcal{E}')^{-1}\zeta, z) = 0\}$. From this definition we find the expression of N'_ζ and N'_z as $N'_\zeta(\hat{\zeta}^P, z) = (S\mathcal{E}')^{-\top}[\partial g(\hat{Y}^P)/\partial y_e^P]^\top = (S\mathcal{E}')^{-\top} N_{y_e}(\hat{Y}^P)$, $N'_z(\hat{\zeta}^P, z) = N_z(\hat{Y}^P)$. The expression of the projection (35) in

ζ coordinates is found by noting that

$$\dot{\hat{\zeta}}^P = S\mathcal{E}'\dot{\hat{y}}_e^P = \begin{cases} S\mathcal{E}'\dot{\hat{y}}_e|_{\hat{x}^P} - (S\mathcal{E}')^{-\top}\dfrac{N_{y_e}\left(N_{y_e}^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z^\top \dot{z}\right)}{N_{y_e}^\top \Gamma N_{y_e}} & \text{if } N_{y_e}^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z^\top \dot{z} \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ S\mathcal{E}'\dot{\hat{y}}_e|_{\hat{x}^P} & \text{otherwise} \end{cases} \tag{37}$$

and then substituting $N'_\zeta = (S\mathcal{E}')^{-\top}N_{y_e}$, $N'_z = N_z$, and $\dot{\hat{y}}_e|_{\hat{x}^P} = (S\mathcal{E}')^{-1}\dot{\hat{\zeta}}|_{\hat{x}^P}$, with obvious definition of $\dot{\hat{\zeta}}|_{\hat{x}^P}$, to find that

$$\dot{\hat{\zeta}}^P = \begin{cases} \dot{\hat{\zeta}}|_{\hat{x}^P} - \dfrac{N'_\zeta\left({N'_\zeta}^\top \dot{\hat{\zeta}}|_{\hat{x}^P} + {N'_z}^\top \dot{z}\right)}{{N'_\zeta}^\top N'_\zeta} & \text{if } {N'_\zeta}^\top \dot{\hat{\zeta}}|_{\hat{x}^P} + {N'_z}^\top \dot{z} \geq 0 \text{ and } [\hat{\zeta}^{P\top}, z^\top]^\top \in \partial\mathcal{C}' \\ \dot{\hat{\zeta}}|_{\hat{x}^P} & \text{otherwise} \end{cases} \tag{38}$$

Next, we show that the boundary of the domain $\mathcal{C}'$ is positively invariant with respect to (38). In order to do that, consider the continuously differentiable function $V_{\mathcal{C}'} = g\left((S\mathcal{E}')^{-1}\zeta, z\right)$ and calculate its time derivative along the trajectory of (38) when $[\hat{\zeta}^{P\top}, z^\top]^\top \in \partial\mathcal{C}'$,

$$\begin{aligned} \dot{V}_{\mathcal{C}'} &= N'_\zeta(\hat{\zeta}^P, z)^\top \dot{\hat{\zeta}}^P + N'_z(\hat{\zeta}^P, z)\dot{z} && (39) \\ &= {N'_\zeta}^\top \dot{\hat{\zeta}}|_{\hat{x}^P} - \frac{{N'_\zeta}^\top N'_\zeta\left({N'_\zeta}^\top \dot{\hat{\zeta}}|_{\hat{x}^P} + {N'_z}^\top \dot{z}\right)}{{N'_\zeta}^\top N'_\zeta} + N'_z\dot{z} && (40) \\ &= 0 && (41) \end{aligned}$$

thus showing that the trajectory $[\hat{\zeta}^{P\top}(t)^\top, z^\top(t))]^\top$ cannot cross the boundary of $\mathcal{C}'$, which in turn implies that $[\hat{y}_e^{P\top}(t), z^\top(t)]^\top$ cannot cross $\partial\mathcal{C}$ and, thus, $\hat{x}^P(t) \in \mathcal{H}^{-1}(\mathcal{C})$ for all t.

Next, to prove part (ii) of the theorem, we want to show that if $X(t) = [x^\top(t), z^\top(t)]^\top$ is contained in a positively invariant, compact set Ω for all $t \geq 0$, then inequality (10) holds for all $t \geq 0$ with $\hat{y}_e(t)$ replaced by $\hat{y}_e^P(t)$, provided that $v(t)$ is uniformly bounded. We start by noting that $y_e(t) = \mathcal{H}(x(t), z(t))$ is contained in the compact set $\mathcal{H}(\Omega)$ for all $t \geq 0$ and $z(t)$ is contained in the compact set $\Omega^z = \{z \in \mathbb{R}^{n_u} \mid X(t) \in \Omega\}$ for all $t \geq 0$. Furthermore, using part (i) of this lemma and part (iv) in A3 we have that

$$[\hat{y}_e^{P\top}(t), z(t)^\top]^\top \in \bar{\mathcal{C}} = \left(\bigcup\nolimits_{z \in \Omega^z} \mathcal{C}^z\right) \times \Omega^z, \text{ for all } t \geq 0,$$

where $\bar{\mathcal{C}}$ is a compact set. Now, part (ii) is proved by noticing that inequality (10) follows directly from the facts above, the boundedness of $v(t)$, and the local Lipschitz continuity of α and β. As already mentioned earlier, the local Lipschitz continuity of α and β alone is not sufficient to establish (10). Part (i) of this lemma is the key missing feature: it proves that $\hat{y}_e^P(t)$ is contained in a compact set whose size is independent of ρ. This makes it possible to establish (10), where the Lipschitz constant γ is independent of ρ.

The proof of part (iii) is based on the knowledge of a Lyapunov function for the observer in $\tilde{\nu}$ coordinates (see (20)). Letting $\tilde{\zeta} = S\tilde{\nu}$, we have that, in new coordinates, $V_o = \tilde{\nu}^\top P\tilde{\nu} = (\tilde{\nu}^\top S)(S\tilde{\nu}) = \tilde{\zeta}^\top\tilde{\zeta}$. Now let $V_o^P = {\tilde{\zeta}^P}^\top \tilde{\zeta}^P$ be a Lyapunov function candidate for the projected observer error dynamics in transformed coordinates and recall that, by assumption, $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C}$, and thus $[x^\top, z^\top]^\top \in \Omega_{c_2}$. The latter fact implies that $[{y_e}^\top, z^\top]^\top \in \mathcal{C}$ or, what is the same, $[\zeta^\top, z^\top]^\top \in \mathcal{C}'$. From (38), when $[\hat{\zeta}^{P\top}, z^\top]^\top$ is in the interior of $\mathcal{C}'$, or $[\hat{\zeta}^{P\top}, z^\top]^\top$ is on the boundary of $\mathcal{C}'$ and $N_\zeta'^\top \dot{\hat{\zeta}}|_{\hat{x}^P} + N_z'^\top \dot{z} < 0$ (i.e., the *unprojected* update is pointed to the interior of $\mathcal{C}'$), we have that $\dot{V}_o^P = \dot{V}_o < 0$. Let us now consider all the remaining cases, i.e., $[\hat{\zeta}^{P\top}, z^\top]^\top \in \partial\mathcal{C}$ and $N_\zeta'^\top \dot{\hat{\zeta}}|_{\hat{x}^P} + N_z'^\top \dot{z} \geq 0$,

$$\dot{V}_o^P = 2\tilde{\zeta}^P \dot{\tilde{\zeta}}^P = 2\tilde{\zeta}^{P\top} \dot{\hat{\zeta}}^P = 2\tilde{\zeta}^{P\top} \left[\dot{\hat{\zeta}}|_{\hat{x}^P} - \dot{\zeta} - p(\hat{\zeta}^P, \dot{\hat{\zeta}}|_{\hat{x}^P}, z, \dot{z}) N_\zeta'(\hat{\zeta}^P, z)\right] = \dot{V}_o(\tilde{\zeta}^P) - 2p\tilde{\zeta}^{P\top} N_\zeta' \tag{42}$$

where $p(\hat{\zeta}^P, \dot{\hat{\zeta}}|_{\hat{x}^P}, z, \dot{z}) = \dfrac{N_\zeta'^\top \dot{\hat{\zeta}}|_{\hat{x}^P} + N_z'^\top \dot{z}}{N_\zeta'^\top N_\zeta'}$ is nonnegative since, by assumption, $N_\zeta'^\top \dot{\hat{\zeta}}|_{\hat{x}^P} + N_z'^\top \dot{z} \geq 0$. Using the fact that $[\zeta^\top, z^\top]^\top \in \mathcal{C}'$ and that $[\hat{\zeta}^{P\top}, z^\top]^\top$ lies on the boundary of $\mathcal{C}'$, we have that the difference vector $[\hat{\zeta}^P - \zeta, 0^\top]^\top$ points outside of $\mathcal{C}'$ or, equivalently, $\hat{\zeta}^P - \zeta$ points outside of the slice $\mathcal{C}'^z = \{\zeta \in \mathbb{R}^n \mid [\zeta^\top, z^\top]^\top \in \mathcal{C}'\}$. Using the definition of $\mathcal{C}'$, we have that the set $\mathcal{C}'^z$ is the image of the convex compact set $\mathcal{C}^z$, defined in A3, under the linear map $S\mathcal{E}'$ and is therefore compact and convex as well. Combining these two facts we have that $\tilde{\zeta}^{P\top} N_\zeta' \geq 0$, thus proving that $\dot{V}_o^P \leq \dot{V}_o < 0$, which concludes the proof of part (iii).

■

Remark 8: In order to avoid the discontinuity in the right hand side of (35) one can employ the smooth projection idea introduced in [19]. In this case, part (i) of Lemma 1 would have to be modified to

$$\hat{x}^P(t) \in \mathcal{H}^{-1}(\bar{\mathcal{C}}), \tag{43}$$

and condition (33) in A3 would have to be replaced by the following

$$\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C} \subset \bar{\mathcal{C}} \subset \mathcal{F}(\mathcal{O}) \tag{44}$$

where $\mathcal{C}$ is a set satisfying (i)-(iv) in A3 and $\bar{\mathcal{C}} \supset \mathcal{C}$ is any neighborhood of $\mathcal{C}$ with the same topological properties.

Remark 9: From the proof of Lemma 1 we conclude that (35) performs a projection for $\hat{x}$ over the set $\mathcal{H}^{-1}(\mathcal{C})$ which, in general, is unknown since we do not know $\mathcal{H}^{-1}$, and is generally *not* convex (see Figure 1). It is interesting to note that applying a standard gradient projection for $\hat{x}$ over an arbitrary convex domain does not necessarily preserve the convergence result (ii) in Theorem 1.

4.2 Closed-Loop Stability

To perform output feedback control we replace the state feedback law $v = \phi(x, z)$ with $\hat{v} = \phi(\hat{x}^P, z)$ which, by the smoothness of ϕ and the fact that $\hat{x}^P$ is guaranteed to belong to the compact set $\mathcal{H}^{-1}(\mathcal{C})$, is bounded provided that z is confined to within a compact set. In the following we will show that $\hat{v}$ makes the origin of (5) asymptotically stable and that Ω_{c_1} is contained in its region of attraction, for all $0 < c_1 < c_2$[3]. The proof is divided in three steps:

[3]Recall that c_2 is a positive constant satisfying A3 and hence its size is constrained by the topology of the observable set $\mathcal{O}$.

1. (Lemma 2). *Invariance of Ω_{c_2} and uniform ultimate boundedness:* Using the arbitrarily fast rate of convergence of the observer (see part (ii) in Theorem 1), we show that any trajectory originating in Ω_{c_1} cannot exit the set Ω_{c_2} and converges in finite time to an arbitrarily small neighborhood of the origin. Here, Lemma 1 plays an important role, in that it guarantees that the peaking phenomenon is eliminated and thus it does not affect $\hat{v}$, allowing us to use the same idea found in [1] to prove stability.

2. (Lemma 3). *Asymptotic stability of the origin:* By using Lemma 2 and the exponential stability of the observer estimate, we prove that the origin of the closed-loop system is asymptotically stable.

3. (Theorem 2). *Closed-loop stability:* Finally, by putting together the results of Lemma 2 and 3, we conclude the closed-loop stability proof.

Let $\tilde{x} \triangleq \hat{x} - x$, and $\tilde{x}^P \triangleq \hat{x}^P - x$, and note that part (ii) of Lemma 1 applies and $\tilde{x}^P \to 0$ as $t \to \infty$ with arbitrarily fast rate, as long as $X \in \Omega_{c_2}$. Moreover, the smoothness of the control law implies that

$$\|\phi(\hat{x}^P, z) - \phi(x, z)\| \leq \bar{\gamma}\|\tilde{x}^P\| \tag{45}$$

for all $X \in \Omega_{c_2}$, $\hat{x}^P \in \mathcal{H}^{-1}(\mathcal{C})$, and some $\bar{\gamma} > 0$. Assume that $\hat{x}^P(0) \in \mathcal{H}^{-1}(\mathcal{C})$, and let A be a positive constant satisfying $\|\partial V/\partial X\| \leq A$ for all X in Ω_{c_2} (its existence is guaranteed by V being continuously differentiable). Now, we can state the following lemma.

Lemma 2 *Suppose that $X(0) \in \Omega_{c_1}$, $\hat{x}^P(0) \in \mathcal{H}^{-1}(\mathcal{C})$, consider the set Ω_{d_ϵ}, where $d_\epsilon = \alpha_2 \circ \alpha_3^{-1}(\mu\, A\, \bar{\gamma}\, \epsilon)$, and choose $\epsilon > 0$ and $\mu > 1$ such that $d_\epsilon < c_1$. Then, there exists a positive scalar $\rho^*, 0 < \rho^* \leq 1$, such that, for all $\rho \in (0, \rho^*]$, the closed-loop system trajectories remain confined in Ω_{c_2}, the set $\Omega_{d_\epsilon} \subset \Omega_{c_1}$ is positively invariant, and is reached in finite time.*

Proof. Since $V(X(0)) \leq c_1 < c_2$, there exists a time $T_1 > 0$ such that $V(X(t)) \leq c_2$, for all $t \in [0, T_1)$. Choose T_0 such that $0 < T_0 < T_1$. Then, using this fact and recalling that, from part (i) in Lemma 1, $\hat{x}^P \in \mathcal{H}^{-1}(\mathcal{C})$ for all $t \geq 0$, we conclude that, for all $t \in [0, T_1)$,

$$[\hat{x}^{P\top}, z^\top]^\top \in \mathcal{F}^{-1}\left(\bigcup_{z \in \Omega^z_{c_2}} \mathcal{C}^z, \Omega^z_{c_2}\right)$$

which, from part (iv) in A3, is a compact set. Thus, from the smoothness of ϕ, $|v|$ is bounded by a constant independent of ρ, and hence we can apply part (ii) in Theorem 1 and parts (ii), (iii) in Lemma 1 and conclude that for any positive ϵ there exists a positive $\rho^*, 0 < \rho^* \leq 1$ such that, for all $\rho \leq \rho^*$, $\|\tilde{x}^P\| \leq \epsilon, \forall t \in [T_0, T_1)$. Hence, for all $t \in [T_0, T_1)$, we have that $V(X(t)) \leq c_2$ and $\|\tilde{x}^P(t)\| \leq \epsilon$. In order for part (ii) in Theorem 1 to hold for all $t \geq T_0$, X must belong to Ω_{c_2} for all $t \geq 0$. So far we can only guarantee that $X \in \Omega_{c_2}$ for all $t \in [0, T_1)$ and hence the result of Theorem 1 applies in this time interval, only. Next, we will show that $T_1 = \infty$, i.e., Ω_{c_2} is an invariant set, so that the result of Theorem 1 will be guaranteed to hold for all $t \geq 0$.

Consider the Lyapunov function candidate V defined in (30)-(32). Taking its derivative with respect to time and using (45),

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial X}\left[f_e(x,z) + g_e\phi(\hat{x}^P, z)\right] = \frac{\partial V}{\partial X}\left[f_e(x,z) + g_e\phi(x,z)\right] + \frac{\partial V}{\partial X}g_e\left[\phi(\hat{x}^P,z) - \phi(x,z)\right] \\
&\leq -\alpha_3(\|X\|) + \left\|\frac{\partial V}{\partial X}\right\| \|\phi(\hat{x}^P,z) - \phi(x,z)\| \\
&\leq -\alpha_3(\|X\|) + A\bar{\gamma}\|\tilde{x}^P\| \\
&\leq -\alpha_3(\|X\|) + A\bar{\gamma}\epsilon \\
&\leq -\alpha_3 \circ \alpha_2^{-1}(V) + A\bar{\gamma}\epsilon
\end{aligned}
$$

for all $t \in [T_0, T_1)$. When $V \geq d_\epsilon$ we have that

$$
\dot{V} \leq -(\mu - 1)A\bar{\gamma}\epsilon
$$

hence V decays linearly, which in turn implies that $X(t) \in \Omega_{c_2}$ for all $t \geq 0$ and that Ω_{d_ϵ} is reached in finite time.

■

Remark 10: The use of the projection for the observer estimate plays a crucial role in the proof of Lemma 2. As ρ is made smaller, the observer peak may grow larger, thus generating a large control input, which in turn might drive the system states X outside of Ω_{c_1} in shorter time. The boundedness of the control input makes sure that the exit time T_1 is independent of ϵ, since the upper bound on the magnitude of $\hat{v}$ will not depend on ϵ, thus allowing one to choose ϵ independently of T_1.

Lemma 2 proves that all the trajectories starting in Ω_{c_1} will remain confined within Ω_{c_2} and converge to an arbitrarily small neighborhood of the origin in finite time. Now, in order to complete the stability analysis, it remains to show that the origin of the output feedback closed-loop system is asymptotically stable, so that if Ω_ϵ is small enough all the closed-loop system trajectories converge to it.

Lemma 3 *There exists a positive scalar ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ all the trajectories starting inside the compact set $\Delta_\epsilon \stackrel{\triangle}{=} \{[X^\top, \tilde{x}^\top]^\top \mid V \leq d_\epsilon$ and $\|\tilde{x}\| \leq \epsilon\}$ converge asymptotically to the origin.*

Proof. Without loss of generality, assume ϵ is small enough so that $\hat{x} \in \mathcal{H}^{-1}(\mathcal{C})$ and, hence, $\tilde{x}^P = \tilde{x}$. From the proof of Theorem 1 recall that $\tilde{x} = \mathcal{H}^{-1}(\hat{y}_e, z) - \mathcal{H}^{-1}(y_e, z)$. Using A1 we have that the mapping $\mathcal{H}^{-1}$ is locally Lipschitz. Hence, there exists a neighborhood $N_{\tilde{y}_e}$ of the origin such that $\|\tilde{x}\| \leq k_0\|\tilde{y}_e\|$, for all $\tilde{y}_e \in N_{\tilde{y}_e}$, and for some positive constant k_0, which, by (29), implies that the origin of the $\tilde{x}$ system is exponentially stable. By the converse Lyapunov theorem we conclude that there exists a Lyapunov function $V_o'(\tilde{x})$ and positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3$ such that

$$
\begin{gathered}
\bar{c}_1\|\tilde{x}\|^2 \leq V_o' \leq \bar{c}_2\|\tilde{x}\|^2 \\
\dot{V}_o' \leq -\bar{c}_3\|\tilde{x}\|^2
\end{gathered}
$$

Define the positive scalar ϵ^* such that $\|\tilde{x}\| \leq \epsilon^*$ implies $\tilde{y}_e \in N_{\tilde{y}_e}$ (the existence of ϵ^* is a direct consequence of the fact that $\mathcal{H}$ is locally Lipschitz). Next, define the following composite Lyapunov function candidate

$$V_c(X, \tilde{x}) = V(X) + \lambda\sqrt{V_o'(\tilde{x})}, \qquad \lambda > \frac{2\sqrt{\bar{c}_2}\bar{\gamma}A}{\bar{c}_3}$$

then,

$$\begin{aligned} \dot{V}_c &\leq -\alpha_3(\|X\|) + A\bar{\gamma}\|\tilde{x}\| - \frac{\lambda}{2\sqrt{V_o'(\tilde{x})}}\bar{c}_3\|\tilde{x}\|^2 \\ &\leq -\alpha_3(\|X\|) - \left(\frac{\bar{c}_3\lambda}{2\sqrt{\bar{c}_2}} - A\bar{\gamma}\right)\|\tilde{x}\| < 0 \end{aligned}$$

where we have used the fact that $[X^\top, \tilde{x}^\top]^\top \in \Delta_\epsilon$ implies that $X \in \Omega_{c_2}$ (provided ϵ is small enough), and hence $\left\|\frac{\partial V}{\partial X}\right\| \leq A$. Since $\dot{V}_c$ is negative definite, all the $[X^\top, \tilde{x}^\top]^\top$ trajectories starting in Δ_ϵ will converge asymptotically to the origin.

■

We are now ready to state the following closed-loop stability theorem.

Theorem 2 *For the closed-loop system (5), (9), (35), satisfying assumptions A1, A2, and A3, the control law $\hat{v} = \phi(\hat{x}^P, z)$, guarantees that there exists a scalar $\rho^*, 0 < \rho^* \leq 1$, such that, for all $\rho \in (0, \rho^*]$, the set $\Omega_{c_1} \times \mathcal{H}^{-1}(\mathcal{C})$ is contained in the region of attraction of the origin $(x = 0, z = 0, \hat{x} = 0)$.*

Proof. By Lemma 3, there exists $\epsilon^* > 0$ such that, for all $\epsilon \in (0, \epsilon^*]$, Δ_ϵ is a region of attraction for the origin. Use Lemma 2 and the fact that $X(0) \in \Omega_{c_1}$ to find $\rho^*, 0 < \rho^* \leq 1$, so that for all $\rho \in (0, \rho^*]$ the state trajectories enter Δ_ϵ in finite time. This concludes the proof of the theorem.

■

Remark 11: Theorem 2 proves regional stability of the closed-loop system, since given an observability domain $\mathcal{O}$, and provided A3 is satisfied, the control law $\hat{v}$, together with (9) and (35), make the compact set $\Omega_{c_1} \times \mathcal{H}^{-1}(\mathcal{C})$ a domain of attraction for the origin of the closed-loop system. The difference between Theorem 2 and a local stability result lies in the fact that here the domain of attraction for x is at least as large as Ω_{c_1} and not restricted to be a small *unknown* neighborhood of the origin. Further, the size of Ω_{c_1} is independent of ρ and thus the domain of attraction *does not* shrink as the rate of convergence of the observer is made faster (that is, when $\rho \to 0$). This is made possible, in our approach, by the employment of the nonlinear projection (35) which eliminates the peaking phenomenon from the observer states. Recall from Remark 7 that the size of Ω_{c_2} (and hence that of Ω_{c_1}) depends on the size of the set $\mathcal{O}$ (see (33)). If A1 is satisfied globally (as in [8, 9, 7]) and $\mathcal{Y}^{\bar{z}}$ is a convex set for all $\bar{z} \in \mathbb{R}^{n_u}$ (see Remark 7), then Theorem 2 guarantees that the domain of attraction $\mathcal{D}$ of the closed-loop system under state feedback is recovered by the output feedback controller, in that c_2 can be chosen arbitrarily large (see Remark 7) and thus Ω_{c_2} can be made arbitrarily close to $\mathcal{D}$ (this then implies that Ω_{c_1} can be made arbitrarily close to $\mathcal{D}$). Note that the same argument holds even when (1) is not UCO but $\mathcal{O}$ is sufficiently large. If the system is UCO and the origin $x = 0$ is globally stabilizable, then the result of Theorem 2 becomes semi-global, as in [9], *provided* the set $\mathcal{Y}^{\bar{z}}$ is convex for all $\bar{z} \in \mathbb{R}^{n_u}$.

Remark 12: Analogous to the result in [1, 7], Theorem 2 proves a separation principle for nonlinear systems: given a stabilizing state feedback controller, the output feedback controller recovers its performance provided that the parameter ρ is chosen small enough. Furthermore, by slight modification of Theorem 3 in [7], it is easy to show that the closed-loop system trajectories under output feedback approach the trajectories under state feedback as $\rho \to 0$. In conclusion, the output feedback controller presented here achieves the same recovery properties of the one in [7] for the more general class of SISO non UCO systems. We must point out, however, that we assume to have perfect knowledge of the system dynamics, whereas the results in [7] admit model uncertainties. We opted not to include model uncertainties to better illustrate the underlying principles of our approach; it is an easy exercise to show that analogous results to the ones in [7] hold when the same assumptions on the model uncertainty are made.

Remark 13: As mentioned in Remark 7, if the plant is UCO (and hence $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^{n_u}$) and $\mathcal{Y}^{\bar{z}}$ is a convex set for all $\bar{z} \in \mathbb{R}^{n_u}$, assumption A3 is automatically satisfied by a sufficiently large cylindrical set $\mathcal{C}$. Even in this case, if $\mathcal{H}^{-1}$ is not explicitly known and one wants to directly estimate the state of the plant, one should employ the dynamic projection (35) since the standard saturation used, e.g., in [1, 9] can only be applied to a high-gain observer in y_e coordinates. Clearly, the only instance when dynamic projection can be replaced by saturation of the observer estimates is when the observability mapping is the identity.

5 Example

Consider the following input-output linearizable dynamical system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 + x_1)\exp(x_1^2) + u - 1 \\ y &= (x_2 - 1)^2 \end{aligned} \tag{46}$$

The control input appears in the first derivative of the output:

$$\dot{y} = 2(x_2 - 1)(1 + x_1)\exp(x_1^2) + 2(x_2 - 1)(u - 1)$$

Notice, however, that the coefficient multiplying u vanishes when $x_2 = 1$, and hence system (46) does not have a well-defined relative degree everywhere. Since u appears in $\dot{y}$, we have that $n_u = 1$, therefore we add one integrator at the input side,

$$\dot{z}_1 = v, \qquad u = z_1. \tag{47}$$

The mapping $\mathcal{F}$ is given by

$$Y = \begin{bmatrix} y \\ \dot{y} \\ z_1 \end{bmatrix} = \mathcal{F}(x, z_1) = \begin{bmatrix} \mathcal{H}(x, z_1) \\ z_1 \end{bmatrix} = \begin{bmatrix} (x_2 - 1)^2 \\ 2(x_2 - 1)[(1 + x_1)\exp(x_1^2) + (z_1 - 1)] \\ z_1 \end{bmatrix} \tag{48}$$

The first equation in (48) is invertible for all $x_2 < 1$, and its inverse is given by $x_2 = 1 - \sqrt{y}$. Substituting x_2 into the second equation in (48) and isolating the term in x_1, we get

$$(1 + x_1)\exp(x_1^2) = \frac{\dot{y} + 2\sqrt{y}(z_1 - 1)}{-2\sqrt{y}} \tag{49}$$

Since $(1+x_1)\exp(x_1^2)$ is a strictly increasing function, it follows that (49) is invertible for all $x_1 \in \mathbb{R}$, however, an analytical solution to this equation cannot be found. In conclusion, Assumption A1 is satisfied on the domain $\mathcal{O} = \{x \in \mathbb{R}^2 \,|\, x_2 < 1\} \times \mathbb{R}$, but an explicit inverse $[x^\top, {z_1}^\top]^\top = \mathcal{F}^{-1}(y_e, z_1)$ is not known. The fact that system (46) is not UCO, together with the non-existence of an explicit inverse to (48), prevents the application of the output feedback control approaches in [1, 8, 2, 9, 3, 5, 4, 6, 7].

To find a stabilizing state feedback controller, note that the extended system (46), (47) can be feedback linearized by letting $x_3 = (1 + x_1)\exp(x_1^2) + z_1 - 1$ and rewriting the system in new coordinates $x_e \triangleq [x_1, x_2, x_3]^\top$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_2 \exp(x_1^2)(2x_1^2 + 2x_1 + 1) + v \end{aligned} \tag{50}$$

Choose $v = -x_2 \exp(x_1^2)(2x_1^2 + 2x_1 + 1) - Kx_e$, where $K = [1, 3, 3]$, so that the closed-loop system becomes $\dot{x}_e = (A_c - B_c K)\, x_e$ with poles placed at -1. Then, the origin $x_e = 0$ is a globally asymptotically equilibrium point of (50), and Assumption A2 is satisfied with $\mathcal{D} = \mathbb{R}^3$. Let $\bar{P}$ be the solution of the Lyapunov equation associated to $A_c - B_c K$, so that a Lyapunov function for system (50) is $V = x_e^\top \bar{P} x_e$, and any set $\Omega_c \triangleq \big\{x_e \in \mathbb{R}^3 \,|\, V(x_e) \le c\big\}$, with $c > 0$, is contained in the region of attraction for the origin.

Next, we will seek to find a set $\mathcal{C}$ satisfying Assumption A3. To this end, notice that

$$\mathcal{Y} = \mathcal{F}(\mathcal{O}) = \mathcal{F}\left(\{x \in \mathbb{R}^2, z_1 \in \mathbb{R} \,|\, x_2 < 1\}\right) = \{\mathbb{R}^+ - 0\} \times \mathbb{R} \times \mathbb{R}.$$

Next recall from Remark 7 that, since A2 holds globally, $\mathcal{C}$ can be chosen to be the cylinder $D \times \mathbb{R}$, where D is any compact convex set in the upper half plane $\{\mathbb{R}^+ - 0\} \times \mathbb{R}$ containing the point $\mathcal{H}(0,0) = (1,0)$. For the sake of simplicity choose D to be the disk of radius $\omega < 1$ centered at $(1,0)$, so that $\mathcal{C} = \{Y \in \mathbb{R}^3 \,|\, (Y_1 - 1)^2 + Y_2^2 < \omega^2\}$, and $C \subset \mathcal{Y}$, as depicted in Figure 3. We stress that our choice is quite conservative and is made exclusively for the sake of illustration. Once $\mathcal{C}$ has been chosen, the control design is complete and the output feedback controller is given by

$$\dot{z}_1 = -\hat{x}_2^P \exp\left\{(\hat{x}_1^P)^2\right\}[2(\hat{x}_1^P)^2 + 2\hat{x}_1^P + 1] - \hat{x}_1^P - 3\hat{x}_2^P - 3\left[(1 + \hat{x}_1^P)\exp\left\{(\hat{x}_1^P)^2\right\} + z_1 - 1\right], \tag{51}$$

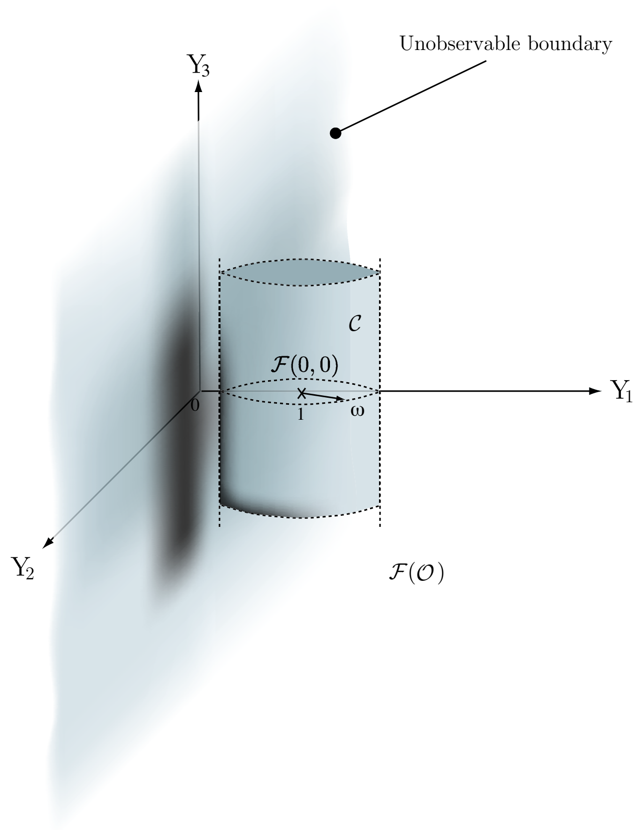


Figure 3: The projection domain $\mathcal{C}$.

where $\hat{x}^P(t)$ is the solution of (35) with

$$\hat{f}(\hat{x}, z, y) = \begin{bmatrix} \hat{x}_2 \\ (1+\hat{x}_1)\exp\left\{(\hat{x}_1)^2\right\} + z_1 - 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 2(\hat{x}_2 - 1) \\ 2(\hat{x}_2-1)\exp\left\{(\hat{x}_1)^2\right\}\left[2(\hat{x}_1)^2 + 2\hat{x}_1 + 1\right] & 2\left[(1+\hat{x}_1)\exp\left\{(\hat{x}_1)^2\right\} + z_1 - 1\right] \end{bmatrix}^{-1}}_{\left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1}} \mathcal{E}^{-1} L[y - (\hat{x}_2 - 1)^2] \tag{52}$$

and

$$\begin{aligned} &\frac{\partial \mathcal{H}}{\partial z} = \begin{bmatrix} 0 \\ 2(\hat{x}_2^P - 1) \end{bmatrix}, \ N_{y_e}(\hat{y}_e^P, z) = \hat{y}_e^P - \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ N_z(\hat{y}_e^P, z) = 0, \\ &\hat{y}_e^P = \begin{bmatrix} (\hat{x}_2^P - 1)^2 \\ 2(\hat{x}_2^P - 1)[(1+\hat{x}_1^P)\exp((\hat{x}_1^P)^2) + (z_1 - 1)] \end{bmatrix}, \ \dot{\hat{y}}_e|_{\hat{x}^P} = \frac{\partial \mathcal{H}}{\partial \hat{x}^P}\hat{f}(\hat{x}^P, z, y) + \frac{\partial \mathcal{H}}{\partial z}\dot{z}. \end{aligned} \tag{53}$$

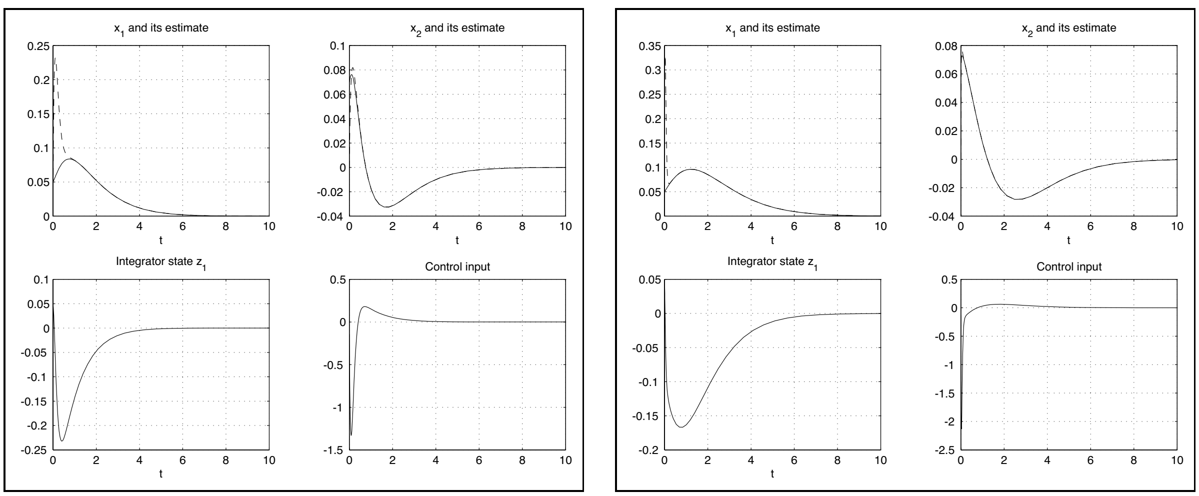
Using controller (51), Theorem 2 guarantees that the origin $x_e = 0$ of the closed-loop system is asymptotically stable and it provides an estimate of its domain of attraction. Specifically, given any positive scalar $c_1 < c_2$, there exists $\rho^* > 0$ such that Ω_{c_1} is contained in the domain of attraction for all $0 < \rho < \rho^*$. In what follows we will find the set Ω_{c_2} satisfying A3. Recalling that, in x_e-coordinates, Ω_{c_2} is expressed as $\{x_e \in \mathbb{R}^2 \,|\, {x_e}^\top \bar{P} x_e\}$, we have that $x_e \in \Omega_{c_2}$ implies $|x_i| \leq (c_2/\lambda_{min}(\bar{P}))$, $i = 1,2,3$, and hence $\Omega_{c_2} \subset \Xi \triangleq \{x_e \in \mathbb{R}^3 \,|\, |x_i| \leq (c_2/\lambda_{min}(\bar{P})), i = 1,2,3\}$. Now let $c_2 < \lambda_{min}(\bar{P})$ and note that, for all $x_e \in y_e$,

$$\left(1 - \frac{c_2}{\lambda_{min}(\bar{P})}\right)^2 \leq |x_2 - 1|^2 \leq \left(1 + \frac{c_2}{\lambda_{min}(\bar{P})}\right)^2$$
$$|2(x_2 - 1)x_3| \leq 2\left(1 + \frac{c_2}{\lambda_{min}(\bar{P})}\right)\frac{c_2}{\lambda_{min}(\bar{P})}.$$

Next, we seek to find c_2 satisfying A3, i.e., such that $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C}$. Using the inequalities above we have that if $c_2 = \min\{\bar{c}_2, \lambda_{min}(\bar{P})\}$, where $\bar{c}_2$ is the largest scalar satisfying

$$\left[\left(1 + \frac{\bar{c}_2}{\lambda_{min}(\bar{P})}\right)^2 - 1\right]^2 + 4\left(1 + \frac{\bar{c}_2}{\lambda_{min}(\bar{P})}\right)^2\left(\frac{\bar{c}_2}{\lambda_{min}(\bar{P})}\right)^2 < \omega^2 \tag{54}$$

(note that $\bar{c}_2 > 0$ satisfying the inequality above always exists) then $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{F}(\Xi) \subset \mathcal{C}$, and hence c_2 satisfies A3.



$\rho = 0.2.$ $\rho = 0.05.$

Figure 4: Closed-loop trajectories under output feedback.

For our simulations we choose $\omega = 0.9$ and, from (54), we get $c_2 = 0.06$. The initial condition of the extended system is set to $x_1(0) = 0.05, x_2(0) = 0.07, x_3(0) = 0.1$ (or $z_1(0) = 0.0474$), which is contained inside Ω_{c_2} so that Theorem 2 can be applied. Finally, we choose the observer gain L to be $[4, 4]^\top$, so that its associated polynomial is Hurwitz with both poles placed at -2. We present four different situations to illustrate four features of our output feedback controller:

1. **Arbitrary fast rate of convergence of the observer**. Figure 4 shows the evolution of the X-trajectory, as well as the control input v, for $\rho = 0.2$ and $\rho = 0.05$. The convergence in the latter case is faster, as predicted by Theorem 1 (see Remark 4).

2. **Observer estimate projection**. Figure 5 shows the evolution of $\hat{x}$ and v for $\rho = 10^{-3}$ with and without projection. The dynamic projection successfully eliminates the peak in the observer states, thus yielding a bounded control input, as predicted by the result of Lemma 1. Figure 6 shows that

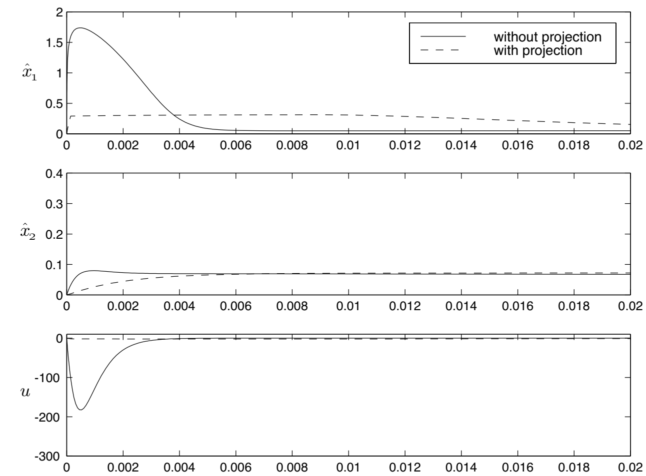


Figure 5: Observer states during the initial peaking phase with and without projection, $\rho = 10^{-3}$.

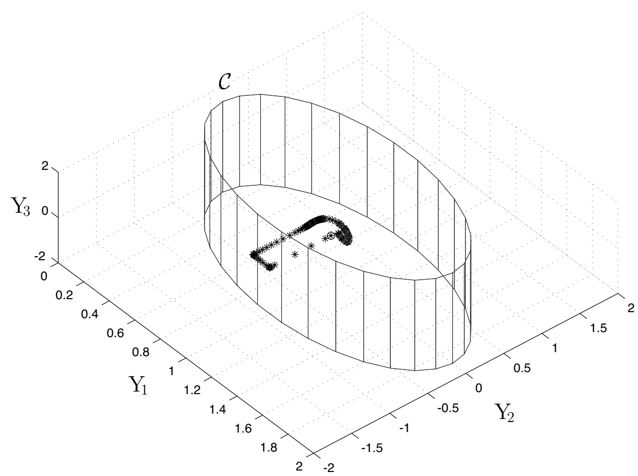


Figure 6: System trajectories in the transformed domain $Y = \mathcal{F}(X)$.

the orbits $Y(t) = \mathcal{F}(X(t))$ are contained within the set $\mathcal{C}$ for all $t \geq 0$, thus confirming the result of Lemma 1. In particular, Figure 6 shows the operation of the projection when the observer trajectory (in Y coordinates) hits the boundary of $\mathcal{C}$: it forces the $\hat{Y}^P$ trajectory to "slide" along the boundary of $\mathcal{C}$ and preserves its convergence characteristics. This is equivalent, in the x domain, to saying that the $\hat{x}^P$ trajectory slides along the boundary of $\mathcal{H}^{-1}(\mathcal{C})$ and converges to x.

3. **Observer estimate projection and closed-loop stability**. In Figure 7 a phase plane plot for x is shown with and without observer projection when $\rho = 10^{-4}$. The small value of ρ generates a significant peak which, if projection is not employed, drives the output feedback trajectories away form the state feedback ones and, in general, may drive the system to instability (see Remark 10). On the other hand, using the observer projection, output feedback and state feedback trajectories are almost indistinguishable.

4. **Trajectory recovery**. The evolution of the X-trajectories for decreasing values of ρ, in Figure 8, shows that the output feedback trajectories approach the state feedback ones as $\rho \to 0$ (see Remark

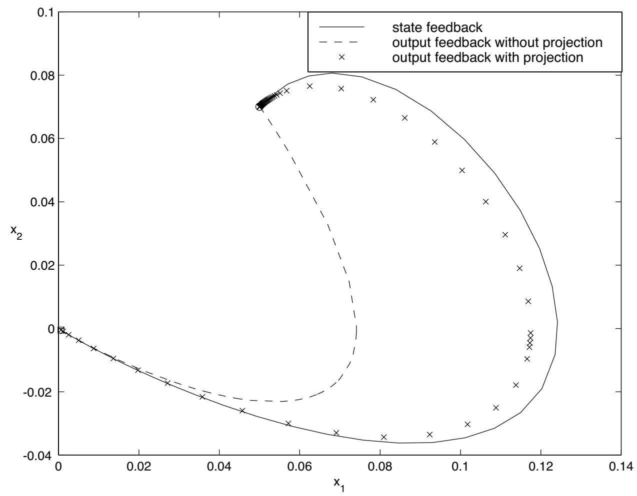


Figure 7: System trajectories with and without projection, $\rho = 10^{-3}$.

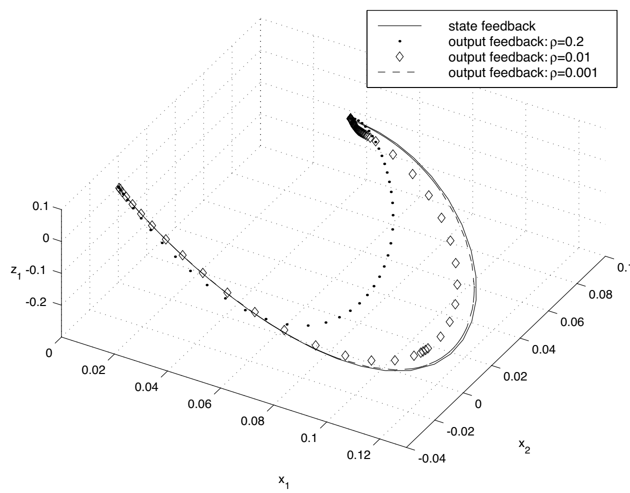


Figure 8: Closed-loop trajectories in the three dimensional space for decreasing values of ρ.

12).

Acknowledgments

The authors would like to thank Mireille Broucke for her helpful suggestions concerning the dynamic projection.

References

[1] F. Esfandiari and H. Khalil, "Output feedback stabilization of fully linearizable systems," *International Journal of Control*, vol. 56, no. 5, pp. 1007–1037, 1992.

[2] H. Khalil and F. Esfandiari, "Semiglobal stabilization of a class of nonlinear systems using output feedback," *IEEE Transactions on Automatic Control*, vol. 38, no. 9, pp. 1412–1415, 1993.

[3] Z. Lin and A. Saberi, "Robust semi-global stabilization of minimum-phase input-output linearizable systems via partial state and output feedback," *IEEE Transactions on Automatic Control*, vol. 40, no. 6, pp. 1029–1041, 1995.

[4] M. Jankovic, "Adaptive output feedback control of non-linear feedback linearizable systems," *International Journal of Adaptive Control and Signal Processing*, vol. 10, pp. 1–18, 1996.

[5] N. Mahmoud and H. Khalil, "Asymptotic regulation of minimum phase nonlinear systems using output feedback," *IEEE Trans. on Automatic Control*, vol. 41, no. 10, pp. 1402–1412, 1996.

[6] N. Mahmoud and H. Khalil, "Robust control for a nonlinear servomechanism problem," *International Journal of Control*, vol. 66, no. 6, pp. 779–802, 1997.

[7] A. Atassi and H. Khalil, "A separation principle for the stabilization of a class of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 44, pp. 1672–1687, September 1999.

[8] A. Tornambè, "Output feedback stabilization of a class of non-minimum phase nonlinear systems," *Systems & Control Letters*, vol. 19, pp. 193–204, 1992.

[9] A. Teel and L. Praly, "Global stabilizability and observability imply semi-global stabilizability by output feedback," *Systems & Control Letters*, vol. 22, pp. 313–325, 1994.

[10] H. Shim and A. R. Teel, "Asymptotic controllability imply semiglobal practical asymptotic stabilizability by sampled-data output feedback." preprint.

[11] E. D. Sontag, "Remarks on stabilization and input to state stability," in *Proceedings of the IEEE Conference on Decision and Control*, (Tampa, FL), pp. 1376–1378, December 1989.

[12] A. Isidori, *Nonlinear Control Systems.* London: Springer-Verlag, Third ed., 1995.

[13] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design.* NY: John Wiley & Sons, Inc., 1995.

[14] G. Ciccarella, M. Dalla Mora, and A. Germani, "A Luenberger-like observer for nonlinear systems," *International Journal of Control*, vol. 57, no. 3, pp. 537–556, 1993.

[15] T. Yoshizawa, *Stability Theory by Lyapunov's Second Method.* The Mathematical Society of Japan, Tokyo, 1966.

[16] J. Kurzweil, "On the inversion of Ljapunov's second theorem on stability of motion," *American Mathematical Society Translations,* Series 2, vol. 24, pp. 19–77, 1956.

[17] H. Khalil, *Nonlinear Systems.* NJ: Prentice-Hall, Second ed., 1996.

[18] P. Ioannou and J. Sun, *Stable and Robust Adaptive Control.* Englewood Cliffs, NJ: Prentice-Hall, 1995.

[19] J. Pomet and L. Praly, "Adaptive nonlinear regulation: Estimation from the Lyapunov equation," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 729–740, 1992.

[20] A. Teel and L. Praly, "Tools for the semiglobal stabilization by partial state and output feedback," *SIAM J. Control Optim.*, vol. 33, no. 5, pp. 1443–1488, 1995.

[21] A. Isidori, A. R. Teel, and L. Praly, "A note on the problem of semiglobal practical stabilization of uncertain nonlinear systems via dynamic output feedback," *Systems & Control Letters*, vol. 39, pp. 165–179, 2000.