Polynomially Solvable Cases of Scheduling Deteriorating Jobs to Minimize Total Completion Time

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Abstract

We consider two single processor scheduling problems in which the processing time p_j of a job is a function of its starting time t. The objective is to minimize the total completion time. In the first case, processing time is a linear function of time, $p_j = 1 + b_j t$, and the set of b_j coefficients is divided into a number of subsets that have the same value of b_j. It is shown that the problem can be polynomially solved if k different types of b_j coefficients exist, k is a fixed number. In the second case, processing time of each job is described by the same function, $p_j = a_j + f(t)$, $f(t)$ is any nonegative, nondecrasing function of t. It is shown that the problem is polynomially solved.

1 Introduction

The aim of the paper is to consider two problems of scheduling of deteriorating jobs on a single processor subject to the total completion time objective. The first problem can be formulated as follows. We are given a set of n nonpreemptable jobs, with no ready times or deadlines. Processing time p_j of job j is a function of time, $p_j = 1 + b_j t$. All jobs are available at time $t = t_0 = 0$ and the objective is minimizing total completion time. The problem will be denoted in the sequel by $1/p_j = 1 + b_j t/\sum C_j$.

In the literature of the subject the total completion time criterion is considered rather rarely. There are known results for simple linear deterioration, $p_j = b_j t$, and the total completion time criterion: single processor case is easy [6], while multiple processor case is NP-hard [1], [3]. Up to now, however, there is known only one result for the problem with equal basic processing times, $p_j = 1 + b_j t$, established by Mosheiov [5]: the optimal schedule is V-shape like, i.e. in the optimal schedule jobs are arranged in decreasing order of b_j's if they are placed before the smallest b_j and in increasing order, if they are placed after such b_j. We show that if there exist "types" of linearly

deteriorating jobs, i.e. sets of b_j's with the same value of the coefficient, and there exists only k such types, k is a fixed number, then the problem can be solved in polynomial time.

The second problem is similar to the first one: all mentioned above assumptions remain valid but the processing time p_j of j-th job is equal to $p_j = a_j + f(t)$, where $f(t)$ is any nondecreasing function, positive for nonnegative t, $f(t) \geq 0$ for $t \geq 0$. We show that the problem can be solved in $O(n \log n)$ time for for a single processor. This problem will be denoted as $1/p_j = a_j + f(t)/\sum C_j$.

2 Results

In the section we show polynomial solvability of scheduling problems of a set of deteriorating jobs with processing times form of $p_j = 1 + b_j t$ (partial linear deterioration) and with processing times form of $p_j = a_j + f(t)$ (simple general deterioration), both on a single processor and subject to the total completion time criterion.

2.1 Partial linear deterioration

Lemma 1 [5] *Let n jobs with processing times form of $p_j = 1 + b_j t$, $j = 1, \ldots, n$, be given. Let $(\pi_1, \ldots, \pi_n)$ be a sequence of the jobs. Then the completion time of job k, $k = \pi_1, \ldots, \pi_n$, is equal to*

$$C_k = \sum_{i=1}^{k} \prod_{j=i+1}^{k} (1 + b_{\pi_j})$$

and the total completion time for the sequence $(\pi_1, \ldots, \pi_n)$ is equal to

$$\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} \sum_{j=1}^{i} \prod_{k=j+1}^{i} (1 + b_{\pi_k}).$$

Proposition 1 *The total completion time for a given permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ can be calculated in $O(n)$ time.*

To prove proposition 1 it is sufficient to note that for a sequence of completion times hold the following recurrence equations:

$$C_{\pi_1} = 1, C_{\pi_k} = (1 + b_{\pi_k}) C_{\pi_{k-1}} + 1$$

for k=2,. . . ,n.

Assume now the set of b_j coefficients is divided into a number of subsets (called *types*) and b_j coefficients belong to the same type if they have the same value. Firstly, note

that if there exist only one type of jobs, i.e. jobs' processing times are form of $p_j = 1+bt$, then the order of jobs does not have meaning and all schedules are optimal.

Thus we assume that there are k different b_j values where k is a given fixed number. Without loss of generality we assume that jobs are numbered in nonincreasing order of their deterioration rates, $b_i \geq b_{i+1}$ for $i = 1, 2, \ldots, k-1$.

As a corollary from theorem about V-shape property [5] we obtain the following.

Lemma 2 *In the optimal schedule jobs with the smallest coefficient, b_k, are scheduled as one group without inserting between them jobs with another b_j coefficients.*

Using a mathematical induction on the number k of types and lemma 2 one can prove the next result.

Proposition 2 *Let the set of b_j coefficients for the problem $1/p_j = 1 + b_j t/\sum C_j$ has k, $k < n$, different values, $b_j \in \{b_1, b_2, \ldots, b_k\}$. Then in the optimal schedule for the problem there exists V-shape according to types of jobs, i.e. jobs of type b_i, $i = 1, 2, \ldots, k-1$ precede in decreasing order jobs of type b_k and then follow jobs of type b_{k-i}, $i = 1, 2, \ldots, k-1$, in increasing order in such a way that jobs of the same type are executed as one group of jobs.*

On the base of proposition 2 one can construct an algorithm for finding the optimal schedule for the problem $1/p_j = 1 + b_j t, b_j \in \{b_1, b_2, \ldots, b_k\}/\sum C_j$, generating all possible candidate schedules that have V-shape property described in proposition 2 and choose the one with minimal total completion time.

Theorem 1 *The problem $1/p_j = 1 + b_j t, b_j \in \{b_1, b_2, \ldots, b_k\}/\sum C_j$ can be solved in $O(n^{k+1})$ time.*

To the proof note that the first step of the above mentioned algorithm needs $O(n^k)$ steps, and each step needs $O(n)$ operations for calculating the total completion time.

2.2 Simple general deterioration

The case when $p_j = a_j + bt$ is a special case of a more general form of deterioration when processing time of each job is described by the same function of job's starting time: $p_j = a_j + f(t)$ where $f(t)$ is nonnegative function for nonnegative values, i.e. when

$$f(t) \geq 0 \quad \text{for} \quad t \geq 0. \tag{2.1}$$

This kind of deterioration has been introduced by Melnikov and Shafransky [4]. If we assume additionally that $f(t)$ is nondecreasing function, i.e.

$$f(t_1) \leq f(t_2) \quad \text{for} \quad t_1 \leq t_2 \tag{2.2}$$

then holds the following.

Proposition 3 *Let $f(t)$ be a function that satisfy conditions* (2.1) *and* (2.2). *Then the problem $1/p_j = a_j + f(t)/\sum C_j$ can be solved in $O(n \log n)$ time by scheduling jobs in the ascending order of their a_j coefficients.*

The proof is based on the lemma [2] about minimizing the sum of products, $\sum_{j=1}^{n} a_j b_j$, of two sequences of numbers. Open question is for which functions, other than nondecreasing, proposition 3 will preserve its validity.

References

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