Critical Values in Asynchronous Random Boolean Networks

Bertrand Mesot and Christof Teuscher

Logic Systems Laboratory, Swiss Federal Institute of Technology (EPFL),
CH-1015 Lausanne, Switzerland
E-mails: bertrand.mesot@epfl.ch, christof@teuscher.ch

Abstract Wherever we see life, we see different kinds of complex networks, reason why they are studied across various fields of science. Random Boolean Networks (RBNs) form a special class in which the links between the nodes and the boolean functions are specified at random. Whereas synchronous RBNs were investigated in detail, there has little been done around their asynchronous counterpart, although there is evidence that most living systems are governed by asynchronous updating. Derrida's annealed approximation predicts a critical connectivity value of $K = 2$ for synchronous RBNs. We present a similar and original approach for asynchronous RBNs and show that they do not possess such a critical connectivity value. The asynchronous and nondeterministic updating scheme introduces perturbations that reach about 25% of the nodes and thus prevents the networks to become stable. Further, our numerical simulations show that asynchronous RBN tend to amplify small and to reduce big perturbations.

1 Introduction

Wherever we see life, we see different kinds of complex networks, reason why they are studied across various fields of science. Many of the natural networks such as ecological food webs, genetic networks, social networks, neural networks, and even the World Wide Web share common and global statistical features and motifs [14, 18, 20].

Among the different kinds of networks, *Random Boolean Networks (RBNs)* (sometimes also called *Kauffman nets or model*) form a special class in which the links between the nodes and the node's boolean transfer functions are specified at random. They are often specified by two parameters: N, the number of nodes and K, the number of incoming links per node (sometimes, K indicates the average number of links). Synchronous RBNs have been seriously investigated by Kauffman [12, 13], Weisbuch [22], and many others as models for biological phenomena such as genetic regulatory networks and embryonic development.

Indeed, randomly connected networks with various kinds of nodes have been analyzed much earlier. The first persons to mention randomly connected networks were Ashby [3] and Allanson [1] who investigated in 1956 networks of dynamical systems. Rozonoér [16] analyzed the properties of networks consisting of elements whose properties depend on parameters chosen at random. In

1971, Amari [2] published a paper on the characteristics of randomly connected threshold-element networks with the intention of understanding some aspects of information processing in nervous systems. He showed that two statistical parameters are sufficient to determine the characteristics of such networks.

Whereas synchronous RBNs as abstract models of specific biological systems were investigated in detail, there has little been done around their asynchronous counterpart, although there is evidence that most living systems are governed by asynchronous updating. Harvey and Bossomaier note, that "[...] for many biological phenomena asynchronous versions are more plausible" [9]. Observed global synchronous behavior in Nature usually simply arises from the local asynchronous behavior.

This paper principally addresses the question whether asynchronous RBNs have a similar phase transition and critical value for K like synchronous RBNs. In particular, we use a similar approach to asynchronous RBNs as Derrida used to find the critical connectivity for synchronous RBNs. Our results are then verified by numerical simulations.

The reminder of the paper is as following: Section 2 introduces the principal characteristics of synchronous and asynchronous RBNs. Section 3 gives an overview on Derrida's annealed approximation that predicts $K = 2$ for synchronous RBNs. Our approach for asynchronous RBNs is presented in Section 4 and the numerical results in Section 5. Section 6 concludes the paper.

2 Synchronous versus Asynchronous Random Boolean Networks

Random boolean networks (RBNs) are usually considered as a more general case of classical cellular automata (CA). In the synchronous version, both are examples of discrete deterministic dynamical systems made up from simple components that process data in parallel. The RBN architecture is in many ways similar to weightless neural networks [21]. Kauffman's studies [13] have revealed surprisingly ordered structures in randomly constructed networks. In particular, the most highly organized behavior appeared to occur in networks where each node receives inputs from two other nodes ($K = 2$). It turned out that the networks exhibit three major regimes of behavior: *ordered* ("solid"), *complex* ("liquid"), and *chaotic* ("gas"). The most complex and interesting dynamics correspond to the liquid interface, the boundary between order and chaos. In the ordered regime, little computation can occur. In the chaotic phase, dynamics are too disordered to be useful. The most important and dominant results of Kauffman's numerical simulations can be summarized as follows [13]: (1) The expected median state cycle length is about $\sqrt{N}$. (2) Most networks have short state cycles, while a few have very long ones. (3) The number of state cycle attractors is about $\sqrt{N}$. (4) The most interesting dynamics appear with an average connectivity of $K = 2$ (the boundary between order and chaos).

Very few work has been done around asynchronous random boolean networks (ARBN), although the updating scheme in discrete systems plays a crucial role

for its properties. Moreover, for many physical and biological phenomena, the assumption of asynchrony seems more plausible. Harvey and Bossomaier [9] have shown that ARBNs behave radically different from the deterministic synchronous version. Di Paolo [15] provided further analysis and mainly investigated rhythmic and non-rhythmic attractors. Although ARBNs cannot exhibit strictly cyclic behavior (due to their random updating scheme), he has shown that they can all the same model rhythmic phenomenon. Recently, Gershenson [7] provided a first classification of the different types of RBNs. The study also revealed that the RBNs point attractors are independent of the updating scheme and that they are more different depending on their determinism rather than depending on their synchronicity.

An *attractor* in a dynamical system is an *equilibrium state*. Following Hopefield [10], the attractors of networks represent a sort of content addressable memory. Each attractor is encompassed by a *basin (domain) of attraction*. A deterministic complex dynamical system with a finite number of states ultimately "settles down" in an attractor after a finite time. If the state vector comes to rest completely, it is called a *fixed point*. If the state vector settles into a periodic motion, it is called a *limited cycle*. Due to their indeterminism, asynchronous RBNs do not have cyclic attractors but only point and loose attractors. Similar to the synchronous version, the number of point attractors is independent of K [9].

Note that there is also a growing interest in asynchronous cellular automata in various problem domains (see for example [4, 5, 17, 19]).

3 Derrida's Annealed Approach to Synchronous RBNs

As stated above, the average connectivity of $K = 2$ presents a critical connectivity for classical synchronous RBNs. This value is obtained by numerical simulations as well as by several theoretical methods. In 1986, Derrida and Pomeau [6] proposed the *annealed approximation* which allowed to predict $K = 2$ as the critical value of K. This section shall briefly recall the basic ideas of Derrida's approach that we then apply in a similar manner in Section 4 to asynchronous RBNs.

Assume that we have a network made up of N nodes, each being randomly connected to K other nodes. Each node can be in one out of two possible states, 0 or 1, and the node's state after the next update is defined by a randomly chosen boolean function that takes the K incoming links as inputs. The *network state* at time t is defined as the vector of node states at time t, which we write as $\mathbf{s}^t$.

The network states $\mathbf{s}^t$ for $t \geqslant 1$ are correlated to the network wiring and the node's boolean functions. Derrida noticed that this is somehow difficult to formalize, which lead him to the correct assumption that, since everything (i.e., wiring, transfer functions) is random in RBNs, randomly generating a new network after each update should not fundamentally affect the overall dynamics of the system. In order to find out the critical value for K, he compared the dynamics of two identically connected networks with state vectors $\mathbf{s}_1^t$ and $\mathbf{s}_2^t$. Let

$\mathbf{s}_2^t$ be a perturbed (i.e., changing the state of a random number of nodes) copy of $\mathbf{s}_1^t$ and let us define d_t as being the normalized *Hamming distance* between $\mathbf{s}_1^t$ and $\mathbf{s}_2^t$. The main question is then as follows: which value of K allows $d_t \to 0$ when $t \to +\infty$?

To answer this question, let a_t be the probability for a node to have the same value in both networks at time t. Let us then make two subsets of nodes called A and B. A contains all nodes that have equal states in $\mathbf{s}_1^t$ and in $\mathbf{s}_2^t$, B contains nodes with unequal states. Obviously, the probability for a node to belong to A at time t is a_t. Moreover, for each node two possibilities arise: (1) all of its input nodes belong to A, and (2) at least one of the input nodes belongs to B. Hence, in the first case, the node's input will be the same in both networks since each input node belongs to A. This does not hold in the second case as at least one of the input nodes belongs to B. For a given node, the probability of having all of its inputs in A is therefore $(a_t)^K$, and $\left(1-(a_t)^K\right)$ otherwise. Consequently, the node's probability of being in the same state in $\mathbf{s}_1^t$ as in $\mathbf{s}_2^t$ at time $t+1$ is 1 in the first case and $1/2$ in the second as we suppose that states 0 and 1 are equally distributed. Indeed, if a node has the same input in both networks, it will surely have the same output and consequently be in the same state in both networks at the next time step. In the second case, inputs are different and outputs will thus be equal with probability $1/2$.

We can therefore describe the evolution of a_t as a function of t by means of the following recursive equation:

$$a_{t+1} = (a_t)^K + \frac{1}{2}\left(1-(a_t)^K\right) = \frac{1+(a_t)^K}{2}\,.$$

By taking into account that $d_t = 1 - a_t$, we get:

$$d_{t+1} = \frac{1-(1-d_t)^K}{2}$$

which describes the evolution of d_t as a function of time t. Figure 1 plots d_{t+1} as a function of d_t for $K = 1, 2, 3, 4, 10$. One can easily see that $K \in \{1, 2\}$ are the two only values that allow $d_t \to 0$ when $t \to +\infty$. Geometrically speaking, the plots for $K = 1$ and $K = 2$ lie below the identity function $d_t = d_{t+1}$ which implies that d_{t+1} tends toward 0 as time increases.

The results suggest that synchronous RBNs with a connectivity of $K = 2$ (the "edge between order and chaos") are particularly resistant to perturbations. This is mainly due to a phase transition in the number of frozen components within a network (see also [13]). Naturally, the question arises whether such a critical value exists in asynchronous RBNs. To the best of our knowledge, our attempt is the first one to investigate this question.

4 Our Approach to Asynchronous RBNs

As mentioned in Section 1, Harvey and Bossomaier claimed that asynchronous systems are biologically more plausible for many phenomena than their synchronous counterpart. However, studying asynchronous RBNs is often all but a

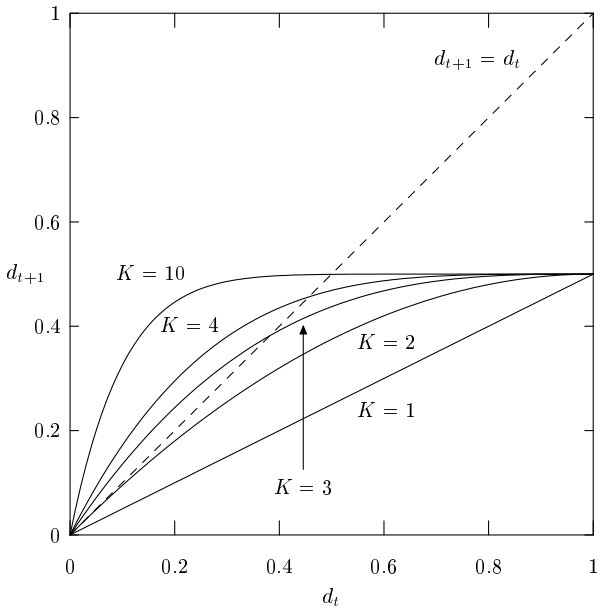

Figure 1. d_{t+1} in function of d_t for $K = 1, 2, 3, 4, 10$. For $K = 1$ and $K = 2$, d_{t+1} tends toward 0 as time increases. For more information, see also Kauffman [13].

trivial task, mainly due to the nondeterminism introduced by the asynchronous node updating scheme.

In this paper, we will only consider the following case of asynchrony: at each time step, 1 to N nodes are randomly selected and synchronously updated. As shown in the previous section, we consider two identically connected networks with different state vectors $\mathbf{s}_1^t$ and $\mathbf{s}_2^t$. Again, consider $\mathbf{s}_2^t$ as being a perturbed copy of $\mathbf{s}_1^t$. In the asynchronous case, three possibilities arise for each node: (1) the node is updated in both networks, similarly to synchronous RBNs; (2) the node is updated in only one of the networks; and (3) the node is not updated at all. We therefore need to calculate the node's probability of being updated. Let m be the number of nodes updated at a given time t. Hence, the node's probability of being updated knowing m is m/N. As $m \in [1, N]$ and as we suppose that all values are equally probable, the probability of being updated becomes:

$$P(\text{being updated}) = \frac{1}{N} \sum_{m=1}^{N} \frac{m}{N} = \frac{N+1}{2N}.$$

Given this, we can now describe the probability of the three above-mentioned situations:

$$p_2 = \left(\frac{N+1}{2N}\right)^2, \quad p_1 = 2\left(\frac{N+1}{2N}\right)\left(1 - \frac{N+1}{2N}\right) \quad \text{and} \quad p_0 = \left(1 - \frac{N+1}{2N}\right)^2$$

where p_i describes the probability that the node is updated in i networks. Note that p_1 is counted twice because there are two possibilities of updating a node in only one of both networks.

Nodes can now again be separated into two subsets A and B with the same meaning as described in Section 3. Hence, if a node has all of its inputs in A and is updated in both networks, it will surely hold the same value in $\mathbf{s}_1^{t+1}$ as in $\mathbf{s}_2^{t+1}$. If only one of both networks updates the node, there will be one chance out of two to be in a different state. Finally, if none of the networks updates the node, it will keep its current state at time $t+1$ and thus hold the same value in both networks if and only if the node belongs to subset A. After adding together all these probabilities we get:

$$(a_t)^K \left[\frac{N+1}{2N} + \left(\frac{N-1}{2N} \right)^2 a_t \right]. \tag{1}$$

This represents the contribution of the nodes having all of their input nodes in A to the overall probability a_{t+1} that a node has the same state in both networks at time $t+1$. To this term we must add the contribution of the nodes that have at least one connection in B. Note that when a node is updated in both networks, the probability of being in the same state at time $t+1$ is not 1 but 1/2, similarly to the synchronous case. We therefore obtain:

$$\left(1 - (a_t)^K\right) \left[\frac{3/2 \cdot N^2 + N - 1/2}{4N^2} + \left(\frac{N-1}{2N} \right)^2 a_t \right]. \tag{2}$$

And finally, the probability a_{t+1} that a node has the same state in both networks at time $t+1$ is obtained by adding 1 and 2 when $N \to +\infty$. Note that $N \to +\infty$ simplifies the writing of the forthcoming equations and is justified by the fact that, as long as we consider probabilities over nodes, a "sufficient" number of them is required. The resulting recursive equation for asynchronous RBNs is thus as follows:

$$a_{t+1} = \frac{1}{8} (a_t)^K + \frac{1}{4} a_t + \frac{3}{8}.$$

As for synchronous RBNs, this leads to the equation that defines the evolution of perturbations in asynchronous RBNs:

$$d_{t+1} = \frac{5}{8} - \frac{1}{8} (1 - d_t)^K - \frac{1}{4} (1 - d_t). \tag{3}$$

Figure 2 shows the plots of Equation 3 for $K = 1, 2, 3, 4, 10$. The following observations can be made:

1. Critical values for K do not seem to exist as no plot is strictly below the identity function $d_{t+1} = d_t$.
2. Each plot starts from one and the same point, situated at $d_{t+1} = 0.25$. This means that two networks with the same initial state will become different on 25% of their nodes at the next time step. Hence, $d_t = 0 \Rightarrow d_{t+1} = 0$ is no longer valid.

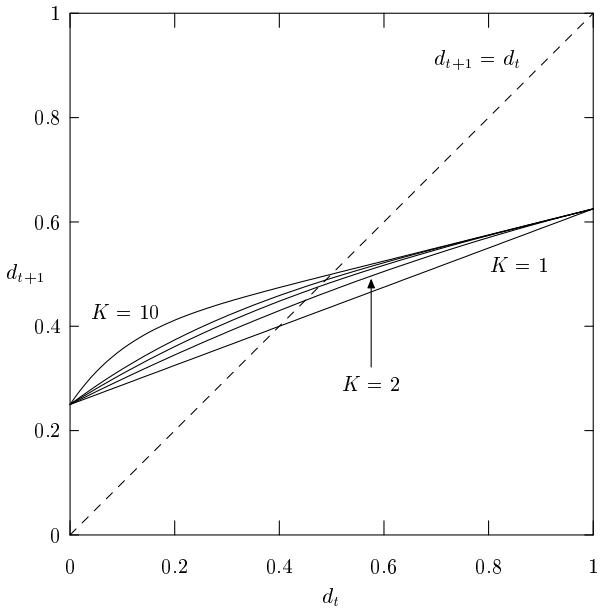

Figure 2. d_{t+1} in function of d_t for $K = 1, 2, 3, 4, 10$. Note that no critical value of K seems to exist.

From the point of view of the above presented theoretical analysis, it follows that asynchronous RBNs do not seem to be as tolerant to perturbations as synchronous RBNs. In fact when d_t is lower than 0.5, d_{t+1} is always bigger than d_t, independently of K. This implies that, instead of reducing perturbations, asynchronous RBNs create new ones.

Remember that Derrida allowed his networks to change at each time step. This simplification has proved to be correct for synchronous networks, i.e., not affecting the overall network dynamics, however, one might ask whether this hypothesis is correct for asynchronous RBNs too. The next section shall address this question by means of numerical simulations.

5 Numerical Results

Figure 3 shows numerical results for $K = 1, 2, 3, 4, 10$ obtained with networks of $N = 200$ nodes. For each value of d_t between 1 and 200, 200 randomly generated pairs of network states were tested during $t = 600$ time steps. Then, for each value of d_t, the mean value of d_{t+1} was computed.

Compared to Figure 2, Figure 3 shows mainly two differences. The plot obtained for $K = 1$ lies very close to the identity function $d_{t+1} = d_t$. Hence, the theoretical results do not correspond to the numerical simulations for that case. A possible explanation is that $K = 1$ networks are highly ordered ("solid") and

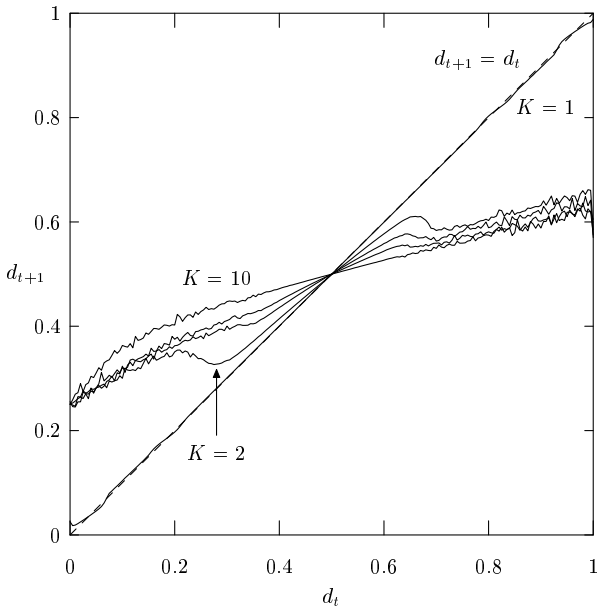

Figure 3. Numerical results obtained for $K = 1, 2, 3, 4, 10$ with asynchronous RBNs made up of 200 nodes.

that they possess a large number of short attractors. Indeed, two identical networks with their state vectors at distance d_t will very quickly end up into different attractors that are separated approximately by the same distance. Remember, however, that our theoretical model is rewired constantly and that therefore no attractors are possible, hence the difference between the simulations and reality.

The second important difference is that our theoretical model also partly fails to predict the behavior of the $K = 2$ plot for $d_t \in [0.2, 0.7]$. The numerical results show that networks with two incoming connections per node can also be very stable within this interval as the plot lies rather close to the identity function.

Finally, the mean error between the theoretical model and the numerical simulations is about 3.3% for $K = 2$, 1.6% for $K = 3$ and falls below 1% for $K > 3$. Hence, the bigger K becomes, the better the model fits to reality.

6 Conclusion and Future Work

We have investigated the dynamic behavior of asynchronous RBNs by means of a method inspired by Derrida's annealed approximation. The main question was whether there is a similar phase transition and critical value for K in asynchronous as in synchronous RBNs.

Our original theoretical approach and numerical simulations revealed that asynchronous RBNs do not have a critical connectivity value similar to syn-

chronous RBNs for K. The asynchronous and nondeterministic updating scheme introduces perturbations that reach about 25% of the nodes and thus prevents the networks to become stable. Although there were some small discrepancies between the theoretical model and the numerical simulations, we can say that our approach to asynchronous RBNs predicts the most important characteristics of their overall dynamics. From the numerical simulations we can conclude that asynchronous RBNs tend to amplify small perturbations, to reduce big ones and to keep them constant when they are located in a region around $d_t = 0.5$. Furthermore, our findings confirm what Harvey and Bossomaier [9] have shown: asynchronous RBNs behave radically different from the deterministic synchronous version.

Synchronous RBNs have mainly been used as abstract models of specific biological systems, however, some other applications exist. In an interesting attempt, for example, Hurford [11] used synchronous RBNs to cast several essential properties of natural languages. He modeled a language as an attractor of a boolean network. To the best of our knowledge no useful and practical applications seems to exist for asynchronous RBNs and one might question whether they are really biologically plausible models. Synchronous updating as used in synchronous RBNs is not usually seen in Nature, although synchronous behavior might arise from the synchronization of asynchronous elements (e.g., [8]). Strict synchrony as well as nondeterministic asynchrony present two extremes—neither is usually observed in Nature. Biological networks do not make use of any global clock (i.e., synchronizing) signal, but usually immediately react to perturbations. In genetic regulatory networks, for example, a change in a gene's activation state may instantly imply other state changes in a deterministic way elsewhere in the network. From this point of view, synchronous RBNs are closer to reality than the asynchronous RBNs studied in this paper. Based on our results, we seriously doubt whether purely asynchronous RBNs are of any interest in modeling biological systems. There exist, however, many variants of asynchronous, deterministic, and lossless information transfer methods (e.g., asynchronous logic based on Muller-C elements) that do not even make use of local clock signals.

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