Adjacent Vertices Fault Tolerance Hamiltonian Laceability of Star Graphs *

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Abstract

Let S_n be an n-dimensional Star graph. In this paper, we show that $S_n - F$ is Hamiltonian laceable where F is the set of $f \leq (n-4)$ pairs of adjacent faulty vertices, $S_n - F$ is Hamiltonian where F is the set of $f \leq (n-3)$ pairs of adjacent faulty vertices. We also show that $S_n - F$ is hyper-Hamiltonian laceable where F is the set of $f \leq (n-4)$ pairs of adjacent faulty vertices. Applying these results, we also construct the fault-free cycle with length $n! - 2f' + 2$ in $S_n - F'$ where F' is the faulty vertices set with at least a black vertex and a white vertex for $|F'| = f' \leq n-3$.

1 Introduction and Definitions

The star graph [1] has been recognized as an attractive alternative to hypercube. It possesses many favorable topological properties such as recursiveness, symmetry, maximal fault tolerance, sub-logarithmic degree and diameter.

The vertices of the n-dimensional star graph S_n are all the permutations of $\{1, 2, \cdots, n\}$. A vertex $d_1 d_2 \cdots d_{i-1} d_i d_{i+1} \cdots d_n$ is adjacent to $d_i d_2 \cdots d_{i-1} d_1 d_{i+1} \cdots d_n$ for $2 \leq i \leq n$. The graph S_4 is illustrated in Figure 1. The degree and diameter of S_n are $n-1$ and $\lfloor 3(n-1)/2 \rfloor$ [6], respectively. The star graph S_n is strongly hierarchical and symmetric, that means S_n can be decomposed into n components which are isomorphic to S_{n-1} by fixing the kth number of vertex for $2 \leq k \leq n$.

A *Hamiltonian cycle* (*path*) is a cycle(path) that visits every vertex exactly once. A bipartite graph $G = (V_b \cup V_w, E)$ is *Hamiltonian* if G contains a Hamiltonian cycle. A bipartite graph

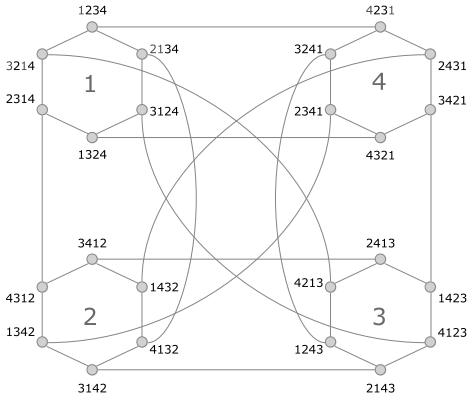

Figure 1: Illustration of S_4.

$G = (V_b \cup V_w, E)$ is *Hamiltonian laceable* if there exists a Hamiltonian path between every pair of vertices s and t for $s \in V_b$ and $t \in V_w$. A bipartite graph $G = (V_b \cup V_w, E)$ is *hyper-Hamiltonian laceable* if for every $\nu \in V_i$, $i = b$ or w, there exists a Hamiltonian path of $G - \nu$ between every pair of vertices of V_j for $V_i \neq V_j$. In [4], Li et al. showed that S_n is $n-3$ edges Hamiltonian laceable and $n-4$ edges hyper-Hamiltonian laceable.

There is no literature about vertex fault tolerance Hamiltonian properties of star graph for the number of faulty vertices more than one. Some authors concerns embedding fault-free cycles and paths for star graphs with faulty vertices. The upper bound of longest fault-free cycle of $S_n - F$ is $n! - 2f$ where $f = \max\{|F \cap V_w|, |F \cap V_b|\}$. In [2, 7], the authors showed that a fault-free cycle of length at least $n! - 2f_v$ can be constructed with f_v faulty vertices. This length is optimal when all the faulty vertices have the same color. However, there exist longer cycles when there are faulty vertices with different color.

In [3], Hung et al. studied the adjacent vertices fault tolerance Hamiltonian laceability

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of hypercube. Let F be the set of k pairs of adjacent vertices. We can denote F as $\{a_1, b_1, a_2, b_2, \cdots, a_k, b_k | a_i \in V_w, b_i \in V_b$ and $(a_i, b_i) \in E$ for $1 \leq i \leq k\}$. A bipartite graph $G = (V_b \cup V_w, E)$ is *f-adjacency Hamiltonian* if $G - F$ is Hamiltonian $\forall f$ pairs of adjacent vertices set F of G. A bipartite graph $G = (V_b \cup V_w, E)$ is *f-adjacency Hamiltonian laceable* if $G - F$ is Hamiltonian laceable $\forall f$ pairs of adjacent vertices set F of G. A bipartite graph $G = (V_b \cup V_w, E)$ is *f-adjacency hyper-Hamiltonian laceable* if $G - F$ is hyper-Hamiltonian laceable $\forall f$ pairs of adjacent vertices set F of G. In this paper, we will show that S_n is f-adjacency Hamiltonian for $f \leq n-3$, f-adjacency Hamiltonian laceable for $f \leq n-4$, and f-adjacency hyper-Hamiltonian laceable for $f \leq n-4$. Applying these results, there exists a fault-free cycle of length $n! - 2|F_v| + 2$ in $S_n - F_v$ where F_v is the set of faulty vertices with at least one black vertex and one white vertex.

The rest of this paper is organized as follows. In Section 2, we will prove the adjacency fault tolerance for Hamiltonian laceability of star graphs. The longest fault-free cycle will be constructed in Section 3. Finally, we give some conclusions in Section 4.

2 The Adjacency Hamiltonian Laceablility of Star Graphs

Let S^i_{n-1} be the subgraph of S_n induced by the vertices whose last bit is i. Let F denote the set of adjacently faulty vertices of S_n, $F^i = F \cap V(S^i_{n-1})$, f^i denote the number of pair of F^i. Let N_n be the set of $\{1, 2, \cdots, n\}$, and M_n be any nonempty subset of N_n. Let $S^{M_n}_{n-1}$ be the subgraphs of S_n induced by $\cup_{i \in M_n} V(S^i_{n-1})$. And let $k(u)$ be the kth bit of a vertex u, and $u' \in S^i_{n-1}$ be adjacent to $u \in S^j_{n-1}$ for $i \neq j$.

In [5], Lin et al. proved the following two lemmas.

Lemma 1 *There are $(n-2)!$ edges between S^i_{n-1} and S^j_{n-1} where $i \neq j, 1 \leq i \leq n$ and $1 \leq j \leq n$*

Lemma 2 *If x and $y \in S^i_{n-1}$ which $1 \leq i \leq n$, and $d(x, y) \leq 2$, then $n(x) \neq n(y)$.*

We will prove the following lemmas.

Lemma 3 *The S_4 is 1-adjacency Hamiltonian.*

Proof: Since S_n is edge symmetric, we only need to consider vertices 1234 and 3214 are faulty. The Hamiltonian cycle of $S_4 - \{1234, 3214\}$ is $\langle 2134$, 4132, 1432, 3412, 4312, 2314, 1324, 4321, 3421, 2431, 4231, 3241, 2341, 1342, 3142, 2143, 1243, 4213, 2413, 1423, 4123, 3124, $2134\rangle$. Thus, S_4 is 1-adjacency Hamiltonian.

Lemma 4 *For $n \geq 5$, given M_n, suppose $|F^i| \leq (2n-10)$, and $S^i_{n-1} - F^i$ is Hamiltonian laceable for $i \in M_n$. For arbitrary $s \in (V_w \cap V(S^{i_1}_{n-1}))$ and $t \in (V_b \cap V(S^{i_2}_{n-1}))$, $i_1, i_2 \in M_n$ and $i_1 \neq i_2$, there exists a Hamiltonian path between s and t in $S^{M_n}_{n-1} - (\cup_{i \in M_n} F^i)$.*

Proof: Let $M_n = \{i_1, i_2, \cdots, i_k\}$, for $2 \leq k \leq n$ and $1 \leq i_j \leq n$ for all $1 \leq j \leq k$. Without loss of generality, we can assume that $s = s_{i_1} \in (V_w \cap V(S^{i_1}_{n-1}))$ and $t = t_{i_k} \in (V_b \cap V(S^{i_k}_{n-1}))$. Since $\frac{(n-2)!}{2} \geq 2n-9$, there exist fault-free edges $(t_{i_j}, s_{i_{j+1}})$ of $S^{M_n}_{n-1}$ where $t_{i_j} \in (V_b \cap V(S^{i_j}_{n-1}))$ and $s_{i_{j+1}} \in (V_w \cap V(S^{i_{j+1}}_{n-1}))$ for $1 \leq j \leq k-1$. Since $S^i_{n-1} - F^i$ is Hamiltonian laceable, there exist a Hamiltonian path $P(s_{i_j}, t_{i_j})$ between s_{i_j} and t_{i_j} of $S^{i_j}_{n-1}$ for $1 \leq j \leq k$. Thus, $\langle s = s_{i_1} \overset{P(s_{i_1}, t_{i_1})}{\longrightarrow} t_{i_1}, s_{i_2} \overset{P(s_{i_2}, t_{i_2})}{\longrightarrow} t_{i_2}, \cdots, s_{i_k} \overset{P(s_{i_k}, t_{i_k})}{\longrightarrow} t_{i_k} = t\rangle$ is a Hamiltonian path of $S^{M_n}_{n-1}$.

Lemma 5 *For $n \geq 5$, if S_{n-1} is f-adjacency Hamiltonian laceable for $0 \leq f \leq n-5$ and f-adjacency Hamiltonian for $0 \leq f \leq n-4$, then S_n is f-adjacency Hamiltonian for $0 \leq f \leq n-3$.*

Proof: Without loss of generality, we can assume that $f^i \geq f^{i+1}$ for $1 \leq i \leq n-1$, and every pair of adjacently faulty vertices $\{a_i, b_i\}$ is in some substar S^i_{n-1}. We will construct the Hamiltonian cycle of $S_n - F$ with the following cases.

Case 1: $f^1 \leq n-4$.

Since $f^1 \geq f^2 \geq \cdots \geq f^n$, $f^1 + f^2 + \cdots + f^n = n-3$ and $n \geq 5$, $f^1, f^2 \leq n-4$ and $f^3, f^4, \cdots, f^n \leq n-5$. Thus, $S^1_{n-1} - F^1$ and $S^2_{n-1} - F^2$ are Hamiltonian and S^i_{n-1} is Hamiltonian laceable for $3 \leq i \leq n$. Let (u, u') be a fault-free edge for $u \in (V_b \cap V(S^1_{n-1}))$ and $u' \in (V_w \cap V(S^2_{n-1}))$. We can denote the Hamiltonian cycle of S^1_{n-1} and S^2_{n-1} as $\langle u \overset{P(u,x)}{\longrightarrow} x, u\rangle$ and $\langle u', y_1 \overset{P(y_1,y_2)}{\longrightarrow} y_2, u'\rangle$, respectively. Without loss of generality, we can assume that x' is in S^3_{n-1}. Applying **Lemma 2**, we can obtain that $n(y_1) \neq n(y_2)$. Thus, one of y'_1 and y'_2 is not in $S^{\{1,3\}}_{n-1}$. We can assume that $y'_1 \in S^{\{4,5,\cdots,n\}}_{n-1}$. Applying **Lemma 4**, we can construct a Hamiltonian path $P(x', y'_1)$ between x and y'_1 of

$S_{n-1}^{\{3,4,\cdots,n\}}$. Therefore, $\langle u \overset{P(u,x)}{\longrightarrow} x, x' \overset{P(x',y_1')}{\longrightarrow} y_1', y_1 \overset{P(y_1,y_2)}{\longrightarrow} y_2, u', u\rangle$ is a Hamiltonian cycle of $S_n - F$, as illustrated in Figure 2(a).

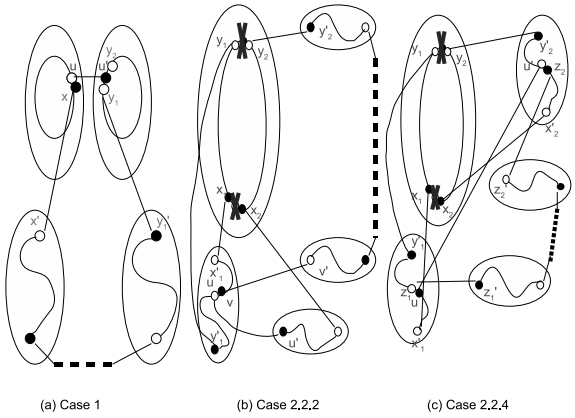

Figure 2: Illustration of **Lemma 5** .

Case 2: $f^1 = n - 3$.

Since S_{n-1} is $(n-4)$-adjacency Hamiltonian, there exists a Hamiltonian cycle C of $S_{n-1}^1 - (F - \{a_1, b_1\})$ for $a_1 \in V_w, b_1 \in V_b$.

Case 2.1: $(a_1, b_1) \in C$.

We can denote the C as $\langle y, a_1, b_1, x, \overset{P(x,y)}{\longrightarrow} y\rangle$.

Suppose that x' and y' are in different substar. Applying **Lemma 4**, we can construct a Hamiltonian path $P(y', x')$ of $S_{n-1}^{\{2,3,\cdots,n\}}$. Thus, $\langle x \overset{P(x,y)}{\longrightarrow} y, y' \overset{P(y',x')}{\longrightarrow} x', x\rangle$ is a Hamiltonian cycle of $S_n - F$.

Suppose that x' and y' are in the same substar. Without loss of generality, we can assume that $x', y' \in S_{n-1}^2$. Let $\langle y' \overset{P(y',v)}{\longrightarrow} v, u, x'\rangle$ be the Hamiltonian path of S_{n-1}^2. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(v', u')$ of $S_{n-1}^{\{3,4,\cdots,n\}}$. Thus, $\langle x \overset{P(x,y)}{\longrightarrow} y, y' \overset{P(y',v)}{\longrightarrow} v, v' \overset{P(v',u')}{\longrightarrow} u', u, x', x\rangle$ is a Hamiltonian cycle of $S_n - F$.

Case 2.2: $(a_1, b_1) \notin C$.

We can denote the Hamiltonian cycle C as $\langle y_1, b_1, y_2 \overset{P(y_2,x_2)}{\longrightarrow} x_2, a_1, x_1 \overset{P(x_1,y_1)}{\longrightarrow} y_1\rangle$.

Case 2.2.1: $n(x_1), n(x_2), n(y_1), n(y_2)$ are mutually different.

Without loss of generality, we can assume that $n(x_1) = 2$ and $n(y_2) = 3$. Let (u, u') be a fault-free edge where $u \in (S_{n-1}^2 \cap V_b)$ and $u' \in (S_{n-1}^3 \cap V_w)$. Let $P(u, x_1')$ and $P(y_2', u')$ be Hamiltonian paths of S_{n-1}^2 and S_{n-1}^3, respectively. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(y_1', x_2')$ of $S_{n-1}^{\{4,5,\cdots,n\}}$. Thus, $\langle x_1 \overset{P(x_1,y_1)}{\longrightarrow} y_1, y_1' \overset{P(y_1',x_2')}{\longrightarrow} x_2', x_2 \overset{P(x_2,y_2)}{\longrightarrow} y_2, y_2' \overset{P(y_2',u')}{\longrightarrow} u', u \overset{P(u,x_1')x_1'}{\longrightarrow} x_1', x_1\rangle$ is a Hamiltonian cycle of $S_n - F$.

Case 2.2.2: $(n(x_1) = n(y_1)$ and $n(x_2) \neq n(y_2))$ or $(n(x_2) = n(y_2)$ and $n(x_1) \neq n(y_1))$.

Without loss of generality, we can assume that $n(x_1) = n(y_1) = 2$ and $n(x_2) = 3 \neq n(y_2)$. Since S_{n-1}^2 is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle y_1' \overset{P(y_1',v)}{\longrightarrow} v, u \overset{P(u,x_1')}{\longrightarrow} x_1'\rangle$ of S_{n-1}^2 where $v \in V_b$ and $n(v) = 3$ and $n(u) \neq 1$, $n(u) \neq n(y_2)$. Let $P(x_2', v')$ be the Hamiltonian path of $S_{n-1}^3 - F^3$. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(y_2', u')$ of $S_{n-1}^{\{4,5,\cdots,n\}}$. Thus, $\langle x_1 \overset{P(x_1,y_1)}{\longrightarrow} y_1, y_1' \overset{P(y_1',v)}{\longrightarrow} v, v' \overset{P(v',x_2')}{\longrightarrow} x_2', x_2 \overset{P(x_2,y_2)}{\longrightarrow} y_2, y_2' \overset{P(y_2',u')}{\longrightarrow} u', u \overset{P(u,x_1')}{\longrightarrow} x_1', x_1\rangle$ is a Hamiltonian cycle of $S_n - F$, as illustrated in Figure 2(b).

Case 2.2.3: $(n(x_1) = n(y_2)$ and $n(x_2) \neq n(y_1))$ or $(n(x_2) = n(y_1)$ and $n(x_1) \neq n(y_2))$.

Without loss of generality, we can assume that $n(x_1) = n(y_2) = 2$. Let $P(x_1', y_2')$ be the Hamiltonian path of S_{n-1}^2. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(x_2', y_1')$ of $S_{n-1}^{\{3,4,\cdots,n\}}$. Thus, $\langle x_1, x_1' \overset{P(x_1',y_2')}{\longrightarrow} y_2', y_2 \overset{P(y_2,x_2)}{\longrightarrow} x_2, x_2' \overset{P(x_2',y_1')}{\longrightarrow} y_1', y_1 \overset{P(y_1,x_1)}{\longrightarrow} x_1\rangle$ is a Hamiltonian cycle of $S_n - F$.

Case 2.2.4: $n(x_1) = n(y_1)$ and $n(x_2) = n(y_2)$.

Without loss of generality, we can assume that $n(x_1) = n(y_1) = 2$ and $n(x_2) = n(y_2) = 3$. Let (u, u') be a fault-free edge where $u \in (S_{n-1}^2 \cap V_b)$ and $u' \in (S_{n-1}^3 \cap V_w)$. Since S_{n-1} is $(n-4)$ edges

Hamiltonian laceable, there exist Hamiltonian paths $\langle y_1' \overset{P(y_1',z_1)}{\longrightarrow} z_1, u \overset{P(u,x_1')}{\longrightarrow} x_1' \rangle$ of S_{n-1}^2 for $n(z_1) \neq 1$ and $\langle y_2' \overset{P(y_2',u')}{\longrightarrow} u', z_2 \overset{P(z_2,x_2')}{\longrightarrow} x_2' \rangle$ of S_{n-1}^3 for $n(z_2) \neq 1$, $n(z_1) \neq n(z_2)$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(z_1', z_2')$ of $S_{n-1}^{\{4,5,\cdots,n\}}$. Thus, $\langle x_1 \overset{P(x_1,y_1)}{\longrightarrow} y_1, y_1' \overset{P(y_1',z_1)}{\longrightarrow} z_1, z_1' \overset{P(z_1',z_2')}{\longrightarrow} z_2', z_2 \overset{P(z_2,x_2')}{\longrightarrow} , x_2', x_2 \overset{P(x_2,y_2)}{\longrightarrow} y_2, y_2' \overset{P(y_2',u')}{\longrightarrow} u', u \overset{P(u,x_1')}{\longrightarrow} x_1', x_1 \rangle$ is a Hamiltonian cycle of $S_n - F$, as illustrated in Figure 2(c).

Case 2.2.5: $n(x_1) = n(y_2)$ and $n(x_2) = n(y_1)$.
Without loss of generality, we can assume that $n(x_2) = n(y_1) = 2$ and $n(x_1) = n(y_2) = 3$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle x_1' \overset{P(x_1',u)}{\longrightarrow} u, v \overset{P(v,y_2')}{\longrightarrow} y_2' \rangle$ for $n(u) \neq 1, 2$ and $n(v) \neq 1, 2$. Applying **Lemma 4**. we can obtain the Hamiltonian path $P(u', v')$ of $S_{n-1}^{\{4,5,\cdots,n\}}$. Thus, $\langle x_2, x_2' \overset{P(x_2',y_1')}{\longrightarrow} y_1', y_1 \overset{P(y_1,x_1)}{\longrightarrow} x_1, x_1' \overset{P(x_1',u)}{\longrightarrow} u, u' \overset{P(u',v')}{\longrightarrow} v', v \overset{P(v,y_2')}{\longrightarrow} y_2', y_2 \overset{P(y_2,x_2)}{\longrightarrow} x_2 \rangle$ is a Hamiltonian cycle of $S_n - F$.

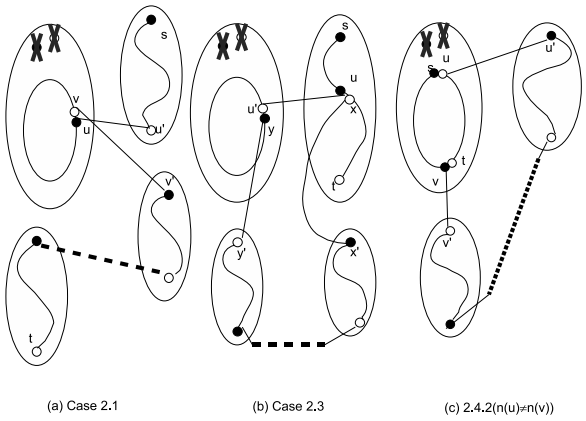

Figure 3: Illustration of **Lemma 6**.

Lemma 6 *For $n \geq 5$, if S_{n-1} is f-adjacency Hamiltonian laceable for $f \leq n-5$ and f-adjacency Hamiltonian for $f \leq n-4$, then S_n is f-adjacency Hamiltonian laceable for $f \leq n-4$.*

Proof: Let $F = \{a_i, b_i |$ for $1 \leq i \leq n-4\}$ be the set of adjacently faulty vertices. Without loss of generality, we can assume that every pair of adjacently faulty vertices $\{a_i, b_i\}$ is in some substar S_{n-1}^i, and $s \in V_b, t \in V_w$, $f^1 \geq f^2 \geq \cdots \geq f_n$. We will construct the Hamiltonian path of $S_n - F$ between s and t,with the following cases.

Case 1: $f^1 \leq n-5$.

Case 1.1: $s \in S_{n-1}^i$, $t \in S_{n-1}^j$, for $i \neq j$.
Applying **Lemma 4**, we can obtain the Hamiltonian path $P(s,t)$ of $S_{n-1}^{\{1,2,\cdots,n\}}$.

Case 1.2: $s, t \in S_{n-1}^i$ for some $1 \leq i \leq n$.
Without loss of generality, we can assume that $i = 1$. Let $\langle s \overset{P(s,u)}{\longrightarrow} u, v \overset{P(v,t)}{\longrightarrow} t \rangle$ be a Hamiltonian path of S_{n-1}^1 for $u', v' \notin F$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', v')$ of $S_{n-1}^{\{2,3,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',v')}{\longrightarrow} v', v \overset{P(v,t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$ between s and t.

Case 2: $f^1 = n-4$.
Let $s \in S_{n-1}^i$ and $t \in S_{n-1}^j$. Since S_{n-1} is $(n-4)$-adjacency Hamiltonian, there is a Hamiltonian cycle C in S_{n-1}^1.

Case 2.1: $i \neq j$, and $i \neq 1$, $j \neq 1$.
Without loss of generality, we can assume that $i = 2, j = 3$. We can denote C as $\langle u \overset{P(u,v)}{\longrightarrow} v, u \rangle$ where $u \in V_b, v \in V_w$ and $n(u) = 2, n(v) \neq 3$. Let $P(s, u')$ be the Hamiltonian path of S_{n-1}^2. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(v', t)$ of $S_{n-1}^{\{3,4,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u')}{\longrightarrow} u', u \overset{P(u,v)}{\longrightarrow} v, v' \overset{P(v',t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 3(a).

Case 2.2: $i = 1, j \neq 1$ or $i \neq 1, j = 1$.
Without loss of generality, we can assume that $i = 1$ and $j = 2$. We can denote C as $\langle s \overset{P(s,u)}{\longrightarrow} u, s \rangle$ for $n(u) \neq 2$. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(u', t)$ of $S_{n-1}^{\{3,4,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.3: $i = j \neq 1$.
Without loss of generality, we can assume that $i = j = 2$. Let $\langle s \overset{P(s,u)}{\longrightarrow} u, x \overset{P(x,t)}{\longrightarrow} t \rangle$ be a Hamiltonian path of

S^2_{n-1} for $u \in V_b, n(u) = 1, u' \notin F$. We can denote C as $\langle u' \xrightarrow{P(u',y)} y, u' \rangle$ for $n(y) \neq n(x)$. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(y', x')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',y)} y, y' \xrightarrow{P(y',x')} x', x \xrightarrow{P(x,t)} t \rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 3(b).

Case 2.4: $i = j = 1$.

Case 2.4.1: $(s, t) \in C$.

We can denote C as $\langle s \xrightarrow{P(s,u)} u, v \xrightarrow{P(v,t)} t \rangle$. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(u', v')$ of $S^{\{2,3,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',v')} v', v \xrightarrow{P(v,t)} t \rangle$ is a Hamiltonian path of $S_n - F$ between s and t.

Case 2.4.2: $(s, t) \notin C$.

We can denote C as $\langle s \xrightarrow{P(s,v)} v, t \xrightarrow{P(t,u)} u, s \rangle$.

Suppose $n(u) = n(v)$. Without loss of generality, we can assume that $n(u) = n(v) = 2$. Let $\langle v' \xrightarrow{P(v',y)} y, x \xrightarrow{P(x,u')} u' \rangle$ be a Hamiltonian path of S^2_{n-1} for $n(x) \neq 1, n(y) \neq 1$. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(y', x')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,v)} v, v' \xrightarrow{P(v',y)} y, y' \xrightarrow{P(y',x')} x', x \xrightarrow{P(x,u')} u', u \xrightarrow{P(u,t)} t \rangle$ is a Hamiltonian path of $S_n - F$.

Suppose $n(u) \neq n(v)$. Applying **Lemma 4**, we can obtain the Hamiltonian path $P(v', u')$ of $S^{\{2,3,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,v)} v, v' \xrightarrow{P(v',u')} u', u, \xrightarrow{P(u,t)} t \rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 3(c).

Theorem 1 *For $n \geq 4$, S_n is f-adjacently Hamiltonian for $f \leq n-3$, and S_n is f-adjacently Hamiltonian laceable for $f \leq n-4$.*

Proof: We prove this theorem by induction on n. The induction base $n = 4$. Since S_n is $(n-3)$ edges Hamiltonian laceable, this theorem is true for $f = 0$. Applying **Lemma 3**, we can obtain S_4 is 1-adjacency-Hamiltonian and Hamiltonian laceable. Thus, this theorem is true for $n = 4$. Applying **Lemma 5** and **Lemma 6**, we prove the induction step.

Theorem 2 *For $n \geq 4$, S_n is f-adjacency hyper-Hamiltonian laceable for $0 \leq f \leq n-4$.*

Proof: We prove this theorem by induction on n. Since S_n is $(n-4)$ edges hyper-Hamiltonian laceable, this theorem is true for $n = 4$ and $f = 0$. By symmetry of star graphs, we can assume that every pair of adjacently faulty vertices $\{a_i, b_i\}$ is in substar S^i_{n-1}. Let $s \in S^j_{n-1}$ and $t \in S^k_{n-1}$ be two white vertices, $\alpha \in V_b$ be the other faulty vertex. We will construct the Hamiltonian path of $S_n - F - \{\alpha\}$ between s and t with the following cases.

Case 1: $f^i \leq n-5$ for $1 \leq i \leq n$ and $\alpha \in S^1_{n-1}$.

Case 1.1: $j \neq k, j \neq 1, k \neq 1$.

Without loss of generality, we can assume that $j = 2, k = 3$. Let (u, u') be a fault-free edge for $u \in (V_b \cap S^2_{n-1})$ and $u' \in (V_w \cap S^1_{n-1})$ and (x, x') be a fault-free edge for $x \in (V_w \cap S^1_{n-1})$ and $x' \in (V_b \cap S^{\{4,5,\cdots,n\}}_{n-1})$. Let $P(s, u)$ and $P(u', x)$ be Hamiltonian paths of S^2_{n-1} and S^1_{n-1}, respectively. Applying **Lemma 4**, we can construct a Hamiltonian path $P(x', t)$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',x)} x, x' \xrightarrow{P(x',t)} t \rangle$ is a Hamiltonian path of $S_n - F$.

Case 1.2: $j = 1 \neq k$ or $k = 1 \neq j$.

Without loss of generality, we can assume that $j = 1, k = 2$. Let (u, u') be fault-free edge for $u \in (V_w \cap S^1_{n-1})$ and $u' \in (V_b \cap S^{\{3,4,\cdots,n\}}_{n-1})$. Let $P(s, u)$ be a Hamiltonian path of S^1_{n-1}. Applying **Lemma 4**, we can construct a Hamiltonian path $P(u', t)$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',t)} t \rangle$ is a Hamiltonian path of $S_n - F$.

Case 1.3: $j = k \neq 1$.

Without loss of generality, we can assume that $j = k = 2$. Let (u, u') be a fault-free edge for $u \in (V_b \cap S^2_{n-1})$ and $u' \in (V_w \cap S^1_{n-1})$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle t \xrightarrow{P(t,x)} x, s \xrightarrow{P(s,u)} u \rangle$, for $x' \notin F$.

Case 1.3.1: $n(x) \neq 1$.

We assume that $n(x) = 3$. Let (v, v') be a fault-free edge for $v \in (V_w \cap S^1_{n-1})$ and $v' \in (V_b \cap S^{\{4,5,\cdots,n\}}_{n-1})$. Let $P(u', v)$ be a Hamiltonian path of S^1_{n-1}. Applying **Lemma 4**, we can obtain a

Hamiltonian path $P(v', x')$ of $S_{n-1}^{\{3,4,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',v)}{\longrightarrow} v, v' \overset{P(v',x')}{\longrightarrow} x', x \overset{P(x,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 1.3.2: $n(x) = 1$.

Let $\langle u' \overset{P(u',y)}{\longrightarrow} y, z \overset{P(z,x')}{\longrightarrow} x'\rangle$ be a Hamiltonian path of S_{n-1}^1 for $n(y), n(z) \neq 2$.. $y', z' \notin F$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(y', z')$ of $S_{n-1}^{\{3,4,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',y)}{\longrightarrow} y, y' \overset{P(y',z')}{\longrightarrow} z', z \overset{P(z,x')}{\longrightarrow} x', x \overset{P(x,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 1.4: $j = k = 1$.

Let $\langle s \overset{P(s,u)}{\longrightarrow} u, v \overset{P(v,t)}{\longrightarrow} t\rangle$ be a Hamiltonian path of S_{n-1}^1 for $u', v' \notin F$. Applying **Lemma 4**, we can construct a Hamiltonian path $P(u', v')$ of $S_{n-1}^{\{2,3,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',v')}{\longrightarrow} v', v \overset{P(v,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2: $f^1 = n - 4$.

Let $\alpha \in S_{n-1}^m$. There is a Hamiltonian cycle C of $S_{n-1}^1 - F^1$. The S_{n-1}^k is Hamiltonian laceable for $2 \leq k \leq n$.

Case 2.1: $j, k, m > 1$ are mutually different.

Without loss of generality, we can assume that $j = 2, m = 3, k = 4$. Let (u, u') be a fault-free edge for $u \in (V_w \cap S_{n-1}^3)$ and $u' \in (V_b \cap S_{n-1}^1)$, and (u', x) be a fault-free edge in C.

Case 2.1.1: $n(x) = 2$ or $n(x) = 4$.

We assume that $n(x) = 2$. Let (y, y') be a fault-free edge for $y \in (V_w \cap S_{n-1}^3)$ and $y' \in (V_b \cap S_{n-1}^{\{5,6,\cdots,n\}})$. Let $P(s, x')$ and $P(u, y)$ be Hamiltonian paths of S_{n-1}^2 and S_{n-1}^3. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(y', t)$ of $S_{n-1}^{\{4,5,\cdots,n\}}$. Thus, $\langle s \overset{P(s,x')}{\longrightarrow} x', x \overset{P(x,u')}{\longrightarrow} u', u \overset{P(u,y)}{\longrightarrow} y, y' \overset{P(y',t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 4(a).

Case 2.1.2: $n(x) \neq 2$ and $n(x) \neq 4$.

Let (y, y') be a fault-free edge for $y \in (V_w \cap S_{n-1}^3)$ and $y' \in (V_b \cap S_{n-1}^2)$. Let $P(s, y')$ and $P(y, u)$ be Hamiltonian paths of S_{n-1}^2 and S_{n-1}^3. Applying **Lemma 4**, we can construct a Hamiltonian path $P(x', t)$ of $S_{n-1}^{\{4,5,\cdots,n\}}$. Thus, $\langle s \overset{P(s,y')}{\longrightarrow} y', y \overset{P(y,u)}{\longrightarrow} u, u' \overset{P(u',x)}{\longrightarrow} x, x' \overset{P(x',t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

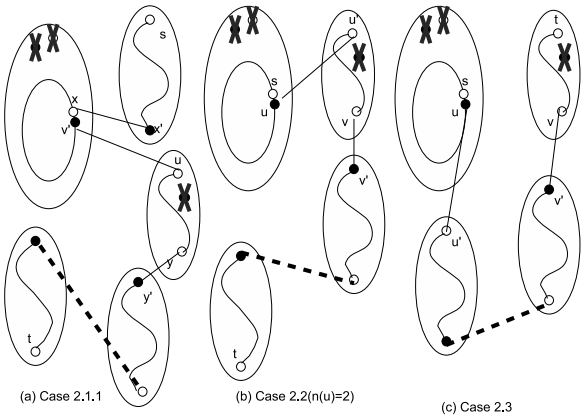

Figure 4: Illustration of **Theorem 2**.

Case 2.2: $j = 1, k \neq m$ and $k, m > 1$ or $k = 1, m \neq j$ and $j, m > 1$.

Without loss of generality, we can assume that $j = 1, m = 2, k = 3$. We denote C as $\langle s \overset{P(s,u)}{\longrightarrow} u, s\rangle$ for $n(u) \neq 3$. Suppose $n(u) = 2$. Let (v, v') be a fault-free edge for $v \in (V_w \cap S_{n-1}^2)$ and $v' \in (V_b \cap S_{n-1}^{\{4,5,\cdots,n\}})$. Let $P(u', v)$ be a Hamiltonian path of S_{n-1}^2. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(v', t)$ of $S_{n-1}^{\{3,4,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',v)}{\longrightarrow} v, v' \overset{P(v',t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$, as illustrated Figure 4(b).

Suppose $n(u) \neq 2$. Let (v, v') be a fault-free edge for $v \in (V_b \cap S_{n-1}^3)$ and $v' \in (V_w \cap S_{n-1}^2)$. And let (x, x') be a fault-free edge for $x \in (V_w \cap S_{n-1}^2)$ and $n(x) \geq 4$, $n(x) \neq n(u)$. Let $P(x, v')$ and $P(v, t)$ be Hamiltonian paths of S_{n-1}^2 and S_{n-1}^3. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', x')$ of $S_{n-1}^{\{4,5,\cdots,n\}}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',x')}{\longrightarrow} x', x \overset{P(x,v')}{\longrightarrow} v', v \overset{P(v,t)}{\longrightarrow} t\rangle$ be a Hamiltonian path of $S_n - F$.

Case 2.3: $j = 1$, $k = m > 1$ or $k = 1$, $j = m > 1$.

Without loss of generality, we can assume that $j = 1, k = m = 2$. Let

(s,u) be a fault-free edge in C where $n(u) \neq 2$, (v,v') be a fault-free edge where $v \in (V_w \cap S^2_{n-1})$ and $n(v) > 2, n(v) \neq n(u)$. Let $P(v,t)$ be a Hamiltonian path of S^2_{n-1}. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u',v')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',v')}{\longrightarrow} v', v \overset{P(v,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 4(c).

Case 2.4: $m = 1$ and $j \neq k \neq 1$.

We can denote C as $\langle \alpha, u \overset{P(u,v)}{\longrightarrow} v, \alpha\rangle$ Without loss of generality, we can assume that $j = 2, k = 3$.

Case 2.4.1: $n(u), n(v) \geq 4$.

Without loss of generality, we can assume that $n(u) = n$. Applying **Lemma 4**, we can obtain two Hamiltonian paths $P(s,u'), P(v',t)$ of $S^{\{2,n\}}_{n-1}$ and $S^{\{3,4,\cdots,n-1\}}_{n-1}$. Thus, $\langle s \overset{P(s,u')}{\longrightarrow} u', u \overset{P(u,v)}{\longrightarrow} v, v' \overset{P(v',t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 5(a).

Case 2.4.2: $n(u) \leq 3, n(v) \geq 4$ or $n(u) \geq 4, n(u) \leq 3$.

Without loss of generality, we can assume that $n(u) = 2, n(v) = 4$. There is a Hamiltonian path $P(s,u')$ of S^2_{n-1}. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(v',t)$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u')}{\longrightarrow} u', u, \overset{P(u,v)}{\longrightarrow} v, v' \overset{P(v',t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.4.3: $n(u), n(v) \leq 3$.

Without loss of generality, we can assume $n(u) = 2$ and $n(v) = 3$. Let $P(s,u')$ be a Hamiltonian path of S^2_{n-1}. Let $\langle v' \overset{P(v',x)}{\longrightarrow} x, y \overset{P(y,t)}{\longrightarrow} t\rangle$ be a Hamiltonian path of S^3_{n-1} for $n(x), n(y) \geq 4$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(x',y')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u')}{\longrightarrow} u', u \overset{P(u,v)}{\longrightarrow} v, v' \overset{P(v',x)}{\longrightarrow} x, x' \overset{P(x',y')}{\longrightarrow} y', y \overset{P(y,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.5: $m = 1$ and $j = k \neq 1$.

Without loss of generality, we can assume that $j = k = 2$. We can denote C as $\langle \alpha, u \overset{P(u,v)}{\longrightarrow} v, \alpha\rangle$.

Case 2.5.1: $n(u) = 2$ or $n(v) = 2$.

Without loss of generality, we can assume that $n(u) = 2, n(v) = 3$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle t \overset{P(t,x)}{\longrightarrow} x, s \overset{P(s,u')}{\longrightarrow} u'\rangle$ for $n(x) \neq 1$.

Suppose $n(x) \neq n(v)$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(v',x')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u')}{\longrightarrow} u', u \overset{P(u,v)}{\longrightarrow} v, v' \overset{P(v',x')}{\longrightarrow} x', x \overset{P(x,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Suppose $n(x) = n(v)$. There is a Hamiltonian path $\langle v' \overset{P(v',y)}{\longrightarrow} y, z \overset{P(z,x')}{\longrightarrow} x'\rangle$ of S^3_{n-1} for $n(y), n(z) \geq 4$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(y',z')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u')}{\longrightarrow} u', u \overset{P(u,v)}{\longrightarrow} v, v' \overset{P(v',y)}{\longrightarrow} y, y' \overset{P(y',z')}{\longrightarrow} z', z \overset{P(z,x')}{\longrightarrow} x', x \overset{P(x,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

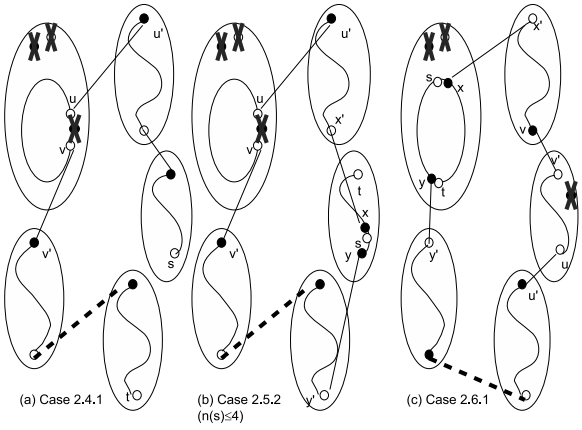

Figure 5: Illustration of **Theorem 2**.

Case 2.5.2: $n(u), n(v) \neq 2$.

Without loss of generality, we can assume that $n(u) = 3, n(v) = 4$.

Suppose $n(s) \geq 5$. Let $y \in (V_b \cap S^2_{n-1})$ for $n(y) = 5$. When $n(s) \leq 4$. Let $y \in S^2_{n-1}$ be a neighbor of s for $n(y) = 5$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle t \overset{P(t,x)}{\longrightarrow} x, s \overset{P(s,y)}{\longrightarrow} y\rangle$ for $n(x) = 3$ or 4. We can assume $n(x) = 3$. Let $P(u',x')$ be a Hamiltonian path of S^3_{n-1}. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(y',v')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,y)}{\longrightarrow} y, y' \overset{P(y',v')}{\longrightarrow} v', v \overset{P(v,u)}{\longrightarrow}$

$u, u' \xrightarrow{P(u',x')} x', x \xrightarrow{P(x,t)} t\rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 5(b).

Case 2.6: $j = k = 1$ and $m \neq 1$.

Without loss of generality, we can assume that $m = 2$. We can denote the cycle C as $\langle s \xrightarrow{P(s,y)} y, t \xrightarrow{P(t,x)} x, s\rangle$.

Case 2.6.1: $n(x) \neq n(y)$ and $n(x), n(y) \geq 3$.

Without loss of generality, we can assume that $n(x) = 3$ and $n(y) = 4$. Let (v, v') be a fault-free edge which $v \in (V_b \cap S^3_{n-1})$ and $v' \in (V_w \cap S^2_{n-1})$. There is a Hamiltonian path $P(u, v')$ of S^2_{n-1} for $u \in (V_w \cap S^2_{n-1})$ and $n(u) \geq 5$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(y', u')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,y)} y, y' \xrightarrow{P(y',u')} u', u \xrightarrow{P(u,v')} v', v \xrightarrow{P(v,x')} x', x \xrightarrow{P(x,t)} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.6.2: $n(x) = n(y) \neq 2$.

Without loss of generality, we can assume that $n(x) = n(y) = 3$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there is a Hamiltonian $P(y', u) = \langle y' \xrightarrow{P(y'v)} v, x' \xrightarrow{P(x',u)} u\rangle$ for $n(u) = 2$ and $n(v) \geq 4$ of S^3_{n-1}. We can assume that $n(v) = 4$. Let $P(u', z)$ be a Hamiltonian path of S^2_{n-1} for $z \in V_w$ and $n(z) \geq 5$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(z', v')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,y)} y, y' \xrightarrow{P(y',v)} v, v' \xrightarrow{P(v',z')} z', z \xrightarrow{P(z,u')} u', u \xrightarrow{P(u,x')} x', x \xrightarrow{P(x,t)} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.6.3: $n(x) = 2 \neq n(y)$.

Without loss of generality, we can assume that $n(y) = 3$. Let (u, u') be fault free edge which $u \in (V_w \cap S^2_{n-1})$ and $u' \in (V_b \cap S^{\{4,5,\cdots,n\}}_{n-1})$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', y')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,y)} y, y' \xrightarrow{P(y',u')} u', u \xrightarrow{P(u,x')} x', x \xrightarrow{P(x,t)} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.6.4: $n(x) = n(y) = 2$.

There is a Hamiltonian path $\langle y' \xrightarrow{P(y',u)} u, v \xrightarrow{P(v,x')} x'\rangle$ of S^2_{n-1} for $n(u), n(v) \geq 3$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', v')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,y)} y, y' \xrightarrow{P(y',u)} u, u' \xrightarrow{P(u',v')} v', v \xrightarrow{P(v,x')} x', x \xrightarrow{P(x,t)} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.7: $j = m = 1$, $k \neq 1$ or $k = m = 1$, $j \neq 1$.

Without loss of generality, we can assume that $k = 2$.

Case 2.7.1: s is adjacent to α and $(s, \alpha) \in C$.

We denote C as $\langle s \xrightarrow{P(s,u)} u, \alpha, s\rangle$.

Suppose $n(u) \neq 2$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', t)$ of $S^{\{2,3,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',t)} t\rangle$ is a Hamiltonian path of $S_n - F$.

Suppose $n(u) = 2$. There is a Hamiltonian path $\langle u' \xrightarrow{P(u',x)} x, y \xrightarrow{P(y,t)} t\rangle$ of S^2_{n-1} for $n(x), n(y) \geq 3$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(x', y')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',x)} x, x' \xrightarrow{P(x',y')} y', y \xrightarrow{P(y,t)} t\rangle$ is Hamiltonian path of $S_n - F$.

Case 2.7.2: (s, α) is not an edge of C.

We can denote C as $\langle s \xrightarrow{P(s,u)} u, \alpha, v \xrightarrow{P(v,x)} x, s\rangle$ for $n(x) \geq 3$.

Case 2.7.2.1: $n(u), n(v), n(x)$ are mutually different, $n(u), n(v) \geq 3$.

Without loss of generality, we can assume that $n(u) = 4$, $n(x) = 5$ and $n(v) = 3$. Let (y, y') is fault-free edge which $y \in (V_w \cap S^3_{n-1})$ and $y' \in (V_b \cap S^2_{n-1})$. Let $P(v', y)$ and $P(y', t)$ be Hamiltonian paths of S^3_{n-1} and S^2_{n-1}, respectively. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', x')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',x')} x', x \xrightarrow{P(x,v)} v, v' \xrightarrow{P(v',y)} y, y' \xrightarrow{P(y',t)} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.7.2.2: $n(u) = n(x) \neq n(v) \geq 3$.

Without loss of generality, we can assume that $n(u) = n(x) = n$ and $n(v) = 3$. Let $P(u', x')$ be a Hamiltonian path of S^n_{n-1}. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(v', t)$ of $S^{\{2,3,\cdots,n-1\}}_{n-1}$. Thus, $\langle s \xrightarrow{P(s,u)} u, u' \xrightarrow{P(u',x')} x', x \xrightarrow{P(x,v)} v, v' \xrightarrow{P(v',t)} t\rangle$ is a Hamiltonian path of $S_n - F$ between s and t.

Case 2.7.2.3: $n(u) = n(x) \neq n(v) = 2$.

Without loss of generality, we can as-

sume that $n(u) = n(x) = 3$. There is a Hamiltonian path $\langle u' \overset{P(u',y)}{\longrightarrow} y, z \overset{P(z,x')}{\longrightarrow} x' \rangle$ of S^3_{n-1}. Let $P(v', t)$ be a Hamiltonian path of S^2_{n-1} for $n(y), n(z) \geq 4$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(y', z')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',y)}{\longrightarrow} y, y' \overset{P(y',z')}{\longrightarrow} z', z \overset{P(z,x')}{\longrightarrow} x', x \overset{P(x,v)}{\longrightarrow} v, v' \overset{P(v',t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.7.2.4: $n(x) = n(v) \neq n(u) \geq 3$.
Without loss of generality, we can assume that $n(x) = n(v) = 3$, $n(u) = 4$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $P(x', v') = \langle x' \overset{P(x',z)}{\longrightarrow} z, y \overset{P(y,v')}{\longrightarrow} v' \rangle$ for $z \in V_w, n(z) = 2$ and $n(y) \geq 5$. Let $P(z', t)$ be a Hamiltonian path of S^2_{n-1}. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', y')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',y')}{\longrightarrow} y', y \overset{P(y,v')}{\longrightarrow} v', v \overset{P(v,x)}{\longrightarrow} x, x' \overset{P(x',z)}{\longrightarrow} z, z' \overset{P(z',t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.7.2.5: $n(x) = n(v) \neq n(u) = 2$.
Without loss of generality, we can assume that $n(x) = n(v) = 3$. Let (y, y') be a fault-free edge for $y \in (V_w \cap S^3_{n-1})$, $y' \in (V_b \cap S^2_{n-1})$ and $d(y', u') > 1$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle x' \overset{P(x',y)}{\longrightarrow} y, z \overset{P(z,v')}{\longrightarrow} v' \rangle$ of S^3_{n-1} for $n(z) \geq 4$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle u' \overset{P(u',w_1)}{\longrightarrow} w_1, y', w_2 \overset{P(w_2,t)}{\longrightarrow} t \rangle$ of S^2_{n-1} for $n(w_1), n(w_2) \geq 4$. Since $n(w_1) \neq n(w_2)$, we can assume $n(w_1) \neq n(z)$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(w_1', z')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,u)}{\longrightarrow} u, u' \overset{P(u',w_1)}{\longrightarrow} w_1, w_1' \overset{P(w_1',z')}{\longrightarrow} z', z \overset{P(z,v')}{\longrightarrow} v', v \overset{P(v,x)}{\longrightarrow} x, x' \overset{P(x',y)}{\longrightarrow} y, y', w_2 \overset{P(w_2,t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 6(*a*).

Case 2.8: $j = k = m = 1$.
Let $a_1 \in V_w$ and $b_1 \in V_b$ be one pair of adjacently faulty vertices of F. Since S_{n-1} is $(n-5)$-adjacency hyper-Hamiltonian laceable, there exists a Hamiltonian path $P(s, t)$ of $S^1_{n-1} - (F - \{a_1, b_1\}) - \{\alpha\}$.

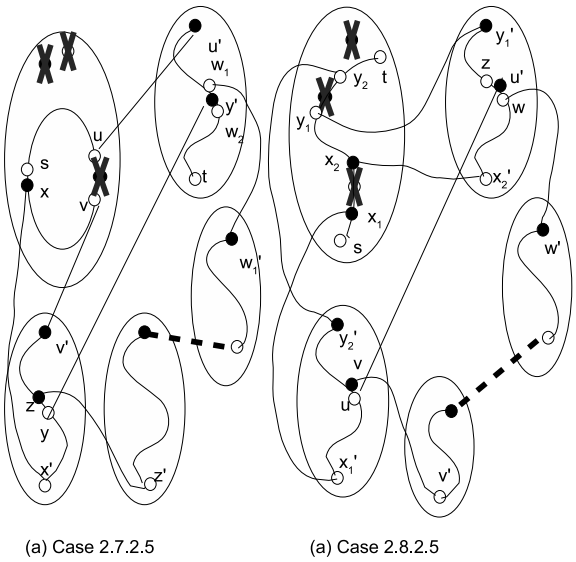

Figure 6: Illustration of **Theorem 2**.

Case 2.8.1: $(a_1, b_1) \in P(s, t)$.
We can denote $P(s, t)$ as $\langle s \overset{P(s,x)}{\longrightarrow} x, a_1, b_1, y \overset{P(y,t)}{\longrightarrow} t \rangle$.
Suppose $n(x) \neq n(y)$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(x', y')$ of $S^{\{2,3,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,x)}{\longrightarrow} x, x' \overset{P(x',y')}{\longrightarrow} y', y \overset{P(y,t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$.
Suppose $n(x) = n(y)$, We can assume that $n(x) = n(y) = 2$. There is a Hamiltonian path $\langle x' \overset{P(x',z)}{\longrightarrow} z, w \overset{P(w,y')}{\longrightarrow} y' \rangle$ of S^2_{n-1} for $n(z), n(w) > 2$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(z', p')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,x)}{\longrightarrow} x, x' \overset{P(x',z)}{\longrightarrow} z, z' \overset{P(z',w')}{\longrightarrow} w', w \overset{P(w,y')}{\longrightarrow} y', y \overset{P(y,t)}{\longrightarrow} t \rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.8.2: (a_1, b_1) is not an edge of $P(s, t)$.
We can denote $P(s, t)$ as $\langle s \overset{P(s,x_1)}{\longrightarrow} x_1, a_1, x_2 \overset{P(x_2,y_1)}{\longrightarrow} y_1, b_1, y_2 \overset{P(y_2,t)}{\longrightarrow} t \rangle$.

Case 2.8.2.1: $n(x_1), n(x_2), n(y_1), n(y_2)$ are mutually different.
Without loss of generality, we can assume that $n(x_1) = 2, n(x_2) = 4, n(y_1) = 3, n(y_2) = 5$. Applying **Lemma 4**, we can obtain a

Hamiltonian path $P(x'_1, y'_1)$ of $S^{\{2,3\}}_{n-1}$ and $P(x'_2, y'_2)$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,x_1)}{\longrightarrow} x_1, x'_1 \overset{P(x'_1,y'_1)}{\longrightarrow} y'_1, y_1 \overset{P(y_1,x_2)}{\longrightarrow} x_2, x'_2 \overset{P(x'_2,y'_2)}{\longrightarrow} y'_2, y_2 \overset{P(y_2,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.8.2.2: $n(x_1) = n(y_1)$ and $n(x_2) \neq n(y_2)$.
Without loss of generality, we can assume that $n(x_1) = n(y_1) = 2$ and $n(x_2) = 3, n(y_2) = 4$. There is a Hamiltonian path $P(x'_1, y'_1)$ of S^2_{n-1}. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(x'_2, y'_2)$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,x_1)}{\longrightarrow} x_1, x'_1 \overset{P(x'_1,y'_1)}{\longrightarrow} y'_1, y_1 \overset{P(y_1,x_2)}{\longrightarrow} x_2, x'_2 \overset{P(x'_2,y'_2)}{\longrightarrow} y'_2, y_2 \overset{P(y_2,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.8.2.3: $n(x_1) = n(y_1)$ and $n(x_2) = n(y_2)$.
Without loss of generality, we can assume that $n(x_1) = n(y_1) = 2$ and $n(x_2) = n(y_2) = 3$. Let $P(x'_1, y'_1)$ be a Hamiltonian path of S^2_{n-1}. Let $\langle x'_2 \overset{P(x'_2,u)}{\longrightarrow} u, v \overset{P(v,y'_2)}{\longrightarrow} y'_2\rangle$ be a Hamiltonian path of S^3_{n-1} for $n(u), n(v) \geq 3$. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(u', v')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,x_1)}{\longrightarrow} x_1, x'_1 \overset{P(x'_1,y'_1)}{\longrightarrow} y'_1, y_1 \overset{P(y_1,x_2)}{\longrightarrow} x_2, x'_2 \overset{P(x'_2,u)}{\longrightarrow} u, u' \overset{P(u',v')}{\longrightarrow} v', v \overset{P(v,y'_2)}{\longrightarrow} y'_2, y_2 \overset{P(y_2,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.8.2.4: $n(x_1) = n(y_2)$ and $n(x_2) \neq n(y_1)$.
Without loss of generality, we can assume that $n(x_1) = n(y_2) = 2, n(x_2) = 3, n(y_1) = 4$. Since S_{n-1} is $(n-4)$ edges Hamiltonian laceable, there exists a Hamiltonian path $\langle x'_1 \overset{P(x'_1,v_1)}{\longrightarrow} v_1, v, v_2 \overset{P(v_2,y'_2)}{\longrightarrow} y'_2\rangle$ of S^2_{n-1} for $u \in V_w, n(u) = 3$ and $n(v_1), n(v_2) \geq 4$. Since $n(v_1) \neq n(v_2)$, we can assume that $n(v_2) \neq n(y_1)$. Let $P(u', x'_2)$ be a Hamiltonian path of S^3_{n-1}. Applying **Lemma 4**, we can construct a Hamiltonian path $P(y'_1, v'_2)$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,x_1)}{\longrightarrow} x_1, x'_1 \overset{P(x'_1,v_1)}{\longrightarrow} v_1, u, u' \overset{P(u',x'_2)}{\longrightarrow} x'_2, x_2 \overset{P(x_2,y_1)}{\longrightarrow} y_1, y'_1 \overset{P(y'_1,v'_2)}{\longrightarrow} v'_2, v_2 \overset{P(v_2,y'_2)}{\longrightarrow} y'_2, y_2 \overset{P(y_2,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$.

Case 2.8.2.5: $n(x_1) = n(y_2)$ and $n(x_2) = n(y_1)$.
Without loss of generality, we can assume that $n(x_1) = n(y_2) = 2, n(x_2) = n(y_1) = 3$. Let $\langle x'_1 \overset{P(x'_1,u)}{\longrightarrow} u, v \overset{P(v,y'_2)}{\longrightarrow} y'_2\rangle$ be a Hamiltonian path of S^2_{n-1} for $n(u) = 3, n(v) \geq 4$. Let $\langle x'_2 \overset{P(x'_2,w)}{\longrightarrow} w, u', z \overset{P(z,y'_1)}{\longrightarrow} y'_1\rangle$ be a Hamiltonian path of S^3_{n-1} for $n(w), n(z) \geq 4$. Since $n(w) \neq n(z)$, we can assume that $n(w) \neq n(u)$. Applying **Lemma 4**, we can construct a Hamiltonian path $P(w', v')$ of $S^{\{4,5,\cdots,n\}}_{n-1}$. Thus, $\langle s \overset{P(s,x_1)}{\longrightarrow} x_1, x'_1 \overset{P(x'_1,u)}{\longrightarrow} u, u', z \overset{P(z,y'_1)}{\longrightarrow} y'_1, y_1 \overset{P(y_1,x_2)}{\longrightarrow} x_2, x'_2 \overset{P(x'_2,w)}{\longrightarrow} w, w' \overset{P(w',v')}{\longrightarrow} v', v \overset{P(v,y'_2)}{\longrightarrow} y'_2, y_2 \overset{P(y_2,t)}{\longrightarrow} t\rangle$ is a Hamiltonian path of $S_n - F$, as illustrated in Figure 6(b).

3 Ring Embedding in Star Graphs with Faulty Vertices

In this section, we will construct the fault-free cycle of S_n with some faulty vertices.

Theorem 3 *There exists a fault-free cycle of length at least* $n! - 2|F_v| + 2$ *in* $S_n - F_v$ *where* F_v *is the set of faulty vertices for* $|F_v \cap V_b| \geq 1$, $|F_v \cap V_w| \geq 1$, $|F_v| \leq (n-3)$ *and* $n \geq 5$.

Proof: Let $x \in V_b$ and $y \in V_w$ be two faulty vertices of F_v. By symmetry of Star graph, we can arrange vertices x and y in different S_{n-1} subgraphs. Without loss of generality, we can assume that $x \in S^1_{n-1}$ and $y \in S^2_{n-1}$. Let $F_v - \{x, y\} = \{a_1, a_2, \cdots, a_{f-2}\}$ and $F'_v = \{b_i | (b_i, a_1)$ be an edge for $a_i \in F_v - \{x, y\}$, b_i and a_i are in the same subgraph$\}$. Let s, t be white vertices of S^1_{n-1} where $s, t, s', t' \notin (F_v \cup F'_v)$, $n(s) = 2, n(t) \geq 3$. Let z be a black vertex of S^2_{n-1} where $z, z' \notin (F_v \cup F'_v)$, $n(z) \geq 3, n(z) \neq n(t)$. Since S_{n-1} is $(n-5)$-adjacency hyper-Hamiltonian laceable, $S^1_{n-1}(S^2_{n-1}) - F_v - F'_v$ is hyper-Hamiltonian laceable. There exist two Hamiltonian path $P(s, t)$ and $P(z, s')$ of S^1_{n-1} and S^2_{n-1}, respectively. Applying **Lemma 4**, we can obtain a Hamiltonian path $P(t', z')$ of $S^{\{3,4,\cdots,n\}}_{n-1}$. Therefore, $\langle s \overset{P(s,t)}{\longrightarrow} t, t' \overset{P(t',z')}{\longrightarrow} z', z \overset{P(z,s')}{\longrightarrow} s', s\rangle$ is a Hamiltonian cy-

cle of $S_n - F_v - F'_v$. The length of this cycle is $n! - 2|F_v| + 2$.

4 Conclusion

We have showed that the n-dimensional star graph is $(n-4)$-adjacency Hamiltonian laceable, $(n-3)$-adjacency Hamiltonian and $(n-4)$-adjacency hyper-Hamiltonian laceable. Applying these results, we construct a fault-free cycle with length $n! - 2|F_v| + 2$ where F_v at least contains a black vertex and a white vertex for $|F_v| \leq n-3$. We will investigate the longest path with faulty vertices of star graphs.

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