On the postulation of s^d fat points in $\mathbb{P}^d$ *

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Abstract

In connection with his counter-example to the fourteenth problem of Hilbert, Nagata formulated a conjecture concerning the postulation of r fat points of the same multiplicity in $\mathbb{P}^2$ and proved it when r is a square. Iarrobino formulated a similar conjecture in $\mathbb{P}^d$. We prove Iarrobino's conjecture when r is a d-th power.

1 Introduction

What is the dimension $l(d, \delta, \mu_1, \ldots, \mu_r)$ of the sub-vector space of $k[X_0, \ldots, X_d]$ containing the homogeneous polynomials of degree δ that vanish at general points $p_1, \ldots, p_r \in \mathbb{P}^d$ with order $\mu_1, \ldots, \mu_r$? This question remains open as soon as $d \geq 2$ and has numerous consequences (see [1], [3], [8], [5], [7] for instance).

When the dimension of the ambiant projective space is $d = 2$ and the number of points is $r \leq 9$, the answer is well known [6]. As for the remaining cases $r > 9$, Nagata formulated a conjecture in connection with his counter-example to the fourteenth problem of Hilbert:

$$l(2, \delta, \underbrace{\mu, \ldots, \mu}_{r\ times}) = l(2, \delta, \mu^r) = 0 \text{ if } \delta \leq \sqrt{r}\mu$$

This conjecture is of particular interest since it crystallizes the difficulties. Indeed, the expected dimension $l(2, \delta, \mu^r)$ is $max(0, v(2, \delta, \mu^r))$ where

$$v(2, \delta, \mu^r) = \frac{(\delta + 2).(\delta + 1)}{2} - r.\frac{\mu.(\mu + 1)}{2}$$

is the so-called virtual dimension. With any known method, the hardest cases are the cases with r fixed, $\mu >> r$ and the degree δ is such that the virtual dimension is zero. An immediate estimate shows that the critical δ for which the virtual dimension is zero is asymptotically equivalent to $\sqrt{r}\mu$. It follows that the hardest cases correspond to Nagata's conjecture. Nagata proved himself this conjecture when r is a square.

*AMS Classification: 14C05; 14J17

Leaving the two-dimensional case for the general case, there is still a conjecture for the dimension $l(d,\delta,\mu_1,\dots,\mu_r)$, due to Iarrobino [4]. Facing the critical cases too, he derived from his conjecture a generalisation of Nagata's conjecture:

Conjecture 1. *Let (r,d) be a couple of integers with*

- $d \geq 2$
- $r \geq max(d+5, 2^d)$
- $(r,d) \notin \{(7,2),(8,2),(9,3),(14,4)\}$.

If $\delta < \sqrt[d]{r}\mu$ then $l(d,\delta,\mu^r)=0$.

In the 2-dimensional case however, this is not exactly Nagata's conjecture. Indeed, Nagata's conjecture is very slightly stronger, since the condition on δ is $\delta \leq \sqrt{r}\mu$, not $\delta < \sqrt{r}\mu$, and this difference turned out to be very important in the applications (in Nagata's counter-example to the fourteenth problem of Hilbert, or in [1] for instance). Replacing carelessly the strict inequality by a large inequality is not possible since the cases $(r,d)=(2^d,d)$ and $(r,d)=(9,2)$ would obviously contradict the statement. Nevertheless, excluding these cases, one can formulate the conjecture as follows:

Conjecture 2. *Let (r,d) be a couple of integers with*

- $d \geq 2$
- $r \geq max(d+5, 2^d+1)$
- $(r,d) \notin \{(7,2),(8,2),(9,2),(9,3),(14,4)\}$.

If $\delta \leq \sqrt[d]{r}\mu$ then $l(d,\delta,\mu^r)=0$.

Let us call this conjecture the large critical conjecture in opposition to the conjecture by Iarrobino which we shall call the strict critical conjecture.

The goal of this paper is to prove that the large critical conjecture holds when the number of points is a power of the dimension of the ambiant projective space:

Theorem 3. *Suppose that the characteristic of the base field k is zero. Let $d \geq 2$ be an integer, r be an integer such that $r = s^d$ for some s. Suppose moreover that $(r,d) \notin \{(1,d),(2^d,d),(9,2)\}$. Then:*
$l(d,\delta,\mu^r)=0$ if $\delta \leq s\mu$.

Remark 4. *It seems that theorem 3 leaves the cases $(r,d) = (2^d,d)$ and $(r,d) = (9,2)$ untreated. However, these two cases are completly understood. Indeed, by [6] for $(r,d)=(9,2)$ and by proposition 16 for $(r,d)=(2^d,d)$, we have $l(d,\delta,\mu^r) = max(0, \binom{\delta+d}{d} - r.\binom{d+\mu-1}{d})$.*

If the characteristic of the base field is arbitrary, we can forget the parts of the proof which use the hypothesis on the characteristic and we still have the strict critical conjecture:

Theorem 5. *Let $d \geq 2$ be an integer and let r be a d^{th}-power. If $\delta < \sqrt[d]{r}\mu$ then $l(d,\delta,\mu^r)=0$.*

2 Stratifications on the Hilbert scheme

In this section, we explain the strategy of the proof: we define locally closed subschemes $C(E_1,\dots,E_i)$ of the Hilbert scheme $Hilb(\mathbb{P}^d)$ and we reduce the proof to an incidence between these subschemes.

Monomial subschemes

A staircase E in $\mathbb{N}^d$ is a subset whose complementary $\mathbb{N}^d - E$ verifies

$$(\mathbb{N}^d - E) + \mathbb{N}^d \subset \mathbb{N}^d - E.$$

A staircase E being fixed, let $I^E \subset k[[x_1,\dots,x_d]]$ (resp. $I^E \subset k[x_1,\dots,x_d]$) be the ideal whose elements are the series (resp. the polynomials)

$$\sum c_{\alpha_1\alpha_2\dots\alpha_d} x_1^{\alpha_1} x_2^{\alpha_2}\dots x_d^{\alpha_d} = \sum c_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$$

verifying $c_{\underline{\alpha}} = 0$ if $\underline{\alpha} \in E$. A zero-dimensional subscheme Z of $\mathbb{P}^d$ supported by a point q is said to be monomial with staircase E if it is defined by the ideal I^E in a suitable formal neighborhood $Spec\ k[[x_1,\dots,x_d]] \hookrightarrow \mathbb{P}^d$ of q.
A fat point of multiplicity m is by definition a monomial subscheme defined by the regular staircase R_m:

$$R_m := \{(\alpha_1,\dots,\alpha_d)\ s.t.\ \alpha_1 + \dots + \alpha_d < m\}.$$

Subschemes of $Hilb(\mathbb{P}^d)$

If $E_1,\dots,E_i$ are finite staircases in $\mathbb{N}^d$, we denote by $C(E_1,\dots,E_i)$ the reduced subscheme of $Hilb\ \mathbb{P}^d$ whose points parametrize the subschemes Z of $\mathbb{P}^d$ which are the disjoint union of i distinct monomial subschemes with staircases $E_1,\dots,E_i$. In symbols $Z = \coprod Z_j$, where Z_j is monomial with staircase E_j. It is known by [2] that $C(E_1,\dots,E_i) \subset Hilb\mathbb{P}^d$ is a locally closed irreducible subscheme. In particular it has a generic point G, which parametrizes a subscheme Z_G whose ideal is denoted by I_{Z_G}. We denote by $l(d,\delta,E_1,\dots,E_i) = h^0(I_{Z_G}(\delta))$ the number of independant hypersurfaces of degree δ in $\mathbb{P}^d$ containing Z_G.

Iarrobino's conjecture and incidence between strata

The theorem we want to prove can obviously be reformulated as:

Theorem 6. *Let $r = s^d$ and $\delta \le s\mu$. Then:*
$l(d,\delta,\underbrace{R_\mu,\dots,R_\mu}_{r\ times}) = 0$ *if* $(s,d) \notin \{(1,d),(2,d),(3,2)\}$ *and if the characteristic of the base field is zero.*

The following proposition reduces the proof of the theorem to the computation of the closure of $C(R_\mu,\dots,R_\mu)$.

Proposition 7. *Let* $E_1, \ldots, E_i \subset \mathbb{N}^d$ *be staircases. Suppose that there exists a staircase* F *with* $F \supset R_{\delta+1}$ *and* $C(F) \subset \overline{C(E_1, \ldots, E_i)}$*, then* $l(d, \delta, E_1, \ldots, E_i) = 0$.

Proof: by semi-continuity of the cohomology $l(d, \delta, E_1, \ldots, E_i) \leq l(d, \delta, F)$ and $l(d, \delta, F) \leq l(d, \delta, R_{\delta+1})$ since $F \supset R_{\delta+1}$. Since obviously $l(d, \delta, R_{\delta+1}) = 0$, the vanishing of $l(d, \delta, E_1, \ldots, E_i)$ follows from the last two inequalities. ■

3 Elementary Incidences

The previous section explained that the theorems would follow from incidences between the various subschemes $C(E_1, \ldots, E_j)$. The goal of this section is to exhibit such incidences.

Let $E \subset \mathbb{N}^d$ be a finite staircase and $i \in \{1, \ldots, d\}$ be an integer. There exists a unique "height" function

$$h_{E,i} : \mathbb{N}^{d-1} \to \mathbb{N}$$

such that

$$(a_1, \ldots, a_d) \in E \Leftrightarrow a_i < h_{E,i}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d)$$

Conversely, a function h is the height function of some staircase if and only if $h(a+b) \leq h(a)$ for any $(a, b) \in \mathbb{N}^{d-1} \times \mathbb{N}^{d-1}$. If $E_1, \ldots, E_j$ are staircases, the sum of $E_1, \ldots, E_j$ along the i^{th} coordinate is the staircase $S_i(E_1, \ldots, E_j)$ characterized by its height function

$$h_{S_i(E_1, \ldots, E_j),i} = \sum_{k=1}^{j} h_{E_k,i}.$$

Proposition 8. *Let* $E_1, \ldots, E_j$ *be staircases and* $k \in \{1, \ldots, j\}$*. Then* $\overline{C(E_1, \ldots, E_j)} \supset C(S_i(E_1, \ldots E_k), E_{k+1}, \ldots, E_j)$.

Proof: this is an straightforward generalisation of [2], proposition 5.1.2. ■

Let $(a_1, \ldots, a_d) \in (\mathbb{N}^*)^d$ and let E be a staircase. We denote by $(a_1, \ldots, a_d).E$ the staircase "obtained from E" by the linear map

$$(x_1, \ldots, x_d) \mapsto (a_1 x_1, \ldots, a_d x_d).$$

Concretely, this is the smallest staircase satisfying the relation:

$$(m_1, \ldots, m_d) \in E \Rightarrow (a_1(m_1+1)-1, \ldots, a_d(m_d+1)-1) \in (a_1, \ldots, a_d).E$$

This is a staircase of cardinal $a_1.a_2 \ldots a_d.\#E$. We denote by $a.E$ the staircase $(a, a, \ldots, a).E$.

Proposition 9. *Let $E, E_1, \dots, E_j$ be staircases. Then:*
$\overline{C(\underbrace{E,\dots,E}_{\prod a_i \ times}, E_1, \dots, E_j)} \supset C((a_1, \dots, a_d).E, E_1, \dots, E_j)$.

Proof: by induction on the number of a_i's which are not equal to one. If all the a_i's but one are equal to one, the statement follows from the previous proposition since

$$(1, \dots, 1, a_i, 1, \dots, 1).E = S_i(\underbrace{E, \dots, E}_{a_i \text{ times}}).$$

For the general case, one can suppose by symmetry that $a_1 \neq 1$. Applying several times -namely $a_2 a_3 \dots a_d$ times- this first step, we get

$$\overline{C(\underbrace{E,\dots,E}_{\prod a_i \ times}, E_1, \dots, E_j)} \supset C(\underbrace{(a_1, 1, \dots, 1).E, \dots, (a_1, 1, \dots, 1).E}_{a_2 \dots a_d \text{ times}}, E_1, \dots, E_j)$$

and, by induction,

$$\overline{C(\underbrace{(a_1, 1, \dots, 1).E, \dots, (a_1, 1, \dots, 1).E}_{a_2 \dots a_d \text{ times}}, E_1, \dots, E_j)}$$

contains

$$C((1, a_2, \dots, a_d).(a_1, 1, \dots, 1).E, E_1, \dots, E_j) = C((a_1, \dots, a_d).E, E_1, \dots, E_j).$$

The expected inclusion follows immediatly. ■

In particular, when $a_1 = a_2 = \dots = a_d = s$, we get:

Proposition 10. *Let $E, E_1, \dots, E_j$ be staircases. Then:*
$\overline{C(\underbrace{E,\dots,E}_{s^d \ times}, E_1, \dots, E_j)} \supset C(s.E, E_1, \dots, E_j)$.

Definition 11. *Let $\Delta = (\delta_1, \dots, \delta_d)$ be a primitive vector in $\mathbb{Z}^d$ such that there exist i, j satisfying $\delta_i \delta_j < 0$. Let $E \subset \mathbb{N}^d$ be a subset. We denote by $\Delta(E) \subset \mathbb{N}^d$ the unique subset verifying the following two conditions:*

- *for any line L in $\mathbb{R}^d$ with direction Δ, the sets $E \cap L$ and $\Delta(E) \cap L$ are equipotent*
- *$\forall i \in \mathbb{N}$, $\forall (n,p) \in (\mathbb{N}^d)^2$, $n \in \Delta(E)$ and $p = n + i\Delta \Rightarrow p \in \Delta(E)$*

To be more explicit, the set $L \cap \mathbb{N}^d$ is finite by hypothesis on Δ. If $m_1 < m_2 < \dots < m_j$ are its elements, ordered by the relation

$$(<) \qquad m_{i_1} < m_{i_2} \Leftrightarrow \exists i \in \mathbb{N},\ m_{i_1} = m_{i_2} + i\Delta,$$

then $\Delta(E) \cap L = \{m_1, \dots, m_k\}$, where $k = \#(E \cap L)$.

Proposition 12. *Let* $\Delta = (\delta_1, \ldots, \delta_d) \in \mathbb{Z}^d$ *be a vector such that*

- $\exists i,\ \delta_i = 1$,
- $\forall k,\ k \neq i \Rightarrow \delta_k \leq 0$,
- $\exists j \neq i, \delta_j \neq 0$.

Then for every staircase E, $\Delta(E)$ *is a staircase. Moreover, for every set of staircases* $E, E_1, \ldots, E_j$, *we have in characteristic zero the incidence:* $\overline{C(E, E_1, \ldots, E_j)} \supset C(\Delta(E), E_1, \ldots, E_j)$

Proof: suppose by symmetry that $\delta_1 = 1$. Let

$$\begin{array}{rcl} \Phi : k[x_1, \ldots, x_d] & \to & k[x_1, \ldots, x_d][t, \frac{1}{t}] \\ x_1 & \mapsto & tx_1 + x_2^{-\delta_2} x_3^{-\delta_3} \ldots x_d^{-\delta_d} \\ x_i & \mapsto & x_i \text{ if } i \neq 1. \end{array}$$

The ideal

$$I(t) = k[x_1, \ldots, x_d][t, \frac{1}{t}]\Phi(I^E)$$

defines a subscheme

$$F \subset (\mathbb{A}^1 - \{0\}) \times \mathbb{A}^d$$

whose fiber over each $t \in \mathbb{A}^1 - \{0\}$ is a monomial subscheme with staircase E. In particular, F is flat over $\mathbb{A}^1 - \{0\}$. The closure $\overline{F} \subset \mathbb{A}^1 \times \mathbb{A}^d$ is defined by the ideal $J(t) = I(t) \cap k[x_1, \ldots, x_d, t]$ and it is flat over $\mathbb{A}^1$.

We want to prove the equality $J(0) = I^{\Delta(E)}$, using a natural graduation.

Let $\varphi_1, \ldots, \varphi_{d-1} : \mathbb{Z}^d \to \mathbb{Z}$ be independant linear forms which vanish on Δ. Consider the multi-graduation D defined by:

$$\begin{array}{rcl} D : \text{Monomials of } k[x_1, \ldots, x_d] & \to & \mathbb{Z}^{d-1} \\ \underline{x}^{\underline{\alpha}} & \mapsto & (\varphi_1(\underline{\alpha}), \ldots, \varphi_{d-1}(\underline{\alpha})) \end{array}$$

The conditions on Δ imply that, for all $\underline{z} = (z_1, \ldots, z_{d-1}) \in \mathbb{Z}^{d-1}$, the sub-vector space $k[x_1, \ldots, x_d]_{\underline{z}} \subset k[x_1, \ldots, x_d]$ containing the elements of degree $\underline{z}$ has finite dimension. Note that $J(t)$ is a graded ideal ie.

$$J(t) = \oplus_{\underline{z} \in \mathbb{Z}^{d-1}} J_{\underline{z}}(t)$$

where

$$J_{\underline{z}}(t) = J(t) \cap k[x_1, \ldots, x_d]_{\underline{z}}[t].$$

In particular, to compute $J(0) = \lim_{t \to 0} J(t)$, it suffices to compute the limit of its graded parts in the grassmannians $G(l, k[x_1, \ldots, x_d]_{\underline{z}})$, where $l = \dim J_{\underline{z}}(t), t \neq 0$. Let $m_1 < \cdots < m_k$ be the monomials of $k[x_1, \ldots, x_d]_{\underline{z}}$, where the order is given by the relation $(<)$ above. Let us admit temporarily the inclusion

$$(*)\ m_{k-l+1}, m_{k-l+2}, \ldots, m_k \in J_{\underline{z}}(0).$$

Then $J_{\underline{z}}(0)$ is the vector space generated by $m_{k-l+1}, m_{k-l+2}, \ldots, m_k$ for dimensional reasons and $J(0) = I^{\Delta(E)}$ since these two graded ideals have the same graded parts. In particular $J(0)$ is an ideal generated by monomials and the set $\Delta(E)$ of monomials which are not in $J(0)$ is a staircase. Moreover, replacing the coordinates $x_1, \ldots, x_d$ of $\mathbb{A}^d$ by any local system of coordinates, one shows by the same computation that any closed point of $C(\Delta(E), E_1, \ldots, E_j)$ is a limit of points which are in $\overline{C(E, E_1, \ldots, E_j)}$. This gives the incidence between the strata.

It remains to show $(*)$. Let $n_1 = x^{\underline{\alpha}(1)}, \ldots, n_l = x^{\underline{\alpha}(l)}$ be the monomials of $I^E \cap k[x_1, \ldots, x_d]_{\underline{z}}$, where $\underline{\alpha}(i) = (\alpha_1(i), \ldots, \alpha_d(i))$. The ideal $I(t)$ contains the monomials

$$\Phi(n_i) \quad = \quad (tx_1 + x_2^{-\delta_2} x_3^{-\delta_3} \ldots x_d^{-\delta_d})^{\alpha_1(i)} x_2^{\alpha_2(i)} \ldots x_d^{\alpha_d(i)}.$$

Since the degree of m_i in x_1 is $k - i$, this equality can be rewritten as:

$$\Phi(n_i) = \sum_{j=0}^{\alpha_1(i)} \binom{\alpha_1(i)}{j} t^j m_{k-j} \quad = \quad \sum_{j=0}^{k-1} \binom{\alpha_1(i)}{j} t^j m_{k-j}$$

with the usual convention $\binom{\alpha_1(i)}{j} = 0$ if $j > \alpha_1(i)$. If N and M are the column matrices whose entries are respectively $\Phi(n_i)$, $i \in \{1, \ldots, l\}$, and $t^j m_{k-j}$, $j \in \{0, \ldots, k-1\}$, if P is the matrix whose coefficient P_{ij} is $\binom{\alpha_1(i)}{j}$, the above equality writes down $N = PM$. Take the first l columns of P to get a square matrix

$$Q = \begin{pmatrix} 1 & \alpha_1(1) & \binom{\alpha_1(1)}{2} & \ldots & \binom{\alpha_1(1)}{l-1} \\ \ldots & \ldots & & & \\ 1 & \alpha_1(l) & \binom{\alpha_1(l)}{2} & \ldots & \binom{\alpha_1(l)}{l-1} \end{pmatrix}.$$

Since the coefficients in the third column are polynomials of degree 2 in α_1, one can replace the third column by a linear combination of the first three columns so that the i^{th} element in the third column becomes $\alpha_1(i)^2$. Similarly, after suitable operations on the columns, the i^{th} element in the fourth, fifth column ... becomes $\alpha_1(i)^3, \alpha_1(i)^4, \ldots$. The resulting matrix is a Van Der Monde matrix in the $\alpha_1(i)$'s. In characteristic zero, its determinant is not zero since the $\alpha_1(i)$'s are distinct. In particular Q is invertible.

The ideal $I(t)$ contains the elements which are the coefficients of the matrix $Q^{-1}N = Q^{-1}PM$. Using that the identity is a submatrix of $Q^{-1}P$ by construction, the i^{th} element in this column matrix is $c_i(t) = t^{i-1} m_{k-i+1} + R$ where R is a polynomial dividible by t^i. Thus, $\frac{c_i(t)}{t^{i-1}} \in J(t)$ and, as expected, $J(0)$ contains $\frac{c_i(t)}{t^{i-1}}(0) = m_{k-i+1}$ for $i \in \{1, \ldots, l\}$. ∎

We give now some combinatorial properties of the map $E \mapsto \Delta(E)$ that we will use later on.

Lemma 13. *Let E and F be two subsets of $\mathbb{N}^d$ and $\Delta = (\delta_1, \ldots, \delta_d) \in \mathbb{Z}^d$ be a direction satisfying the properties of the preceding proposition. Suppose that for every line L with direction Δ, we have the inequality on cardinals:*

$$\#\{E \cap L\} \geq \#\{F \cap L\}$$

then $\Delta(E) \supset \Delta(F)$.

Proof: we must show for everery line L the inclusion $\Delta(E) \cap L \ \supset \Delta(F) \cap L$. This is obvious since, using the m_i's introduced after definition 11, $\Delta(E) \cap L = \{m_1, \ldots, m_{\#\{E \cap L\}}\}$ and $\Delta(F) \cap L = \{m_1, \ldots, m_{\#\{F \cap L\}}\}$. ■

Applying this lemma to the following E and to $F = R_\mu$, noticing that $\Delta(R_\mu) = R_\mu$, we get:

Lemma 14. *Let R_μ be a regular staircase, $m \in R_\mu$, $P \subset \mathbb{N}^d$ a subset such that $P \cap R_\mu = \emptyset$ and $E = R_\mu \cup P - \{m\}$. If there exists $i \in \mathbb{Z}$ such that $m + i\Delta \in P$, then $\Delta(E) \supset R_\mu$.*

4 Conclusion of the proofs

4.1 Proof of theorems 3 and 5

Let us denote the stratum $C(E_1, \ldots, E_1, \ldots, E_r, \ldots, E_r)$ by $C(E_1^{n_1}, \ldots, E_r^{n_r})$ where n_i is the number of copies of E_i. According to proposition 7, to conclude the proof of theorem 3 (resp. of theorem 5), we must show that, for $s \geq 3$, $d \geq 2$ and $(s, d) \neq (3, 2)$ (resp. for $s \geq 1$, $d \geq 2$) $\overline{C(R_\mu^{s^d})} \supset C(E)$ for some staircase E containing $R_{s\mu+1}$ (resp. containing $R_{s\mu}$). By proposition 10,

$$\overline{C(R_\mu^{s^d})} \supset C(s.R_\mu).$$

Since $s.R_\mu \supset R_{s\mu}$, this concludes the proof of theorem 5. As for theorem 3, taking for E the staircase $\Delta_d(\Delta_{d-1}(\ldots \Delta_1(s.R_\mu)))$ constructed in the following lemma, we have

$$\overline{C(s.R_\mu)} \supset C(E)$$

by proposition 12. The required inclusion $\overline{C(R_\mu^{s^d})} \supset C(E)$ follows immediatly from the last two displayed inclusions. ■

Lemma 15. *Let (s, d) be a couple of integers with $d \geq 2$,$s \geq 3$, and $(s, d) \neq (3, 2)$. Then there exists $(\Delta_d, \ldots, \Delta_1) \in (\mathbb{Z}^d)^d$ such that*

- *$\forall i$, Δ_i verifies the conditions of proposition 12*
- *$\forall \mu > 0$, $\Delta_d(\Delta_{d-1}(\ldots(\Delta_1(s.R_\mu)))) \supset R_{s\mu+1}$.*

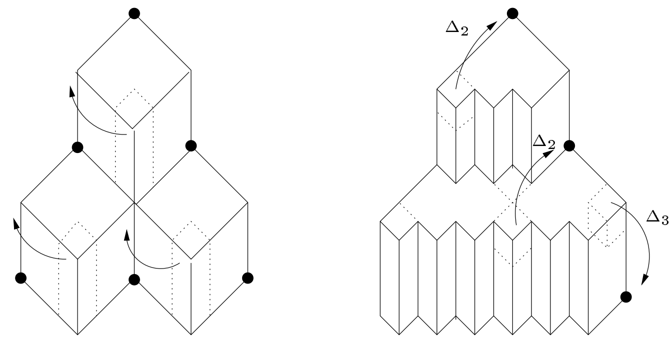
Proof: as we will proceed by induction on the dimension d, we precise our notations and we denote by $R_i(d)$ the regular staircase R_i in $\mathbb{N}^d$. Considering the couples (s, d) involved in the proposition, we have to initialize the induction with the cases $(s > 3, d = 2)$ and $(s = 3, d = 3)$.
Initial cases. If $d = 2, s > 3$, then one can take $\Delta_1 = (1, -s + 1)$ and $\Delta_2 = (-s + 2, 1)$. When $s = 3, d = 3$, we must find $\Delta_1, \Delta_2, \Delta_3$ such that

$$\Delta_3(\Delta_2(\Delta_1(3.R_\mu(3)))) \supset R_{3\mu+1}(3).$$

The $\frac{(\mu+1)(\mu+2)}{2}$ elements of the difference

$$R_{3\mu+1}(3) - 3.R_\mu(3) = \{(3x, 3y, 3z),\ x + y + z = \mu\}$$

are shown in the following figure with $\mu = 2$. Taking $\Delta_1 = (1, -2, 0)$, we have:



$3 * R_\mu$ $\quad$ $\Delta_1(3 * R_\mu)$

$\bullet \in R_{3\mu+1} - 3 * R_\mu$ $\quad$ $\bullet \in R_{3\mu+1} - \Delta_1(3 * R_\mu)$

$$R_{3\mu+1}(3) - \Delta_1(3.R_\mu(3)) = \{(0, 3y, 3z),\ y + z = \mu\}$$

Finally, taking $\Delta_2 = (-3, 0, 1)$ and $\Delta_3 = (0, 1, -2)$,

$$\Delta_3(\Delta_2(\Delta_1(3.R_\mu(3)))) \supset R_{3\mu+1}(3),$$

as expected.
Step from $d-1$ to d. Let T_i be the "i^{th} slice" of $s.R_\mu(d)$, i.e.

$$T_i := \{m \in \mathbb{N}^{d-1}\ s.t.\ (i, m) \in s.R_\mu(d)\}.$$

Then $T_i = s.R_{\nu(i)}(d-1)$ with $\nu(i) = max(0, \mu - [\frac{i}{s}])$, [.] standing for the integral part. When $i < s\mu$, $\nu(i) > 0$ and we can apply induction to T_i. Using moreover that $s\nu(i) \geq s\mu - i$, we get elements $\gamma_1, \ldots, \gamma_{d-1} \in \mathbb{N}^{d-1}$ such that,

$$T'_i = \gamma_{d-1}(\ldots(\ \gamma_1(T_i))) \supset R_{s\nu(i)+1}(d-1) \supset R_{s\mu+1-i}(d-1).$$

Let $\Delta_i = (0, \gamma_i) \in \mathbb{N}^d$. The i^{th} slice of the staircase

$$F = \Delta_{d-1}(\ldots(\Delta_1(s.R_\mu(d))))$$

is T'_i. Summing up, for $i < s\mu$, the i^{th} slice of F strictly contains the i^{th} slice $R_{s\mu+1-i}(d-1)$ of $R_{s\mu+1}(d)$. In particular, F contains all the d-tuples whose sum is $s\mu$ except $(s\mu, 0, 0, \ldots, 0)$.

It remains to find Δ_d such that $\Delta_d(F) \supset R_{s\mu+1}(d)$ by an application of lemma 14.
Note that

$$T_{s\mu-1} = s.R_1(d-1) \supset R_s(d-1) \supset R_3(d-1).$$

It follows that

$$T'_{s\mu-1} \supset \gamma_{d-1}(\ldots(\gamma_1(R_3(d-1)))) = R_3(d-1)$$

and that the element $z = (s\mu - 1, 2, 0, \ldots, 0)$ is in F. Let $\Delta_d = (1, -2, 0, \ldots, 0)$. Applying lemma 14 with $m = (s\mu, 0, 0, \ldots, 0)$, $E = F$, $P = F - R_{s\mu+1}(d)$, $\Delta = \Delta_d$, $s\mu + 1$ instead of μ, $m + i\Delta = z$, we get the expected inclusion $\Delta_d(F) \supset R_{s\mu+1}(d)$. ■

4.2 The case $r = 2^d$

The goal of this section is to compute the postulation of 2^d fat points of multiplicity μ in $\mathbb{P}^d$, stated in remark 4:

Proposition 16. *Let $r = 2^d$ and $v(d, \delta, \mu^r) = \binom{\delta+d}{d} - r.\binom{d+\mu-1}{d}$. Then: $l(d, \delta, \mu^r) = max(0, v(d, \delta, \mu^r))$.*

Proof: if Z_G is the generic union of 2^d fat points of multiplicity μ, the vector space $H^0(I_{Z_G}(\delta))$ being the kernel of the restriction morphism:

$$H^0(\mathcal{O}_{\mathbb{P}^d}(\delta)) \to H^0(\mathcal{O}_{Z_G}(\delta)),$$

its dimension $l(d, \delta, \mu^r)$ is at least

$$v(d, \delta, \mu^r) = h^0(\mathcal{O}_{\mathbb{P}^d}(\delta)) - h^0(\mathcal{O}_{Z_G}(\delta)).$$

To prove the reverse inequality $l(d, \delta, \mu^r) \leq max(0, v(d, \delta, \mu^r))$, since $\overline{C(R_\mu^{2^d})} \supset C(2.R_\mu)$ by corollary 10, it suffices by semi-continuity to exhibit a subscheme Z in $C(2.R_\mu)$ such that

$$h^0(I_Z(\delta)) = max(0, v(d, \delta, \mu^r))$$

for all δ. Let $\mathbb{A}^d = Spec\ k[x_1, \ldots, x_d] \subset \mathbb{P}^d$ be an affine space and Z be the subscheme of $\mathbb{A}^d$ whose ideal is $I^{2.R_\mu}$. By deshomogeneisation, the vector space $H^0(\mathcal{O}_{\mathbb{P}^d}(\delta))$ is in bijection with the subspace $S_\delta \subset k[x_1, \ldots, x_d]$ containing the polynomials of degree at most δ, and $H^0(I_Z(\delta))$ corresponds to $I^{2.R_\mu} \cap S_\delta$. Now, dim $I^{2.R_\mu} \cap S_\delta$ is the number of monomials in $R_{\delta+1}$ which are not in $2.R_\mu$. Since

$$R_{2\mu} \subset 2.R_\mu \subset R_{2\mu+1},$$

this number is 0 if $\delta \leq 2\mu - 1$ and $h^0(\mathcal{O}_{\mathbb{P}^d}(\delta)) - \#(2.R_\mu)$ if $\delta \geq 2\mu$. ■

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