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Separation of Variables and Exact Solutions of Generalized Nonlinear Klein-Gordon Equations

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In this paper, the generalized conditional symmetry approach is developed to study the separation of variables for generalized nonlinear Klein-Gordon equations. We derive a complete list of canonical forms for a generalized nonlinear Klein-Gordon equation and a system of generalized nonlinear Klein-Gordon equations that submit to separation of variables in some coordinates. As a result, some exact solutions to the Bullough-Dodd equation, Liouville equation, Sine-Gordon equation and Sinh-Gordon equation are obtained. A symmetry group interpretation of the known results concerning separation of variables with the scalar Klein-Gordon equation is also given.

§1. Introduction

The symmetry group method is an effective and powerful method to find exact solutions and symmetry reductions of partial differential equations (PDEs) arising in physics and applied sciences. One of the important applications of the classical method due to Lie [1)-5)] is the separation of variables for linear PDEs. [6)] It is certain that functionally separable solutions and group-invariant solutions form two classes of important exact solutions of PDEs. They usually reflect some basic and significant properties of PDEs.

In this paper, we are concerned with the separation of variables for generalized nonlinear Klein-Gordon equations, which have widespread applications in physics and wave propagation. A solution of a PDE with two independent variables t and x is said to be functionally separable if $u = h(\phi(x) + \psi(t))$ for some single variable functions h, ϕ and ψ. The method used here is the generalized conditional symmetry (GCS) approach [7)-9)] (also referred to as the conditional Lie-Bäcklund symmetry method). The GCS is a natural generalization of the generalized symmetry [1), 2)] (or Lie-Bäcklund symmetry [3), 10)]) in the same way that the conditional symmetry [11)] is a generalization of the Lie point symmetry. This method has been applied successfully to find some interesting exact solutions and symmetry reductions of certain nonlinear PDEs, [7)-9), 12)-16)] and these solutions generally cannot be derived with the classical and the conditional symmetry methods.

Consider a system of kth-order PDEs with two independent variables t and x and m dependent variables $u = (u^1, u^2, \cdots, u^m)$:

$$H_\alpha(t, x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) = 0, \quad 1 \le \alpha \le m. \tag{1}$$

Here $u_{(j)}$ denotes all jth-order partial derivatives of u with respect to t and x, the functions H_α are some smooth functions of the indicated variables.

Let

$$V=\sum_{\alpha=1}^{m}F^{\alpha}(t,x,u,u_{(1)},\cdots)\frac{\partial}{\partial u^{\alpha}} \tag{2}$$

be an evolutionary vector field, where the functions F^{α}, $\alpha=1,2,\cdots,m$, are smooth functions of $t,x,u,u_{(1)},\cdots$.

Definition 1. An evolutionary vector field (2) is said to be a generalized symmetry of (1) if

$$V^{(k)}(H_{\alpha})|_{E}=0, \tag{3}$$

where E denotes the solution manifold of the system (1), and $V^{(k)}=\sum\limits_{\alpha,|J|\le k}D_JF^{\alpha}\frac{\partial}{\partial u_J^{\alpha}}$, where $D_J=\frac{\partial^{j_1}}{\partial t^{j_1}}\frac{\partial^{j_2}}{\partial x^{j_2}}$, $u_J=\frac{\partial^{j_1+j_2}u}{\partial t^{j_1}\partial x^{j_2}}$, $J=(j_1,j_2)$.

Definition 2. An evolutionary vector field (2) is said to be a generalized conditional symmetry of (1) if

$$V^{(k)}(H_{\alpha})|_{E\cap W}=0, \tag{4}$$

where W is a kth-order system of (1) obtained by appending the invariant surface conditions

$$F^{\alpha}=0,\quad \alpha=1,2,\cdots,m \tag{5}$$

and their partial derivatives with respect to x.

It follows from Definition 2 that (1) admits the GCS (2) if and only if

$$H'_{\alpha}\vec{F}|_{E\cap W}=0,\quad 1\le\alpha\le m. \tag{6}$$

Here $\vec{F}=(F^1,F^2,\cdots,F^m)$, and $H'_{\alpha}\vec{F}$ denotes the Gâteaux derivative of the vector functions H_{α} in the direction $\vec{F}$, defined by

$$H'_{\alpha}\vec{F}=\frac{d}{d\epsilon}H_{\alpha}(t,x,u^{\alpha}+\epsilon F^{\alpha},u_t^{\alpha}+\epsilon D_tF^{\alpha},u_x^{\alpha}+\epsilon D_xF^{\alpha},\cdots),$$

where D_x and D_t are respectively the total derivative operators with respect to x and t, defined by

$$D_x=\frac{\partial}{\partial x}+u_x^{\alpha}\frac{\partial}{\partial u^{\alpha}}+u_{(1)x}^{\alpha}\frac{\partial}{\partial u_{(1)}^{\alpha}}+\cdots,$$

$$D_t=\frac{\partial}{\partial t}+u_t^{\alpha}\frac{\partial}{\partial u^{\alpha}}+u_{(1)t}^{\alpha}\frac{\partial}{\partial u_{(1)}^{\alpha}}+\cdots.$$

A simple but useful observation is that if (1) has an additive separable solution $u=\phi(x)+\psi(t)$, then u satisfies the constraint $u_{xt}=0$. Moreover, we have the following theorem. [16]

Theorem 1. Equation (1) possesses the functionally separable solution

$$f(u) = \phi(x) + \psi(t) \tag{7}$$

if and only if it admits the GCS

$$V = h(u, u_t, u_x, u_{xt})\frac{\partial}{\partial u}, \tag{8}$$

where

$$h(u, u_t, u_x, u_{xt}) = u_{xt} + \frac{f''}{f'}u_t u_x.$$

This paper is organized as follows. In §2, we discuss the separation of variables and exact solutions of the generalized nonlinear Klein-Gordon equation

$$\Box u \equiv u_{tt} - u_{xx} = F(u, u_t, u_x). \tag{9}$$

Some functionally separable solutions of (9) are obtained in §3. In §4, we consider the separation of variables of a system of generalized nonlinear Klein-Gordon equations,

$$\begin{aligned} \Box u &= F(u, v, u_x, v_x), \\ \Box v &= G(u, v, u_x, v_x), \end{aligned} \tag{10}$$

where F and G are some smooth functions of the indicated variables in a well-defined domain and $|F_v| + |F_{v_x}| + |G_u| + |G_{u_x}| \neq 0$. The system (10) and some of its special cases have important applications in quantum field theory, wave propagation and integrable systems.[20)-22)] We classify all the possibilities for which the system (10) admits functionally separable solutions. Section 5 contains some concluding remarks.

§2. Separation of variables for Eq. (9)

The symmetry reduction of (9) with $F_{u_x} = F_{u_t} = 0$ is discussed in Ref. 5). In this section, we concentrate on the separation of variables for the generalized nonlinear Klein-Gordon equation (9) by using the GCS approach. In the case that $F(u, u_t, u_x)$ depends on neither u_t nor u_x, the separation of variables of (9) has been discussed by several authors using several different kinds of methods.[17)-19)] We are now interested in the generalized separable solutions of the form (7).

From Theorem 1, we know that (9) admits the separable solution (7) if and only if it admits the GCS

$$V = (u_{xt} + g(u)u_x u_t)\frac{\partial}{\partial u}, \tag{11}$$

that is

$$\begin{aligned} &V^{(2)}(\Box u - F(u, u_t, u_x)) \\ &\quad = D_t^2\eta - D_x^2\eta - F_u\eta - F_{u_t}D_t\eta - F_{u_x}D_x\eta \\ &\quad = D_t^2\eta - F_{u_t}D_t\eta = 0, \end{aligned} \tag{12}$$

whenever $D_x^i\eta = 0$ $(i = 0, 1, 2, \cdots)$ and $\Box u = F(u, u_t, u_x)$, where

$$\eta = u_{xt} + g(u)u_x u_t.$$

A straightforward calculation leads to

$$\begin{aligned} D_t^2\eta - F_{u_t}D_t\eta \\ &= g^2F_{u_tu_x}u_t^2u_x^2 + (g'' - 2gg')(u_xu_t^3 - u_tu_x^3) + FF_{uu_t}u_x \\ &\quad - (gF_{uu_t} + g'F_{u_t})u_xu_t^2 - (g'F_{u_x} + gF_{uu_x})u_x^2u_t + F_{u_tu_x}u_{xx}^2 \\ &\quad + [FF_{u_tu_x} - g(F_{u_tu_t} + F_{u_xu_x})u_tu_x + F_{uu_t}u_x + F_{uu_x}u_t]u_{xx} \\ &\quad + [F_{uu} + gF_u - gFF_{u_tu_t} + (3g' - 2g^2)F]u_xu_t, \end{aligned} \tag{13}$$

where the prime denotes differentiation with respect to the indicated single variable. Setting (13) equal to zero yields the system

$$\begin{aligned} &F_{u_tu_x} = 0, \\ &F_{uu_t}u_x + F_{uu_x}u_t - g(F_{u_tu_t} + F_{u_xu_x})u_tu_x = 0, \\ &[(g'' - 2gg')(u_t^2 - u_x^2) - (gF_{uu_x} + g'F_{u_x})u_x - (gF_{uu_t} + g'F_{u_t})u_t \\ &- gFF_{u_tu_t} + F_{uu} + gF_u + (3g' - 2g^2)F]u_t + FF_{uu_t} = 0. \end{aligned} \tag{14}$$

To obtain solutions of (14), we distinguish three cases, as we now discuss.

2.1. $F_{u_t} = 0,\ F_{u_x} \neq 0.$

In this case, (14) is equivalent to the system

$$g'' - 2gg' = 0, \tag{15a}$$
$$F_{uu_x} - gu_xF_{u_xu_x} = 0, \tag{15b}$$
$$(g'F_{u_x} + gF_{uu_x})u_x - F_{uu} - gF_u - (3g' - 2g^2)F = 0. \tag{15c}$$

Differentiating (15c) with respect to u_x and using (15b), we find

$$g' - g^2 = 0. \tag{16}$$

This equation has the solutions

$$g = 0 \quad \text{and} \quad g = -\frac{1}{u + c},$$

where c is an arbitrary constant and can be chosen to be 0 through the translation of u. Two subcases then arise:

(A) $g = -1/u$.
Substituting g into (15), we get

$$F = (c_1 + c_2 \ln u)u + uG(\frac{u_x}{u}), \tag{17}$$

in which c_1 and c_2 are arbitrary constants and G is an arbitrary function of u_x/u . Solving $u_{xt} - (u_x u_t)/u = 0$, we obtain some multiplicable separable solutions,

$$u = \phi(x)\psi(t), \tag{18}$$

where ϕ and ψ satisfy the system of second ODEs

$$\phi'' + (c_1 + c_2 \ln \phi)\phi + \phi G(\frac{\phi'}{\phi}) = \lambda\phi,$$
$$\psi'' - c_2\psi \ln \psi = \lambda\psi. \tag{19}$$

In this paper, λ always denotes the separation constant.

(B) $g = 0$.
We have

$$F = c_1 + c_2 u + Q(u_x), \tag{20}$$

where Q is an arbitrary function of u_x. In this subcase, the equation has the additive separable solutions

$$u = \phi(x) + \psi(t), \tag{21}$$

where ϕ and ψ satisfy

$$\phi'' + Q(\phi') + c_2\phi + c_1 - \lambda = 0,$$
$$\psi'' - c_2\psi - \lambda = 0. \tag{22}$$

2.2. $F_{u_t} = F_{u_x} = 0$

In this case, $g(u)$ and $F(u)$ satisfy (15a) and the equation

$$F'' + gF' + (3g' - 2g^2)F = 0. \tag{23}$$

Solving this equation, we obtain the following.

Theorem 2. The equation

$$\Box u = F(u) \tag{24}$$

admits nontrivial separable solutions of the form (7) if and only if it is locally equivalent to one of the following equations, up to equivalence under translation and dilatation of u:

(1) $\Box u = c_1 + c_2 u$;

(2) $\Box u = c_1 e^u + c_2 e^{-2u}$;

(3) $\Box u = (c_1 + c_2 \ln u)u$;

(4) $\Box u = c_1 \sin(2u) + c_2[2\cos u - \sin(2u) \ln|\sec u + \tan u|]$;

(5) $\Box u = c_1 \sinh(2u) + c_2[2\sinh u + \sinh(2u) \ln|(\sinh u)^{-1} - \coth u|]$;

(6) $\Box u = c_1 \sinh(2u) + c_2[2\cosh u + \sinh(2u) \arctan \sinh u]$. (25)

It is of interest to note that (2) in Theorem 2 is the well-known Bullough-Dodd equation. For $c_2 = 0$, it reduces to the Liouville equation. Equation (4) with $c_2 = 0$ is the Sine-Gordon equation, and (5) with $c_2 = 0$ is the Sinh-Gordon equation. Separable solutions of the equations in (25) can be obtained by solving (25) together with $\sigma = u_{xt} + g(u)u_x u_t = 0$, which is given in §3.

2.3. $F_{u_t} \neq 0$, $F_{u_x} \neq 0$.

In this case, we note that (9) is invariant under the transformation $u \to v = h(u)$, so it is sufficient to seek the additive separable solution (21) of (9), i.e. $g = 0$. Solving the system (14) with $g = 0$, we have the following.

Theorem 3. Equation (9) with $F_{u_t} F_{u_x} \neq 0$ admits the functionally separable solution (7) if and only if it is equivalent to one of the following equations:

$$\begin{aligned}
&(1)\ \ \Box u = u_x^2 - u_t^2 + c_1 u + c_2;\\
&(2)\ \ \Box u = u^{-1}(u_t^2 - u_x^2) + c_1 u^{-1} + c_2 u^2;\\
&(3)\ \ \Box u = (u_x^2 - u_t^2)\tan u + (c_1 u + c_2)\tan u + c_1;\\
&(4)\ \ \Box u = (u_t^2 - u_x^2)\coth u - (c_1 u + c_2)\coth u + c_1;\\
&(5)\ \ \Box u = (u_t^2 - u_x^2)\tanh u - (c_1 u + c_2)\tanh u + c_1.
\end{aligned} \tag{26}$$

The equations in Theorem 3 admit the additive separable solution (21) with $\phi(x)$ and $\psi(t)$ satisfying the following systems:

(1)

$$\begin{aligned}
\phi'' + \phi'^2 + c_1\phi + c_2 &= \lambda,\\
\psi'' + \psi'^2 - c_1\psi &= \lambda.
\end{aligned} \tag{27a}$$

(2)

$$\begin{aligned}
\phi'^2 + 2c_2\phi^3 - B\phi^2 + A\phi - c_1 &= \lambda,\\
\psi'^2 - 2c_2\psi^3 - B\psi^2 - A\psi &= \lambda.
\end{aligned} \tag{27b}$$

(3)

$$\begin{aligned}
\phi'^2 &= -c_1\phi + \lambda\sin(2\phi) + B\cos(2\phi) + A - c_2,\\
\psi'^2 &= c_1\psi + \lambda\sin(2\psi) - B\cos(2\psi) + A.
\end{aligned} \tag{27c}$$

(4)

$$\begin{aligned}
\phi'^2 &= \lambda e^{2\phi} + Be^{-2\phi} - c_1\phi - A - c_2,\\
\psi'^2 &= \lambda e^{-2\psi} + Be^{2\psi} + c_1\psi - A.
\end{aligned} \tag{27d}$$

(5)

$$\phi'^2 = \lambda e^{2\phi} + Be^{-2\phi} - c_1\phi - A - c_2,$$
$$\psi'^2 = -\lambda e^{-2\psi} - Be^{2\psi} + c_1\psi - A. \tag{27e}$$

§3. Exact solutions of equations in (25)

In this section, we construct exact solutions of the equations in (25). These solutions include time and space periodic solutions and blow-up solutions. In particular, we obtain some new exact solutions to the Bullough-Dodd equation, Liouville equation, Sine-Gordon equation and Sinh-Gordon equation.

Example 1. The Bullough-Dodd equation,

$$\Box u = c_1 e^u + c_2 e^{-2u}, \tag{28}$$

admits the separable solution

$$u = \ln[\phi(x) + \psi(t)], \tag{29}$$

where $\phi(x)$ and $\psi(t)$ satisfy the system

$$\begin{aligned}\phi'^2 &= -2c_1\phi^3 + \lambda\phi^2 - 2A\phi + B + c_2 \equiv L,\\ \psi'^2 &= 2c_1\psi^3 + \lambda\psi^2 + 2A\psi + B \equiv \tilde{L}.\end{aligned} \tag{30}$$

Solutions of the system (30) for arbitrary parameters λ, A, B, c_1 and c_2 have been obtained and are listed in Table I. In Table I, t_0 and x_0 are constants: $p = \sqrt{(\alpha-\beta)^2+\gamma^2}$, $\tilde{p} = \sqrt{(\tilde{\alpha}-\tilde{\beta})^2+\tilde{\gamma}^2}$. The quantities $k = \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}$, $r = \sqrt{\frac{p-\beta-\alpha}{2p}}$, and $\tilde{k} = \sqrt{\frac{\tilde{\alpha}-\tilde{\beta}}{\tilde{\alpha}-\tilde{\gamma}}}$ are the modulus of the Jacobi elliptic functions sn and cn.

From Table I we find that the Bullough-Dodd equation admits temporally and spatially periodic solutions as well as blow-up solutions, and the Liouville equation also has temporally and spatially periodic solutions. The behavior of the three solutions of (28) is presented in Figs. 1–3.

Example 2. The equation

$$\Box u = (c_1 + c_2 \ln u)u, \tag{31}$$

admits exact solutions of the form

$$u = \exp[\phi(x) + \psi(t)], \tag{32}$$

where ϕ and ψ satisfy

$$\phi'^2 = Ae^{-2\phi} - c_2\phi + \frac{1}{2}c_2 - c_1 + \lambda,$$
$$\psi'^2 = Be^{-2\psi} + c_2\psi - \frac{1}{2}c_2 + \lambda. \tag{33}$$

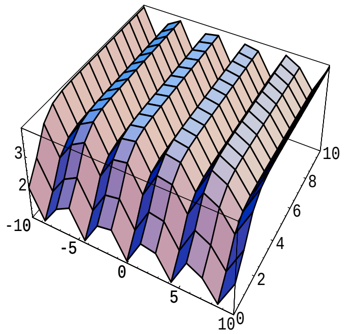

Fig. 1. Typical variation of a solution to (28) with $c_1 = 1$ and $c_2 = 4\sqrt{3}/9$, corresponding to the parameter values $\lambda = 12$, $A = 11$, $B = 12 - 4\sqrt{3}/9$, $\alpha = 3$, $\beta = 2$, $\gamma = 1$, $\tilde{\alpha} = -2-2\sqrt{3}/3$ and $\tilde{\beta} = -2+\sqrt{3}/3$ (entry 18 in Table I).

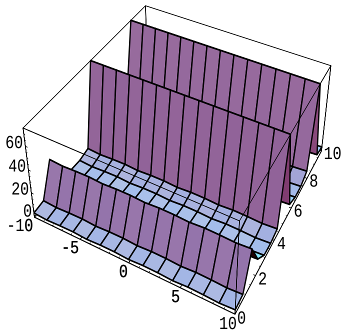

Fig. 2. Typical variation of a solution to (28) with $c_1 = 1$ and $c_2 = 4\sqrt{3}/9$, corresponding to the parameter values $\lambda = 12$, $A = 11$, $B = 12 - 4\sqrt{3}/9$, $\alpha = 3$, $\beta = 2$, $\gamma = 1$, $\tilde{\alpha} = -2+2\sqrt{3}/3$ and $\tilde{\beta} = -2-\sqrt{3}/3$ (entry 19 in Table I).

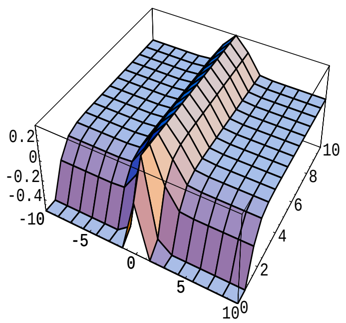

Fig. 3. Typical variation of a solution to (28) with $c_1 = 0$ and $c_2 = 1$, corresponding to the parameter values $\lambda = B = 1$, $A = 2$ (entry 1 in Table I).

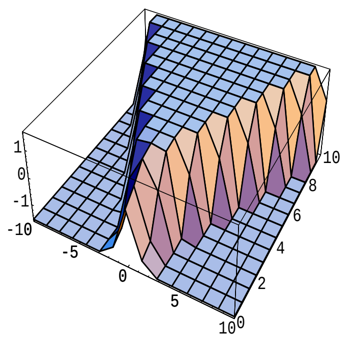

Fig. 4. Typical variation of a solution to (34) with $c_1 = 1$ and $c_2 = 2$, corresponding to the parameter values $\lambda = B = 0$, $A = 4$.

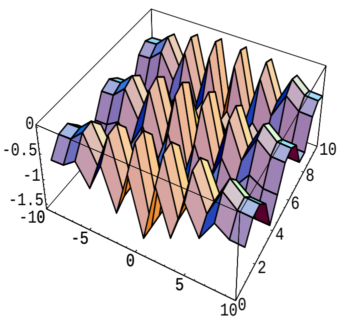

Fig. 5. Typical variation of a solution to (34) with $c_1 = 1$ and $c_2 = 0$, corresponding to the parameter values $\lambda = 2$, $B = -1$ and $A = 1$.

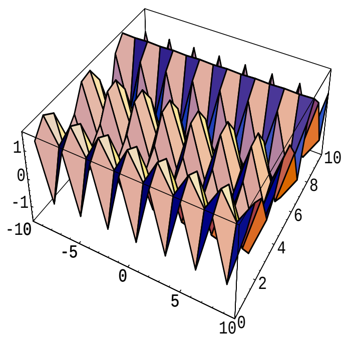

Fig. 6. Typical variation of a solution to (34) with $c_1 = 1$ and $c_2 = 0$, corresponding to the parameter values $\lambda = B = -1$ and $A = 4$.

Table I. Exact solutions of system (30).

Parameters	$\phi(x)$	$\psi(t)$
$c_1=0,\lambda>0$, $A^2-\lambda(B+c_2)>0$, $A^2-\lambda B>0$	$\frac{\sqrt{A^2-\lambda(B+c_2)}}{\lambda}\cosh\sqrt{\lambda}(x+x_0)+\frac{A}{\lambda}$	$\frac{\sqrt{A^2-\lambda B}}{\lambda}\cosh\sqrt{\lambda}(t-t_0)-\frac{A}{\lambda}$
$c_1=0,\lambda>0$, $A^2-\lambda(B+c_2)>0$, $A^2-\lambda B<0$	$\frac{\sqrt{A^2-\lambda(B+c_2)}}{\lambda}\cosh\sqrt{\lambda}(x+x_0)+\frac{A}{\lambda}$	$\frac{\sqrt{\lambda B-A^2}}{\lambda}\sinh\sqrt{\lambda}(t-t_0)-\frac{A}{\lambda}$
$c_1=0,\lambda>0$, $c_2<0, A^2=\lambda B$	$\frac{\sqrt{-\lambda c_2}}{\lambda}\cosh\sqrt{\lambda}(x+x_0)+\frac{A}{\lambda}$	$e^{\sqrt{\lambda}(t-t_0)}-\frac{A}{\lambda}$
$c_1=0,\lambda<0$, $A^2-\lambda(B+c_2)>0$, $A^2-\lambda B>0$	$\frac{\sqrt{A^2-\lambda(B+c_2)}}{\lambda}\sin\sqrt{-\lambda}(x+x_0)+\frac{A}{\lambda}$	$\frac{\sqrt{A^2-\lambda B}}{\lambda}\sin\sqrt{-\lambda}(t-t_0)-\frac{A}{\lambda}$
$c_1=0,\lambda>0$, $A^2-\lambda(B+c_2)<0$, $A^2-\lambda B>0$	$\frac{\sqrt{\lambda(B+c_2)-A^2}}{\lambda}\sinh\sqrt{\lambda}(x+x_0)+\frac{A}{\lambda}$	$\frac{\sqrt{A^2-\lambda B}}{\lambda}\cosh\sqrt{\lambda}(t-t_0)-\frac{A}{\lambda}$
$c_1=0,\lambda,c_2>0$, $A^2-\lambda(B+c_2)=0$	$e^{\sqrt{\lambda}(x+x_0)}+\frac{A}{\lambda}$	$\sqrt{\frac{c_2}{\lambda}}\cosh\sqrt{\lambda}(t-t_0)-\frac{A}{\lambda}$
$c_1=\lambda=0, A\neq 0$	$-\frac{A}{2}(x+x_0)^2+\frac{B+c_2}{2A}$	$\frac{A}{2}(t-t_0)^2-\frac{B}{2A}$
$c_2=0,c_1>0,\alpha>\beta>\gamma$, $L=-2c_1(\phi-\alpha)(\phi-\beta)(\phi-\gamma)$	$\frac{(\alpha-\gamma)\beta-\gamma(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}{\alpha-\gamma-(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}$	$(\beta-\gamma)\mathrm{cn}^{-2}[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(t-t_0),k]-\beta$
$c_2=0,c_1>0$, $L=-2c_1(\phi-\alpha)((\phi-\beta)^2+\gamma^2)$,	$\alpha-\frac{p(1-\mathrm{cn}[\sqrt{2pc_1}(x+x_0),r])^2}{\mathrm{sn}^2[\sqrt{2pc_1}(x+x_0),r]}$	$\frac{p(1-\mathrm{cn}[\sqrt{2pc_1}(t-t_0),r])^2}{\mathrm{sn}^2[\sqrt{2pc_1}(t-t_0),r]}-\alpha$
$c_2=0,c_1>0$, $L=-2c_1(\phi-\alpha)^3$	$\alpha-\frac{2}{c_1(x+x_0)^2}$	$\frac{2}{c_1(t-t_0)^2}-\alpha$
$c_2=0,c_1>0$, $L=-2c_1(\phi-\alpha)(\phi-\beta)^2,\alpha>\beta$	$(\beta-\alpha)\coth^2\frac{\sqrt{2c_1(\alpha-\beta)}}{2}(x+x_0)+\alpha$	$(\alpha-\beta)\coth^2\frac{\sqrt{2c_1(\alpha-\beta)}}{2}(t-t_0)-\alpha$
$c_2=0,c_1>0$, $L=-2c_1(\phi-\alpha)(\phi-\beta)^2,\alpha>\beta$	$(\beta-\alpha)\tanh^2\frac{\sqrt{2c_1(\alpha-\beta)}}{2}(x+x_0)+\alpha$	$(\alpha-\beta)\tanh^2\frac{\sqrt{2c_1(\alpha-\beta)}}{2}(t-t_0)-\alpha$
$c_2=0,c_1>0$, $L=-2c_1(\phi-\alpha)(\phi-\beta)^2,\alpha<\beta$	$(\alpha-\beta)\tan^2\frac{\sqrt{2c_1(\beta-\alpha)}}{2}(x+x_0)+\alpha$	$(\beta-\alpha)\tan^2\frac{\sqrt{2c_1(\beta-\alpha)}}{2}(t-t_0)-\alpha$

(continued)

It is impossible to find all solutions of (33). But for $A=B=0$, we obtain an exact solution of (31) given by

$$u=\exp[\frac{c_2}{4}\left((t-t_0)^2-(x+x_0)^2\right)+1-\frac{c_1}{c_2}].$$

Example 3. The generalized Sine-Gordon equation,

$$\Box u=c_1\sin(2u)+c_2[2\cos u-\sin(2u)\ln|\sec u+\tan u|], \tag{34}$$

has exact solutions of the form

$$u=\arcsin\tanh[\phi(x)+\psi(t)], \tag{35}$$

where ϕ and ψ satisfy

$$\phi'^2=\lambda e^{-2\phi}+Be^{2\phi}-2c_2\phi+A,$$

$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)(\phi-\beta)(\phi-\gamma), \tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})(\psi-\tilde{\gamma})$	$\frac{(\alpha-\gamma)\beta-\gamma(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}{\alpha-\gamma-(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}$	$(\tilde{\alpha}-\tilde{\beta})\mathrm{cn}^{-2}[\sqrt{\frac{c_1(\tilde{\alpha}-\tilde{\gamma})}{2}}(t-t_0),k]+\tilde{\beta}$
$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)(\phi-\beta)(\phi-\gamma), \tilde{L} = 2c_1(\psi-\tilde{\alpha})[(\psi-\tilde{\beta})^2+\tilde{\gamma}^2]$	$\frac{(\alpha-\gamma)\beta-\gamma(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}{\alpha-\gamma-(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}$	$\frac{\tilde{p}(1-\mathrm{cn}[\sqrt{2\tilde{p}c_1}(t-t_0),k])^2}{\mathrm{sn}^2[\sqrt{2\tilde{p}c_1}(t-t_0),k]}+\tilde{\alpha}$
$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)(\phi-\beta)(\phi-\gamma), \tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})^2, \tilde{\beta} > \tilde{\alpha}$	$\frac{(\alpha-\gamma)\beta-\gamma(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}{\alpha-\gamma-(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}$	$(\tilde{\beta}-\tilde{\alpha})\coth^2\sqrt{\frac{2c_1(\tilde{\beta}-\tilde{\alpha})}{2}}(t-t_0)+\tilde{\alpha}$
$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)(\phi-\beta)(\phi-\gamma), \tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})^2, \tilde{\beta} > \tilde{\alpha}$	$\frac{(\alpha-\gamma)\beta-\gamma(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}{\alpha-\gamma-(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}$	$(\tilde{\beta}-\tilde{\alpha})\tanh^2\sqrt{\frac{2c_1(\tilde{\beta}-\tilde{\alpha})}{2}}(t-t_0)+\tilde{\alpha}$
$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)(\phi-\beta)(\phi-\gamma), \tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})^2, \tilde{\alpha} > \tilde{\beta}$	$\frac{(\alpha-\gamma)\beta-\gamma(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}{\alpha-\gamma-(\alpha-\beta)\mathrm{sn}^2[\sqrt{\frac{c_1(\alpha-\gamma)}{2}}(x+x_0),k]}$	$(\tilde{\alpha}-\tilde{\beta})\tan^2\sqrt{\frac{2c_1(\tilde{\alpha}-\tilde{\beta})}{2}}(t-t_0)+\tilde{\alpha}$
$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)[(\phi-\beta)^2+\gamma^2], \tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})(\psi-\tilde{\gamma})$	$\alpha-\frac{p(1-\mathrm{cn}[\sqrt{2pc_1}(x+x_0),r])^2}{\mathrm{sn}^2[\sqrt{2pc_1}(x+x_0),r]}$	$(\tilde{\alpha}-\tilde{\beta})\mathrm{cn}^{-2}[\sqrt{\frac{c_1(\tilde{\alpha}-\tilde{\gamma})}{2}}(t-t_0),r]+\tilde{\beta}$
$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)[(\phi-\beta)^2+\gamma^2], \tilde{L} = 2c_1(\psi-\tilde{\alpha})[(\psi-\tilde{\beta})^2+\tilde{\gamma}^2]$	$\alpha-\frac{p(1-\mathrm{cn}[\sqrt{2pc_1}(x+x_0),r])^2}{\mathrm{sn}^2[\sqrt{2pc_1}(x+x_0),r]}$	$\frac{\tilde{p}(1-\mathrm{cn}[\sqrt{2\tilde{p}c_1}(t-t_0),r])^2}{\mathrm{sn}^2[\sqrt{2\tilde{p}c_1}(t-t_0),r]}+\tilde{\alpha}$
$c_2 \neq 0, c_1 > 0, L = -2c_1(\phi-\alpha)[(\phi-\beta)^2+\gamma^2], \tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})^2, \tilde{\beta} > \tilde{\alpha}$	$\alpha-\frac{p(1-\mathrm{cn}[\sqrt{2pc_1}(x+x_0),r])^2}{\mathrm{sn}^2[\sqrt{2pc_1}(x+x_0),r]}$	$(\tilde{\beta}-\tilde{\alpha})\coth^2\sqrt{\frac{2c_1(\tilde{\beta}-\tilde{\alpha})}{2}}(t-t_0)+\tilde{\alpha}$

(continued)

$$\psi'^2 = -\lambda e^{2\psi} - Be^{-2\psi} + 2c_2\psi + A - 2c_1. \tag{36}$$

Some special exact solutions of (36) with $c_1 > 0$ are listed in Table II. In Table II, α, β, ξ and η are some positive constants with the constraints $\alpha\beta\xi\eta = 1$, $X = \sqrt{-B(\alpha^2+\beta^2)}(x+x_0)$, $T = \sqrt{-B(\xi^2+\eta^2)}\xi^{-1}\eta^{-1}(t-t_0)$, $k = \frac{\sqrt{\alpha^2-\beta^2}}{\alpha}$, $r = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$, and $\tilde{k} = \frac{\sqrt{\xi^2-\eta^2}}{\xi}$.

From Table II we find that the Sine-Gordon equation admits temporally and spatially periodic solutions as well as blow-up solutions. The behavior of three solutions to (34) is presented in Figs. 4–6.

$c_2 \neq 0, c_1 > 0$, $L = -2c_1(\phi - \alpha)[(\phi-\beta)^2+\gamma^2]$, $\tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})^2$, $\tilde{\beta} > \tilde{\alpha}$	$\alpha - \frac{p(1-\mathrm{cn}[\sqrt{2pc_1}(x+x_0),r])^2}{\mathrm{sn}^2[\sqrt{2pc_1}(x+x_0),r]}$	$(\tilde{\beta} - \tilde{\alpha})\tanh^2\sqrt{\frac{2c_1(\tilde{\beta}-\tilde{\alpha})}{2}}(t-t_0) + \tilde{\alpha}$
$c_2 \neq 0, c_1 > 0$, $L = -2c_1(\phi - \alpha)[(\phi-\beta)^2+\gamma^2]$, $\tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})^2$, $\tilde{\alpha} > \tilde{\beta}$	$\alpha - \frac{p(1-\mathrm{cn}[\sqrt{2pc_1}(x+x_0),r])^2}{\mathrm{sn}^2[\sqrt{2pc_1}(x+x_0),r]}$	$(\tilde{\alpha} - \tilde{\beta})\tan^2\sqrt{\frac{2c_1(\tilde{\alpha}-\tilde{\beta})}{2}}(t-t_0) + \tilde{\alpha}$
$c_2 \neq 0, c_1 > 0$, $L = -2c_1(\phi - \alpha)(\phi-\beta)^2$, $\tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})(\psi-\tilde{\gamma})$, $\alpha > \beta$	$(\beta - \alpha)\coth^2\frac{\sqrt{2c_1(\alpha-\beta)}}{2}(x+x_0) + \alpha$	$(\tilde{\alpha} - \tilde{\beta})\mathrm{cn}^{-2}[\sqrt{\frac{c_1(\tilde{\alpha}-\tilde{\gamma})}{2}}(t-t_0), \tilde{k}] + \tilde{\beta}$
$c_2 \neq 0, c_1 > 0$, $L = -2c_1(\phi - \alpha)(\phi-\beta)^2$, $\tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})(\psi-\tilde{\gamma})$, $\alpha > \beta$	$(\beta - \alpha)\tanh^2\frac{\sqrt{2c_1(\alpha-\beta)}}{2}(x+x_0) + \alpha$	$(\tilde{\alpha} - \tilde{\beta})\mathrm{cn}^{-2}[\sqrt{\frac{c_1(\tilde{\alpha}-\tilde{\gamma})}{2}}(t-t_0), \tilde{k}] + \tilde{\beta}$
$c_2 \neq 0, c_1 > 0$, $L = -2c_1(\phi - \alpha)(\phi-\beta)^2$, $\tilde{L} = 2c_1(\psi-\tilde{\alpha})(\psi-\tilde{\beta})(\psi-\tilde{\gamma})$, $\alpha < \beta$	$(\alpha - \beta)\tan^2\frac{\sqrt{2c_1(\beta-\alpha)}}{2}(x+x_0) + \alpha$	$(\tilde{\alpha} - \tilde{\beta})\mathrm{cn}^{-2}[\sqrt{\frac{c_1(\tilde{\alpha}-\tilde{\gamma})}{2}}(t-t_0), \tilde{k}] + \tilde{\beta}$

Example 4. The equation

$$\Box u = c_1 \sinh(2u) + c_2[2\sinh u + \sinh(2u)\ln|(\sinh u)^{-1} - \coth u|] \quad (37)$$

admits the separable solution

$$u = \operatorname{arccosh}\coth[\phi(x) + \psi(t)], \quad (38)$$

where ϕ and ψ satisfy

$$\begin{aligned} \phi'^2 &= \lambda e^{2\phi} + Be^{-2\phi} + 2c_2\phi + A, \\ \psi'^2 &= \lambda e^{-2\psi} + Be^{2\psi} - 2c_2\psi + A + 2c_1. \end{aligned} \quad (39)$$

Solutions of the system (39) can be constructed in the same manner as for the system (36). We omit them here.

Example 5. The equation

$$\Box u = c_1 \sinh(2u) + c_2[2\cosh u + \sinh(2u)\arctan\sinh u] \quad (40)$$

admits the separable solution

$$u = \operatorname{arcsinh}\tan[\phi(x) + \psi(t)], \quad (41)$$

where $\phi(x)$ and $\psi(t)$ satisfy the system

$$\begin{aligned} \phi'^2 &= A\sin(2\phi) + B\cos(2\phi) - 2c_2\phi + \lambda - 2c_1, \\ \psi'^2 &= A\sin(2\psi) - B\cos(2\psi) + 2c_2\psi + \lambda. \end{aligned} \quad (42)$$

Table II. Exact solutions of system (36).

Parameters	$\phi(x)$	$\psi(t)$
$c_2=\lambda=0, B>0, A-2c_1, A>0$	$-\frac{1}{2}\ln\frac{B\sinh^2\sqrt{A}(x+x_0)}{A}$	$\frac{1}{2}\ln\frac{B\cosh^2\sqrt{A-2c_1}(t-t_0)}{A-2c_1}$
$c_2=\lambda=0, B<0, A-2c_1>0, A>0$	$-\frac{1}{2}\ln[-\frac{B\cosh^2\sqrt{A}(x+x_0)}{A}]$	$\frac{1}{2}\ln[-\frac{B\sinh^2\sqrt{A-2c_1}(t-t_0)}{A-2c_1}]$
$c_2=\lambda=0, B<0, A-2c_1<0, A>0$	$-\frac{1}{2}\ln[-\frac{B\cosh^2\sqrt{A}(x+x_0)}{A}]$	$\frac{1}{2}\ln\frac{B\cos^2\sqrt{2c_1-A}(t-t_0)}{A-2c_1}$
$B=\lambda=0,\ c_2\neq 0$	$-\frac{1}{2}c_2(x+x_0)^2+\frac{A}{2c_2}$	$\frac{1}{2}c_2(t-t_0)^2-\frac{A-2c_1}{2c_2}$
$c_2=0, B>0, A=B(\alpha^2+\beta^2),\lambda=B\alpha^2\beta^2,A-2c_1=B\alpha^2\beta^2(\xi^2+\eta^2)$	$\ln\lvert\frac{\beta\mathrm{sn}[\alpha\sqrt{B}(x+x_0),k]}{\mathrm{cn}[\alpha\sqrt{B}(x+x_0),k]}\rvert$	$\ln\frac{\xi\eta}{\sqrt{\xi^2-(\xi^2-\eta^2)\mathrm{sn}^2[\xi\sqrt{B}\alpha\beta(t-t_0),k]}}$
$c_2=0, B>0, A=B(\alpha^2-\beta^2),\lambda=-B\alpha^2\beta^2,A-2c_1=B\alpha^2\beta^2(\xi^2-\eta^2)$	$\ln\lvert\frac{\beta}{\mathrm{cn}[\sqrt{B(\alpha^2+\beta^2)}(x+x_0),r]}\rvert$	$\ln\lvert\frac{\eta}{\mathrm{cn}[\sqrt{B(\xi^2+\eta^2)}\xi^{-1}\eta^{-1}(t-t_0),r]}\rvert$
$c_2=0, B<0, A=-B(\alpha^2-\beta^2),\lambda=-B\alpha^2\beta^2,A-2c_1=B\alpha^2\beta^2(\xi^2-\eta^2)$	$\ln\frac{\alpha\beta\mathrm{sn}[X,r]}{\sqrt{\alpha^2+\beta^2-\alpha^2\mathrm{sn}^2[X,r]}}$	$\ln\frac{\xi\eta\ \mathrm{sn}[T,r]}{\sqrt{\xi^2+\eta^2-\eta^2\mathrm{sn}^2[T,r]}}$
$c_2=0, B<0,A=-B(\alpha^2+\beta^2),\lambda=B\alpha^2\beta^2,A-2c_1=-2B\alpha\beta$	$\ln\frac{\alpha\beta}{\sqrt{\alpha^2-(\alpha^2-\beta^2)\mathrm{sn}^2[\alpha\sqrt{-B}(x+x_0),k]}}$	$\ln\lvert\sqrt{\frac{2c_1-A}{2\lambda}}\tan\sqrt{\frac{A-2c_1}{2}}(t-t_0)\rvert$
$c_2=0, B<0,A=-B(\alpha^2+\beta^2),\lambda=B\alpha^2\beta^2,A-2c_1=2B\alpha\beta$	$\ln\frac{\alpha\beta}{\sqrt{\alpha^2-(\alpha^2-\beta^2)\mathrm{sn}^2[\alpha\sqrt{-B}(x+x_0),k]}}$	$\ln\lvert\sqrt{\frac{A-2c_1}{2\lambda}}\coth\sqrt{\frac{2c_1-A}{2}}(t-t_0)\rvert$
$c_2=0, B<0,A=-B(\alpha^2+\beta^2),\lambda=B\alpha^2\beta^2,A-2c_1=2B\alpha\beta$	$\ln\frac{\alpha\beta}{\sqrt{\alpha^2-(\alpha^2-\beta^2)\mathrm{sn}^2[\alpha\sqrt{-B}(x+x_0),k]}}$	$\ln\lvert\sqrt{\frac{A-2c_1}{2\lambda}}\tanh\sqrt{\frac{2c_1-A}{2}}(t-t_0)\rvert$
$c_2=0,B=\lambda\xi^2\eta^2>0,A-2c_1=\lambda(\xi^2+\eta^2),A>0$	$\ln\lvert\sqrt{\frac{A}{2B}}\tan\sqrt{\frac{A}{2}}(x+x_0)\rvert$	$\ln\frac{\xi\eta}{\sqrt{\xi^2-(\xi^2-\eta^2)\mathrm{sn}^2[\xi\sqrt{\lambda}(t-t_0),\tilde{k}]}}$
$c_2=0,B=\lambda\xi^2\eta^2>0,A-2c_1=\lambda(\xi^2+\eta^2),A<0$	$\ln\lvert\sqrt{-\frac{A}{2B}}\coth\sqrt{-\frac{A}{2}}(x+x_0)\rvert$	$\ln\frac{\xi\eta}{\sqrt{\xi^2-(\xi^2-\eta^2)\mathrm{sn}^2[\xi\sqrt{\lambda}(t-t_0),\tilde{k}]}}$
$c_2=0,B=\lambda\xi^2\eta^2>0,A-2c_1=\lambda(\xi^2+\eta^2),A<0$	$\ln\lvert\sqrt{-\frac{A}{2B}}\tanh\sqrt{-\frac{A}{2}}(x+x_0)\rvert$	$\ln\frac{\xi\eta}{\sqrt{\xi^2-(\xi^2-\eta^2)\mathrm{sn}^2[\xi\sqrt{\lambda}(t-t_0),\tilde{k}]}}$

Some exact solutions of (42) are given by the following.

(5.1) For $A=B=0$:

$$\phi(x)=-\frac{c_2}{2}(x+x_0)^2+\frac{\lambda-2c_1}{2c_2},$$

$$\psi(t)=\frac{c_2}{2}(t-t_0)^2-\frac{\lambda}{2c_2}.$$

(5.2) For $c_2 = 0,\ A \neq 0$:

$$\phi(x) = \arctan[J^{-1}(\lambda - 2c_1 - B, 2A, 2(\lambda - 2c_1), 2A, B + \lambda - 2c_1)(x + x_0)],$$
$$\psi(t) = \arctan[J^{-1}(\lambda + B, 2A, 2\lambda, 2A, \lambda - B)(t - t_0)].$$

(5.3) For $A = c_2 = 0,\ B = \lambda - 2c_1 > 0$:

$$\phi(x) = \arctan \sinh \sqrt{2B}(x + x_0),$$
$$\psi(t) = \arctan[J^{-1}(\lambda + B, 0, 2\lambda, 0, \lambda - B)(t - t_0)].$$

(5.4) For $A = c_2 = 0,\ B = -\lambda < 0$:

$$\phi(x) = \arctan[J^{-1}(2\lambda - 2c_1, 0, 2(\lambda - 2c_1), 0, -2c_1)(x + x_0)],$$
$$\psi(t) = \arctan \sinh \sqrt{2\lambda}(t - t_0).$$

Here $J(A, B, C, D, E)(y)$ is the elliptic integral defined by

$$J(A, B, C, D, E)(y) = \int^y \frac{ds}{\sqrt{As^4 + Bs^3 + Cs^2 + Ds + E}}, \tag{43}$$

and $J^{-1}(A, B, C, D, E)$ is the inverse function of $J(A, B, C, D, E)$.

§4. Separation of variables for the system (10)

We now consider the separation of variables for the system (10). We seek functionally separable solutions of the form

$$\begin{aligned} \tilde{f}(u) &= \phi_1(x) + \psi_1(t), \\ \tilde{g}(v) &= \phi_2(x) + \psi_2(t). \end{aligned} \tag{44}$$

In view of Theorem 1, the system (10) admits the separable solution (44) if and only if it admits the GCS

$$V = (u_{xt} + f(u)u_x u_t)\frac{\partial}{\partial u} + (v_{xt} + g(v)v_x v_t)\frac{\partial}{\partial v}, \tag{45}$$

where $f = \tilde{f}''/\tilde{f}'$ and $g = \tilde{g}''/\tilde{g}'$. Equivalently, we have

$$V^{(2)}(u_{tt} - u_{xx} - F)|_{E_1 \cap W_1} = V^{(2)}(v_{tt} - v_{xx} - G)|_{E_1 \cap W_1} = 0, \tag{46}$$

where E_1 is the solution manifold of the system (10), W_1 is the set of the prolongation of invariant surface conditions, e.g. $D_x^i(u_{xt} + f(u)u_x u_t) = D_x^i(v_{xt} + g(v)v_x v_t) = 0$, with $i = 0, 1, 2, \cdots$.

A straightforward but cumbersome calculation yields

$$V^{(2)}(u_{tt} - u_{xx} - F)|_{E_1 \cap W_1} = (F_{uv_x} - fu_x F_{u_x v_x} + fF_{v_x})u_t v_{xx}$$
$$+ (F_{vv_x} - gv_x F_{v_x v_x} - gF_{v_x})v_t v_{xx} + (F_{uu_x} - fu_x F_{u_x u_x})u_t u_{xx}$$
$$+ (F_{vu_x} - gv_x F_{u_x v_x})v_t u_{xx} + (f'' - 2ff')(u_x u_t^3 - u_t u_x^3) - [(f'F_{u_x} + fF_{uu_x})u_x^2$$
$$- (F_{uu} + fF_u + (3f' - 2f^2)F)u_x - (F_{uv} - fu_x F_{vu_x} + fF_v)v_x]u_t$$
$$+ [(F_{uv} - gv_x F_{uv_x} - gfv_x F_{v_x} + fF_v)u_x - v_x^2(g' - g^2)F_{v_x}$$
$$+ (F_{vv} - gv_x F_{vv_x} - gF_v)v_x]v_t. \tag{47a}$$

$$V^{(2)}(v_{tt} - v_{xx} - G)|_{E_1 \cap W_1} = (G_{vu_x} - gv_x G_{u_x v_x} + gG_{u_x})v_t u_{xx}$$
$$+ (G_{uu_x} - fu_x G_{u_x u_x} - fG_{u_x})u_t u_{xx} + (G_{vv_x} - gv_x G_{v_x v_x})v_t v_{xx}$$
$$+ (G_{uv_x} - fu_x G_{u_x v_x})u_t v_{xx} + (g'' - 2gg')(v_x v_t^3 - v_t v_x^3) - [(g'G_{v_x} + gG_{vv_x})v_x^2$$
$$- (G_{vv} + gG_v + (3g' - 2g^2)G)v_x - (G_{uv} - gv_x G_{uv_x} + gG_u)u_x]v_t$$
$$+ [(G_{uv} - fu_x G_{vu_x} + gG_u - fgu_x G_{u_x})v_x - u_x^2(f' - f^2)G_{u_x}$$
$$+ (G_{uu} - fu_x G_{uu_x} - fG_u)u_x]u_t. \tag{47b}$$

The vanishing of expressions (47a) and (47b) implies that f, g, F and G satisfy the system

$$f'' - 2ff' = 0, \tag{48a}$$
$$g'' - 2gg' = 0, \tag{48b}$$
$$F_u - fu_x F_{u_x} + fF = f_1(u, v), \tag{48c}$$
$$F_v - gv_x F_{v_x} = f_2(u, v), \tag{48d}$$
$$f_{1v} v_x + [f_{1u} + 2(f' - f^2)F]u_x = 0, \tag{48e}$$
$$(f_{2u} + ff_2)u_x + (f_{2v} - gf_2)v_x = 0, \tag{48f}$$
$$G_v - gv_x G_{v_x} + gG = g_1(u, v), \tag{48g}$$
$$G_u - fu_x G_{u_x} = g_2(u, v), \tag{48h}$$
$$g_{1u} u_x + [g_{1v} + 2(g' - g^2)G]v_x = 0, \tag{48i}$$
$$(g_{2v} + gg_2)v_x + (g_{2u} - fg_2)u_x = 0, \tag{48j}$$

as can be shown after some simplifications, where f_i and g_i $(i = 1, 2)$ are functions of u and v, to be determined.

It is readily seen that if $F(u, v)$, $G(u, v)$, $f(u)$ and $g(v)$ are solutions of the system (48), then

$$\hat{F}(u, v, u_x, v_x) = G(v, u, v_x, u_x),$$
$$\hat{G}(u, v, u_x, v_x) = F(v, u, v_x, u_x),$$
$$\hat{f}(v) = g(v),$$
$$\hat{g}(u) = f(u) \tag{49}$$

are also solutions of (48) for some f_i and g_i $(i = 1, 2)$. Hence we only consider the solutions of system (48) which are not equivalent under the transformations (49). It is easy to see that (48a) and (48b) are solved respectively by

$$f = a, -u^{-1}, \tan u, -\coth u, -\tanh u, \tag{50a}$$

and

$$g = b, -v^{-1}, \tan v, -\coth v, -\tanh v, \tag{50b}$$

where a and b are arbitrary constants. Choosing any two solutions f and g from (50a) and (50b) and inserting them into the other equations of system (48), we can obtain all solutions of system (48). For example, we consider $f = -u^{-1}$ and $g = \tan v$. First, substituting f and g into (48f) and solving, we find

$$f_2 = \frac{a_2 u}{\cos v}. \tag{51}$$

From this point, a_i $(i = 1, 2, \cdots)$ are considered to be arbitrary constants. Noting that $f' - f^2 = 0$, from (48e), we find

$$f_1 = a_1. \tag{52}$$

Now, substituting $g = \tan v$ and f_2 into (48d), and solving for F, we have

$$F = a_2 u \ln|\sec v + \tan v| + F_1(u, u_x, v_x \sec v), \tag{53}$$

where F_1 is a function of the indicated variables, to be determined. Substituting (53) into (48c), F_1 then satisfies

$$F_{1u} + u^{-1} u_x F_{1u_x} - u^{-1} F_1 = a_1, \tag{54}$$

which is solved by

$$F_1 = a_1 u \ln u + uH(\frac{u_x}{u}, v_x \sec v), \tag{55}$$

where H is an arbitrary function of the indicated variables. Hence we obtain

$$F = a_1 u \ln u + a_2 u \ln|\sec v + \tan v| + uH(\frac{u_x}{u}, v_x \sec v). \tag{56}$$

To determine G, substituting f and g into (48j) and solving, we obtain

$$g_2 = \frac{b_0 \cos v}{u}. \tag{57}$$

(48i) then implies

$$G = -\frac{g_{1u}}{2}\frac{u_x}{v_x} - \frac{g_{1v}}{2}. \tag{58}$$

The substitution of (57) and (58) into (48g) and (48h) yields $g_{1u} = 0$ and G satisfying

$$G_{vv} + \tan v G_v + (2 + \sec^2 v)G = 0, \tag{59}$$

which has the general solution

$$G = b_1 \sin(2v) + b_2(2\cos v - \sin(2v)\ln|\sec v + \tan v|). \tag{60}$$

If f and g are taken to be any of the last three solutions of (50a) and (50b), the system (48) does not admit solutions with the constraint $|F_v|+|F_{v_x}|+|G_u|+|G_{u_x}| \neq 0$. Now, we have the following theorem.

Theorem 4. The system (10) admits the separable solutions (44) if and only if it is locally equivalent to one of the following systems:

$$\begin{aligned}
&(1)\ \ \Box u = a_1 u + a_2 e^v + H(u_x, e^v v_x),\\
&\qquad \Box v = b_1 e^v + b_2 e^{-2v}.\\
&(2)\ \ \Box u = a_1 u + a_2 \ln v + H_1(u_x, v^{-1}v_x),\\
&\qquad \Box v = b_1 v\ln v + b_2 uv + vH_2(u_x, v^{-1}v_x).\\
&(3)\ \ \Box u = a_1 e^u + a_2 e^{-2u},\\
&\qquad \Box v = b_1 v \ln v + b_2 v e^u + vH(e^u u_x, v^{-1}v_x).
\end{aligned}$$

$$\begin{aligned}
&(4)\ \ \Box u = a_1 u + a_2 \ln|\sec v + \tan v| + H(u_x, (\sec v)v_x),\\
&\qquad \Box v = b_1 \sin(2v) + b_2(2\cos v - \sin(2v)\ln|\sec v + \tan v|).\\
&(5)\ \ \Box u = a_1 u + a_2 \ln|\coth v - \sinh^{-1} v| + H(u_x, (\sinh^{-1} v)v_x),\\
&\qquad \Box v = b_1 \sinh(2v) + b_2(2\sinh v + \sinh(2v)\ln|\coth v - \sinh^{-1} v|).\\
&(6)\ \ \Box u = a_1 u + a_2 \arctan\sinh v + H(u_x, (\cosh^{-1} v)v_x),\\
&\qquad \Box v = b_1 \sinh(2v) + b_2(2\cosh v + \sinh(2v)\arctan\sinh v).\\
&(7)\ \ \Box u = a_1 u\ln u + a_2 u \ln|\sec v + \tan v| + uH(u^{-1}u_x, (\sec v)v_x),\\
&\qquad \Box v = b_1 \sin(2v) + b_2(2\cos v - \sin(2v)\ln|\sec v + \tan v|).
\end{aligned}$$

$$\begin{aligned}
&(8)\ \ \Box u = a_1 u\ln u + a_2 u\ln|\coth v - \sinh^{-1} v| + uH(u^{-1}u_x, (\sinh^{-1} v)v_x),\\
&\qquad \Box v = b_1 \sinh(2v) + b_2(2\sinh v + \sinh(2v)\ln|\coth v - \sinh^{-1} v|).\\
&(9)\ \ \Box u = a_1 u \ln u + a_2 u \arctan\sinh v + uH(u^{-1}u_x, (\cosh^{-1} v)v_x),\\
&\qquad \Box v = b_1 \sinh(2v) + b_2(2\cosh v + \sinh(2v)\arctan\sinh v).
\end{aligned}$$

In the above equations, H, H_1 and H_2 are arbitrary functions of the indicated variables.

Theorem 5. Equations (1)–(9) in Theorem 4 respectively admit the following separable solutions.

(1) $u = \phi_1(x) + \psi_1(t)$, $v = \ln(\phi_2(x) + \psi_2(t))$, where ϕ_i and ψ_i $(i = 1, 2)$ satisfy the system

$$\begin{aligned}
&\phi_1'' + a_2\phi_2 + H(\phi_1', \phi_2') + a_1\phi_1 = \lambda_1,\\
&\psi_1'' - a_2\psi_2 - a_1\psi_1 = \lambda_1,\\
&\phi_2'^2 = -2b_1\phi_2^3 + \lambda_2\phi_2^2 - A\phi_2 + B + b_2,\\
&\psi_2'^2 = 2b_1\psi_2^3 + \lambda_2\psi_2^2 + A\psi_2 + B.
\end{aligned} \tag{61a}$$

Hereafter λ_i $(i = 1, 2)$ are considered to be separation constants and A and B arbitrary constants.

(2) $u = \phi_1(x) + \psi_1(t)$, $v = e^{\phi_2(x)+\psi_2(t)}$, with

$$\begin{aligned} &\phi_1'' + a_1\phi_1 + a_2\phi_2 + H_1(\phi_1', \phi_2') = \lambda_1,\\ &\psi_1'' - a_1\psi_1 - a_2\psi_2 = \lambda_1,\\ &\phi_2'' + \phi_1'^2 + b_1\phi_2 + b_2\phi_1 + H_2(\phi_1', \phi_2') = \lambda_2,\\ &\psi_2'' + \psi_2'^2 - b_1\psi_2 - b_2\psi_1 = \lambda_2. \end{aligned} \tag{61b}$$

(3) $u = \ln(\phi_1(x) + \psi_1(t))$, $v = e^{\phi_2(x)+\psi_2(t)}$, with

$$\begin{aligned} &\phi_1'^2 = -2a_1\phi_1^3 + \lambda_1\phi_1^2 - A\phi_1 + B + a_2,\\ &\psi_1'^2 = 2a_1\psi_1^3 + \lambda_1\psi_1^2 + A\psi_1 + B,\\ &\phi_2'' = -\phi_2'^2 - b_2\phi_1 - b_1\phi_2 - H(\phi_1', \phi_2') + \lambda_2,\\ &\psi_2'' = -\psi_2'^2 + b_1\psi_2 + b_2\psi_1 + \lambda_2. \end{aligned} \tag{61c}$$

(4) $u = \phi_1(x) + \psi_1(t)$, $v = \arcsin\tanh(\phi_2(x) + \psi_2(t))$, with

$$\begin{aligned} &\phi_1'' = -a_1\phi_1 - a_2\phi_2 - H(\phi_1', \phi_2') + \lambda_1,\\ &\psi_1'' = a_1\psi_1 + a_2\psi_2 + \lambda_1,\\ &\phi_2'^2 = \lambda_2 e^{-2\phi_2} + Be^{2\phi_2} - 2b_2\phi_2 + A,\\ &\psi_2'^2 = -\lambda_2 e^{2\psi_2} - Be^{-2\psi_2} + 2b_2\psi_2 + A - 2b_1. \end{aligned} \tag{61d}$$

(5) $u = \phi_1(x) + \psi_1(t)$, $v = \operatorname{arccosh}\coth(\phi_2(x) + \psi_2(t))$, with

$$\begin{aligned} &\phi_1'' = -a_1\phi_1 - a_2\phi_2 - H(\phi_1', -\phi_2') + \lambda_1,\\ &\psi_1'' = a_1\psi_1 + a_2\psi_2 + \lambda_1,\\ &\phi_2'^2 = \lambda_2 e^{2\phi_2} + Be^{-2\phi_2} + 2b_2\phi_2 + A,\\ &\psi_2'^2 = \lambda_2 e^{-2\psi_2} + Be^{2\psi_2} - 2b_2\psi_2 + A + 2b_1. \end{aligned} \tag{61e}$$

(6) $u = \phi_1(x) + \psi_1(t)$, $v = \operatorname{arsinh}\tan(\phi_2(x) + \psi_2(t))$, with

$$\begin{aligned} &\phi_1'' = -a_1\phi_1 - a_2\phi_2 - H(\phi_1', \phi_2') + \lambda_1,\\ &\psi_1'' = a_1\psi_1 + a_2\psi_2 + \lambda_1,\\ &\phi_2'^2 = A\cos(2\phi_2) + B\sin(2\phi_2) - 2b_2\phi_2 + \lambda_2 - 2b_1,\\ &\psi_2'^2 = -A\cos(2\psi_2) + B\sin(2\psi_2) + 2b_2\psi_2 + \lambda_2. \end{aligned} \tag{61f}$$

(7) $u = e^{\phi_1(x)+\psi_1(t)}$, $v = \arcsin\tanh(\phi_2(x) + \psi_2(t))$, with

$$\begin{aligned} &\phi_1'' = -\phi_1'^2 - a_1\phi_1 - a_2\phi_2 - H(\phi_1', \phi_2') + \lambda_1,\\ &\psi_1'' = -\psi_1'^2 + a_1\psi_1 + a_2\psi_2 + \lambda_1,\\ &\phi_2'^2 = \lambda_2 e^{-2\phi_2} + Be^{2\phi_2} - 2b_2\phi_2 + A,\\ &\psi_2'^2 = -\lambda_2 e^{2\psi_2} - Be^{-2\psi_2} + 2b_2\psi_2 + A - 2b_1. \end{aligned} \tag{61g}$$

(8) $u = e^{\phi_1(x)+\psi_1(t)}$, $v = \operatorname{arccosh}\coth(\phi_2(x) + \psi_2(t))$, with

$$\begin{aligned} &\phi_1'' = -{\phi_1}'^2 - a_1\phi_1 - a_2\phi_2 - H(\phi_1', -\phi_2') + \lambda_1,\\ &\psi_1'' = -\psi_1'^2 + a_1\psi_1 + a_2\psi_2 + \lambda_1,\\ &\phi_2'^2 = \lambda_2 e^{2\phi_2} + Be^{-2\phi_2} + 2b_2\phi_2 + A,\\ &\psi_2'^2 = \lambda_2 e^{-2\psi_2} + Be^{2\psi_2} - 2b_2\psi_2 + A + 2b_1. \end{aligned} \tag{61h}$$

(9) $u = e^{\phi_1(x)+\psi_1(t)}$, $v = \operatorname{arcsinh}\tan(\phi_2(x)+\psi_2(t))$, with

$$\begin{aligned}
\phi_1'' &= -\phi_1'^2 - a_1\phi_1 - a_2\phi_2 - H(\phi_1', \phi_2') + \lambda_1,\\
\psi_1'' &= -\psi_1'^2 + a_1\psi_1 + a_2\psi_2 + \lambda_1,\\
\phi_2'^2 &= A\cos(2\phi_2) + B\sin(2\phi_2) - 2b_2\phi_2 + \lambda_2 - 2b_1,\\
\psi_2'^2 &= -A\cos(2\psi_2) + B\sin(2\psi_2) + 2b_2\psi_2 + \lambda_2.
\end{aligned} \tag{61i}$$

We use the system (6) of Theorem 5 as an example to prove this theorem. For the system (6) with $f = 0$, $g = -\tanh v$, solving $u_{xt} = 0$ and $v_{xt} - (\tanh v)v_x v_t = 0$, we have

$$u = \phi_1(x) + \psi_1(t), \quad v = \operatorname{arcsinh}\tan(\phi_2(x) + \psi_2(t)). \tag{62}$$

Differentiating the second equation of (62) with respect to x, we obtain

$$(\cosh^{-1} v)v_x = \phi_2'. \tag{63}$$

Substitution of (62) and (63) into the system (6) of Theorem 4 yields ϕ_i and ψ_i $(i = 1, 2)$ satisfying the system

$$\psi_1'' = \phi_1'' + a_1(\phi_1 + \psi_1) + a_2(\phi_2 + \psi_2) + H(\phi_1', \phi_2'), \tag{64a}$$

$$\psi_2'' + (\sinh v)\psi_2'^2 = \phi_2'' + (\sinh v)\phi_2'^2 + 2b_1 \sinh v + 2b_2(\sinh v)(\phi_2 + \psi_2) + 2b_2. \tag{64b}$$

Thus there exists a separation constant λ_1 such that

$$\begin{aligned}
&\psi_1'' - a_1\psi_1 - a_2\psi_2 = \lambda_1,\\
&\phi_1'' + a_1\phi_1 + a_2\phi_2 + H(\phi_1', \phi_2') = \lambda_1.
\end{aligned} \tag{65}$$

We rewrite (64b) as

$$\cos(\phi_2 + \psi_2)(\psi_2'' - \phi_2'' - 2b_2) + \sin(\phi_2 + \psi_2)[\psi_2'^2 - \phi_2'^2 - 2b_2(\phi_2 + \psi_2) - 2b_1] = 0. \tag{66}$$

Differentiating (66) with respect to t yields

$$\begin{aligned}
&\cos(\phi_2 + \psi_2)[\psi_2''' + \psi_2'^3 - \psi_2'\phi_2'^2 - 2b_2(\phi_2 + \psi_2)\psi_2' - 2b_1\psi_2']\\
&\quad + \sin(\phi_2 + \psi_2)[\phi_2'\psi_2'' + \psi_2'\phi_2''] = 0.
\end{aligned} \tag{67}$$

Differentiating (67) with respect to x and using (66), we obtain

$$\psi_2'\phi_2''' - \phi_2'\psi_2''' - 2\phi_2'\psi_2'^3 + 2\psi_2'\phi_2'^3 + 4b_2(\phi_2 + \psi_2)\phi_2'\psi_2' + 4b_1\phi_2'\psi_2' = 0. \tag{68}$$

Dividing equation (68) by $\phi_2'\psi_2'$ gives

$$\frac{\phi_2'''}{\phi_2'} - \frac{\psi_2'''}{\psi_2'} - 2\psi_2'^2 + 2\phi_2'^2 + 4b_2(\phi_2 + \psi_2) + 4b_1 = 0, \tag{69}$$

which implies there exists a separation constant $\tilde{\lambda}_2$ such that

$$\begin{aligned}&\phi_2''' + 2\phi_2'^3 + 4b_2\phi_2\phi_2' + 4b_1\phi_2' = \tilde{\lambda}_2\phi_2',\\ &\psi_2''' + 2\psi_2'^3 - 4b_2\psi_2\psi_2' = \tilde{\lambda}_2\psi_2'.\end{aligned} \tag{70}$$

Integration of (70) yields

$$\phi_2'^2 = A_1\cos(2\phi_2) + A_2\sin(2\phi_2) - 2b_2\phi_2 + \frac{\tilde{\lambda}_2}{2} - 2b_1, \tag{71a}$$

$$\psi_2'^2 = B_1\cos(2\psi_2) + B_2\sin(2\psi_2) + 2b_2\psi_2 + \frac{\tilde{\lambda}_2}{2}, \tag{71b}$$

where A_i and B_i $(i = 1, 2)$ are integration constants. Differentiating (71a) and (71b) with respect to x and t, respectively, we have

$$\begin{aligned}&\phi_2'' = -A_1\sin(2\phi_2) + A_2\cos(2\phi_2) - b_2,\\ &\psi_2'' = -B_1\sin(2\psi_2) + B_2\cos(2\psi_2) + b_2.\end{aligned} \tag{72}$$

Substitution of the systems (71) and (72) into (66) leads to

$$(B_1 + A_1)\sin(\phi_2 - \psi_2) + (B_2 - A_2)\cos(\phi_2 - \psi_2) = 0, \tag{73}$$

which gives

$$B_1 = -A_1 = -A, \quad B_2 = A_2 = B. \tag{74}$$

Then, (61f) is obtained by using the new parameter $\lambda_2 = \tilde{\lambda}_2/2$. The other systems can be derived in the same manner.

§5. Concluding remarks

In this paper, we have employed the GCS approach in the analysis of the separation of variables for generalized nonlinear Klein-Gordon equations. We have derived a complete list of canonical forms for a generalized nonlinear Klein-Gordon equation and a system of generalized nonlinear Klein-Gordon equations that submit to separation of variables in some coordinates. A symmetry group interpretation of the known results [17)-19)] concerning separation of variables with the scalar Klein-Gordon equation was given. Exact solutions of the resulting equations were reduced to solve systems of first and second-order ordinary differential equations. As a by-product, some new exact solutions to the Bullough-Dodd model, Sine-Gordon equation and Sinh-Gordon equation were obtained.

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