Center and diameter problems in plane triangulations and quadrangulations

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Abstract. In this note, we present first linear time algorithms for computing the center and the diameter of several classes of face regular plane graphs: triangulations with inner vertices of degree ≥ 6, quadrangulations with inner vertices of degree ≥ 4 and the subgraphs of the regular hexagonal grid bounded by a simple circuit of this grid.

1. Introduction

This paper describes linear time algorithms for computing diameters and centers of several classes of face regular plane graphs. Namely, we consider plane triangulations with inner vertices of degree at least six (called *trigraphs*) and plane quadrangulations with inner vertices of degree at least four (called *squaregraphs*). Particular cases of these graphs are the subgraphs of the regular triangular and square grids which are induced by the vertices lying on a simple circuit and inside the region bounded by this circuit. They are called *triangular* and *square systems*, respectively (square systems are also known as *polyominoes*). As a byproduct, we obtain linear time algorithms for the same problems on *benzenoids*, alias, *hexagonal systems* (subgraphs of the regular hexagonal grid bounded by a simple circuit) and the graphs resulting from squaregraphs by transforming each inner face into a 4-clique (called *kinggraphs*). The latter class covers all subgraphs of the King grid $\mathbb{Z}_8$ bounded by a simple circuit. Notice that trigraphs, squaregraphs, and kinggraphs are particular instances of, respectively, bridged, median, and Helly graphs, three classes of graphs playing an important role in metric graph theory.

The diameter and center problems are basic problems in graph theory and computational geometry. They naturally arise in communication and transportation networks, robot-motion planning but also in several other areas. We emphasize on subclasses of planar graphs since for planar graphs the practical applications seem of most importance. The starting point of our research was a question: how to find the center of benzenoid systems efficiently. Benzenoids represent a significant class of chemical graphs and their encoding constitutes an important subject of research in computational chemistry. One canonical way of such an encoding is to label the carbon atoms level-wise starting from the center. In his H. Skolnik award lecture, A. Balaban noticed that "Finding the centers for polycyclic graphs is an open problem, and, if solved, it could provide a useful basis for chemical information systems" [2]. Trying to find an efficient algorithm for the center problem on benzenoids, we noticed that its solution can be obtained from the solution of the same problem on two triangular systems inferred from the initial benzenoid.

Here we design a general approach for computing the centers which can be applied not only to triangular systems but also to all trigraphs, squaregraphs and kinggraphs. Additionally, we characterize centers of trigraphs and kinggraphs (answering a question posed in [9]); the centers of squaregraphs were characterized in [11] and, independently, in [9] for the particular case of the square systems. Some properties of centers of square systems have been given in [10].

Our method is based on the following. First we compute the diameter and a diametral pair of vertices of respective graphs by using row-wise maxima search of [1] in a totally monotone matrix (this approach was already employed in [7] for computing the geodesic diameter of a simple polygon in linear time). The correctness relays on the result of [8, 5] that in all those graphs the furthest neighbours of a vertex are located on the outer face. This approach would result in a linear time algorithm, provided one can compute after a linear time preprocessing step the entries (which are distances between boundary vertices) of the matrix under search in constant time. This can be done for both trigraphs and squaregraphs due to their nice metric properties.

Having a diametral pair, a region containing at least one central vertex is located and preprocessed in such a way that all vertices of minimum eccentricity in this region can be found in linear time. Then, using the established structure of the center, the remaining part of the center is built up. This generic method is applied to trigraphs and squaregraphs only. The centers and diameters of hexagonal systems and kinggraphs are derived by simply employing their relation to triangular systems and squaregraphs, respectively.

The paper is organized as follows. In the next section, we define the center and the diameter problems, recall some necessary notions, and formulate the basic properties of main classes of graphs (due to space limitations, we postpone their proof to the full version). In Section 3, we describe the general scheme for the diameter problem, and specify it for each of the classes of graphs. Finally, Section 4 provides linear time algorithms for the center problem.

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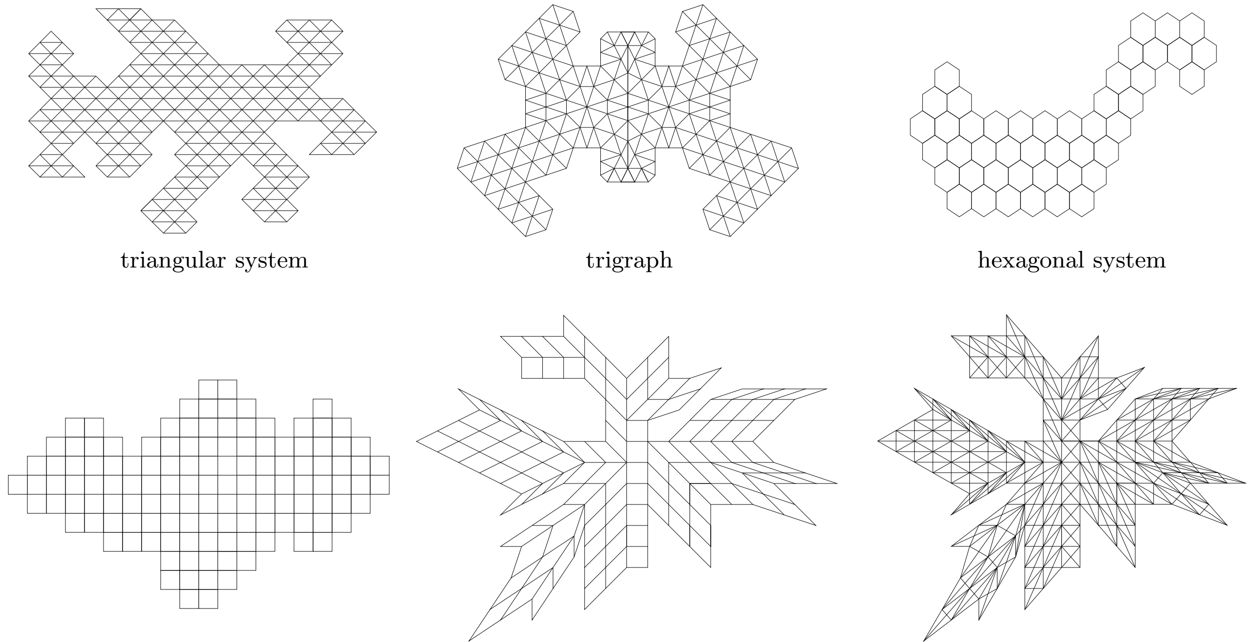

Figure 1

2. Preliminaries

2.1. Definitions and notations. All graphs $G = (V, E)$ occurring in this note are connected, finite, and undirected. The *length* of a path from a vertex v to a vertex u of G is the number of edges in the path. The *distance* $d(u, v)$ (or $d_G(u, v)$) is the length of a shortest (u, v)-path and the *interval* $I(u, v)$ between these vertices is the set $I(u, v) = \{w \in V : d(u, v) = d(u, w) + d(w, v)\}$. Set $L_i := \{w \in I(u, v) : d(u, w) = i\}$ and call L_i a *level set* of $I(u, v)$. A subset $S \subseteq V$ is called *convex* if $I(u, v) \subseteq S$ whenever $u, v \in S$, and *gated* if for each $v \notin S$ there exists a (necessarily unique) vertex $v' \in S$ (the *gate* of v in S) such that $v' \in I(v, u)$ for every $u \in S$. A subgraph H of G is an *isometric subgraph* if $d_H(u, v) = d_G(u, v)$ for any vertices u, v of H. The *ball* with center v and radius r is denoted by $B_r(v)$ (set $N(v) := B_1(v) \setminus \{v\}$). More generally, the *r-neighbourhood* of a set $S \subseteq V$ is the set $B_r(S) = \{v \in V : d(v, S) \leq r\}$, where $d(v, S) = \min\{d(v, u) : u \in S\}$ is the distance from the vertex v to the set S. Denote by $\pi(v, S) = \{u \in S : d(v, u) = d(v, S)\}$ the *projection* of v on the set S.

The *eccentricity* $e(v)$ of a vertex v is the maximum distance from v to a vertex in G. (For a subset S, its eccentricity is $e(S) = max_{v \in V} min_{u \in S} d(v, u)$.) Denote by $F(v)$ the set of all *furthest furthest* of v, i.e., $F(v) = \{u \in V : d(v, u) = e(v)\}$. The *radius* $r(G)$ is the minimum eccentricity of a vertex in G and the *diameter* $d(G)$ the maximum eccentricity. The *center* $C(G)$ of G is the subgraph induced by the set of all *central vertices*, i.e., vertices whose eccentricities are equal to $r(G)$.

Denote by $\mathcal{T}$, $\mathcal{Q}$, $\mathcal{K}$, $\mathcal{H}$ the classes of trigraphs, squaregraphs, kinggraphs, hexagonal systems, respectively and their members by T, Q, K, H (for examples see Fig.1). For a graph G from one of these classes, let ∂G be the bounding cycle of the external face of G. The papers [8, 5] present the following property of such a G:

(P1) [8, 5] *For any vertex v of G, the set $F(v)$ of furthest neighbours belongs to ∂G.*

2.2. Trigraphs. We recall some properties of trigraphs (for some of them and references see [3]).

(PT1) *The balls and the r-neighbourhoods of convex sets of T are convex; in particular, the centers of trigraphs are convex.*

(PT2) *T is 4-clique free.*

(PT3) *The metric projection $\pi(x, S)$ of a vertex x on a convex set S of T is convex; moreover, for each $y \in S$ there is a shortest (x, y)-path which goes via $\pi(x, S)$.*

Three vertices u, v, w of a graph G are said to form a *metric triangle uvw* if the intervals $I(u, v), I(v, w)$, and $I(w, u)$ pairwise intersect only in the common end vertices. If $d(u, v) = d(v, w) = d(w, u) = k$, then this metric triangle is called *equilateral* of *size k*.

(PT4) *Metric triangles of T are equilateral; moreover, their convex hulls are isomorphic to metric triangles of the same size of the triangular grid.*

In the sequel we will use the same name for metric triangles and their convex hulls.

(PT5) *Every interval $I(u, v)$ of T is convex; additionally, $I(u, v)$ can be represented as an isometric subgraph of the triangular grid.*

(PT6) *If two adjacent vertices x, y of T are equidistant from a vertex v, then there exists a common neighbour of x and y one step closer to v.*
(PT7) *T does not contain three pairwise adjacent vertices having the same distance to a fourth vertex.*

By a *convex cut* of T we will mean a convex path c of T whose both endvertices lie on ∂T. Every convex path $c_0 = (u_0, u', \ldots, v', v_0)$ extends to a convex cut. Indeed, if, say, u_0 is an inner vertex, then extend c_0 by adding an edge u_0u'', such that $B_1(u') \cap B_1(u'') = \{u_0\}$ (this is always possible, because u_0 has at least six neighbours). Continuing this way, we will arrive at a locally-convex path c with both ends on ∂T. Since in trigraphs locally-convex connected subsets are convex [3], c is a convex cut extending c_0.

We say that a convex cut $c = (u, \ldots, v)$ separates two vertices x and y, if x and y lie in different connected components of the graph $T \backslash c$. From (PT3) one concludes that $d(x, y) \geq d(x, c) + d(c, y)$. The projection $\pi(x, c)$ of a vertex x on c is a subpath of c, which we denote by $[u_x, v_x]$. One can show that there exists a unique furthest from x vertex x_c which belongs to the intersection $I(x, u_x) \cap I(x, v_x)$. Then $u_x x_c v_x$ is a metric triangle, moreover, x_c lies on a shortest path between x and every vertex of c. A *histogram* H_c of a convex cut c is the union of all metric triangles having one side on c. It can be shown that every histogram H_c of T is an isometric subgraph of T and of the triangular grid (hence we can refer to the vertices of H_c as *convex, concave,* and *regular*). Moreover, if T does not contain inner vertices of degree 7, then H_c is convex.

2.3. Triangular and hexagonal systems. Let H and T be a hexagonal and a triangular system. Let E_1, E_2, and E_3 denote the edges of H (or T) of a given direction. For $i = 1, 2, 3$, let H_i be the graph obtained from H by deleting all edges of E_i. Analogously, let T_i be the graph obtained from T by deleting all the edges which do not belong to E_i. Note that the connected components of the graphs H_i and T_i are paths. Define a graph A_i whose vertices are connected components of H_i (or T_i) and where two such components P' and P'' are adjacent in A_i iff there exist two adjacent in H (resp., T) vertices, one in P' and another in P''. Since H and T are simply connected subgraphs of the hexagonal and triangular grids, every A_i is a tree. This yields to the following canonical embedding α of H and T into the Cartesian product $A = A_1 \times A_2 \times A_3$: for every vertex v of H and T set $\alpha(v) = (P, Q, R)$, where P, Q, and R are the connected components of the graphs H_1, H_2, H_3 (resp., T_1, T_2, and T_3) sharing v.
(PHT1) [6] *For each vertices $u', v' \in H$ and $u'', v'' \in T$ one has $d_H(u', v') = d_A(\alpha(u'), \alpha(v'))$ and $2d_T(u'', v'') = d_A(\alpha(u''), \alpha(v''))$. The trees A_1, A_2, A_3 and the labels of the vertices of H and T can be computed in linear time.*
(Analogously, square systems embeds isometrically into the Cartesian product of two trees.) Using this structure, after an $O(n)$ preprocessing, one can answer in $O(1)$ time per query questions of the form "What is the distance between vertices x and y of H (or T)?". Indeed, (PHT1) reduces this question to three similar problems on tree-factors, where we can use the algorithm for computing nearest common ancestors $nca(x, y)$ in the tree A_i rooted at r_i in $O(1)$ time. Since $d_{A_i}(x, y) = d_{A_i}(x, r_i) + d_{A_i}(y, r_i) - 2d_{A_i}(r_i, nca(x, y))$, we can find the distance between x and y in $O(1)$.

2.4. Squaregraphs and kinggraphs. Let Q and K be a squaregraph and a kinggraph.
(PQ1) *Convex and gated sets of Q are the same. Intervals and balls of radius 2 of Q are gated, moreover, the intervals of Q can be embedded isometrically into the square grid.*

From the definition of kinggraphs it immediately follows that all maximal cliques of K are of size 4 and the intersection of two K_4 is empty, or a vertex, or an edge. Therefore, an edge of K belongs to maximum two K_4s and any K_3 extends in a unique way to a K_4. We continue with a basic property of kinggraphs.
(PK1) [4] *Every collection of pairwise intersecting balls of K has a nonempty intersection.*

This Helly property is a powerful tool for kinggraphs. For example, one can easily show that the property (PT6) is verified by kinggraphs. Other consequences are the following properties of kinggraphs.
(PK2) *If three vertices of a 4-clique of K are equidistant from a vertex v then the fourth vertex of this clique is one step closer to v.*
(PK3) *The projection of a vertex x on an interval $I(u, v)$ is a vertex or an edge; for each $y \in I(u, v)$ there is a shortest (x, y)-path which goes via this projection.*

There is a standard transformation, associating with each K a quadrangulation $Q(K)$: insert vertices at crossing points of K, create edges of length 1, and remove the superfluous edges. Clearly, $d_{Q(K)}(u, v) = 2d_K(u, v)$ any u, v of K, in particular $d(Q(K)) = 2d(K)$. Notice also that the eccentricity in $Q(K)$ of a vertex v of K is precisely twice its eccentricity in K.
(PK4) *$Q(K)$ is a squaregraph.*

2.5. Computing distances and projections to a convex set. Let $S \subset V$ be a convex set of a trigraph $T = (V, E)$. We describe a generic procedure for computing the distances and the projections of all vertices of T to S (this applies to squaregraphs as well). Let ∂S be the outer face of the plane graph induced by S. Since S is convex, by (PT3) the projection $\pi(v, S)$

of each vertex $v \in V \setminus S$ is a convex path of ∂S which we will denote by $[u_i, u_j]$, meaning a path of ∂S from u_i to u_j in counterclockwise order.

First, using a modification of *Breadth-First-Search (BFS)*, one can partition the region $(V \setminus S) \cup \partial S$ into $k = e(S) + 1$ level sets (spheres) $\partial S, S_1, S_2, ..., S_{k-1}$, where $S_i = \{v \in V : d(v, S) = i\}$ $(i \geq 1)$ is the boundary of the ith neighbourhood of set S. Clearly, S_{i+1} can be easily derived from S_i. One can modify this algorithm, in order to compute for each vertex v its projection $\pi(v, S)$. We initialize a BFS by letting $S_0 := \partial S$ and $\pi(v, S) = \{v\}$ for each $v \in \partial S$. Then, when a new vertex v is added to S_i, the projections of its neighbours from S_{i-1} have been already computed. Since the $(i-1)$-neighbourhood of S is convex and G does not contain 4-cliques, v has one or two neighbours in S_{i-1}. Set $\pi(v, S)$ to be the union of the projections of these neighbours. Clearly, all distances $d(v, S)$ and all projections $\pi(v, S)$ $(v \in V)$ can be found in total linear time.

If the set S in question is a convex cut c, the previous procedure can be modified to find the histogram H_c. Namely, for each vertex x compute the length of its projection $\pi(x, c) = [u_x, v_x]$. Then $u_x x v_x$ is a metric triangle iff $d(u_x, x) = d(u_x, v_x)$, because the metric triangles of T are equilateral.

3. Diameter Problem

3.1. The method. The algorithm for computing the diameter and a diametral pair uses matrix-searching of totally monotone matrices [1]. The idea to use matrix-searching to compute the diameter of a simple polygon was employed by Hershberger and Suri in [7]. A matrix D is called *totally monotone* if $D(i, k) < D(i, l)$ implies $D(j, k) < D(j, l)$ for any $1 \leq i < j < n$ and $1 \leq k < l < m$. Aggarwal et al. [1] established that the row-wise maxima of a totally monotone $n \times m$ matrix can be found with only $O(n + m)$ comparisons and evaluations of the matrix entries. The matrix is defined implicitly – an entry is evaluated only when needed by the algorithm. If evaluating an entry takes $O(f(n, m))$ time, then the complexity of the algorithm is $O((n + m)f(n, m))$.

An example of a totally monotone matrix is obtained if one considers two disjoint paths $P' = (u_1, \ldots, u_n)$ and $P'' = (v_1, \ldots, v_m)$ on the boundary ∂G of a plane graph G; the vertices in each path are ordered counterclockwise. Define the matrix D by letting $D(i, j) = d(u_i, v_j)$. One can easily see that D is totally monotone. Indeed, pick the vertices u_i, u_j and v_k, v_l such that $d(u_i, v_k) < d(u_i, v_l)$ and $1 \leq i < j \leq n$ and $1 \leq k < l \leq m$. Pick a shortest path between u_i and v_k and a shortest path between u_j and v_l. By Jordan's curve theorem, these paths intersect in a vertex x (it may happen that x coincides with one of four specified vertices). By the triangle inequality, $d(u_i, v_l) \leq d(u_i, x) + d(x, v_l)$ and $d(u_j, v_k) \leq d(u_j, x) + d(x, v_k)$. Summing up these inequalities, we deduce that $d(u_i, v_l) + d(u_j, v_k) \leq d(u_i, v_k) + d(u_j, v_l)$. Since $d(u_i, v_k) < d(u_i, v_l)$, from previous inequality one concludes that $d(u_j, v_k) < d(u_j, v_l)$, establishing the total monotonicity of D.

Now, we will exploit this idea to find a diametral pair of vertices in considered classes of plane graphs. In these cases, (P1) implies that every vertex has all its furthest neighbours located on the outer face. In particular, the diameter is always realized by two boundary vertices. We will adapt a lemma from [12] to reduce the diameter problem to an instance of maxima-finding in a totally monotone matrix. Let G be a plane graph obeying (P1) and let v, w be two vertices of ∂G such that $w \in F(v)$. Removing these vertices, the cycle ∂G divides into two disjoint paths P' and P''.

Lemma 3.1. $d(G) = max\{\ e(w),\ max\{d(p, q) : p \in P', q \in P''\}\ \}$.

Proof. From the choice $w \in F(v)$ we have $d(v, w) = e(v) \leq e(w)$. Hence, if $e(v) = d(G)$, then $e(w) = d(G)$, too. Pick a diametral pair p, q and assume by way of contradiction that both vertices p and q belong to the same path P' or P'', say $p, q \in P'$. Let p lie in $P' \bigcup \{v\}$ between v and q. As above, one concludes that $d(w, v) + d(p, q) \leq d(w, p) + d(q, v)$. Since $d(v, q) \leq d(w, v) = e(v)$, necessarily $d(w, p) \geq d(p, q) = d(G)$ and hence $e(w) = d(G)$. □

Thus, the diameter problem in such plane graphs reduces to the problem of computing row-wise minima in a totally monotone matrix D defined for P' and P''. If G is a triangular, hexagonal or square system, then after a linear preprocessing (consisting in embedding of respective graphs into the product of three or two trees), the distance between any two given vertices can be computed in constant time. Therefore, any entry of D can be computed in time $O(1)$ and hence, using the algorithm from [1], in total $O(n + m)$ time one can compute a furthest neighbour for each vertex of the path P' in the path P''. As a consequence, the diameter of G can be computed in linear time.

3.2. Diameters of trigraphs and squaregraphs. Let now $T \in \mathcal{T}$ and $Q \in \mathcal{Q}$, and let the paths P' and P'' of T (resp., Q) be defined as above. To get a linear algorithm for computing $d(T)$ and $d(Q)$, all one needs to show is that after a linear time preprocessing it is possible to report the distance $d(p, q)$ between any $p \in P'$ and $q \in P''$ in $O(1)$. We first show this for T.

In a preprocessing step we will find the interval $I(v, w)$, compute the distances $d(p, I(v, w))$ and $d(q, I(v, w))$ for all $p \in P'$ and $q \in P''$ and the projections of these vertices on $I(v, w)$ (denote them by

$\pi(p)$ and $\pi(q)$, respectively). Since $I(v,w)$ is convex, all those distances and projections can be computed in total linear time. Let X be a region of T bounded by interval $I(v,w)$ and path P' (note that X does not contain any inner vertex of $I(v,w)$). Analogously, let Y be the region bounded by $I(v,w)$ and P''. Since $I(v,w)$ is convex, both sets $X \bigcup I(v,w)$ and $Y \bigcup I(v,w)$ are convex as well. Consequently, for each $p \in P'$ and $q \in P''$, by (PT3), one has

$$d(p,q) = d(p,\pi(p)) + d(\pi(p),q) =$$
$$d(p,\pi(p)) + d(\pi(p),\pi(q)) + d(\pi(q),q).$$

We already know the distances $d(p,\pi(p))$ and $d(q,\pi(q))$. So, it remains to compute $d(\pi(p),\pi(q))$ in constant time.

By (PT5), the interval $I(v,w)$ embeds isometrically into the triangular grid. For this, we endow the triangular grid with a coordinate system $(\overrightarrow{vs},\overrightarrow{vt})$, where $s,t \in I(v,w)$ are pairwise adjacent neighbours of v, such that every vertex $u \in I(v,w)$ has positive coordinates $x(u)$ and $y(u)$. Consider the level sets of $I(v,w)$. The structure of $I(v,w)$ implies that $|L_{i-1}|-1 \le |L_i| \le |L_{i-1}|+1$ for every $i = 1,...,d(v,w)$. For $i = 0,1,...,d(v,w)-1$, we embed the level L_{i+1} as a contiguous segment of grid-points of the line $x+y = i+1$. Let $u_i = (i-l,l)$ and $v_i = (m,i-m)$ be the end-vertices of L_i. Then the end-vertices of L_{i+1} are the points with the coordinates $(i-l+1,l)$ and $(m,i-m+1)$ if $|L_{i+1}| = |L_i|+1$, $(i-l+1,l)$ and $(m+1,i-m)$ if $|L_{i+1}| = |L_i|$, and $(i-l,l+1)$ and $(m+1,i-m)$ if $|L_{i+1}| = |L_i|-1$. One can easily check that indeed the distance between two vertices of $I(v,w)$ is the same in T and in the triangular grid.

Now, we show that every projection $\pi(p)$ on $I(v,w)$ consists of one or two incident segments (sharing a concave vertex of $I(v,w)$) of the triangular grid. For this, it suffices to establish that $\pi(p)$ does not contain in its interior a convex vertex of $I(v,w)$. Suppose this is not the case, and let $b \in \pi(p)$ be such a vertex. Let a and c be the neighbours of b in $\pi(p)$. Then there is a vertex $q \in I(v,w)$ adjacent to a,b,c. Since $k = d(p,a) = d(p,b) = d(p,c)$, by (PT6) there is a vertex p' adjacent to a,b and a vertex p'' adjacent to b,c at distance $k-1$ to p. Since $B_{k-1}(p)$ is convex, p' and p'' either are adjacent or coincide. In both cases we conclude that b is an inner vertex of T of degree < 6, a contradiction.

Hence we reduced the initial problem of computing the distance between two projections on $I(v,w)$ to that of computing the distance between two segments s_1 and s_2 of the triangular grid, given by their end-vertices. This can be done in constant time by distinguishing the cases when s_1 and s_2 lie on parallel or intersecting lines. To find the distance $d(\pi(p),\pi(q))$ one has to compute the distance between each pair of segments constituting $\pi(p)$ and $\pi(q)$. Since every projection comprises maximum two segments, we conclude that the required distance $d(\pi(p),\pi(q))$ can be found in constant time, thus establishing that the diameter of T can be determined in linear time.

The diameter and a diametral pair of a squaregraph Q can be found analogously, even easier. In this case, the interval $I(v,w)$ and the sets X and Y are gated and the gates of all vertices of Q in $I(v,w)$ can be computed in linear time. Again, in order to compute $d(p,q)$ for $p \in P', q \in P''$, it suffices to find $d(\pi(p),\pi(q))$. For this, we simply embed isometrically $I(v,w)$ into the rectilinear grid (this can be done in the same way as for trigraphs) and compute $d(\pi(p),\pi(q))$ by the respective formula. In order to find the diameter of a kinggraph K, simply construct the squaregraph $Q(K)$ and use the equality $d(Q(K)) = 2d(K)$. Summarizing, we have established the following result.

Theorem 3.1. *The diameter of a graph from $\mathcal{T}$, $\mathcal{Q}$, $\mathcal{K}$, $\mathcal{H}$ can be computed in linear time.*

4. Center Problem

4.1. Trigraphs.

The structure of the center. The center of a graph G is the intersection of the balls of radius $r(G)$. If these balls are convex (as in the case of T), the center is convex as well. The following result gives the structure of possible centers of trigraphs.

Lemma 4.1. *The center of a trigraph T is a 3-sun, a convex path, or a convex strip (see Fig.2).*

Proof. First, we show that $C(T)$ does not contain (a) a subgraph in the form of two 3-cycles with one vertex in common and (b) a convex subgraph consisting of a 3-cycle and a pendant edge. To show (a), assume by way of contradiction that $C(T)$ contains the 3-cycles $\tau' = (x,y',y'')$ and $\tau'' = (x,z',z'')$. Pick $v \in F(x)$. By (PT7), v cannot be equidistant to all vertices of τ' and τ'', hence v is at distance $r(T)-1$ to a vertex from each cycle. These two vertices belong to $I(x,v)$, therefore by (PT5) they must be adjacent, which is impossible. Analogously, let $C(T)$ contain a convex subgraph in the form of a 3-cycle (x,y,z) plus an edge xw. Again, taking $v \in F(x)$ one concludes that $d(y,v) = r(T)-1$ or $d(z,v) = r(T)-1$, say first. If $d(w,v) = r(T)-1$, then $y,w \in I(x,v)$, and they must be adjacent, which is impossible. Hence $d(w,v) = r(G)$ and by (TP6) there is a common neighbour u of x and w one step closer to v. Since $u,y \in I(x,v)$, they are adjacent. But then $u \in I(w,y)$, in contradiction with the convexity of the set $\{x,y,z,w\}$. This establishes (b).

These properties impose severe constraints on the shape of $C(T)$. First, from (a) one concludes that $C(T)$ is an outerplanar graph. A straightforward analysis using both (a) and (b) shows that if $C(T)$ contains a 3-sun, then it coincides with this 3-sun. Otherwise, if $C(T)$ contains a 3-cycle but is not a 3-sun, a case analysis shows that $C(T)$ is a strip. □

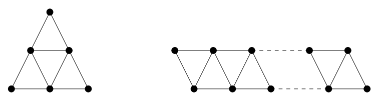

Figure 2

Locating a histogram H_c intersecting the center. Let u, v be a pair of diametral vertices computed in Section 3. If $d(G) = d(u,v)$ is even, say $d(G) = 2k$, then $B(u,k) \cap B(v,k)$ is a convex path P. If $d(G) = 2k-1$, then $B(u,k) \cap B(v,k)$ is a convex strip bounded by two convex paths P and P'. As noticed in 2.2, every such convex path can be extended to a convex cut c of T (if T is a triangular system, this extension is unique, unless the initial path consists of a single vertex). If the respective path consists of one vertex, then take a cut c passing via this vertex and not containing other vertices of $I(u,v)$.

Lemma 4.2. $e(c) \leq r(T)$.

Proof. Obviously $k \leq r(T)$, where k is defined as above, because $d(G) \leq 2r(G)$ for a graph G of even diameter and $d(G) \leq 2r(G) - 1$ for a graph G of odd diameter. From the definition of the cut c one concludes also that $d(u,c) \leq k$, $d(v,c) \leq k$, and $d(u,v) = d(u,c) + d(c,v)$. Now, suppose by way of contradiction that there is a vertex x such that $d(x,c) > r(T)$. Assume, without loss of generality, that c separates the vertices x and u. From (PT3) one concludes that

$$\begin{aligned} d(u,x) &\geq d(u,c) + d(c,x) > d(u,c) + r(T) \\ &\geq d(u,c) + k \geq d(u,c) + d(c,v) = d(u,v), \end{aligned}$$

contrary to the assumption that u, v is a diametral pair. □

Lemma 4.3. *The histogram H_c contains at least one central vertex of T.*

Proof. Suppose by way of contradiction that $H_c \cap C(T) = \emptyset$. Pick a central vertex x closest to c. Consider the projection $[u_x, v_x]$ of x on c and the metric triangle $(u_x x_c v_x)$ as defined above. Pick a neighbour x' of x on a shortest (x, x_c)-path. Since $d(x',c) < d(x,c)$, from the choice of x one concludes that x' is not central. Let $y \in F(x')$. Since $d(x',y) > r(T) = e(x)$, one concludes that $d(x',y) = d(x,y) + 1 = r(T) + 1$. From Lemma 4.2 we have $d(y,c) \leq r(T)$. Pick a vertex z from the projection of y on c. From the definition of x_c we deduce that x_c (and therefore x') lies on a shortest (x,z)-path. Since $x, z \in B(y, r(T))$ and $x' \notin B(y, r(T))$, we get a contradiction with the convexity of balls of T. Hence H_c intersects the center. □

One can present examples of trigraphs in which the histogram H_c does not contain the whole center.

Computing a representative set for H_c. Let H_c be the histogram defined by the cut c. In this section we describe a procedure for computing the projections of all vertices of T on H_c. To simplify the presentation, we assume that c is a horizontal path whose leftmost vertex is denoted by $u_1 := u$. The vertices of ∂H_c are ordered counterclockwise starting from u_1, such that $\partial H_c = (u_1, u_2, ..., u_p)$. If T does not contain inner vertices of degree 7, then H_c is a convex set, whence the projection of each vertex $v \in T \setminus H_c$ is a convex path of ∂H_c which we will denote by $\pi(v) := [u_i, u_j]$. Moreover, as shown in Section 2.5, all distances $d(v, H_c)$ and all projections $\pi(v)$ can be found in total linear time, because the property (PT3) holds.

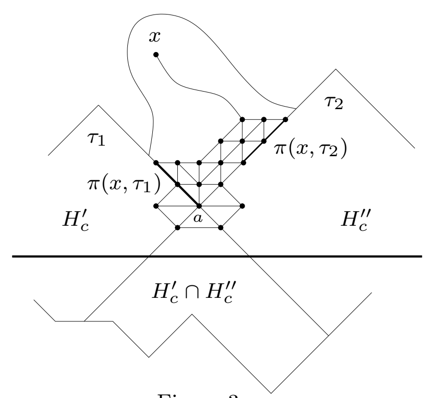

Figure 3

If T contains inner vertices of degree 7, H_c is no longer convex. However, for each vertex x of T, there exist at most two metric triangles τ_1 and τ_2 of H_c, such that for any vertex y of H_c there is a shortest (x,y)-path intersecting $\pi(x,\tau_1) \cup \pi(x,\tau_2)$ (the proof is deferred to the full version). The sets $H'_c(x) = \{y \in H_c : I(y,x) \cap \pi(x,\tau_1) \neq \emptyset\}$ and $H''_c(x) = \{y \in H_c : I(y,x) \cap \pi(x,\tau_2) \neq \emptyset\}$ form two well-defined sub-histograms (see Fig.3). In order to find the vertices of T having their first and second projections one proceeds in the following way. Take each pair of consecutive metric triangles τ_1 and τ_2 of H_c such that their boundaries intersect in an inner vertex a of degree 7 (see Fig.3). The vertex a is adjacent to a vertex $a' \notin \tau_1 \cup \tau_2$, such that a and a' share common neighbours in $\partial\tau_1$ and $\partial\tau_2$. Then all vertices which will have their first and second projections in τ_1 and τ_2 are precisely the vertices of the connected component of the graph $T \setminus H_c$ which contains the vertex a' (again, the proof is given in the full version). Having this component constructed, we simply compute the projections of its vertices x on τ_1 and then on τ_2. We abbreviate both $\pi(x, \tau_1)$ and $\pi(x, \tau_2)$ by $\pi(x)$, call them the first and the second projection of x and store them in two lists (if a vertex has a unique projection, then its projection is included in both lists).

Actually, only the projections of a small number of vertices of T (in fact, of ∂T) are relevant for computing the central vertices in H_c. A subset $R \subseteq T$ is called *representative* for H_c if whenever a vertex $x \in H_c$ is at distance $\leq k$ from all vertices of R then $e(x) \leq k$. We will construct a representative set $R \subseteq \partial T$ (since for each $x \in H_c$ and each $y \notin \partial T$ there exists a vertex $y' \in \partial T$ such that $d(y', x) = d(y', y) + d(y, x)$, whence R consists of boundary vertices only).

To construct R, we sweep simultaneously the lists of first and second projections of boundary vertices (a vertex v_i may appear twice in R). Each time, among two projections in head of lists we treat that which appears first in the counterclockwise traversal of ∂H_c. Its corresponding vertex v_i is inserted in R provided this projection is different from the projection of the vertex v_k last inserted in R or $d(v_i, \pi(v_i)) > d(v_k, \pi(v_k))$ holds. In the second case, additionally we remove v_k from R. For each vertex v_i of R we maintain the following information: the respective projection on H_c and the distance from v_i to this projection. This guarantees that the vertices of R are ordered according to the occurrence of their projections on ∂H_c.

Lemma 4.4. *R is a representative set for H_c such that $|R| \leq 2|\partial H_c| - 1$.*

Proof. Suppose by way of contradiction that there exist vertices $x \in H_c$ and $v \in \partial T$ such that $d(x, v) > k \geq d(x, w)$ for all $w \in R$. According to the algorithm, there exists a vertex $u \in R$ dominating v, that is $\pi(v) = \pi(u)$ and $d(v, \pi(v)) \leq d(u, \pi(u))$. By (PT3) we have $d(v, x) = d(v, \pi(v)) + d(\pi(v), x)$ and $d(u, x) = d(u, \pi(u)) + d(\pi(u), x)$, yielding $d(u, x) \geq d(v, x)$, which contradicts our assumption. Hence R is representative for H_c.

To establish the second assertion, first we show that if $\pi(v_s) \subset \pi(v_t)$ for $v_s, v_t \in R$, then the paths $\pi(v_s) = [u_i, u_j]$ and $\pi(v_t) = [u_k, u_l]$ have a common endpoint. Indeed, let $v'_s u_i u_j$ and $v'_t u_k u_l$ be the respective metric triangles of T. If u_i and u_j are inner vertices of the path $[u_k, u_l]$, then v'_s will be an inner vertex of $v'_t u_k u_l$ and every shortest path between v_s and v'_s will intersect either $[v'_t, u_k]$ or $[v'_t, u_l]$, and one of the vertices u_k or u_l must belong to $\pi(v_s)$, whence $\pi(v_s)$ and $\pi(v_t)$ share a common end. From this one concludes that the sequence $\{i + j : \pi(u) = [u_i, u_j] \text{ and } u \in R\}$ consists of pairwise distinct numbers each between 1 and $2|\partial H_c| - 1$, whence $|R| \leq 2|\partial H_c| - 1$. □

Computing $C(T) \cap H_c$ and $C(T)$. First, we describe how to compute $H_c \cap C(T)$. For this, we use an embedding of H_c into the triangular grid such that c embeds as a horizontal path and endow H_c with a coordinate system. Divide H_c into convex (in the usual sense) quadrangles $Q_1, \ldots, Q_r$ by using the cuts of H_c which pass via convex and concave vertices of H_c (eventually, some quadrangles may degenerate into triangles). Denote by a'_i, b'_i, a''_i, b''_i the sides of Q_i starting from the left top side (see Fig.4).

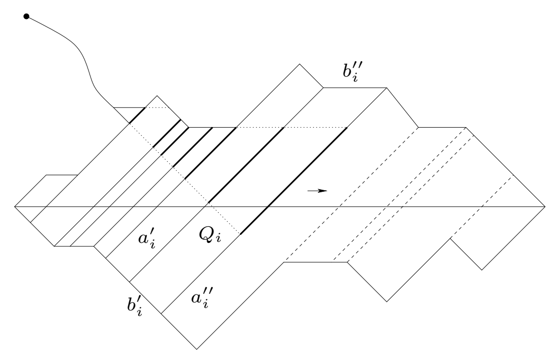

Figure 4

At the next step, for each Q_i we compute a representative set R_i. This is done in two stages. First compute the subset R'_i of vertices of R_i whose projections on Q_i belong to $a'_i \cup b'_i$, and then the subset R''_i of vertices of R_i whose projections on Q_i belong to $a''_i \cup b''_i$. For each $i = 1, ..., r$, initially R'_i is the subset of vertices of R whose projection on H_c belong to $a'_i \cup b'_i$. If $i \geq 2$, then add to this list the vertex at maximum distance from Q_i such that its projection on H_c lies on $a''_{i-1} - a'_i$ (hence its projection on Q_i is the corner vertex formed by a'_i and b'_i). Analogously, add to R'_i the vertex at maximum distance from Q_i such that its projection on H_c lies on b''_{i-1} (hence its projection on Q_i is the corner vertex formed by a'_i and b''_i). Notice that in both cases the projections in question appear consecutively on R, which allows to find all of them in one sweep of R. Finally, add to R'_i a subset of R'_{i-1} selected in the following way. Notice that H_c is embedded in the triangular grid. Using its intrinsic coordinate system, it is easy to deduce the projection of a vertex on $a'_i \cup b'_i$ from its projection on $a'_{i-1} \cup b'_{i-1}$ (and its distance to Q_i). Now, as we did for R, we simply sweep the list R'_{i-1} and remove all its vertices whose projection on $a'_i \cup b'_i$ coincides with a projection of another vertex having a larger distance to Q_i. The list R''_i is computed analogously. Finally, set $R_i := R'_i \cup R''_i$.

Lemma 4.5. *R_i is a representative set for Q_i such that $|R_i| \leq 2|\partial Q_i| - 1$.*

The proof is similar to the proof of Lemma 4.4 and is deferred to the full version. The complexity of this algorithm is proportional to the sum of lengths of considered lists. As we noticed above, the list R is scanned twice. During the first phase, each list R_i' is scanned once and as we did for the size of R, one can show that $|R_i'| \leq 2|a_i' \cup b_i'|$. Analogously, $|R_i''| \leq 2|a_i'' \cup b_i''|$. Hence the total number of operations to compute the sets R_i is linear in the size of H_c.

It remains to find the vertices of least eccentricity inside each Q_i. For each vertex $w \in R_i$, the intersection of Q_i with the kth neighbourhood of the (convex) path $\pi(w, Q_i)$ is defined by three inequalities (written in our system of coordinates). Each of these inequalities is of one of the following types: $x \leq r_i - d_1$, $x \geq d_2 - r_i$, $x-y \leq r_i - d_3$, $x-y \geq d_4 - r_i$, $y \leq r_i - d_5$, $y \geq d_6 - r_i$, where $d_1, ..., d_6$ are constant values, which can be easily computed from the projection of w and where r_i is the minimum eccentricity of a vertex of Q_i. While sweeping the list R_i we add a new inequality to the system if it is not dominated by an already existing inequality (this test can be performed in constant time). At the end we will obtain a system with at most 6 inequalities from which one can deduce in constant time what is the smallest r_i such that the resulting system is feasible.

Thus we have a linear time algorithm for computing the set $H_c \cap C(T)$ (which is non-empty by Lemma 4.3). We leave the completion of $C(T)$ for all trigraphs to the full version, and here outline how to compute the center of a triangular system T. For this, pick a vertex $w \in H_c \cap C(T)$. First find all central vertices from $B_2(w)$ by simply computing their eccentricities ($B_2(w)$ contains a constant number of vertices). This allows to decide if $C(T)$ is a 3-sun. If not, then take the three convex cuts c_1, c_2, c_3 passing via w. For each cut c_i $(i = 1, 2, 3)$ construct the histogram H_{c_i} and compute $H_{c_i} \cap C(T)$. From Lemma 4.1 we infer that in this way we can generate the whole center of T.

4.2. Hexagonal systems. Viewing a hexagonal system H as a bipartite graph $(V_1 \dot\cup V_2, E)$, one can define two triangular systems $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$: two vertices $u, v \in V_i$ are adjacent in T_i iff $d_H(u, v) = 2$ (see Fig.5). Obviously, $d_H(u, v) = 2d_{T_i}(u, v)$ holds for any two vertices $u, v \in V_i$. Let r_i and C_i be the radius and the center of T_i $(i = 1, 2)$.

If $r_1 \neq r_2$ we assert that $r(H) = 2\min\{r_1, r_2\} + 1$ and $C(H) = C_1$, if $r_1 < r_2$, and $C(H) = C_2$, otherwise. Indeed, let say $r_1 < r_2$. The eccentricity in H of a vertex of V_2 is at least $2r_2 \geq 2r_1 + 2$. Now, pick $v_1 \in V_1$. This vertex is adjacent to a vertex v_2 of V_2. Since $e(v_2) \geq 2r_1+2$, one deduces that $e(v_1) \geq 2r_1+1$. On the other hand, the eccentricity in H of each vertex of C_1 is $2r_1 + 1$, yielding that $C(H) = C_1$.

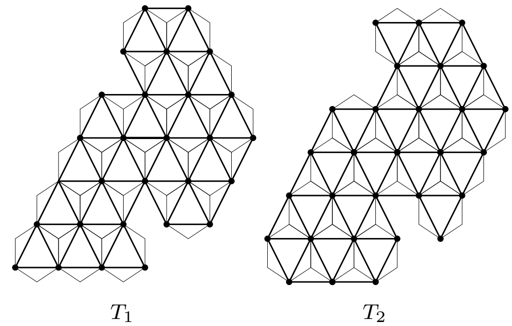

Figure 5

Now suppose that $r_1 = r_2$. Denote this number by r and set $C := C_1 \cup C_2$. Since the eccentricity of each vertex outside C is at least $2r + 2$, while the eccentricity of each vertex of C is at most $2r + 1$, $C(H)$ is located in C. Thus one has to compute the eccentricities of vertices of C and to select those with minimum eccentricity. If C_i is a 3-sun or a subgraph of a 3-sun, then the eccentricities of all its vertices can be computed in total linear time. Otherwise, if C_i is a strip with ≥ 6 vertices or a convex path with ≥ 4 vertices, then we compute the eccentricities of its end-vertices. In order to compute the eccentricities of the remaining vertices of C, we distinguish several possibilities. First, notice that if a vertex $c \in C$ has in H a neighbour c' outside C, then $e(c) = 2r + 1$: indeed, then $|e(c) - e(c')| \leq 1$ and $e(c') \geq 2r+2$, while $2r \leq e(c) \leq 2r + 1$. Thus, if every $c \in C$ has a neighbour outside C, then $r(H) = 2r + 1$ and $C(H) = C$. It remains to consider the vertices $c \in C$ whose all neighbours in H belong to C (denote this subset of C by C_0).

Now, if there exists a vertex $c \in C_0$ with $|N(c)| = 3$ and $N(c) \subset C$, then we assert that $r(H) = 2r$ and $c \in C(H)$. Indeed, let $c \in T_1$ and denote its neighbours by v_1, v_2, v_3. Pick $u \in V_2$. Since $v_1, v_2, v_3 \in C_2$, the distance in T_2 from u to each of these vertices is at most r. From property (PT7) of trigraphs at least one of these distances is $\leq r - 1$, whence $d(c, u) \leq 2(r-1) + 1 \leq 2r$. Since the distance from c to every vertex of V_1 is at most $2r$, we conclude that $e(c) = 2r$ and $c \in C(H)$. We claim that in this case, all vertices $c' \in C_0$ obeying $|N(c')| = 2$ and $N(c') \subseteq C$ have eccentricity $2r + 1$, hence they are not central (denote the set of such c' by C_0'). Since C_2 contains the 3-cycle $\{v_1, v_2, v_3\}$ of T_2, our constraints impose that C_2 is a strip and $|C_2| \geq 6$.

First assume that C_1 is a convex path of T_1. Then one can easily see that $C_0' \subseteq C_2$. Pick $c' \in C_0'$ and set $N(c') = \{u_1, u_2\} \subseteq C_1$. Denote by u_3 the third vertex of T_1 from the hexagon containing u_1 and u_2. Since $u_3 \notin C_1$, the eccentricity in T_1 of the pair $\{u_1, u_2\}$ cannot be $r - 1$, whence the eccentricity of c' in H is at least $2r+1$. Now assume that C_1 is also a strip with at least 6 vertices. Then C_2 will also contain vertices of degree 3 with all neighbours in C_1, whence C_1 and C_2 can be treated in the same way. So, pick a vertex $c' \in C'_0 \cap C_2$ and set $N(c') = \{u_1, u_2\} \subseteq C_1$. Let $\tau_1 = \{u_1, u_2, u_3\}$ be a face of C_1 containing the edge u_1u_2 and $\tau_2 = \{u_1, t_1, t_2\}$ the face of C_1 such that $\tau_1 \cap \tau_2 = \{u_1\}$. Suppose by way of contradiction that the eccentricity of c' in H is $2r$. This implies that the eccentricity in T_1 of its neighbourhood $\{u_1, u_2\}$ must be equal to $r-1$. Take w of T_1 at maximum distance from u_1. Then we must have $u_2 \in I(u_1, w)$ in T_1. By (PT5), the interval $I(u_1, w)$ in T_1 cannot contain simultaneously u_2 and a vertex t_1 or t_2. Since $t_1, t_2 \in C_1$, we conclude that in T_1 all three vertices u_1, t_1, t_2 are located at distance r from w, contradicting (PT7). This establishes that C'_0 does not contain central vertices of H.

Finally assume that neither C_1 nor C_2 contain vertices of degree 3 with all three neighbours in C, i.e., that $C'_0 = C_0$ holds. Then one can easily see that both C_1 and C_2 are paths. Moreover, using similar arguments as above one can show that $e(u) = 2r+1$ for all $u \in C_1 \cup C_2$, whence $r(H) = 2r+1$ and $C(H) = C$. This establishes that in all cases $C(H)$ can be derived in linear time from the centers of T_1 and T_2.

4.3. Squaregraphs. First we establish a relationship between radii and diameters of squaregraphs Q.

Lemma 4.5. $d(Q) \geq 2r(Q) - 2$.

Proof. The proof is a consequence of the following property: *given a vertex v and a subset of vertices S of Q, if $I(v,x) \cap I(v,y) \neq \{v\}$ for each pair $x, y \in S$, then $\cap_{x \in S} I(v,x) \neq \{v\}$.* Indeed, pick a central vertex v of Q and as S take the set of all vertices of Q having distance $r(Q)$ or $r(Q)-1$ to v. If $\cap_{x \in S} I(v,x) \neq \{v\}$, then pick a neighbour v_0 of v from this intersection. Since $d(v_0, x) \leq r(Q) - 1$ for every $x \in S$ and $d(v_0, y) \leq d(v,y) + 1 \leq r(Q) - 2 + 1 = r(Q) - 1$ for any other vertex y, we conclude that $e(v_0) \leq r(Q) - 1$, a contradiction. Thus $\cap_{x \in S} I(v,x) = \{v\}$ and by our claim there exist two vertices $x, y \in S$ such that $I(v,x) \cap I(v,y) = \{v\}$. Since the intervals of Q are gated, this implies that v is the gate of y in $I(v,x)$, whence $d(x,y) = d(x,v) + d(v,y) \geq r(Q) - 1 + r(Q) - 1 = 2r(Q) - 2$, yielding $d(Q) \geq 2r(Q) - 2$.

It remains to establish the claim. For this proceed by induction on the size of S. By the induction hypothesis, for each $y \in S$ one can find a vertex $z_y \in N(v) \cap (\cap_{x \in S \setminus \{y\}} I(v,x))$. Assume that all such vertices z_y are pairwise distinct. Pick $a, b, c \in S$ and consider the vertices z_a, z_b, z_c. Denote by g_a the gate of a in $I(z_b, z_c)$, by g_b the gate of b in $I(z_a, z_c)$, and by g_c the gate of c in $I(z_a, z_b)$. Since $z_b, z_c \in I(v,a)$, g_a is different from v. Hence the vertices g_a, g_b, g_c are pairwise distinct and different from v. But then $(z_a, g_c, z_b, g_a, z_c, g_b)$ is a cycle of Q separating v from the rest of the graph. This implies that v is an inner vertex of degree 3, a contradiction. □

Let u, v be a pair of diametral vertices of Q. If $d(Q) = d(u,v)$ is odd, say $d(Q) = 2k-1$, then Lemma 4.5 implies that $r(Q) = k$. In this case, the whole center is located in $I(u,v)$, namely in two levels L_{k-1} and L_k. Indeed, every other vertex w of $I(u,v)$ has distance $\geq k+1$ to one of the end-vertices of $I(u,v)$. If $w \notin I(u,v)$ and $\pi(w)$ is its gate in $I(u,v)$, and, say, $d(\pi(w), u) \geq k$, then $d(w,u) = d(w, \pi(w)) + d(\pi(w), u) \geq k+1$. Thus $C(Q) \subseteq L_{k-1} \cup L_k$.

On the other hand, if $d(Q) = 2k$, then, by Lemma 4.5, $r(Q)$ may take two values k and $k+1$. If $r(Q) = k$, then, as in previous case, one concludes that $C(Q) \subseteq L_k$. Finally, if $r(Q) = k+1$, analogously one can prove that $C(Q) \subseteq L_{k-1} \cup L_k \cup L_{k+1} \cup N(a) \cup N(b)$, where $\{a, b\} = L_k \cap \partial I(u,v)$. Since we do not know the exact radius, first we should test if $r(Q)$ is k. For this, we must find $C := L_k \cap (\cap_{w \in V} B_k(w))$. If C is empty, then $r(Q) = k+1$, otherwise $r(Q) = k$ and $C(Q) = C$.

Hence, in order to find the center of Q, for each of the levels L in question we have to accomplish the same task: find the vertices of L which belong to all balls of Q of given radius m ($m = k$ in the first two cases and $m = k+1$ in the third case). In the third case one has to answer a similar question for $N(a)$ and $N(b)$.

To answer the first question, we embed isometrically the interval $I(u,v)$ into the rectilinear grid such that the image of u is the point $(0,0)$ (this can be done in linear time). Assume, without loss of generality, that L is the kth level of $I(u,v)$, i.e., all vertices of L lie on the line $x+y=k$. More precisely, L consists of all grid points on this line comprised between the points $((x(a), y(a))$ and $((x(b), y(b))$. One can easily see that the interval $I(a,b)$ consists of all grid-points comprised in the isothetic rectangle spanned by the points $(x(a), y(a))$ and $(x(b), y(b))$ (in fact, this rectangle is a square). Using the algorithm from Subsection 2.5, compute the projections $\pi(w)$ and distances of all vertices $w \in V$ to $I(a,b)$. Notice that

$$L \cap B_m(w) = L \cap B_{m'}(\pi(w)), \qquad (*)$$

where $m' := m - d(w, \pi(w))$. Instead of finding $L \cap B_{m'}(\pi(w))$ we compute the intersection I_w of the segment $[a,b]$ with the rectilinear ball of radius m' centered at $\pi(w)$ (for a given w, this can be done in constant time). Now, let $I := \cap_{w \in V} I_u$ (this intersection can be found in $O(|V|)$ time). In view of $(*)$, $L \cap (\cap_{w \in V} B_m(w))$ consists of all grid-points of the segment I.

To answer a similar question for $N(a)$, notice that the ball $B_2(a)$ is convex, therefore gated. Hence, we compute the gates and the distances of vertices of Q to $B_2(a)$ and use $(*)$ with L replaced with $N(a)$. If $d(\pi(w),a)=2$, then $N(a)\cap B_{m'}(\pi(w))=N(a)$ if $m'\geq 3$ and consists of two vertices if $m'=1$ or 2. Analogously, if $\pi(w)\in N(a)$, then $N(a)\cap B_{m'}(\pi(w))=N(a)$ if $m'\geq 2$, otherwise this intersection coincides with $\pi(w)$. Obviously, all this justifies a simple linear algorithm for computing $N(a)\cap(\cap_{w\in V}B_m(w)) = N(a)\cap(\cap_{w\in V}B_{m'}(\pi(w)))$, leading to a linear algorithm for computing $C(Q)$.

4.4. Kinggraphs. Let K be a kinggraph.

Lemma 4.6. $d(K)\geq 2r(K)-1$.

Proof. Suppose by way of contradiction that for each pair of vertices u,v of K we have $d(u,v)\leq 2r(K)-2$. Then the balls of radius $r(K)-1$ centered at u and v intersect. By the Helly property (PK1) all balls of radius $r(K)-1$ have a vertex in common. Obviously, the eccentricity of this vertex is at most $\leq r(K)-1$, a contradiction. □

Let u,v be a pair of diametral vertices of K. Notice that u,v is also a diametral pair of the squaregraph $Q(K)$. Since the interval $I(u,v)$ in $Q(K)$ can be embedded isometrically into the square grid, the interval with the same end-vertices of K can be embedded isometrically into the King grid $\mathbb{Z}_8$. Hence the level sets L_i of $I(u,v)$ in K are isometric paths of K (denote their end-vertices by a_i and b_i). Notice additionally that L_i are also level sets of the interval $I(u,v)$ in $Q(K)$, namely its $2i$th level.

If the diameter of K is even, say $d(K)=2k$, then Lemma 4.6 implies that $r(K)=k$. Then one can easily see that $C(K)$ is an isometric subpath of the kth level of $I(u,v)$. If $d(K)=2k-1$, then $r(K)=k$ by Lemma 4.6. We assert that $C(K)\subseteq(L_{k-1}\cup L_k)\cup(N(a_{k-1})\cap N(a_k))\cup(N(b_{k-1})\cap N(b_k))$. Indeed, pick a vertex $c\in C(K)$. If $c\in I(u,v)$, then necessarily $c\in L_{k-1}\cup L_k$. Otherwise, if $c\notin I(u,v)$, from property (PK3) and since $e(c)\leq k$ we conclude that c is adjacent either to a_{k-1} and a_k or to b_{k-1} and b_k, establishing our assertion. Notice that $(N(a_{k-1})\cap N(a_k))\setminus I(u,v)$ and $(N(b_{k-1})\cap N(b_k))\setminus I(u,v)$ consist of a vertex or two adjacent vertices. The eccentricity of each of these vertices can be computed directly. In order to select the central vertices from L_{k-1} and L_k, construct the square graph $Q(K)$ and find the intersection of all balls of radius $2k$ with those levels. This can be done in linear time as shown in Subsection 4.3.

To precise the structure of $C(K)$ in the second case, notice that if a K_3 belongs to $C(K)$, then the unique K_4 containing it also belongs to $C(K)$. This is a direct consequence of the property (PK2). Two vertices $x\in L_{k-1}, y\in L_k$ at distance 2 have precisely two common neighbours z_1,z_2. We assert that, if $x,y\in C(K)$, then $z_1,z_2\in C(K)$ as well. Since $C(K)$ is an isometric subgraph of K, at least one of two common neighbours, say $z_1\in L_{k-1}$, belongs to $C(K)$. Suppose $z_2\notin C(K)$ and pick u at distance $r(K)+1$ from z_2. Since z_2 is adjacent to x,y,z_1 all these vertices have distance $r(K)$ to u. Let z_3 be the fourth vertex of the 4-clique containing y,z_1, and z_2. By (PT6), y and z_1 have a common neighbour at distance $r(K)-1$ to u. It is easy to see that this can be only the vertex z_3. Since z_2 and z_3 are adjacent, $d(z_2,u)\leq r(K)$ and a contradiction with our choice of u arises. Thus $z_2\in C(K)$. Analogously, one can show that if $x,y\in C(K)\cap L_k$ and $d(x,y)=2$, then the common neighbour of x and y from L_k belongs to the center. Thus, $L_{k-1}\cap C(K)$ and $L_k\cap C(K)$ are isometric paths of K. Altogether this shows that $C(K)$ is a chain of 4-cliques.

Lemma 4.7. *$C(K)$ is an isometric path if $d(K)=2r(K)$ and an isometric chain of K_4s if $d(K)=2r(K)-1$. Moreover, any isometric path and any chain of K_4s can be realized as a center of some kinggraph.*

Summarizing, here is the main result of this note.

Theorem 4.1. *The center of a graph from $\mathcal{T},\mathcal{H},\mathcal{Q},\mathcal{K}$ can be computed in linear time.*

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