Logical Construction of Final Coalgebras

Luigi Santocanale [1]

LaBRI, Université Bordeaux 1
santocan@labri.fr

Abstract

We prove that every finitary polynomial endofunctor of a category $\mathcal{C}$ has a final coalgebra, provided that $\mathcal{C}$ is locally Cartesian closed, it has finite coproducts and is an extensive category, it has a natural number object.

1 Introduction

In this note we prove that every polynomial endofunctor

$$P(X) = \sum_{i=1,\dots,n} \Omega_i \times X^{A_i} \tag{1}$$

of a given category $\mathcal{C}$ has a final coalgebra, assuming that (1) $\mathcal{C}$ is locally Cartesian closed, (2) it has finite coproducts and is an extensive category, (3) it has a natural number object. It readily follows that the functor P generates a cofree comonad and that it is completely iterative.

Our proof is inspired by the many set theoretic representations – see [9] – of the final P-coalgebra: for example, if A is a large enough alphabet, this final coalgebra can be represented as a set of pairs (I, t), $I \subseteq A^*$ being a nonempty lower ideal w.r.t. the prefix order of the free monoid, and t being a labeling of words in I subject to typing constraints. Each such pair can also be identified with a particular *complete* A-branching tree over some set of labels, a tree being complete if each node has exactly one subtree in the direction a, for $a \in A$. We illustrate this idea with the next example.

Example 1.1 The set of infinite terms over the signature $\Omega = \{f, g\}$ – where f is a unary function symbol and g is a binary function symbol – is the final coalgebra of the functor $P(X) = \{f\} \times X + \{g\} \times X^2$. Let A be the direction alphabet $\{(f,1),(g,1),(g,2)\}$ and let Ω' be the signature Ω augmented with

[1] Research supported by the European Community through a Marie Curie Individual Fellowship.

This is a preliminary version. The final version will be published in Electronic Notes in Theoretical Computer Science
URL: www.elsevier.nl/locate/entcs

a new symbol $\perp$. Then the infinite terms over the signature $\Omega = \{f, g\}$ are in bijection with those complete A-branching trees labeled in Ω' satisfying the following clauses:

(i) If a node is labeled by f, then the son along the direction $(f, 1)$ is not labeled by the symbol $\perp$ while all the subtrees in the directions $(g, 1), (g, 2)$ are constantly labeled by $\perp$.

(ii) If a node is labeled by g, then the subtree along the direction $(f, 1)$ is constantly labeled by the symbol $\perp$ and the sons in the directions $(g, 1), (g, 2)$ are not labeled by $\perp$.

(iii) The root of the tree is not labeled by the symbol $\perp$.

A complete tree satisfying these conditions is represented in figure 1.

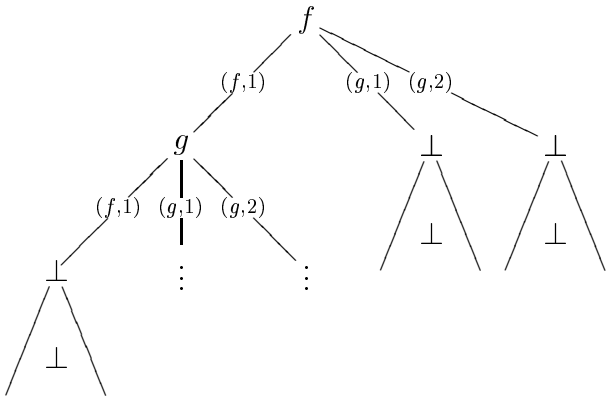


Fig. 1. Ideals as complete trees

The final P-coalgebra is therefore a subset of the object X^{A^*} of complete A-branching trees with labels in some set X; and the object X^{A^*} is itself a final coalgebra of some other functor. We shall investigate the process by which a final coalgebra of P is extracted from the final coalgebra of another functor by means of logical operations. We shall show that this process can be carried within the weak logic corresponding to the categorical properties (1)-(3). The reader should remark from the beginning the structure of the clauses (i)-(iii) of Example 1.1 and be aware that the use of classical logic is restricted to the constructive Boolean logic of extensive categories (see for example [7]).

Our interest in this problem stems from a general investigation [8,19] on the relationships between induction and coinduction, i.e. initial algebras and final coalgebras of functors. In [8] we proved that, given an adjoint pair of endofunctors $F \dashv G$, if a free F-algebra functor $\widehat{F}$ exists and has a right adjoint $\widetilde{G}$, then $\widetilde{G}$ is a cofree G-coalgebra functor. For example, if A^* is the free monoid generated by A in a Cartesian closed category, then $A^* \times X$ is the free $A \times (-)$-algebra generated by X, and the function space X^{A^*} turns out to be the cofree $(-)^A$-coalgebra over X. Briefly, some final coalgebras arise from duality. A question suggested from the set theoretic example but left open in

[8] is whether and under which conditions the duality is the starting point for constructing other final coalgebras.

Among the categories satisfying conditions (1)-(3) are elementary toposes with a natural number object. For Grothendieck toposes – which as categories are locally presentable [3] – standard tools can be used to show that a polynomial endofunctor has a final coalgebra: this functor is accessible and its final coalgebra can be constructed as the inverse limit of the terminal chain (the chain of iterated functorial applications that begins at the terminal object [5,2,23]). Many toposes of interest in computer science are not complete; among them we list the effective topos [12] and the free topos generated by a countable language [17]. Apparently the construction/representation of a final coalgebra as an inverse limit is not available since it depends on completeness. Nonetheless it is shown in [14] that these toposes admit final coalgebras of partial product functors, see [11]; the polynomial functors that we consider are indeed examples of partial product functors. These final coalgebras are constructed as *internal* limits of the terminal chain; the proof of their existence depends on the development of internal category theory [16] and on the theory of iterative data in an elementary topos [15] by which it is shown that some external iterative processes can be internalized.

Our goal is to present and analyze a construction of the final P-coalgebra alternative to the terminal chain construction. This construction depends only on a restricted (negative) fragment of the topos theoretic structure: we do not need a subobject classifier nor a factorization system corresponding to existential quantification. This remark could be exploited to further investigate preservation of final coalgebras under morphisms of toposes. Moreover, we are interested in modeling the arithmetic of typed programming languages in structured categories. To this goal the topos structure is often considered too strong and the weaker properties (1)-(3) are preferred, see for example [13] and, more recently, [1].

In the development we shall adopt the framework of [8] and prove a more general result: given adjoint pairs $F_i \dashv G_i$, $i = 1, \ldots, n$, an endofunctor of the form

$$P(X) = \sum_{i=1,\ldots,n} \Omega_i \times G_i(X)$$

has a final coalgebra under the assumption that $\mathcal{C}$ has the properties (1)-(2) and that both a free algebra functor of $\sum_{i=1,\ldots,n} F_i$ and a cofree coalgebra functor of $\prod_{i=1,\ldots,n} G_i$ exist. We recover the previous statement letting $F_i X = A_i \times X$, $G_i X = X^{A_i}$, and recalling that in a locally Cartesian closed category with a natural number object $\mathbb{N}$ (i) a free monoid A^* is computed as the partial product of A with the arrow $\mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N} \xrightarrow{s} \mathbb{N}$, (ii) the functor $A^* \times X$ is a free algebra functor of $A \times X$ (and X^{A^*} is a cofree coalgebra functor of X^A). Using this general form, we can show that functorial systems of equations such

as

$$\left\{ X_k \;=\; \sum_{i=1,\dots,n} \Omega_{i,k} \times X_{f(k)}^{A_{i,k}} \right\}_{k=1,\dots,m}$$

have a greatest solution. For this we observe that conditions (1)-(2) are stable under formation of products of categories and use results from [13].

This note is structured as follows: in section 2 we overview on the mathematical setting and introduce the notation. In section 3 we find sufficient conditions for a subobject of the object of complete trees to be the final coalgebra of the polynomial functor P. In section 4 we construct a subobject and show that the sufficient conditions holds; in this way we prove the main claim. Finally, in section 5, we argue that this construction of a final coalgebra can be lifted to prove the existence of a functor which preserves the terminal object, and argue for the need of extensiveness.

2 Preliminaries

In this note we shall prove the following statement:

Theorem 2.1 *Let $\mathcal{C}$ be a locally Cartesian closed category with finite coproducts and assume $\mathcal{C}$ is an extensive category. Let $F_i \dashv G_i$, $i \in I$, be a finite collection of adjoint endofunctors of a category $\mathcal{C}$ and put*

$$F(X) = \sum_{i \in I} F_i(X)\,, \qquad\qquad G(X) = \prod_{i \in I} G_i(X)\,.$$

If a pair of adjoint endofunctors $\widehat{F} \dashv \widetilde{G}$ – where $\widehat{F}$ is a free F-algebra functor or equivalently $\widetilde{G}$ is a cofree G-coalgebra functor – exists, then the functor

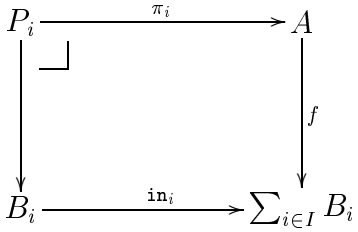
$$P(X) = \sum_{i \in I} \Omega_i \times G_i(X) \tag{2}$$

has a final coalgebra.

We begin explaining the statement and the general setting in which we work; we introduce a useful lemma about universal quantification at the end of this section.

A category $\mathcal{C}$ is locally Cartesian closed if it has a terminal object 1 and each slice category $\mathcal{C}/C$ is a Cartesian closed category [10,18,20]. This implies that $\mathcal{C}$ is itself Cartesian closed, since it is equivalent to the slice category $\mathcal{C}/1$. Moreover $\mathcal{C}$ is a distributive category, meaning that if we construct the

pullbacks



then the diagram $(\pi_i : P_i \longrightarrow A)_{i\in I}$ is again a coproduct diagram. $\mathcal{C}$ is an extensive category if the converse condition holds: if $(\pi_i : P_i \longrightarrow A)_{i\in I}$ is a coproduct diagram and the diagram above commutes, then it is a pullback. Rephrased, the coproduct injections are Cartesian natural transformations.

By saying that a free F-algebra functor exists, we mean that the forgetful functor $U_F : \mathrm{Alg}(F) \longrightarrow \mathcal{C}$ has a left adjoint. Spelled out, for every object X we can find an object $\widehat{F}X$ and a diagram

$$\mathtt{z}_X : X \longrightarrow \widehat{F}X \qquad\qquad \mathtt{s}_X : F\widehat{F}X \longrightarrow \widehat{F}X$$

with the initial property w.r.t. similar diagrams: for every pair (a, f), where $a : X \longrightarrow A$ and $f : FA \longrightarrow A$, there exists a unique arrow $\{\!|a, f|\!\} : \widehat{F}X \longrightarrow A$ such that

$$\mathtt{z}_X \cdot \{\!|a, f|\!\} = a\,, \qquad\qquad \mathtt{s}_X \cdot \{\!|a, f|\!\} = F\{\!|a, f|\!\} \cdot f\,.$$

Similarly, by saying that a cofree G-coalgebra functor exists, we mean that the forgetful functor $U_G : \mathrm{CoAlg}(G) \longrightarrow \mathcal{C}$ has a right adjoint. Spelled out, for every object X we can find an object $\widetilde{G}X$ and a diagram

$$\mathtt{h}_X : \widetilde{G}X \longrightarrow X \qquad\qquad \mathtt{t}_X : \widetilde{G}X \longrightarrow G\widetilde{G}X$$

with the the final property w.r.t. similar diagrams. Clearly, $\widehat{F}$ and $\widetilde{G}$ are functors, obtained by composing the left adjoint with U_F and the right adjoint with U_G; $\mathtt{z}, \mathtt{s}, \mathtt{h}, \mathtt{t}$ are natural transformations.

If a free F-algebra functor $\widehat{F}$ is given and we define

$$\mathtt{i}_X = F\mathtt{z}_X \cdot \mathtt{s}_X : FX \longrightarrow \widehat{F}X\,, \quad \mathtt{m}_X = \{\!|\mathtt{id}_{\widehat{F}X}, \mathtt{s}_X|\!\} : \widehat{F}\widehat{F}X \longrightarrow \widehat{F}X\,,$$

then the tuple $\langle \widehat{F}, \mathtt{i}, \mathtt{z}, \mathtt{m}\rangle$ is the free monad generated by F [4,22]. Dually, if a cofree G-coalgebra functor $\widetilde{G}$ is given, then we can define $\mathtt{p}_X : \widetilde{G}X \longrightarrow GX$ and $\mathtt{d}_X : \widetilde{G}X \longrightarrow \widetilde{G}\widetilde{G}X$ so that $\langle \widetilde{G}, \mathtt{p}, \mathtt{h}, \mathtt{d}\rangle$ is the cofree comonad generated by G.

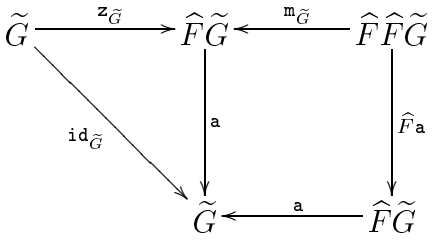
In [8] we proved the following facts. If a pair $(\widehat{F}, \widetilde{G})$ is given, $\widehat{F}$ being a free F-algebra functor and $\widetilde{G}$ being a cofree G-coalgebra functor, then $\widehat{F}$ is left adjoint to $\widetilde{G}$. Conversely, if a free F-algebra functor $\widehat{F}$ is given and has a right adjoint $\widetilde{G}$, then $\widetilde{G}$ can be endowed with the structure of a cofree G-coalgebra functor. Dually, if a cofree G-coalgebra functor $\widetilde{G}$ is given and

has a left adjoint $\widehat{F}$, then $\widehat{F}$ can be can be endowed with the structure of a free F-algebra functor.

Consequently, the structures of the free monad and the cofree comonad are related under transposition. The counit $\mathtt{h}_X$ of the cofree comonad is obtained by transposing $\mathtt{z}_X : X \longrightarrow \widehat{F}X$ while the comultiplication $\mathtt{d}_X : \widetilde{G}X \longrightarrow \widetilde{G}\widetilde{G}X$ is obtained by transposing twice the arrow

$$\mathtt{m}_{\widetilde{G}X} \cdot \mathtt{ev}_X : \widehat{F}\widehat{F}\widetilde{G}X \longrightarrow \widehat{F}\widetilde{G}X \longrightarrow X\,,$$

where $\mathtt{ev}$ is the counit of the adjunction. Transposing once this arrow, we obtain a natural action $\mathtt{a}$ of $\widehat{F}$ over $\widetilde{G}$ satisfying the usual conditions:



In the statement of Theorem 2.1 the functor F is the sum $\sum_{i\in I} F_i$; hence we shall consider several restrictions of the multiplication $\mathtt{m}$ and of the action $\mathtt{a}$:

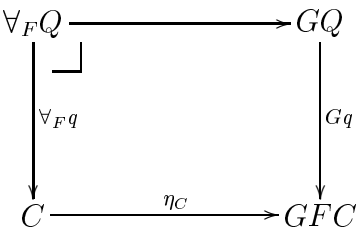
$$\begin{aligned}\mathtt{m}_i &= \mathtt{in}_{F_i\widehat{F}} \cdot \mathtt{i}_{\widehat{F}} \cdot \mathtt{m} \\ &= \mathtt{in}_{F_i\widehat{F}} \cdot \mathtt{s}_{\widehat{F}} : F_i\widehat{F} \longrightarrow \widehat{F}\,,\end{aligned} \qquad \begin{aligned}\mathtt{a}_i &= \mathtt{in}_{F_i\widetilde{G}} \cdot \mathtt{i}_{\widetilde{G}} \cdot \mathtt{a} \\ &= \mathtt{in}_{F_i\widetilde{G}} \cdot \mathtt{s}_{\widetilde{G}} : F_i\widetilde{G} \longrightarrow \widetilde{G}\,.\end{aligned}$$

Thus for each i in the finite set I let Ω_i be a given object of $\mathcal{C}$ and put $\Omega = \sum_{i\in I} \Omega_i$; 1 will be a chosen terminal object of $\mathcal{C}$. We are interested in the object $\widetilde{G}(1+\Omega)$ for which we shall use the abbreviated notation $\widetilde{G}$. If X is a subobject of $\widetilde{G}$, we shall use the notation κ^X to denote the intended monic arrow $\kappa^X : X \longrightarrow \widetilde{G}$. If X and Y are two subobjects of $\widetilde{G}$ such that κ^X factors through κ^Y, then we shall use κ^X_Y for the inclusion of X into Y.

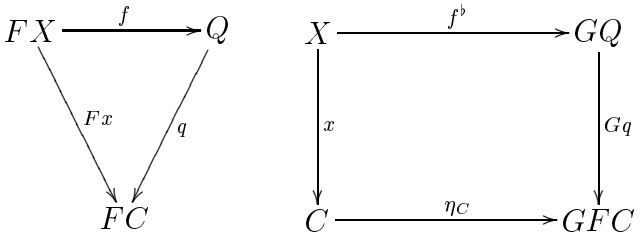
In order to prove the theorem we shall use the following fact: given an endofunctor F of a category $\mathcal{C}$ we can construct a new functor from the slice category $\mathcal{C}/C$ to the slice category $\mathcal{C}/FC$ according to the following rule: an object over C (X, x) (where $x : X \longrightarrow C$) is sent to (FX, Fx) and a map $f : X \longrightarrow Y$ such that $f \cdot y = x$ is sent to Ff.

Lemma 2.2 *If F has a right adjoint G and $\mathcal{C}$ has pullbacks then the functor $F : \mathcal{C}/C \longrightarrow \mathcal{C}/FC$ has also a right adjoint $\forall_F$ which is computed by pulling*

back along the unit of the adjunction:



Proof. The two diagrams



correspond under transposition, so that one commutes if and only if the other does. □

Observe that if q is monic then Gq is also monic, and therefore $\forall_F q$ is monic.

In the proof we have used the notation $f^\flat$ for the transpose of $f : FX \longrightarrow Y$. We shall use the same notation in the rest of the paper, and the notation $g^\#$ for the transpose of an arrow $g : X \longrightarrow GY$.

3 Representation of the Category of P-Coalgebras

In this section we give an adequate form to the category of P-coalgebras, P being the functor defined in (2). That is, we shall define a category $\mathcal{D}$ and argue that this category is equivalent to $\mathrm{CoAlg}(P)$. This is done to have explicitly available the reasoning by cases arising from pulling back against coproduct injections.

- An object of $\mathcal{D}$ is a tuple $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ such that
  - $(\alpha_i : A_i \longrightarrow A)_{i\in I}$ is a coproduct diagram in $\mathcal{C}$,
  - for each $i \in I$, $h_i : A_i \longrightarrow \Omega_i$ and $s_i : F_i A_i \longrightarrow A$.
- An arrow f from $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ to $(\beta_i : B_i \longrightarrow B, h'_i, s'_i)_{i\in I}$ is an arrow $f : A \longrightarrow B$ in $\mathcal{C}$ such that
  - $\alpha_i \cdot f$ factors through β_i, say $\alpha_i \cdot f = f_i \cdot \beta_i$,
  - for each $i \in I$ the following equations hold:

$$h_i = f_i \cdot h'_i\,, \qquad\qquad s_i \cdot f = F_i f_i \cdot s'_i\,.$$

Given an object $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ of $\mathcal{D}$ a coalgebra is constructed on A as follows:

$$\sum_{i\in I}\langle h_i, s_i^\flat\rangle : A \longrightarrow \sum_{i\in I}\Omega_i \times G_i A\,.$$

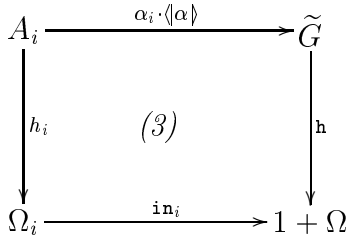
It is easily verified that an arrow f in $\mathcal{D}$ is a coalgebra morphism as well, thus we have defined a functor $C : \mathcal{D} \longrightarrow \mathrm{CoAlg}(P)$. Given a coalgebra $\beta : B \longrightarrow \sum_{i\in I}\Omega_i \times G_i B$, we construct a coproduct diagram $\beta_i : B_i \longrightarrow B$ and arrows $\langle h_i, t_i\rangle : B_i \longrightarrow \Omega_i \times G_i B$ pulling back along injections $\mathtt{in}_i : \Omega_i \times G_i B \longrightarrow \sum_{i\in I}\Omega_i \times G_i B$; we obtain an object of $\mathcal{D}$ by transposing the t_i. The object $(\beta_i : B_i \longrightarrow B, h_i, t_i^\#)$ has the universal property needed to define a functor right adjoint to C. Moreover, when the functor C is applied to it, we obtain back (β, B). In order to conclude that C is an equivalence, it is enough to argue that each object of $\mathcal{D}$ is isomorphic to an object coming from $\mathrm{CoAlg}(P)$: for this we use the fact that $\mathcal{C}$ is an extensive category.

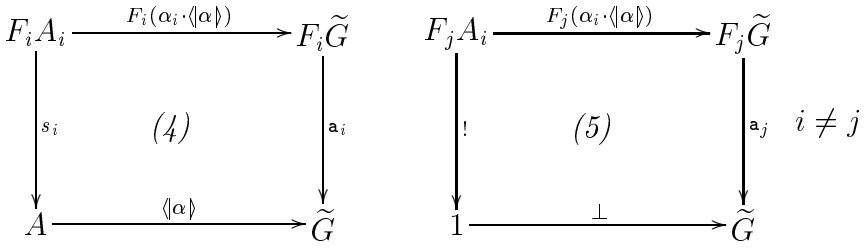
Recall that the category of F-algebras is isomorphic to the category of G-coalgebras and that the F-algebra $\langle \widetilde{G}, \mathtt{h}, \mathtt{s}_{\widetilde{G}}\rangle$ is cofreely generated by $1+\Omega$. There is an F-algebra structure $! : F1 \longrightarrow 1$ and we let $\bot : 1 \longrightarrow \widetilde{G}$ be the unique arrow such that

$$\bot \cdot \mathtt{h} = \mathtt{in}_l : 1 \longrightarrow 1+\Omega$$
$$F\bot \cdot \mathtt{s}_{\widetilde{G}} = ! \cdot \bot : F1 \longrightarrow \widetilde{G}\,.$$

Using this arrow we discover the following universal property:

Proposition 3.1 *For every object $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ of $\mathcal{D}$ there exists a unique arrow $\langle\!|\alpha|\!\rangle$ such that the diagrams*





commute.

Proof. Given an object $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ of $\mathcal{D}$ there is an F-algebra whose carrier is the object $1 + A$ and whose structure is

$$\begin{aligned}\left\{! \cdot \mathtt{in}_l, s_{i,j}\right\} : & F(1 + A) \\ & \cong F1 + \sum_{i\in I} F_i(\sum_{j\in I} A_j) \\ & \cong F1 + \sum_{i,j\in I} F_i A_j \longrightarrow 1 + A\end{aligned}$$

where $! \cdot \mathtt{in}_l : F1 \longrightarrow 1 \longrightarrow 1 + A$, and the $s_{i,j}$ are defined as follows:

$$s_{i,j} = \begin{cases} s_i \cdot \mathtt{in}_r \colon F_i A_i \longrightarrow A \longrightarrow 1 + A\,,\ i = j\,, \\ ! \cdot \mathtt{in}_l \colon F_j A_i \longrightarrow 1 \longrightarrow 1 + A\,,\ \text{otherwise.} \end{cases}$$

The F-algebra we have constructed is over $1+\Omega$ and the map $\left\{\bot, \langle\!|\alpha|\!\rangle\right\}$ is the unique arrow to the F-algebra cofreely generated by $1 + \Omega$. □

Proposition 3.2 *Suppose that we can find an object $(\kappa_D^{D_i} : D_i \longrightarrow D, h_i, s_i)_{i\in I}$ of $\mathcal{D}$ whose canonical arrow $\langle\!|\kappa|\!\rangle : D \longrightarrow \widetilde{G}$ is monic and such that, for every object $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ of $\mathcal{D}$, the canonical arrow $\langle\!|\alpha|\!\rangle$ factors through D fibrewise: that is, for each $i \in I$ we can write $\alpha_i \cdot \langle\!|\alpha|\!\rangle = \langle\!\langle\alpha\rangle\!\rangle_i \cdot \kappa_D^{D_i} \cdot \langle\!|\kappa|\!\rangle$ for a necessarily unique arrow $\langle\!\langle\alpha\rangle\!\rangle_i$. Then this object is a terminal object in $\mathcal{D}$.*

The arrow $\langle\!\langle\alpha\rangle\!\rangle_i$ is unique since in a distributive category coproduct injections are monic, see for example [6]; it follows that the arrows $\kappa_D^{D_i} \cdot \langle\!|\kappa|\!\rangle$ are monic.

Proof. Write κ^{D_i} for $\kappa_D^{D_i} \cdot \langle\!|\kappa|\!\rangle$, so that the equations

$$h_i \cdot \mathtt{in} = \kappa^{D_i} \cdot \mathtt{h} \tag{6}$$

$$s_i \cdot \langle\!|\kappa|\!\rangle = F_i\kappa^{D_i} \cdot \mathtt{a}_i \tag{7}$$

$$! \cdot \bot = F_j\kappa^{D_i} \cdot \mathtt{a}_j \quad i \neq j \tag{8}$$

hold. Let $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ be an object of $\mathcal{D}$, put $\langle\!\langle\alpha\rangle\!\rangle = \sum_{i\in I}\langle\!\langle\alpha\rangle\!\rangle_i$, and observe that $\langle\!|\alpha|\!\rangle = \langle\!\langle\alpha\rangle\!\rangle \cdot \langle\!|\kappa|\!\rangle$. In order to argue that $s_i \cdot \langle\!\langle\alpha\rangle\!\rangle = F_i\langle\!\langle\alpha\rangle\!\rangle_i \cdot s_i$ use the fact that $\langle\!|\kappa|\!\rangle$ is monic and argue that $s_i \cdot \langle\!\langle\alpha\rangle\!\rangle \cdot \langle\!|\kappa|\!\rangle = F_i\langle\!\langle\alpha\rangle\!\rangle_i \cdot s_i \cdot \langle\!|\kappa|\!\rangle$:

$$\begin{aligned} s_i \cdot \langle\!\langle\alpha\rangle\!\rangle \cdot \langle\!|\kappa|\!\rangle &= s_i \cdot \langle\!|\alpha|\!\rangle \\ &= F_i(\alpha_i \cdot \langle\!|\alpha|\!\rangle) \cdot \mathtt{a}_i \\ &= F_i\langle\!\langle\alpha\rangle\!\rangle_i \cdot F_i\kappa^{D_i} \cdot \mathtt{a}_i \\ &= F_i(\langle\!\langle\alpha\rangle\!\rangle_i) \cdot s_i \cdot \langle\!|\kappa|\!\rangle \qquad \text{by (7)} \end{aligned}$$

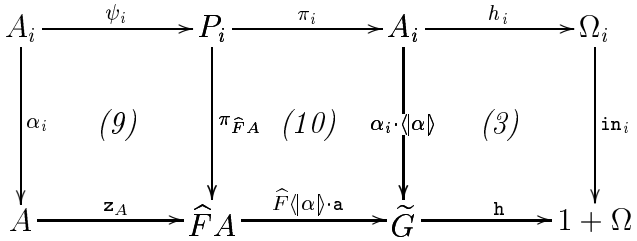
Similarly, if $f : A \longrightarrow D$ is another morphism in $\mathcal{D}$, it is enough to show that $f \cdot \langle\!|\kappa|\!\rangle$ has the properties that uniquely determine $\langle\!|\alpha|\!\rangle$, so that $f \cdot \langle\!|\kappa|\!\rangle =$

$\langle\!|\alpha|\!\rangle = \langle\!\langle\alpha\rangle\!\rangle \cdot \langle\!|\kappa|\!\rangle$ implies $f = \langle\!\langle\alpha\rangle\!\rangle$. This is easily achieved using equations (6), (7), and (8). □

According to the proposition, in order to construct a terminal object in $\mathcal{D}$ it is enough to find a subobject $\kappa^D : D \longrightarrow \widetilde{G}$ with a structure of an object in $\mathcal{D}$ such that the arrows $\langle\!|\alpha|\!\rangle$ factor through D fibrewise and equations (6), (7), and (8) hold, with $\langle\!|\kappa|\!\rangle$ replaced by κ^D. We shall construct such a subobject in the next section. To end this section, we present a useful characterization of the maps $\langle\!|\alpha|\!\rangle$.

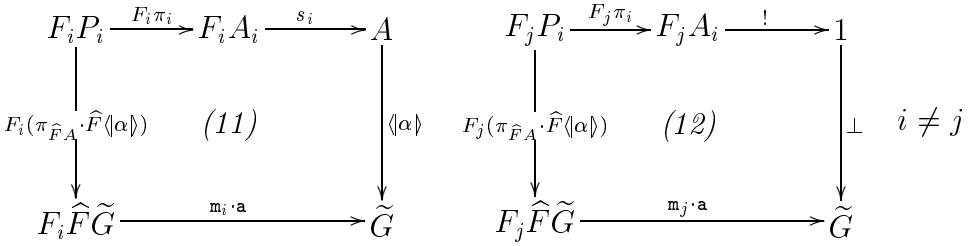
Proposition 3.3 *Given an object $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ of $\mathcal{D}$, an arrow $\langle\!|\alpha|\!\rangle$ makes diagrams (3), (4), and (5) commutative if and only if the following conditions hold:*

(i) *We can find commutative diagrams*



where the pasting of (10) and (3) is a pullback and $\psi_i \cdot \pi_i = \mathtt{id}_{A_i}$.

(ii) *The diagrams*

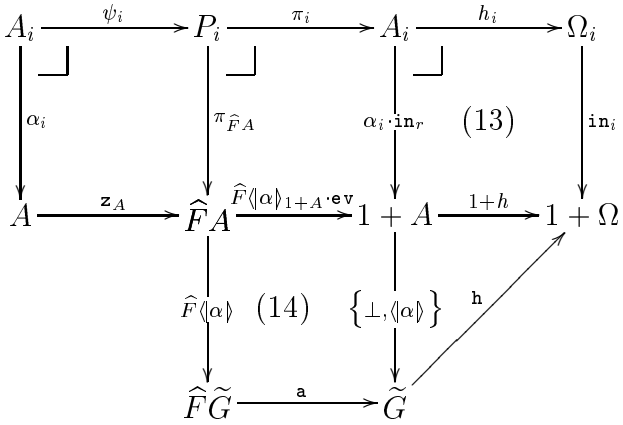


commute.

Proof. We argue first that $\langle\!|\alpha|\!\rangle$ has these properties. Let $h = \sum_{i\in I} h_i$ and observe that the F-algebra we have constructed in the proof of Proposition 3.1 is over $1 + \Omega$ by means of the arrow $1 + h : 1 + A \longrightarrow 1 + \Omega$. This algebra is also over $1 + A$, and as we have defined a map $\langle\!|\alpha|\!\rangle : A \longrightarrow \widetilde{G}(1+\Omega)$ we can define a map $\langle\!|\alpha|\!\rangle_{1+A} : A \longrightarrow \widetilde{G}(1 + A)$ as well; the two maps are related by $\langle\!|\alpha|\!\rangle = \langle\!|\alpha|\!\rangle_{1+A} \cdot \widetilde{G}(1 + h)$. In the diagram below the square (13) is a pullback, since we are working in an extensive category; the other two squares in the top row are pullbacks by construction. Thus every composite square in the

SANTOCANALE

top row is also a pullback.



We claim that $\psi_i \cdot \pi_i = \mathtt{id}_{A_i}$: observe that $\mathtt{z}_A \cdot \widehat{F}\langle\!|\alpha|\!\rangle_{1+A} \cdot \mathtt{ev} = \mathtt{in}_r$ and that the commutative diagram corresponding to the relation $\alpha_i \cdot \mathtt{in}_r = \mathtt{id}_{A_i} \cdot (\alpha_i \cdot \mathtt{in}_r)$ is a pullback since $\mathtt{in}_r$ and α_i are monic arrows.

We need to show that diagram (14) commutes. For this, recall that the category of F-algebras is isomorphic to the category of Eilenberg-Moore coalgebras for the comonad $\widetilde{G}$, see [8]; the map $\{\bot, \langle\!|\alpha|\!\rangle_{1+A}\} : 1+A \longrightarrow \widetilde{G}(1+A)$ is therefore an Eilenberg-Moore coalgebra for the comonad $\widetilde{G}$ since, under this isomorphism, it corresponds to the algebra constructed in Proposition 3.1. Therefore:

$$\begin{aligned}
\langle\!|\alpha|\!\rangle_{1+A} \cdot \widetilde{G}\{\bot, \langle\!|\alpha|\!\rangle\} &= \langle\!|\alpha|\!\rangle_{1+A} \cdot \widetilde{G}\{\bot, \langle\!|\alpha|\!\rangle_{1+A}\} \cdot \widetilde{G}\widetilde{G}(1+h) \\
&= \langle\!|\alpha|\!\rangle_{1+A} \cdot \mathtt{d}_{1+A} \cdot \widetilde{G}\widetilde{G}(1+h) \\
&= \langle\!|\alpha|\!\rangle_{1+A} \cdot \widetilde{G}(1+h) \cdot \mathtt{d}_{1+\Omega} = \langle\!|\alpha|\!\rangle \cdot \mathtt{d}_{1+\Omega}\,.
\end{aligned}$$

Transposing this relation we deduce that diagram (14) commutes.

The proof that $\langle\!|\alpha|\!\rangle$ satisfies the other two relations is now straightforward:

$$\begin{aligned}
F_i(\pi_{\widehat{F}A} \cdot \widehat{F}\langle\!|\alpha|\!\rangle) \cdot \mathtt{m}_i \cdot \mathtt{a} &= F_i(\pi_{\widehat{F}A} \cdot \widehat{F}\langle\!|\alpha|\!\rangle) \cdot F_i\mathtt{a} \cdot \mathtt{a}_i \\
&= F_i(\pi_i \cdot \alpha_i \cdot \langle\!|\alpha|\!\rangle) \cdot \mathtt{a}_i && \text{by (10)} \\
&= F_i\pi_i \cdot s_i \cdot \langle\!|\alpha|\!\rangle\,. && \text{by (4)}
\end{aligned}$$

The other relation is proved similarly.

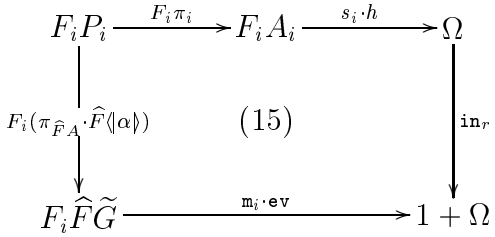
Conversely, if $\langle\!|\alpha|\!\rangle$ satisfies the conditions of the Proposition, then also diagrams (4) and (5) commute. For example:

$$\begin{aligned}
s_i \cdot \langle\!|\alpha|\!\rangle = F_i\psi_i \cdot F_i\pi_i \cdot s_i \cdot \langle\!|\alpha|\!\rangle &= F_i\psi_i \cdot F_i(\pi_{\widehat{F}A} \cdot \widehat{F}\langle\!|\alpha|\!\rangle) \cdot \mathtt{m}_i \cdot \mathtt{a} && \text{by (11)} \\
&= F_i(\alpha_i \cdot \mathtt{z}_A \cdot \widehat{F}\langle\!|\alpha|\!\rangle) \cdot \mathtt{m}_i \cdot \mathtt{a} && \text{by (9)} \\
&= F_i(\alpha_i \cdot \langle\!|\alpha|\!\rangle) \cdot F_i\mathtt{z}_{\widetilde{G}} \cdot \mathtt{m}_i \cdot \mathtt{a} \;= F_i(\alpha_i \cdot \langle\!|\alpha|\!\rangle) \cdot \mathtt{a}_i\,.
\end{aligned}$$

□

11

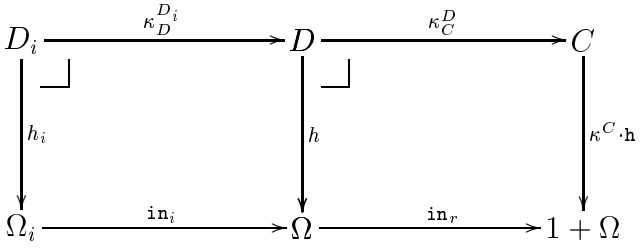
In particular we see that the diagram



commutes. For this it is enough to past the commutative diagram corresponding to the relation $\langle\!|\alpha|\!\rangle \cdot \mathtt{h} = h \cdot \mathtt{in}_r$ on the right of diagram (12).

4 The Construction

In this section we construct a subobject of $\widetilde{G}$ satisfying the constraints given in Proposition 3.2. The strategy is as follows: we shall construct a subobject $\kappa^C : C \longrightarrow \widetilde{G}$ with some properties; we leave for the moment unspecified these properties, we just think of C as being a modification of $\widetilde{G}$. Then we shall define the subobject $\kappa^D : D \longrightarrow \widetilde{G}$ and the coproduct diagram $(\kappa_D^{D_i} : D_i \longrightarrow D)_{i \in I}$ by means of the following pullbacks:

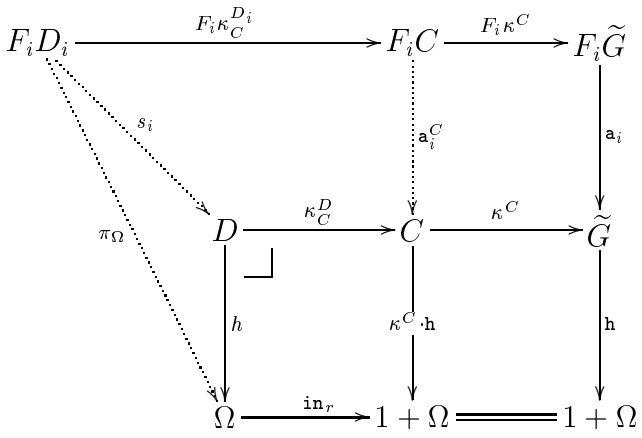


To understand the definition of D, consider the case when $\mathcal{C}$ is the category of sets and functions, $A = \sum_{i \in I} A_i$, $F_i(X) = A_i \times X$, $\widehat{F}(X) = A^* \times X$, and $\widetilde{G}(X) = X^{A^*}$. Then D is the set of all trees $t \in (1+\Omega)^{A^*}$, having the property C, such that the root of the tree is not labeled by $\perp$, i.e. $t(\varepsilon) \in \Omega$. In this way we are mimicking clause (iii) from Example 1.1.

Next we need to put a structure of an object of $\mathcal{D}$ on the top of the coproduct diagram $(\kappa_D^{D_i} : D_i \longrightarrow D)_{i \in I}$. Observe that the $h_i : D_i \longrightarrow \Omega_i$ come with the definition of the coproduct; thus we only need to construct

SANTOCANALE

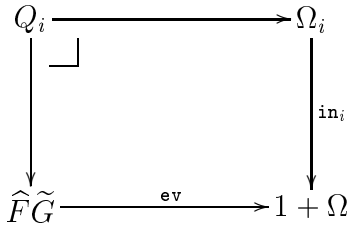
$s_i : F_i D_i \longrightarrow D$. The recipe is described by means of the following diagram:

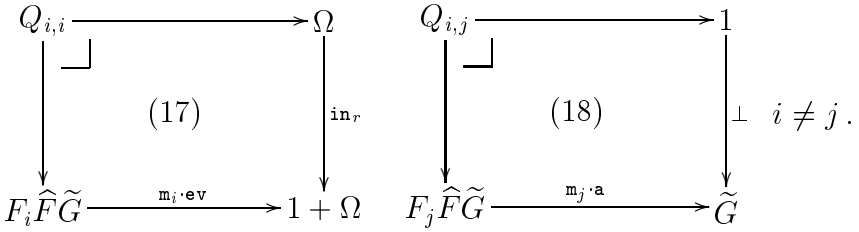

$F_i D_i$ $F_i \kappa_C^{D_i}$ $F_i C$ $F_i \kappa^C$ $F_i \widetilde{G}$
s_i $\mathtt{a}_i^C$ $\mathtt{a}_i$
π_Ω D κ_C^D C κ^C $\widetilde{G}$
h $\kappa^C \cdot \mathtt{h}$ $\mathtt{h}$
Ω $\mathtt{in}_r$ $1 + \Omega$ $1 + \Omega$

(16)

We need the subobject C of $\widetilde{G}$ to be closed under the action of F_i (Lemma 4.1) and the arrow $\widehat{F}\kappa^{D_i} \cdot \mathtt{a}_i \cdot \mathtt{h}$ to factor through $\mathtt{in}_r$ (Lemma 4.2). Having these properties we can define s_i as the canonical arrow from $F_i D_i$ to the pullback D.

We define now the object C. Proposition 3.3 suggests what are its building blocks: these are subobjects Q_i of $\widehat{F}\widetilde{G}$ and $Q_{i,j}$ of $F_j\widehat{F}\widetilde{G}$ defined as the following pullbacks:


Q_i Ω_i
$\mathtt{in}_i$
$\widehat{F}\widetilde{G}$ $\mathtt{ev}$ $1 + \Omega$


$Q_{i,i}$ Ω
(17)
$\mathtt{in}_r$
$F_i\widehat{F}\widetilde{G}$ $\mathtt{m}_i \cdot \mathtt{ev}$ $1 + \Omega$
$Q_{i,j}$ 1
(18)
$\bot$ $i \neq j$.
$F_j\widehat{F}\widetilde{G}$ $\mathtt{m}_j \cdot \mathtt{a}$ $\widetilde{G}$

The Q_i and the $Q_{i,j}$ are indeed subobjects of $\widehat{F}\widetilde{G}$ and $F_j\widehat{F}\widetilde{G}$, respectively: the reason is that coproduct injections are monic in a distributive category; moreover, the relation $\bot \cdot \mathtt{h} = \mathtt{in}_l$ exhibits $\bot$ as a monic. In a Cartesian closed category product and exponentiation give the collection of subobjects of the terminal object the structure of a Brouwerian semilattice. Considering this structure in the slice categories $\mathcal{C}/\widehat{F}\widetilde{G}$ and $\mathcal{C}/\widetilde{G}$ we define:

$$C_i = \forall_{\widehat{F}}(Q_i \to \bigwedge_{j \in I} \forall_{F_j} Q_{i,j}) , \qquad\qquad C = \bigwedge_{i \in I} C_i .$$

13

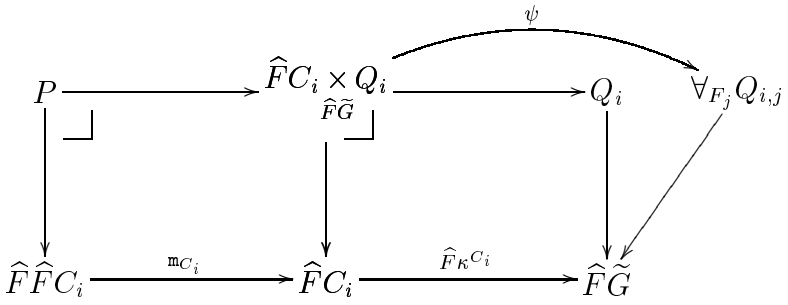
The meaning of the universal quantification $\forall_F$ is explained in Lemma 2.2. The reader can also find there the reason for which all the C_i are indeed subobjects of $\widetilde{G}$.

In order to understand the definition of the C_i, consider again the set theoretic example. Here $Q_{i,i}$ is the set of triples (a, w, t) in $A_i \times A^* \times (1+\Omega)^{A^*}$ such that $t(wa) \in \Omega$; $Q_{i,j}$ is the set of triples (a, w, t) in $A_j \times A^* \times (1+\Omega)^{A^*}$ such that the subtree of t rooted at wa (i.e. the function $\lambda x.t(wax)$) is constantly labeled by $\perp$. Therefore C_i is the collection of trees $t \in (1+\Omega)^{A^*}$ with the following property: *for all $w \in A^*$ such that $t(w) \in \Omega_i$, for all $j \in I$ and $a \in A_j$, if $i = j$ then $t(wa)$ is in Ω, and otherwise, if $i \neq j$, then the subtree of t rooted at wa is constantly labeled by $\perp$.* That is, the definition of the C_i mimics clauses (i)-(ii) of Example 1.1.

Lemma 4.1 *The object C is closed under the action of $\widehat{F}$ and therefore under the actions of F and of each F_i.*

Proof. The second statement follows from the first since we can write $\mathtt{a}_i = \iota_i \cdot \mathtt{a}$, where the $\iota_i : F_i X \longrightarrow \widehat{F}X$ are natural in X.

Thus we shall show that the arrow $\widehat{F}\kappa^{C_i} \cdot \mathtt{a}$ factors through C_i; this will be enough since it is easily verified that the intersection of subobjects closed under the action of $\widehat{F}$ is again closed under this action. By the definition of C_i, this is equivalent to the pullback of Q_i along $\widehat{F}(\widehat{F}\kappa^{C_i} \cdot \mathtt{a})$ to factor through $\forall_{F_j} Q_{i,j}$, for all $j \in I$. To this end, observe first that in the following diagram we can factor the pullback P through $\forall_{F_j} Q_{i,j}$:



Here the arrow ψ corresponds under the adjunctions to the identity of C_i. The result follows since the relations

$$\begin{aligned} \mathtt{m}_{C_i} \cdot \widehat{F}\kappa^{C_i} \cdot \mathtt{ev} &= \widehat{F}\widehat{F}\kappa^{C_i} \cdot \mathtt{m}_{\widetilde{G}} \cdot \mathtt{a} \cdot \mathtt{h} \\ &= \widehat{F}\widehat{F}\kappa^{C_i} \cdot \widehat{F}\mathtt{a} \cdot \mathtt{a} \cdot \mathtt{h} \;= \widehat{F}(\widehat{F}\kappa^{C_i} \cdot \mathtt{a}) \cdot \mathtt{ev} \end{aligned}$$

show that P is indeed the pullback of Q_i along $\widehat{F}(\widehat{F}\kappa^{C_i} \cdot \mathtt{a})$. □

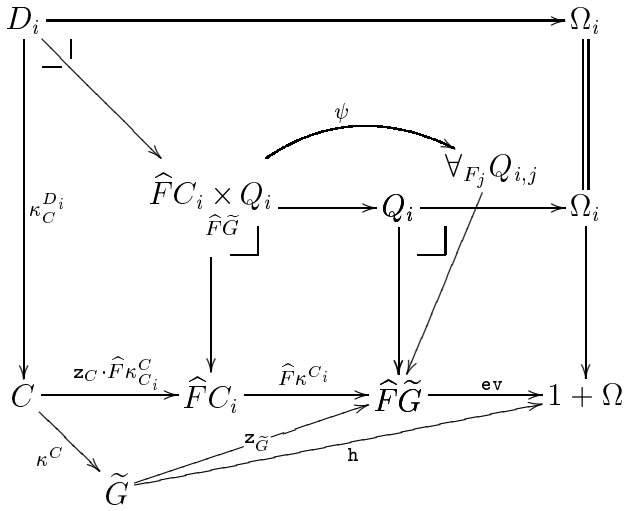
Lemma 4.2 *There is a factorization of the form*

$$F_i\kappa^{D_i} \cdot \mathtt{a}_i \cdot \mathtt{h} = \pi_\Omega \cdot \mathtt{in}_r \,,$$

for some necessarily unique π_Ω; moreover

$$F_j\kappa^{D_i}\cdot \mathtt{a}_j = ! \cdot \bot\,. \tag{19}$$

Proof. The diagram



shows that the arrow $\kappa^{D_i}\cdot \mathtt{z}_{\widetilde{G}}$ can be factored through $\forall_{F_j} Q_{i,j}$ over $\widehat{F}\widetilde{G}$. Transposing this relation, we obtain that $F_j(\kappa^{D_i}\cdot \mathtt{z}_{\widetilde{G}})$ can be factored through $Q_{i,j}$ over $F_j\widehat{F}\widetilde{G}$. Considering the definition of the $Q_{i,j}$ as pullbacks, we obtain that $F_i(\kappa^{D_i}\cdot \mathtt{z}_{\widetilde{G}})\cdot \mathtt{m}_i\cdot \mathtt{ev}$ can be factored through $\mathtt{in}_r : \Omega \longrightarrow 1+\Omega$ and $F_j(\kappa^{D_i}\cdot \mathtt{z}_{\widetilde{G}})\cdot \mathtt{m}_j\cdot \mathtt{a}$ is constant:

$$F_i(\kappa^{D_i}\cdot \mathtt{z}_{\widetilde{G}})\cdot \mathtt{m}_i\cdot \mathtt{ev} = \pi_\Omega\cdot \mathtt{in}_r\,, \qquad F_j(\kappa^{D_i}\cdot \mathtt{z}_{\widetilde{G}})\cdot \mathtt{m}_j\cdot \mathtt{a} = !\cdot\bot\,.$$

Finally, it is enough to observe that $F_i\mathtt{z}_{\widetilde{G}}\cdot \mathtt{m}_i\cdot \mathtt{ev} = \mathtt{a}_i\cdot \mathtt{h}$ and $F_j\mathtt{z}_{\widetilde{G}}\cdot \mathtt{m}_j\cdot \mathtt{a} = \mathtt{a}_j$.□

With the proofs of the two lemmas we have completed the definition of the object $(\kappa_D^{D_i} : D_i \longrightarrow D, h_i, s_i)_{i\in I}$ in $\mathcal{D}$. We observe:

Proposition 4.3 *Equations (6), (7), and (8) holds.*

Proof. Equation (6) follows from the definition of D_i. Equation (7) is the top row of squares in diagram (16). Equation (8) is the relation (19). □

In order to use Proposition 3.2 and conclude that this is a terminal object of $\mathcal{D}$, we still need to prove:

Proposition 4.4 *If $(\alpha_i : A_i \longrightarrow A, h_i, s_i)_{i\in I}$ is an object of $\mathcal{D}$, then the canonical arrow $\langle\!|\alpha|\!\rangle$ factors through D fibrewise.*

Proof. We start showing that $\langle\!|\alpha|\!\rangle$ factors through C, that is, that $\langle\!|\alpha|\!\rangle$ factors through each C_i. Unraveling the definition of C_i, we need to show that $\widehat{F}\langle\!|\alpha|\!\rangle$ factors through $Q_i \to \bigwedge_{j\in I}\forall_{F_j} Q_{i,j}$, or equivalently that the pullback P_i of $\widehat{F}\langle\!|\alpha|\!\rangle\cdot \mathtt{ev}$ along the injection $\Omega_i \longrightarrow 1+\Omega$ factors through each $\forall_{F_j} Q_{i,j}$ over

$\widehat{F}\widetilde{G}$. Transposing we need to show that $F_j(\pi_{\widehat{F}A} \cdot \widehat{F}\langle\!|\alpha|\!\rangle)$ factors through each $Q_{i,j}$ over $F_j\widehat{F}\widetilde{G}$. Considering the definition of the $Q_{i,j}$ as pullbacks (17) and (18), the statement follows from the commuting diagrams (15) and (12).

Finally we must show that $\langle\!|\alpha|\!\rangle$ factors through D fibrewise. It is enough to recall that $\alpha_i \cdot \langle\!|\alpha|\!\rangle \cdot \mathtt{h} = h_i \cdot \mathtt{in}_i$, which implies that $\alpha_i \cdot \langle\!|\alpha|\!\rangle$ factors through D_i. □

Hence we can state:

Theorem 4.5 *The object $(\kappa_D^{D_i} : D_i \longrightarrow D, h_i, s_i)_{i\in I}$ is a terminal object in the category $\mathcal{D}$ and therefore the polynomial functor $P(X)$ has a final coalgebra.*

5 Further Observations

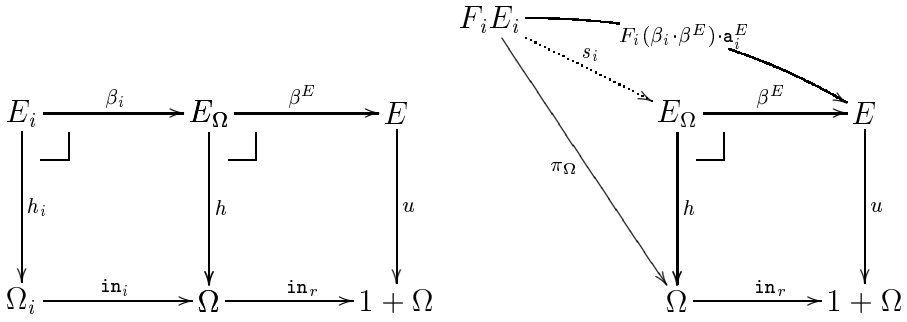
5.1 Rational Reconstruction

Let us denote by $\mathrm{Alg}(F)_{1+\Omega}$ the category of F-algebras over $1+\Omega$, i.e. the category whose objects are pairs formed by an F-algebra and a map from the carrier to $1+\Omega$, with the evident definition of morphisms. In Lemma 4.1 we have observed that C is sub-F-algebra of the final F-algebra over $1+\Omega$; we shall abuse of notation and write C for the object in $\mathrm{Alg}(F)_{1+\Omega}$ whose F-algebra is the restriction of $\mathtt{a}$ to C, and whose map to $1+\Omega$ is $\kappa^C \cdot \mathtt{h}$. In Proposition 3.1 we have defined a functorial correspondence $K : \mathrm{CoAlg}(P) \longrightarrow \mathrm{Alg}(F)_{1+\Omega}$; however, in the proof of Proposition 4.4, it is contained the statement that we can factor K as

$$\mathrm{CoAlg}(P) \xrightarrow{K} \mathrm{Alg}(F)_{1+\Omega}/C \xrightarrow{U} \mathrm{Alg}(F)_{1+\Omega}$$

where $\mathrm{Alg}(F)_{1+\Omega}/C$ is the usual slice category. The forgetful functor U has a right adjoint R_U, which associates to an F-algebra over $1+\Omega$ $((D,\delta),h)$ a canonical F-algebra structure on the pullback $D \underset{1+\Omega}{\times} C$.

The construction presented in section 4 gives rise to a functor L_K left adjoint to K. Indeed, if $((E,\mathtt{a}^E),u)$ is an F-algebra over $1+\Omega$, then we can construct the coproduct diagram $(\beta_i : E_i \longrightarrow E_\Omega)_{i\in I}$ by pulling back along the injections:



Using the fact that $(E, \mathtt{a}^E)$ comes with a map to $(C, \mathtt{a}^C)$ the argument of Lemma 4.2 is adapted to show that $F_i(\beta_i \cdot \beta^E) \cdot \mathtt{a}_i^E \cdot u$ factors through $\mathtt{in}_r$; hence we can construct the s_i as we have done for D. This construction is clearly functorial, and the correspondence

$$\frac{E \longrightarrow 1 + A}{E_\Omega \longrightarrow A}$$

establishes the adjunction $L_K \dashv K$. The functor K is full and faithful and therefore the left adjoint L_K preserves limits of diagrams taking values in $\mathrm{CoAlg}(P)$; the terminal object is one of these limits.

To summarize, we have argued about the existence of a functor

$$\mathrm{Alg}(F)_{1+\Omega} \longrightarrow \mathrm{CoAlg}(P)\,,$$

obtained by composing the functors R_U and L_K; this functor preserves the terminal object.

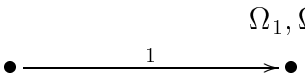
5.2 Extensiveness is Necessary

Since the construction presented in section 4 can be carried in an arbitrary distributive category, it might be asked whether the resulting object has the desired universal property in distributive categories that are not extensive. For example, does this construction define a greatest fixed point in an arbitrary Heyting algebra? The answer is negative. Observe, however, that if $\mathcal{C}$ is a poset then a top element in $\mathcal{D}$ is preserved by the left adjoint $C : \mathcal{D} \longrightarrow \mathrm{CoAlg}(P)$ whose counit is an isomorphism; it does not matter that the category $\mathcal{D}$ is equivalent to $\mathrm{CoAlg}(P)$. Thus we need the assumption of extensiveness only to prove that diagrams (15) and (12) commute. The argument relied on the fact that we found an arrow $\pi_i : P_i \longrightarrow A_i$, for which we needed diagram (13) to be a pullback.

Easy computations of Heyting algebras lead to ask whether the relation

$$\nu_X.(\bigvee_{i\in I}(\,\Omega_i \wedge G_i X\,)) = \Omega \wedge \bigwedge_{i\in I} \widetilde{G}(\Omega_i \to G_i\Omega) \tag{20}$$

holds – we are using here standard notation for fixed points. While the unary version (i.e. when I is a singleton) of this equation holds and is indeed Segerberg's equation axiomatizing PDL [21], the binary version is false. This is shown by considering the transition system



over which we interpret two modal operators $\langle 1\rangle, \langle 2\rangle$ and the two propositional constants Ω_1, Ω_2 in the usual way. If we put $\langle\{1,2\}^*\rangle Y = \mu_X.(Y \vee \langle 1\rangle X \vee \langle 2\rangle X)$, then we observe that the relation

$$\begin{aligned}(\Omega_1 \wedge \Omega_2) \ \vee \ \langle\{1,2\}^*\rangle(\ \neg\Omega_1 \ \wedge \ \langle 1\rangle(\Omega_1 \wedge \Omega_2)\ ) \\ \vee \ \langle\{1,2\}^*\rangle(\ \neg\Omega_2 \ \wedge \ \langle 2\rangle(\Omega_1 \wedge \Omega_2)\ ) \\ \leq \mu_X.(\ (\Omega_1 \vee \langle 1\rangle X) \ \wedge \ (\Omega_2 \vee \langle 2\rangle X)\ )\end{aligned}$$

does not hold in the transition system. This relation is the dual of equation (20).

Acknowledgment

The author is grateful to professor Robin Cockett for useful discussion on the subject.

References

[1] M. Abbot, T. Altenkirch, and N. Ghani. Categories of containers. In *FOSSACS03, Foundations of Software Science and Computation Structures*, LNCS. Springer-Verlag, 2003.

[2] J. Adámek and V. Koubek. On the greatest fixed point of a set functor. *Theor. Comput. Sci.*, (1):57–75, 1995.

[3] J. Adámek and J. Rosický. *Locally presentable and accessible categories.* Cambridge University Press, Cambridge, 1994.

[4] M. Barr. Coequalizers and free triples. *Math. Z.*, 116:307–322, 1970.

[5] M. Barr. Terminal coalgebras in well-founded set theory. *Theoret. Comput. Sci.*, 114(2):299–315, 1993.

[6] J.R.B. Cockett. Introduction to distributive categories. *Math. Struct. in Comp. Science*, 3:277–307, 1993.

[7] J.R.B. Cockett and Stephen Lack. The extensive completion of a distributive category. *Theory Appl. Categ.*, 8:541–554, 2001.

[8] R. Cockett and L. Santocanale. Induction, coinduction and adjoints. To appear in *Electronic Notes in Computer Science*, March 2002.

[9] B. Courcelle. Fundamental properties of infinite trees. *Theoret. Comput. Sci.*, 25(2):95–169, 1983.

[10] B. Day. Limit spaces and closed span categories. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 65–74. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[11] R. Dyckhoff and W. Tholen. Exponentiable morphisms, partial products and pullback complements. *J. Pure Appl. Algebra*, 49:103–116, 1987.

[12] J. M. E. Hyland. The effective topos. In *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, pages 165–216. North-Holland, Amsterdam, 1982.

[13] C. B. Jay and J. R. B. Cockett. Shapely types and shape polymorphism. In *Programming languages and systems—ESOP '94 (Edinburgh, 1994)*, volume 788 of *Lecture Notes in Comput. Sci.*, pages 302–316. Springer, Berlin, 1994.

[14] P. Johnstone, J. Power, T. Tsujishita, H. Watanabe, and J. Worrell. On the structure of categories of coalgebras. *Theoret. Comput. Sci.*, 260(1-2):87–117, 2001. Coalgebraic methods in computer science (Lisbon, 1998).

[15] P. T. Johnstone and G. C. Wraith. Algebraic theories in toposes. In *Indexed categories and their applications*, volume 661 of *Lecture Notes in Math.*, pages 141–242. Springer, Berlin, 1978.

[16] P.T. Johnstone. *Topos Theory*, volume 10 of *L.M.S. Monographs*. Academic Press, 1977.

[17] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*. Cambridge University Press, Cambridge, 1988.

[18] J. Penon. Catégories localement internes. *C. R. Acad. Sci. Paris Sér. A*, 278:1577–1580, 1974.

[19] L. Santocanale. μ-bicomplete categories and parity games. *Theoretical Informatics and Applications*, 36:195–227, September 2002.

[20] R. A. G. Seely. Locally Cartesian closed categories and type theory. *Math. Proc. Cambridge Philos. Soc.*, 95(1):33–48, 1984.

[21] K. Segerberg. A completeness theorem in the modal logic of programs. *Notices Amer. Math. Soc*, 24:A552, 1977. Abstract 77T-E69.

[22] V. Trnková, J. Adámek, V. Koubek, and J. Reiterman. Free algebras, input processes and free monads. *Comment. Math. Univ. Carolinae*, 16:339–351, 1975.

[23] J. Worrell. Terminal sequences for accessible endofunctors. In B. Jacobs and J. Rutten, editors, *Electronic Notes in Theoretical Computer Science*, volume 19. Elsevier Science Publishers, 2000.