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RESEARCH NOTE

Normal-mode splitting due to inner-core anisotropy

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SUMMARY

There is a collection of core-sensitive normal modes that is split much more than predicted from the Earth's rotation, ellipticity, and lateral heterogeneity. *PKIKP* traveltime observations suggest that the Earth's inner core exhibits cylindrical anisotropy about a nearly corotational axis. We investigate the effect of transverse isotropy, which is the simplest type of anisotropy that exhibits cylindrical symmetry, on the free oscillations of the Earth. We demonstrate that transverse isotropy with a symmetry axis parallel to the rotation axis produces splitting of the form $\delta\omega_m = \omega(a' + c'm^2 + dm^4)$, where m denotes the azimuthal order of a specific singlet within a given multiplet with degenerate eigenfrequency ω; the scalars a', c', and d represent the effects of transverse isotropy on a particular normal mode. The effect of a tilt in the anisotropic symmetry axis relative to the axis of rotation can be easily incorporated and leads to non-zonal splitting

Key words: anisotropy, anomalous splitting, core structure, normal modes.

1 INTRODUCTION

Masters & Gilbert (1981) were the first to identify a collection of normal modes that is split much more than predicted from the Earth's rotation, ellipticity, and lateral heterogeneity. Over the last decade this collection has grown to a total of about 20 so-called anomalously split modes (Ritzwoller, Masters & Gilbert 1986, 1988; Woodhouse, Giardini & Li 1986; Giardini, Li & Woodhouse 1988; Li, Giardini & Woodhouse 1991; Widmer, Masters & Gilbert 1992). All these modes are sensitive to core structure and exhibit enhanced, predominantly quadratic, splitting.

There is accumulating evidence from *PKIKP* traveltime observations that the inner core exhibits cylindrical anisotropy about the Earth's rotation axis, although there is disagreement about the level of anisotropy (Morelli, Dziewonski & Woodhouse 1986; Shearer, Toy & Orcutt 1988; Shearer & Toy 1991; Creager 1992; Song & Helmberger 1993; Su & Dziewonski 1995). Recently, Tromp (1993) demonstrated that most of the anomalous splitting of the currently identified normal modes can be explained in terms of inner-core anisotropy that is compatible with traveltime observations. The purpose of this research note is to give the theoretical background for Tromp's results.

2 GENERAL ANISOTROPY

We use degenerate perturbation theory to determine the first-order effects of general anisotropy on the Earth's eigenfrequencies. We make the self-coupling approximation, that is, we assume that a given singlet only couples to singlets within the same multiplet. For a given multiplet with degenerate eigenfrequency ω, the splitting-matrix elements due to a general elastic tensor $\mathbf{\Lambda}$ are given by (Woodhouse & Dahlen 1978)

$$A_{mm'} = (2\omega)^{-1} \int_V \boldsymbol{E}_{m'} : \mathbf{\Lambda} : \boldsymbol{E}_m^* \, d\mathbf{r}^3. \tag{1}$$

The volume of the Earth is denoted by V, and the asterisk denotes complex conjugation. The singlet strain tensor $\boldsymbol{E}_m$ is related to the singlet displacement gradient $\boldsymbol{\nabla}\mathbf{u}_m$ by

$$\boldsymbol{E}_m = \frac{1}{2}[\boldsymbol{\nabla}\mathbf{u}_m + (\boldsymbol{\nabla}\mathbf{u}_m)^T]. \tag{2}$$

The vector fields $\mathbf{u}_m$, $m = -l, \ldots, l$, where l denotes the angular degree and m denotes the azimuthal order, are the displacement eigenfunctions of the $2l+1$ singlets within a given multiplet: they are orthonormalized such that

$$\int_V \rho \mathbf{u}_{m'} \cdot \mathbf{u}_m^* \, d\mathbf{r}^3 = \delta_{mm'}, \tag{3}$$

where ρ denotes the distribution of density within the Earth.

The singlet eigenfunction $\mathbf{u}_m$ can be expressed in terms of radial eigenfunctions $U(r)$, $V(r)$, $W(r)$ and spherical harmonics $Y_{lm}(\hat{\mathbf{r}})$, where r denotes the radius and $\hat{\mathbf{r}}$ denotes points on the unit sphere, as

$$\mathbf{u}_m = U\hat{\mathbf{r}}Y_{lm} + V\nabla_1 Y_{lm} - W\hat{\mathbf{r}} \times \nabla_1 Y_{lm}. \tag{4}$$

The spherical harmonics Y_{lm} are normalized according to the convention of Edmonds (1960), such that

$$\int_\Omega Y_{lm} Y_{lm}^* \, d\Omega = 1, \tag{5}$$

where Ω denotes the unit sphere. The surface gradient ∇_1 is related to the gradient ∇ by

$$\nabla = \hat{\mathbf{r}}\partial_r + r^{-1}\nabla_1. \tag{6}$$

The radial eigenfunctions U, V, and W depend on the angular degree l and overtone number n of the multiplet; for brevity, we omit this dependence on l and n. For spheroidal modes $W=0$, and the normalization (3) becomes

$$\int_0^a \rho[U^2 + l(l+1)V^2]r^2 \, dr = 1, \tag{7}$$

where a denotes the radius of the Earth. For toroidal modes $U=V=0$, and the normalization (3) reduces to

$$l(l+1)\int_0^a \rho W^2 r^2 \, dr = 1. \tag{8}$$

For practical purposes, it is convenient to consider tensors relative to the canonical basis $\hat{\mathbf{e}}_-$, $\hat{\mathbf{e}}_0$, $\hat{\mathbf{e}}_+$, which is related to the spherical basis $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{r}}$ by (Phinney & Burridge 1973)

$$\hat{\mathbf{e}}_- = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\theta}} - i\hat{\boldsymbol{\phi}}), \qquad \hat{\mathbf{e}}_0 = \hat{\mathbf{r}}, \qquad \hat{\mathbf{e}}_+ = -\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}). \tag{9}$$

We use Greek indices α, β, etc. to denote the components of tensors relative to the basis $(\hat{\mathbf{e}}_-, \hat{\mathbf{e}}_0, \hat{\mathbf{e}}_+)$; they can take on the values $-$, 0, and $+$. The Cartesian components of tensors will be indicated by using indices which take on the values 1, 2, or 3. The canonical unit vectors are orthonormal in the sense

$$\hat{\mathbf{e}}_\alpha^* \cdot \hat{\mathbf{e}}_\beta = \delta_{\alpha\beta}. \tag{10}$$

Contractions of tensors are performed by means of the metric tensor, which has components

$$g_{\alpha\beta} = \hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\beta. \tag{11}$$

This implies that $g_{00}=1$, $g_{-+}=g_{+-}=-1$, and $g_{\alpha\beta}=0$ if $\alpha+\beta\neq 0$. We can express a singlet eigenfunction $\mathbf{u}_m$ in terms of its components u^α relative to the canonical basis $\hat{\mathbf{e}}_\alpha$ as

$$\mathbf{u}_m(\mathbf{r}) = u^\alpha(r) Y_{lm}^N(\hat{\mathbf{r}})\hat{\mathbf{e}}_\alpha, \tag{12}$$

where $N=\alpha$. Throughout this paper summation over a repeated Greek sub- and superscript is implied. The generalized spherical harmonics Y_{lm}^N are fully normalized such that

$$\int_\Omega Y_{lm}^N Y_{lm}^{N*} \, d\Omega = 1. \tag{13}$$

Notice that $Y_{lm}^0 = Y_{lm}$. The two descriptions of the singlet displacement field (4) and (12) are, of course, equivalent; the scalar fields u^-, u^0, u^+ are related to the radial eigenfunctions U, V, W by

$$u^\pm = \Omega_l^0(V \pm iW), \qquad u^0 = U, \tag{14}$$

where

$$\Omega_l^N = [\tfrac{1}{2}(l+N)(l-N+1)]^{1/2}. \tag{15}$$

The singlet strain tensor $\boldsymbol{E}_m$ can also be expressed in terms of generalized spherical harmonics:

$$\boldsymbol{E}_m(\mathbf{r}) = E^{\alpha\beta}(r) Y_{lm}^N(\hat{\mathbf{r}})\hat{\mathbf{e}}_\alpha\hat{\mathbf{e}}_\beta, \tag{16}$$

where $N=\alpha+\beta$. In terms of the radial eigenfunctions U, V, W the elements of the singlet strain tensor are given by

$$E^{\pm\pm} = \Omega_l^0\Omega_l^2 r^{-1}(V \pm iW), \tag{17}$$

$$E^{00} = \dot{U}, \tag{18}$$

$$E^{\pm\mp} = -\tfrac{1}{2}F, \tag{19}$$

$$E^{0\pm} = \tfrac{1}{2}\Omega_l^0(X \pm iZ), \tag{20}$$

where a dot $\dot{}$ denotes d/dr and where we have defined

$$F = r^{-1}[2U - l(l+1)V], \tag{21}$$

$$X = \dot{V} + r^{-1}(U - V), \tag{22}$$

$$Z = \dot{W} - r^{-1}W. \tag{23}$$

Finally, we can express the elastic tensor $\boldsymbol{\Lambda}$ in terms of generalized spherical harmonics:

$$\boldsymbol{\Lambda}(\mathbf{r}) = \sum_{s=0}^{\infty}\sum_{t=-s}^{s} \Lambda_{st}^{\alpha\beta\gamma\delta}(r) Y_{st}^N(\hat{\mathbf{r}})\hat{\mathbf{e}}_\alpha\hat{\mathbf{e}}_\beta\hat{\mathbf{e}}_\gamma\hat{\mathbf{e}}_\delta, \tag{24}$$

where $N=\alpha+\beta+\gamma+\delta$. Using the generalized spherical harmonic decompositions of the singlet strain tensor (16) and the elastic tensor (24), the matrix elements (1) become (Mochizuki 1986; Li *et al.*, 1991)

$$A_{mm'} = (2\omega)^{-1} \sum_{s=0,2,4,\ldots} \sum_{t=-s}^{s} {}_s\Gamma_{mm'}^{NN'} \times \int_0^a E^{\alpha\beta}\Lambda_{st}^{\alpha'\beta'\gamma\delta}E^{\gamma\delta*} g_{\alpha\alpha'} g_{\beta\beta'} r^2 \, dr, \tag{25}$$

where $N'=\alpha+\beta$, $N=\gamma+\delta$, and $N-N'=\alpha'+\beta'+\gamma+\delta$. The parameters ${}_s\Gamma_{mm'}^{NN'}$ are given in terms of Wigner 3-j symbols (Edmonds 1960) by

$$\begin{aligned} {}_s\Gamma_{mm'}^{NN'} &= \int_\Omega Y_{lm'}^{N'} Y_{st}^{N-N'} Y_{lm}^{N*} \, d\Omega \\ &= (-1)^{N+m}(2l+1)\left(\frac{2s+1}{4\pi}\right)^{1/2} \\ &\quad \times \begin{pmatrix} l & s & l \\ -N & N-N' & N' \end{pmatrix}\begin{pmatrix} l & s & l \\ -m & t & m' \end{pmatrix}, \end{aligned} \tag{26}$$

where $t = m - m'$. Notice that in this approximation only even angular degrees s contribute to the splitting.

Equation (25) determines the splitting of the Earth's free oscillations due to general anisotropy. In the next section we consider splitting due to transverse isotropy, which is the simplest kind of anisotropy that exhibits cylindrical symmetry.

3 TRANSVERSE ISOTROPY

In this section we consider the effect of transverse anisotropy, which is a relatively simple case of the general anisotropy considered in the previous section, on the Earth's free oscillations. In Section 3.1 we consider transverse isotropy with a symmetry axis parallel to the Earth's rotation axis. The effect of transverse isotropy with a symmetry axis that is tilted relative to the rotation axis is discussed in Section 3.2. We only consider spheroidal modes because the collection of anomalously split modes consists excusively of *PKIKP*-equivalent free oscillations.

Table 1. Generalized spherical harmonic expansion coefficients $\Lambda^{\alpha\beta\gamma\delta}_{st}$ for the elastic tensor $\mathbf{\Lambda}$ defined by eq. (24). In terms of the transversely isotropic elastic parameters A, C, L, N, F the parameters λ_1, λ_2, λ_3, λ_4, λ_5 are given by $\lambda_1 = 6A + C - 4L - 10N + 8F$, $\lambda_2 = A + C + 6L + 5N - 2F$, $\lambda_3 = -6A + C - 4L + 14N + 5F$, $\lambda_4 = A + C + 3L - 7N - 2F$, $\lambda_5 = A + C - 4L - 2F$.

	$s=0$	$s=2$	$s=4$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{0000}_{s0}$	$\frac{1}{15}(\lambda_1+2\lambda_2)$	$\frac{4}{21}(\lambda_3+2\lambda_4)$	$\frac{8}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\pm\mp\mp}_{s0}$	$\frac{2}{15}\lambda_2$	$-\frac{4}{21}\lambda_4$	$\frac{2}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\mp\pm\mp}_{s0}$	$\frac{1}{15}(\lambda_1+\lambda_2)$	$-\frac{2}{21}(\lambda_3+\lambda_4)$	$\frac{2}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\mp00}_{s0}$	$-\frac{1}{15}\lambda_1$	$-\frac{1}{21}\lambda_3$	$\frac{4}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm0\mp0}_{s0}$	$-\frac{1}{15}\lambda_2$	$-\frac{1}{21}\lambda_4$	$\frac{4}{35}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm000}_{s0}$		$\frac{1}{7\sqrt{3}}(\lambda_3+2\lambda_4)$	$\frac{4}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\pm\mp0}_{s0}$		$-\frac{2}{7\sqrt{3}}\lambda_4$	$\frac{2}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\mp\pm0}_{s0}$		$-\frac{1}{7\sqrt{3}}(\lambda_3+\lambda_4)$	$\frac{2}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\pm00}_{s0}$		$\frac{2}{7\sqrt{6}}\lambda_3$	$\frac{4}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm0\pm0}_{s0}$		$\frac{2}{7\sqrt{6}}\lambda_4$	$\frac{4}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\pm\pm\mp}_{s0}$		$-\frac{2}{7\sqrt{6}}(\lambda_3+2\lambda_4)$	$\frac{2}{7\sqrt{10}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\pm\pm0}_{s0}$			$\frac{2}{\sqrt{70}}\lambda_5$
$\sqrt{\frac{2s+1}{4\pi}}\,\Lambda^{\pm\pm\pm\pm}_{s0}$			$\frac{4}{\sqrt{70}}\lambda_5$

3.1 Corotational symmetry axis

Let the z-axis be defined by the Earth's axis of rotation. The following nine Cartesian elements of the elastic tensor $\mathbf{\Lambda}$ are non-zero (Love 1927):

$$\Lambda^{1111} = \Lambda^{2222} = A, \qquad \Lambda^{3333}\ C, \qquad \Lambda^{1133} = \Lambda^{2233} = F, \tag{27}$$

$$\Lambda^{1313} = \Lambda^{2323} = L, \quad \Lambda^{1212} = N, \quad \Lambda^{1122} = A - 2N. \tag{28}$$

Notice that there are only five independent elastic parameters: A, C, F, L, and N.

To obtain the components of the elastic tensor relative to the canonical basis $\hat{\mathbf{e}}_\alpha$, $\alpha = -, 0, +$, we can either use a direct transformation from Cartesian coordinates to canonical coordinates, or convert from Cartesian to spherical coordinates, after which we can use the results in Appendix A of Mochizuki (1986) to obtain the canonical representation. The non-zero generalized spherical harmonic coefficients $\Lambda^{\alpha\beta\gamma\delta}_{st}$ are listed in Table 1. Notice that all non-zero expansion coefficients have azimuthal order $t = 0$: this is to be expected since our transverse isotropy exhibits zonal symmetry. Notice also that the non-zero elements involve angular degrees 0, 2, and 4 only; there are five non-zero coefficients with angular degree 0, 11 non-zero coefficients with angular degree 2, and 13 non-zero coefficients with angular degree 4. The degree 0 coefficients are determined exclusively by the parameters $\lambda_1 = 6A + C - 4L - 10N + 8F$ and $\lambda_2 = A + C + 6L + 5N - 2F$, the degree 2 coefficients are completely determined by the parameters $\lambda_3 = -6A + C - 4L + 14N + 5F$ and $\lambda_4 = A + C + 3L - 7N - 2F$, and the degree 4 coefficients are all completely determined by the parameter $\lambda_5 = A + C - 4L - 2F$.

Generalized Legendre functions P^N_{st} are defined by

$$Y^N_{st}(\theta, \phi) = \left(\frac{2s+1}{4\pi}\right)^{1/2} P^N_{st}(\cos\theta)\exp(it\phi). \tag{29}$$

As a result we have

$$Y^N_{s0}(\theta, \phi) = \left(\frac{2s+1}{4\pi}\right)^{1/2} P^N_{s0}(\cos\theta). \tag{30}$$

The generalized Legendre functions P^N_{s0} which are relevant to the expansion coefficients $\Lambda^{\alpha\beta\gamma\delta}_{s0}$ listed in Table 1 are given in Table 2.

Because the non-zero expansion coefficients have

Table 2. Generalized Legendre functions $P^N_{s0}(x)$, where $x = \cos\theta$, that are relevant in the context of transverse isotropy.

	$s=0$	$s=2$	$s=4$
$N=0$	1	$\frac{1}{2}(3x^2-1)$	$\frac{1}{8}(35x^4-30x^2+3)$
$N=1$		$\frac{1}{2}\sqrt{6}\,x(1-x^2)^{1/2}$	$\frac{1}{4}\sqrt{5}\,(7x^3-3x)(1-x^2)^{1/2}$
$N=2$		$\frac{1}{4}\sqrt{6}\,(1-x^2)$	$\frac{1}{8}\sqrt{10}\,(7x^2-1)(1-x^2)$
$N=3$			$\frac{1}{4}\sqrt{35}\,x(1-x^2)^{3/2}$
$N=4$			$\frac{1}{16}\sqrt{70}\,(1-x^2)^2$

azimuthal degree $t = m - m' = 0$, the splitting matrix (25) becomes

$$A_{mm'} = \delta\omega_m \delta_{mm'}. \tag{31}$$

Using the results

$$\begin{pmatrix} l & 0 & l \\ -m & 0 & m \end{pmatrix} = (-1)^{l+m}\left[\frac{(2l)!}{(2l+1)!}\right]^{1/2}, \tag{32}$$

$$\begin{pmatrix} l & 2 & l \\ -m & 0 & m \end{pmatrix} = (-1)^{l+m}\left[\frac{(2l-2)!}{(2l+3)!}\right]^{1/2} \times 2[3m^2 - l(l+1)], \tag{33}$$

$$\begin{pmatrix} l & 4 & l \\ -m & 0 & m \end{pmatrix} = (-1)^{l+m}\left[\frac{(2l-4)!}{(2l+5)!}\right]^{1/2} \times \{6(l+2)(l+1)l(l-1) + [50 - 60l(l+1)]m^2 + 70m^4\}, \tag{34}$$

it is straightforward to demonstrate that

$$\delta\omega_m = (2\omega)^{-1}\sum_{s=0,2,4}(-1)^m(2l+1)\begin{pmatrix} l & s & l \\ -m & 0 & m \end{pmatrix} I_s = \omega(a' + c'm^2 + dm^4). \tag{35}$$

For spheroidal modes, the scalars a', c', and d are given by

$$a' = (2\omega^2)^{-1}(-1)^l(2l+1)\left\{\left[\frac{(2l)!}{(2l+1)!}\right]^{1/2} I_0 - 2l(l+1)\left[\frac{(2l-2)!}{(2l+3)!}\right]^{1/2} I_2 + 6(l+2)(l+1)l(l-1)\left[\frac{(2l-4)!}{(2l+5)!}\right]^{1/2} I_4\right\}, \tag{36}$$

$$c' = (2\omega^2)^{-1}(-1)^l(2l+1)\left\{6\left[\frac{(2l-2)!}{(2l+3)!}\right]^{1/2} I_2 + [50 - 60(l+1)]\left[\frac{(2l-4)!}{(2l+5)!}\right]^{1/2} I_4\right\}, \tag{37}$$

$$d = (2\omega^2)^{-1}(-1)^l(2l+1)\left\{70\left[\frac{(2l-4)!}{(2l+5)!}\right]^{1/2} I_4\right\}. \tag{38}$$

The integrals I_0, I_2, and I_4 are defined by

$$I_s = \sum_{N=0}^{4}\sum_{i=1}^{i_N}\left(\frac{2s+1}{4\pi}\right)^{1/2}\int_0^a {}_iK_s^N r^2\, dr, \qquad s = 0, 2, 4, \tag{39}$$

where $i_0 = 5$, $i_1 = 3$, $i_2 = 3$, $i_3 = 1$, and $i_4 = 1$. The kernels ${}_iK_s^N$ are given by

$${}_1K_s^0 = \dot{U}^2\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}\Lambda_{s0}^{0000}, \tag{40}$$

$${}_2K_s^0 = 2\Omega_l^0\Omega_l^2\Omega_l^0\Omega_l^2 r^{-2}V^2\begin{pmatrix} l & s & l \\ -2 & 0 & 2 \end{pmatrix}\Lambda_{s0}^{++--}, \tag{41}$$

$${}_3K_s^0 = F^2\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}\Lambda_{s0}^{+-+-}, \tag{42}$$

$${}_4K_s^0 = -2F\dot{U}\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}\Lambda_{s0}^{+-00}, \tag{43}$$

$${}_5K_s^0 = 2\Omega_l^0\Omega_l^0 X^2\begin{pmatrix} l & s & l \\ -1 & 0 & 1 \end{pmatrix}\Lambda_{s0}^{+0-0}, \tag{44}$$

$${}_1K_s^1 = -4\Omega_l^0 X\dot{U}\begin{pmatrix} l & s & l \\ -1 & 1 & 0 \end{pmatrix}\Lambda_{s0}^{+000}, \tag{45}$$

$${}_2K_s^1 = -4\Omega_l^0\Omega_l^2\Omega_l^0 r^{-1}VX\begin{pmatrix} l & s & l \\ -2 & 1 & 1 \end{pmatrix}\Lambda_{s0}^{++-0}, \tag{46}$$

$${}_3K_s^1 = 4\Omega_l^0 XF\begin{pmatrix} l & s & l \\ 0 & 1 & -1 \end{pmatrix}\Lambda_{s0}^{+-+0}, \tag{47}$$

$${}_1K_s^2 = 4\Omega_l^0\Omega_l^2 r^{-1}V\dot{U}\begin{pmatrix} l & s & l \\ -2 & 2 & 0 \end{pmatrix}\Lambda_{s0}^{++00}, \tag{48}$$

$${}_2K_s^2 = 2\Omega_l^0\Omega_l^0 X^2\begin{pmatrix} l & s & l \\ -1 & 2 & -1 \end{pmatrix}\Lambda_{s0}^{+0+0}, \tag{49}$$

$${}_3K_s^2 = -4\Omega_l^0\Omega_l^2 r^{-1}VF\begin{pmatrix} l & s & l \\ -2 & 2 & 0 \end{pmatrix}\Lambda_{s0}^{+++-}, \tag{50}$$

$${}_1K_s^3 = -4\Omega_l^0\Omega_l^2\Omega_l^0 r^{-1}VX\begin{pmatrix} l & s & l \\ -2 & 3 & -1 \end{pmatrix}\Lambda_{s0}^{+++0}, \tag{51}$$

$${}_1K_s^4 = 2\Omega_l^0\Omega_l^2\Omega_l^0\Omega_l^2 r^{-2}V^2\begin{pmatrix} l & s & l \\ -2 & 4 & -2 \end{pmatrix}\Lambda_{s0}^{++++}. \tag{52}$$

The splitting matrix (31) may be expressed in terms of splitting-function coefficients c_{s0} as follows:

$$A_{mm'} = \omega\sum_{s=0,2,4}\gamma_{s0}^{mm'}c_{s0}, \tag{53}$$

where

$$\gamma_{st}^{mm'} = (-1)^m(2l+1)\left(\frac{2s+1}{4\pi}\right)^{1/2} \times \begin{pmatrix} l & s & l \\ -m & t & m' \end{pmatrix}\begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{54}$$

The splitting-function coefficients c_{s0} are given by

$$c_{s0} = (2\omega^2)^{-1}\left(\frac{2s+1}{4\pi}\right)^{-1/2} I_s \Big/ \begin{pmatrix} l & s & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

The splitting function

$$f = \sum_{s=0}\sum_{t=-s}^{s} c_{st}Y_s^t \tag{56}$$

is purely zonal because the coefficients c_{st} are zero if $t \neq 0$.

The relatively simple form of the splitting predicted by eq. (35) exhibits a quadratic as well as a quartic dependence on the azimuthal order m. These splitting characteristics are in accordance with the observed anomalous splitting.

3.2 Tilted symmetry axis

Recent work by Su & Dziewonski (1995) indicates that the symmetry axis of the inner-core anisotropy is slightly tilted relative to the Earth's rotation axis. In this section we investigate the effect of such a tilt on the Earth's normal modes.

Let $\Lambda_{st}^{\prime\alpha\beta\gamma\delta}$ denote the generalized spherical harmonic coefficients of the elastic tensor in the rotating reference frame. The generalized spherical harmonic coefficients of the elastic tensor in a tilted frame whose z-axis coincides with the symmetry axis of the anisotropy are denoted by $\Lambda_{s0}^{\alpha\beta\gamma\delta}$, and are listed in Table 1. The coefficients $\Lambda_{st}'^{\alpha\beta\gamma\delta}$ and $\Lambda_{s0}^{\alpha\beta\gamma\delta}$ may be related to each other by a simple rotation (Edmonds 1960):

$$\Lambda_{st}'^{\alpha\beta\gamma\delta} = \mathscr{D}_{t0}^{(s)} \Lambda_{s0}^{\alpha\beta\gamma\delta}. \tag{57}$$

The matrix elements $\mathscr{D}_{t0}^{(s)}$ describe a rotation of generalized spherical harmonic coefficients from the tilted reference frame to the corotational reference frame. Using eq. (57) in eq. (25) we obtain the following expression for the splitting matrix due to tilted, transverse isotropy:

$$A_{mm'} = (2\omega)^{-1} \sum_{s=0,2,4} \sum_{t=-s}^{s} (-1)^{m}(2l+1)\begin{pmatrix} l & s & l \\ -m & t & m' \end{pmatrix} \mathscr{D}_{t0}^{(s)} I_s$$

$$= \omega \sum_{s=0,2,4} \sum_{t=-s}^{s} \gamma_{st}^{mm'} c'_{st}. \tag{58}$$

The splitting-function coefficients c'_{st} are given in terms of the coefficients c_{s0} by

$$c'_{st} = \mathscr{D}_{t0}^{(s)} c_{s0}, \tag{59}$$

which amounts to a simple harmonic rotation. As a result of this rotation the splitting function (56) exhibits non-zonal behaviour, which is something that may be observed in the data.

3.3 Body-wave velocity perturbations

Let the unit vector $\hat{\mathbf{k}}$ denote the propagation direction of a body wave, and let the unit vector $\hat{\mathbf{p}}$ denote its polarization. Then the perturbation in body-wave velocity δv due to a perturbation in the elastic tensor $\mathbf{\Lambda}$ is determined by

$$\rho\delta v^2 = \hat{\mathbf{k}}\hat{\mathbf{p}} : \mathbf{\Lambda} : \hat{\mathbf{k}}\hat{\mathbf{p}}. \tag{60}$$

Let ξ denote the angle between the direction of a body-wave trajectory and the symmetry axis (z-axis) of the transverse isotropy; this axis may or may not coincide with the Earth's rotation axis. For the transversely isotropic elastic tensor given by eq. (28) the perturbations in P and S velocity are determined by

$$\rho\delta v_P^2 = A + 2(A - F - 2L)\cos^2\xi + (A + C - 2F - 4L)\cos^4\xi, \tag{61}$$

$$\rho\delta v_{S_{\mathrm{me}}}^2 = L + (A + C - 2F - 4L)\cos^2\xi - (A + C - 2F - 4L)\cos^4\xi, \tag{62}$$

$$\rho\delta v_{S_{\mathrm{eq}}}^2 = N + (L - N)\cos^2\xi, \tag{63}$$

where S_{me} denotes the S wave polarized in the meridional plane and S_{eq} denotes the S wave polarized in the equatorial plane. No inner-core shear wave, let alone inner-core shear anisotropy, has ever been unambiguously observed.

4 CONCLUSIONS

The Earth's largest deviations from sphericity are its rotation and ellipticity of figure. The associated splitting is predicted to be of the form (Woodhouse & Dahlen 1978)

$$\omega_m = \omega(1 + a + bm + cm^2). \tag{64}$$

The first-order effects of the Earth's rotation are represented by the parameter b and produce linear splitting as a function of m. The Earth's ellipticity of figure and second-order effects of its rotation are represented by the parameters a and c and produce quadratic splitting in m. The combined splitting due to rotation, ellipticity, and transverse isotropy is obtained by adding eqs (64) and (35):

$$\omega_m = \omega[1 + (a + a') + bm + (c + c')m^2 + dm^4]. \tag{65}$$

Tromp (1993) used eq. (65) to invert for a transversely isotropic inner core model that predicts the observed anomalous splitting reasonably well; this inner-core model is compatible with traveltime observations.

Additional splitting of the Earth's spheroidal free oscillations due to lateral heterogeneity and boundary topography can be incorporated by calculating the eigenvalues of a $(2l+1)(2l+1)$ Hermitian matrix with elements

$$H_{mm'} + A_{mm'}. \tag{66}$$

The matrix elements $H_{mm'}$ are defined by eq. (97) of Woodhouse & Dahlen (1978) and incorporate splitting due to rotation, ellipticity, lateral heterogeneity, and boundary topography. The matrix elements $A_{mm'}$ determine the effects of inner-core anisotropy and are given by eq. (53).

Both Creager (1992) and Su & Dziewonski (1994) report a slight tilt in the symmetry axis of the anisotropy relative to the Earth's axis of rotation. The matrix elements $A_{mm'}$ that describe normal-mode splitting due to tilted transverse isotropy can be determined by a simple rotation of the coefficients of the splitting function and are given by eq. (58).

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REFERENCES

Creager, K.C., 1992. Anisotropy of the inner core from differential travel times of the phases PKP and PKIKP, *Nature*, **356**, 309–314.

Edmonds, R.A., 1960. *Angular Momentum in Quantum Mechanics*, Princeton University Press, Princeton, NJ.

Giardini, D., Li, X.-D. & Woodhouse, J.H., 1988. Splitting functions of long-period normal modes of the Earth, *J. geophys. Res.*, **93**, 13 716–13 742.

Li, X.-D., Giardini, D. & Woodhouse, J.H., 1991. Large-scale three-dimensional even-degree structure of the Earth from splitting of long-period normal modes, *J. geophys. Res.*, **96**, 551–557.

Love, A.E.H., 1927. *A Treatise on the Mathematical Theory of Elasticity*, 4th edn, Cambridge University Press, Cambridge.

Masters, G. & Gilbert, F., 1981. Structure of the inner core inferred from observations of its spheroidal shear modes, *Geophys. Res. Lett.*, **8**, 569–571.

Mochizuki, E., 1986. The free oscillations of an anisotropic and heterogeneous Earth, *Geophys. J. R. astr. Soc.*, **34**, 451–847.

Morelli, A., Dziewonski, A.M. & Woodhouse, J.H., 1986. Anisotropy of the inner core inferred from PKIKP travel times, *Geophys. Res. Lett.*, **13**, 1545–1548.

Phinney, R.A. & Burridge, R., 1973. Representation of the elastic-gravitational excitation of a spherical Earth model by generalized spherical harmonics, *Geophys. J. R. astr. Soc.*, **34**, 451–847.

Ritzwoller, M., Masters, G. & Gilbert, F., 1986. Observations of anomalous splitting and their interpretation in terms of aspherical structure, *J. geophys. Res.*, **91,** 10 203–10 228.

Ritzwoller, M., Masters, G. & Gilbert, F., 1988. Constraining aspherical structure with low-degree interaction coefficients: application to uncoupled multiplets, *J. geophys. Res.*, **93,** 6369–6396.

Shearer, P.M. & Toy, K.M., 1991 PKP(BC) versus PKP(DF) differential travel times and aspherical structure in the Earth's inner core, *J. geophys. Res.*, **96,** 2233–2247.

Shearer, P.M., Toy, K.-M. & Orcutt, J.A., 1988. Axi-symmetric Earth models and inner-core anisotropy, *Nature*, **333,** 228–232.

Song, X.D. & Helmberger, D.V., 1993. Anisotropy of the Earth's inner core, *Geophys. Res. Lett.*, **20,** 2591–2594.

Su, W.J. & Dziewonski, A.M., 1995. Inner core anisotropy in three dimensions, *J. geophys. Res.*, in press.

Tromp, J., 1993. Support for anisotropy of the Earth's inner core, *Nature*, **366,** 678–681.

Widmer, R., Masters, G. & Gilbert, F., 1992. Observably split multiplets—data analysis and interpretation in terms of large-scale aspherical structure, *Geophys. J. Int.*, **111,** 559–576.

Woodhouse, J.H. & Dahlen, F.A., 1978. The effect of a general aspherical perturbation on the free oscillations of the Earth, *Geophys. J. R. astr. Soc.*, **53,** 333–354.

Woodhouse, J.H., Giardini, D. & Li, X.-D., 1986. Evidence for inner core anisotropy from splitting in free oscillation data, *Geophys. Res. Lett.*, **13,** 1549–1552.