On the cohomology algebra of a fiber

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Abstract

Let $f : E \to B$ be a fibration of fiber F. Eilenberg and Moore have proved that there is a natural isomorphism of vector spaces between $H^*(F;\mathbb{F}_p)$ and $\mathrm{Tor}^{C^*(B)}(C^*(E),\mathbb{F}_p)$. Generalizing the rational case proved by Sullivan, Anick [2] proved that if X is a finite r-connected CW-complex of dimension $\leq rp$ then the algebra of singular cochains $C^*(X;\mathbb{F}_p)$ can be replaced by a commutative differential graded algebra $A(X)$ with the same cohomology. Therefore if we suppose that $f : E \hookrightarrow B$ is an inclusion of finite r-connected CW-complexes of dimension $\leq rp$, we obtain an isomorphism of vector spaces between the algebra $H^*(F;\mathbb{F}_p)$ and $\mathrm{Tor}^{A(B)}(A(E),\mathbb{F}_p)$ which has also a natural structure of algebra. Extending the rational case proved by Grivel-Thomas-Halperin [13], we prove that this isomorphism is in fact an isomorphism of algebras. In particular, $H^*(F;\mathbb{F}_p)$ is a divided powers algebra and p^{th} powers vanish in the reduced cohomology $\tilde{H}^*(F;\mathbb{F}_p)$.

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1 Introduction

Let $f : E \to B$ be a continuous map between pointed spaces. The inverse image of the base point $f^{-1}(*)$ is not in general a homotopy invariant of f. But after replacing E up to homotopy by a space X such that the new map $p : X \twoheadrightarrow B$ is a fibration, the space $p^{-1}(*)$, called the *homotopy fiber* of f, becomes a unique homotopy invariant of f. In particular, if $f : E \twoheadrightarrow B$ was initially a fibration, the *fiber* of f, $f^{-1}(*)$, has the homotopy type of the homotopy fiber.

We work over a field $\Bbbk$. The normalized singular cochain functor induces a morphism of differential graded algebras (DGA's) $C^*(f) : C^*(B) \to C^*(E)$. If f is a weak homotopy equivalence then $C^*(f)$ is a DGA morphism such that the map induced in homology $H^*(f)$ is an isomorphism (We say that $C^*(f)$ is a *quasi-isomorphism.*). So if two topological spaces X and Y are weakly homotopy equivalent then $C^*(X)$ and $C^*(Y)$ are linked by a chain of DGA quasi-isomorphisms, and we say that they are *weakly DGA-equivalent.* Note that the weak homotopy type of the DGA $C^*(X)$ is a much stronger homotopic invariant of X than the cohomology algebra $H^*(X)$.

Considering a DGA morphism $A \to M$, the homology of the complex $M \otimes_A \Bbbk$ is not invariant by DGA quasi-isomorphisms. But replacing M by an A-module P "free" in the category of A-modules such that there is a quasi-isomorphism of A-modules $P \overset{\simeq}{\twoheadrightarrow} M$ (We say that P is an *A-semifree resolution of* M.), the homology $H(P \otimes_A \Bbbk)$ becomes an invariant called the *differential torsion product* denoted $\mathrm{Tor}^A(M, \Bbbk)$. This differential torsion product generalizes the standard definition of torsion product in the non-graded non-differential case.

Let F denote the homotopy fiber of $f : E \to B$. The link between topology and algebra is provided by the Eilenberg-Moore formula which gives the isomorphism of graded vector spaces

$$H^*(F) \cong \mathrm{Tor}^{C^*(B)}(C^*(E), \Bbbk).$$

Generally this formula is used implicitly by applying the well-known Eilenberg-Moore spectral sequence. This formula allows the computation of the cohomology of F, $H^*(F)$, as a vector space. On the contrary, we don't know how to compute in general $H^*(F)$ as an algebra. Given a particular map $f : E \to B$ of homotopy fiber F, your best chance for computing the algebra $H^*(F)$ is to apply the formidable machinery of the Eilenberg-Moore or Serre spectral sequences using all their algebraic structure. But it does not always work.

In this article, we are interested in this problem: how to compute the cohomology algebra $H^*(F)$? Other works on the subject are [7] and [21].

When $A \to M$ is a morphism of commutative differential graded algebras (CDGA's), $\mathrm{Tor}^A(M, \Bbbk)$ has a natural structure of algebra ([18] Theorem VIII.2.1 in the non-graded non-differential case). When $\Bbbk = \mathbb{Q}$, Sullivan [22] proved that for any simply-connected topological space X, $C^*(X)$ is naturally weakly DGA-equivalent to a CDGA $A_{PL}(X)$. Replacing $C^*(B)$ and $C^*(E)$ by $A_{PL}(B)$ and $A_{PL}(E)$, $\mathrm{Tor}^{A_{PL}(B)}(A_{PL}(E), \Bbbk)$ has now an algebra structure and a theorem proved by Grivel [11], Thomas (unpublished) and Halperin [13], called the theorem of the model of the fibre showed that this algebra coincides with that of $H^*(F)$. Over a field $\Bbbk$ of characteristic 0, this theorem solves completely the problem of computing the algebra $H^*(F)$.

Over a field $\Bbbk$ of positive characteristic p, extending Sullivan's result, Anick ([2] dualize Proposition 8.7(a)) proved that if X is a finite r-connected CW-complex of dimension $\leq rp$ (We say that X is in the *Anick range.*), $C^*(X)$ is weakly DGA-equivalent to an CDGA $A(X)$ that we will call an *Anick model* of X. A natural question was to generalize the Grivel-Thomas-Halperin theorem in this new context and that is the main result of this paper:

Theorem 9.2 *Assume the characteristic of the field $\Bbbk$ is an odd prime p. Let $f : E \hookrightarrow B$ be an inclusion of CW-complexes with trivial r-skeleton and of dimension $\leq rp$. Let $A(E)$ and $A(B)$ denote their respective Anick models. If F is the homotopy fiber of f then*

$$H^*(F; \Bbbk) \cong \mathrm{Tor}^{A(B)}(A(E), \Bbbk) \textit{ as graded algebras.}$$

The case of the inclusion $* \hookrightarrow B$ has been proved by Halperin in [14]. In fact, he proved that there is an isomorphism of Hopf algebras $H^*(\Omega B; \Bbbk) \cong \mathrm{Tor}^{A(B)}(\Bbbk, \Bbbk)$.

In rational homotopy, the Grivel-Thomas-Halperin theorem, by staying at the level of semifree resolutions without taking their homology, not only gives the cohomology algebra of F but also its weak rational homotopy type: it gives a CDGA weakly CDGA-equivalent to $A_{PL}(F)$, so in particular weakly DGA-equivalent to $C^*(F)$. Our Theorem does not give in general a CDGA weakly DGA-equivalent to $C^*(F)$ (Remark 9.10). However, in this article we will adopt this idea that it is better to work at the level of semi-free resolution and we will not speak about Tor after this introduction. In particular, we will give a formulation of our main theorem as close as possible to the usual formulation for the Grivel-Thomas-Halperin theorem ([10], 15.5).

To prove our theorem, surprisingly, we will not use the previous Eilenberg-Moore formula but another Eilenberg-Moore formula. Consider a G-fibration $\pi : E \twoheadrightarrow X$: it means in particular that π is a fibration whose fiber G is a topological monoid acting on E. Then there is an isomorphism of graded vector spaces

$$H_*(X) \cong \mathrm{Tor}^{C_*(G)}(C_*(E), \mathbb{k}).$$

Let A be a DGA, M an A-module. A general way to compute $\mathrm{Tor}^A(M, \mathbb{k})$ is to consider the bar construction $B(M; A; A)$ which is an A-semifree resolution of M and to take the homology of $B(M; A) := B(M; A; A) \otimes_A \mathbb{k}$. In this second Eilenberg-Moore formula, following the general idea that to manipulate semi-free resolution is better than working with Tor, Félix, Halperin and Thomas remarked that it is more fruitful to consider the bar construction $B(C_*(E); C_*(G))$ instead of its homology $\mathrm{Tor}^{C_*(G)}(C_*(E), \mathbb{k})$. They constructed a natural coalgebra structure on the bar construction $B(C_*(E); C_*(G))$ and proved that the differential graded coalgebra (DGC) $B(C_*(E); C_*(G))$ is weakly DGC-equivalent to the DGC $C_*(X)$ [9].

Let $f : E \to B$ be a continuous map between path connected pointed spaces of homotopy fiber F. Starting Barratt-Puppe sequence, they showed that $B(C_*(F); C_*(\Omega B))$, where the Moore loop space ΩB acts on F by the holonomy action, is weakly DGC-equivalent to $C_*(E)$. Pursuing Barratt-Puppe sequence, we easily see that, when F is path connected, $B(C_*(\Omega B); C_*(\Omega E))$ is weakly DGC equivalent to $C_*(F)$ (Proposition 3.10). That is the starting observation of our paper. We now give the plan.

Section 2. We set up the notations, introduce some definitions and give some elementary properties.

Section 3. We review the work of Felix, Halperin and Thomas in [9]. In particular, we give a simple form of the Félix-Halperin-Thomas diagonal on the bar construction ([9] 4.1), analogous to the definition of the diagonal on $C_*(X)$ ([18] p. 245).

Section 4. We carefully review the notion of homotopy in the category of DGA's and of chain complexes using cylinders: the notion of homotopy does not depend of the cylinder considered, homotopies can be composed with maps, added, DGA homotopies are closely linked with derivations. We conclude by giving two lifting lemmas.

Section 5. We prove that the bar construction transforms homotopies of pairs of DGA's into chain complexes homotopies.

Section 6. Let $f : E \hookrightarrow B$ be an inclusion of simply connected CW-complexes of homotopy fiber F. The natural coalgebra structure on the bar construction $B(C_*(\Omega B); C_*(\Omega E))$ is determined by the Hopf algebras morphism $C_*(\Omega f) : C_*(\Omega E) \to C_*(\Omega B)$. Theorem 6.2 allows us to replace in the bar construction, this strict Hopf algebras morphism by the inclusion $(TX, \partial) \hookrightarrow (TY, \partial)$ between the Adams-Hilton models [1] of E and B, after having first equipped both DGA's (TX, ∂) and (TY, ∂) with a structure of Hopf algebra up to homotopy (HAH) such that the HAH structure on (TY, ∂) extends the one on (TX, ∂). Therefore, Theorem 6.2 gives the isomorphisms of algebras

$$H^*(F) \cong H^*(B(TY; TX)^\vee) \cong H^*((TY \otimes_{TX} \Bbbk)^\vee).$$

Section 7. As an application of Theorem 6.2, we compute the coalgebra $H_*(F_{\Sigma f})$ where $F_{\Sigma f}$ is the homotopy fiber of a suspended map injective in homology (Theorem 7.3).

Section 8. We define the homotopy cofiber of a CDGA morphism $A \to M$: it is a CDGA defined up to weak CDGA-homotopy type, whose homology coincides with the the algebra $\mathrm{Tor}^A(M, \Bbbk)$ ([18] Corollary VIII.2.3). At the level of CDGA's, we rediscovered that the algebra $\mathrm{Tor}^A(M, \Bbbk)$ can be computed either with an A-semifree resolution of M or with an A-semifree resolution of $\Bbbk$ and is invariant by CDGA quasi-isomorphisms.

Section 9. We prove Theorem 9.2 and show how to apply it to compute the cohomology algebra of some fiber.

Section 10. Let F be the homotopy fiber of an inclusion of CW-complexes in the Anick range. Then the cohomology algebra $H^*(F)$ is a divided powers algebra (Theorem 10.8).

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2 Algebraic preliminaries and notation

We work over an arbitrary field $\Bbbk$. References for these algebraic preliminaries are [9], [14], [15], [10], [5] and [2]. We just give our notations and recall the less-known definitions.

The symbol $\cong$ denotes an isomorphism. The homology functor from differential graded objects to graded objects is denoted H. The denomination "chain" will be restricted to objects with a non-negative lower degree and "cochain" to those with a non-negative upper degree. The degree of an element x is denoted $|x|$.

The *suspension* of a graded vector space M is the graded vector space sM such that $(sM)_{i+1} = M_i$.

Let C be an augmented complex. The kernel of the augmentation is denoted $\overline{C}$.

A *differential graded algebra*, or DGA, is a complex A equipped with two morphisms of complexes $\mu : A \otimes A \to A$ and $\eta : \mathbb{k} \to A$ called the *multiplication* and the *unit* such that $\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$ (associativity) and $\mu \circ (\eta \otimes 1) = 1 = \mu \circ (1 \otimes \eta)$ (unitary). The commutator isomorphism $\tau : A \otimes B \overset{\cong}{\to} B \otimes A$ is given by $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$. A *commutative* DGA or CDGA is a DGA such that $\mu \circ \tau = \mu$.

If $\frac{1}{2} \in \mathbb{k}$, a *differential divided powers algebra* or ? *-algebra* is an augmented CDGA A together with the maps

$$\gamma^k : \overline{A}_{2n} \to \overline{A}_{2nk} \quad (k \in \mathbb{N},\ n \in \mathbb{Z})$$

such that:

(i) $\gamma^0(a) = 1;\ \gamma^1(a) = a.$

(ii) $\gamma^i(a)\gamma^j(a) = \dfrac{(i+j)!}{i!j!}\gamma^{i+j}(a).$

(iii) $\gamma^k(a+b) = \sum_{i+j=k} \gamma^i(a)\gamma^j(b).$

(iv) $\gamma^i(ab) = \begin{cases} 0 & \text{if } |a|,\ |b| \text{ odd and } i \geq 2 \\ a^i\gamma^i(b) & \text{if } |a|,\ |b| \in \overline{A}_{even} \end{cases}$

(v) $\gamma^j(\gamma^i(a)) = \frac{(ij)!}{(i!)^j j!}\gamma^{ij}(a)$ if $i,\ j > 0$.

(vi) the differential d satisfies: $d\gamma^k(a) = d(a)\,\gamma^{k-1}(a)$.

Let A, B be two ?-algebras. A ? *-morphism* $f : A \to B$ is a morphism of augmented CDGA's such that $f\gamma^k(a) = \gamma^k f(a)$.

A *derivation* D in a graded algebra A is a linear map of degree $|D|$ such that $Dxy = Dx.y + (-1)^{|D||x|}x.Dy$, $x, y \in A$.

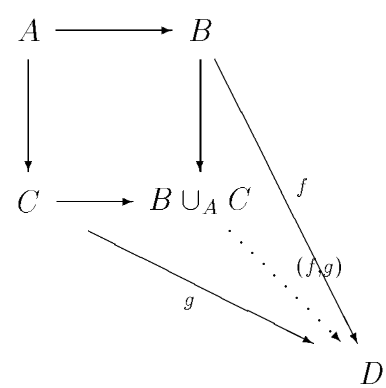
The tensor algebra TV on a complex V is the free DGA on V. The free CDGA on V is denoted ΛV.

The *free divided powers algebra* $?V$ on V is the free commutative graded algebra generated by $\gamma^i(v)$ for $v \in V_{even}$, $i \in \mathbb{N}^*$ and v for $v \in V_{odd}$ divided by the relations $\gamma^i(v)\gamma^j(v) = \dfrac{(i+j)!}{i!j!}\gamma^{i+j}(v)$. Over $\mathbb{Q}$, $?V \cong \Lambda V$ by $\gamma^i(v) \mapsto \dfrac{v^i}{i!}$.

Let A be a CDGA, V and W two graded vector spaces. A $?$-*derivation* in $A \otimes ?W$ is a derivation D such that $D\gamma^k(w) = D(w)\gamma^{k-1}(w)$, $k \geq 1$, $w \in W^{even}$. Any linear map $V \oplus W \to \Lambda V \otimes ?W$ of degree k extends to a unique $?$-derivation over $\Lambda V \otimes ?W$.

Let A be a DGA, M a right A-module, N a left A-module. The tensor product of M and N over A, denoted $M \otimes_A N$, is the complex quotient of $M \otimes N$ by the sub-complex generated by $m.a \otimes n - m \otimes a.n$, $m \in M$, $n \in N$, $a \in A$. If A is augmented, $M \otimes_A \mathbb{k} = M/M \cdot \overline{A}$.

Let $A \to B$, $A \to C$ be two morphisms in a category. If it exists, the push out and the morphism given by the universal property will be denoted as in the commutative diagram:



If it exists, the sum of B and C is denoted $B \amalg C$.

The push out exists in the category of DGA's. The push out is $B \otimes_A C$ in the category of CDGA's. In particular, the tensor product of DGA's is the sum in the category of CDGA's.

A *quasi differential graded coalgebra*, or quasi DGC, is an augmented complex C equipped with a morphism of augmented complexes $\Delta : C \to C \otimes C$ called the *diagonal*. Let C and C' be two quasi DGC's. A morphism of augmented chain complexes $f : C \to C'$ is a *morphism of quasi DGC's* if $\Delta f = (f \otimes f)\Delta$. A *differential graded coalgebra*, or DGC, is a quasi DGC such that $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$ (coassociativity) and $(\varepsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \varepsilon) \circ \Delta$ (counitary). A DGC is *cocommutative* if $\tau \circ \Delta = \Delta$. The dual $Hom(C, \mathbb{k})$ of a DGC C is a DGA denoted $C^\vee$.

A *quasi differential graded Hopf algebra*, or quasi DGH, is an augmented DGA K equipped with a morphism of augmented DGA's $\Delta : K \to K \otimes K$. A *differential graded Hopf algebra*, or DGH, is a quasi DGH such that $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \varepsilon) \circ \Delta$.

The notion of homotopy we use in the category of augmented chain complexes and of augmented DGA's, are recalled in section 4. The symbol $\approx$ stands for homotopic morphisms.

A *coalgebra up to homotopy* is a chain quasi DGC such that $(\Delta \otimes 1) \circ \Delta \approx (1 \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes 1) \circ \Delta \approx 1 \approx (1 \otimes \varepsilon) \circ \Delta$. Let C and C' be two coalgebras up to homotopy. A morphism of augmented chain complexes $f : C \to C'$ is a *morphism of coalgebras up to homotopy* if $\Delta f \approx (f \otimes f)\Delta$.

A *Hopf algebra up to homotopy*, or HAH, is a quasi DGH such that $(\Delta \otimes 1) \circ \Delta \approx (1 \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes 1) \circ \Delta \approx 1 \approx (1 \otimes \varepsilon) \circ \Delta$. Let K, K' be two HAH's. A morphism of augmented DGA's $f : K \to K'$ is a *HAH morphism* if $\Delta f \approx (f \otimes f)\Delta$.

Let K be a quasi DGH. A *quasi left K-coalgebra D* is both a quasi DGC and a left K-module such that the action $K \otimes D \to D$ is a quasi DGC morphism. A *K-coalgebra D* is a coassociative and counitary quasi K-coalgebra. Since $\Delta : K \to K \otimes K$ is a morphism of augmented DGA's then $\mathbb{k}$ is K-coalgebra.

Property 2.1 The tensor product of a quasi right K-coalgebra C and a quasi left K-coalgebra D over K, $C \otimes_K D$, is a quasi DGC. If C and D are coassociative, counitary, cocommutative then so is $C \otimes_K D$.

Further, the following example can be useful for computation:

Example 2.2 Let (A, d) and $(A \amalg TV, D)$ be two counitary quasi DGH's such that the inclusion $(A, d) \hookrightarrow (A \amalg TV, D)$ is a quasi DGH morphism. Then

$$(A \amalg TV, D) \otimes_{(A,d)} (\mathbb{k}, 0) \cong (T(A \otimes V), \overline{D}) \text{ as quasi DGC's.}$$

The diagonal on $T(A \otimes V)$ is the morphism of complexes

$$\Delta_{T(A \otimes V)} : T(A \otimes V) \to T(A \otimes V) \otimes T(A \otimes V)$$

where the image of a typical element $a_1 \otimes v_1 \otimes \cdots \otimes a_n \otimes v_n$ is

$$\sum \quad \pm([a'_1 a_{11} \otimes v_{11} \otimes \cdots \otimes a_{1p} \otimes v_{1p} \otimes a_{1p+1} a'_2 a_{21} \otimes v_{21} \otimes \cdots \otimes v_{np} \varepsilon(a_{np+1})] \\ \otimes [a''_1 b_{11} \otimes w_{11} \otimes \cdots \otimes b_{1q} \otimes w_{1q} \otimes b_{1q+1} a''_2 b_{21} \otimes w_{21} \otimes \cdots \otimes w_{nq} \varepsilon(b_{nq+1})])$$

$$\begin{aligned} \text{if} \quad \Delta_A a_i &= \sum a'_i \otimes a''_i, \\ \Delta_{A\amalg TV} v_i &= \sum (a_{i1} \otimes v_{i1} \otimes \cdots \otimes a_{ip} \otimes v_{ip} \otimes a_{ip+1}) \otimes (b_{i1} \otimes w_{i1} \otimes \cdots \otimes b_{iq} \otimes w_{iq} \otimes b_{iq+1}) \end{aligned}$$

and $\pm$ is the sign of the permutation

$$\begin{pmatrix} a'_1 a''_1 v_{11} \cdots a_{1p} v_{1p} a_{1p+1} b_{11} w_{11} \cdots b_{1q} w_{1q} b_{1q+1} a'_2 a''_2 \cdots a_{n1} v_{n1} \cdots a_{np} v_{np} a_{np+1} b_{n1} w_{n1} \cdots b_{nq} w_{nq} b_{nq+1} \\ a'_1 a_{11} v_{11} \cdots a_{1p} v_{1p} a_{1p+1} a'_2 a_{21} v_{21} \cdots v_{np} a_{np+1} a''_1 b_{11} w_{11} \cdots b_{1q} w_{1q} b_{1q+1} a''_2 b_{21} w_{21} \cdots w_{nq} b_{nq+1} \end{pmatrix}.$$

The differential is given by the formula:

$$\overline{D}(a_1 \otimes v_1 \otimes \cdots \otimes a_n \otimes v_n) = \sum_{i=1}^{n} (-1)^{\eta_i - |a_i|} a_1 \otimes v_1 \otimes \cdots \otimes da_i \otimes \cdots \otimes v_n$$
$$+ \sum_{i=1}^{n-1} (-1)^{\eta_i} \sum a_1 \otimes v_1 \cdots \otimes a_i c_{i1} \otimes u_{i1} \otimes \cdots \otimes c_{ir} \otimes u_{ir} \otimes c_{ir+1} a_{i+1} \otimes v_{i+1}$$
$$+ (-1)^{\eta_n} \sum a_1 \otimes v_1 \otimes \cdots \otimes a_n c_{n1} \otimes u_{n1} \otimes \cdots \otimes c_{nr} \otimes u_{nr} \varepsilon(c_{nr+1})$$

$$\begin{aligned} \text{if} \quad Dv_i &= \sum c_{i1} \otimes u_{i1} \otimes \cdots \otimes c_{ir} \otimes u_{ir} \otimes c_{ir+1} \\ \text{and} \quad \eta_i &= |a_1| + |v_1| + |a_2| + \cdots + |v_{i-1}| + |a_i|. \end{aligned}$$

A *differential graded Lie algebra*, or DGL, is a complex L equipped with a morphism of complexes: $[\ ,\ ] : L \otimes L \to L$ such that for x, y, $z \in L$:

- $[x, y] = -(-1)^{|x||y|}[y, x]$
- $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0$
- $[x, [x, x]] = 0,\ x \in L_{odd}$

The universal enveloping algebra of L is denoted UL.

A quasi-isomorphism is denoted $\xrightarrow{\simeq}$. Two objects A and B in a category $\mathcal{C}$ are *weakly $\mathcal{C}$-equivalent*, denoted $A \sim B$, if they are connected by a chain of $\mathcal{C}$-quasi-isomorphisms of the form:

$$A \xleftarrow{\simeq} A(1) \xrightarrow{\simeq} \cdots\cdots \xleftarrow{\simeq} A(n) \xrightarrow{\simeq} B.$$

If the $\mathcal{C}$-quasi-isomorphisms are natural, we say that A and B are *naturally weakly $\mathcal{C}$-equivalent.*

Let A be an augmented DGA, M a right A-module, N a left A-module. Denote d_1 be the differential of the complex $M \otimes T(s\overline{A}) \otimes N$ obtained by tensorization. We denote the tensor product of the elements $m \in M$, $sa_1 \in s\overline{A}$, ... , $sa_k \in s\overline{A}$ and $n \in N$ by $m[sa_1|\cdots|sa_k]n$. Let d_2 be the differential on the graded vector space $M \otimes T(s\overline{A}) \otimes N$ defined by:

$$\begin{aligned} d_2 m[sa_1|\cdots|sa_k]n \;&=\; (-1)^{|m|} m a_1 [sa_2|\cdots|sa_k]n \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} m[sa_1|\cdots|sa_i a_{i+1}|\cdots|sa_k]n \\ &\quad - (-1)^{\varepsilon_{k-1}} m[sa_1|\cdots|sa_{k-1}]a_k n; \end{aligned}$$

Here $\varepsilon_i = |m| + |sa_1| + \cdots + |sa_i|$.

Remark 2.3 We only find the above formula in the non-graded case in the literature ([18] X.(2.5)). We obtain the appropriate signs by Mac Lane's condensation of complexes of complexes ([18] X.9). If we set $N = \mathbb{k}$, we recover the same formula as in [9] §4.

The *bar construction of A with coefficients in M and N*, denoted $B(M; A; N)$, is the complex $(M \otimes T(s\overline{A}) \otimes N, d_1 + d_2)$. We use mainly $B(M; A) = B(M; A; \mathbb{k})$. The *reduced bar construction of A*, denoted $B(A)$, is $B(\mathbb{k}; A)$.

Let L be a DGL, M a right UL-module. If $\frac{1}{2} \in \mathbb{k}$, $B(M; UL)$ has a subcomplex $C_*(M; L) = (M \otimes ?\, sL, d_1 + d_2)$ [14, §1]. Its dual, denoted $C^*(M; L)$, is called the *Cartan-Chevalley-Eilenberg complex with coefficients in M*. Again $C_*(L)$ denotes $C_*(\mathbb{k}; L)$.

Let A be a DGA. A *semifree extension of an A-module M* is an inclusion of A-modules: $(M, d) \rightarrowtail (M \oplus (A \otimes V), D)$ such that:

- $V = \oplus_{k \in \mathbb{N}} V(k)$ as graded vector space.
- $D : V(k) \to M \oplus (A \otimes V(< k))$, $k \in \mathbb{N}$ where $V(< k) = \oplus_{i=0}^{k-1} V(i)$.

An A-*semifree module* is an A-module $(A\otimes V, D)$ such that $0\rightarrowtail(A\otimes V,D)$ is a semifree extension of 0.

A *free extension* is an inclusion of augmented DGA's: $(A,d)\rightarrowtail(A\amalg TV,D)$ such that $V=\oplus_{k\in\mathbb{N}}V(k)$ and $D:V(k)\to A\amalg TV(<k)$, $k\in\mathbb{N}$.
A *free DGA* is a DGA (TV,D) such that $\Bbbk\rightarrowtail(TV,D)$ is a free extension.

A *relative Sullivan model* is an inclusion of augmented CDGA's: $(A,d)\rightarrowtail(A\otimes\Lambda V,D)$ such that $V=\oplus_{k\in\mathbb{N}}V(k)$ and $D:V(k)\to A\otimes\Lambda V(<k)$, $k\in\mathbb{N}$.
A *Sullivan model* is a CDGA $(\Lambda V,D)$ such that $\Bbbk\rightarrowtail(\Lambda V,D)$ is a relative Sullivan model.
A *Sullivan model of a DGA A* is a Sullivan model $(\Lambda V,D)$ equipped with a DGA quasi-isomorphism $(\Lambda V,D)\overset{\simeq}{\to}A$.

A ?*-free extension* is an inclusion of augmented CDGA's: $(A,d)\rightarrowtail(A\otimes\,?\,V,D)$ such that $V=\oplus_{k\in\mathbb{N}}V(k)$, $D:V(k)\to A\otimes\,?\,V(<k)$, $k\in\mathbb{N}$ and D is a ?-derivation. In particular, if A is a ?-algebra, than the ?-free extension $(A,d)\rightarrowtail(A\otimes\,?\,V,D)$ is a ?-morphism.

Note that the condition on the graded vector space V in these four similar definitions is always satisfied in chain. In particular, an inclusion of chain DGA's $(A,d)\rightarrowtail(A\amalg TV,D)$ is always a free extension.

The complex of *indecomposables* of the augmented DGA A, denoted $Q(A)$ is $\overline{A}/\overline{A}\cdot\overline{A}$. The augmented DGA A is *minimal* if the differential on $Q(A)$ is zero. An inclusion of augmented DGA's $A\hookrightarrow B$ is *minimal* if the augmented DGA $B\cup_A\Bbbk$ is minimal. In particular, an inclusion of augmented CDGA's $A\hookrightarrow B$ is minimal if the augmented CDGA $B\otimes_A\Bbbk$ is minimal.

Property 2.4 (i) If $A\rightarrowtail A\amalg TV$ is a free extension then $A\amalg TV$ is (left and right) A-semifree.

(ii) [10, 14.1] If $A\rightarrowtail A\otimes\Lambda V$ is a relative Sullivan model then $A\otimes\Lambda V$ is A-semifree.

(iii) If $A\rightarrowtail A\otimes\,?\,V$ is a ?-free extension then $A\otimes\,?\,V$ is A-semifree.

The normalized singular chain complex of a topological space X with coefficients in $\Bbbk$ is denoted $C_*(X)$.

Let G be a topological monoid. A right G-*Serre fibration* is a Serre fibration $p:E\twoheadrightarrow B$ such that E is a right G-space, for each $b\in B$ the *fiber* $p^{-1}(b)$ is stable by G and for each $z\in E$ the map $g\mapsto z.g$ is a weak homotopy equivalence from G to $p^{-1}(p(z))$.

Convention 2.5 Let $f : E \to B$ be a continuous map. When we will speak about the homotopy fiber of f, except if specified, we will choose the homotopy fiber where the holonomy acts on the left and denotes it by F_f.

3 The bar construction with coefficients as a DGC

Property 3.1 ([18] X.7.2) Let A (respectively B) be an augmented DGA, M (respectively N) a left A-module (respectively B-module) and P (respectively Q) a right A-module (respectively B-module). Then we have an Alexander-Whitney morphism of complexes

$$AW : B(P \otimes Q; A \otimes B; M \otimes N) \to B(P; A; M) \otimes B(Q; B; N)$$

where the image of a typical element $p \otimes q[s(a_1 \otimes b_1)| \cdots |s(a_k \otimes b_k)]m \otimes n$ is

$$\sum_{i=0}^{k} (-1)^{\zeta_i} p[sa_1| \cdots |sa_i]a_{i+1} \cdots a_k m \otimes qb_1 \cdots b_i[sb_{i+1}| \cdots |sb_k]n.$$

$$\begin{aligned} \text{Here} \quad \zeta_i &= \sum_{j=1}^{k} \left(|q| + \sum_{l=1}^{j-1} |b_l| \right) |a_j| + \left(|q| + \sum_{j=1}^{k} |b_j| \right) |m| \\ &+ \sum_{j=i+1}^{k} (j-i)|a_j| + (k-i)|m| + |i||q| + \sum_{j=1}^{i-1} (i-j)|b_j|. \end{aligned}$$

AW is natural and associative ($AW \circ (AW \otimes id) = AW \circ (id \otimes AW)$).

Remark 2.3 holds here too.

Corollary 3.2 *Let K be a quasi DGH, C a quasi right K-coalgebra, D a quasi left K-coalgebra. Then $B(C; K; D)$ is a quasi DGC with the diagonal*

$$B(C; K; D) \xrightarrow{B(\Delta_C; \Delta_K; \Delta_D)} B(C \otimes C; K \otimes K; D \otimes D) \xrightarrow{AW} B(C; K; D) \otimes B(C; K; D)$$

$$\textit{and the counit} \quad B(C; K; D) \xrightarrow{B(\varepsilon_C; \varepsilon_K; \varepsilon_D)} B(\mathbb{k}; \mathbb{k}; \mathbb{k}) = \mathbb{k}.$$

If K, C and D are coassociative, counitary then so is $B(C; K; D)$. This coalgebra structure on $B(C; K; D)$ is functorial.

Proof. It is obvious with commutative diagrams using AW's associativity, naturality and the functoriality of the bar construction. $\boxed{QED}$

Property 3.3 Moreover, if C is K-semifree then $B(C;K;D) \stackrel{\simeq}{\twoheadrightarrow} C \otimes_K D$ is a quasi-isomorphism of quasi DGC's.

Proof. The morphism of quasi left K-coalgebras $B(K;K;D) \stackrel{\simeq}{\twoheadrightarrow} D$ remains a quasi-isomorphism of quasi DGC's after applying $C \otimes_K -$ ([9] 2.3 (i) and Property 2.1). $\boxed{QED}$

Remark 3.4 When K is a DGH and C is a K-coalgebra, the coalgebra structure on $B(C;K)$ coincides with the one defined in ([9] 4.1). The proof is a tedious calculation. Anyway, we don't need to give it, since we will verify that the following theorem is valid independently of the functorial coalgebra structure chosen on the bar construction, either the one defined by Félix-Halperin-Thomas, or the one defined in Corollary 3.2.

Theorem 3.5 *([9] 5.1) Let $p : E \twoheadrightarrow B$ be a right G-Serre fibration with B path connected. Then there is a natural DGC quasi-isomorphism $B(C_*(E);C_*(G)) \stackrel{\simeq}{\twoheadrightarrow} C_*(B)$.*

Remark 3.6 This natural quasi-isomorphism is the identity on $C_*(E)$ for the $*$-fibration $id : E \to E$.

Proof. As shown in Theorem 8.3 of [10], if $m : M \stackrel{\simeq}{\twoheadrightarrow} C_*(E)$ is a right $C_*(G)$-semifree resolution of $C_*(E)$ then we have the commuting diagram of complexes.

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{m} & C_*(E) \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle C_*(p)} \\ M \otimes_{C_*(G)} \mathbb{k} & \xrightarrow[\overline{m}]{\simeq} & C_*(B) \end{array}$$

In particular, we can take $M = B(C_*(E);C_*(G);C_*(G))$. Since m, $C_*(p)$ and ζ are DGC morphisms and ζ is an epimorphism, $\overline{m}$ is a DGC morphism too. $\boxed{QED}$

Remark 3.7 Suppose further that G acts from the left on a space Y and that the map $q : E \times_G Y \twoheadrightarrow B$ defined by $q(z, y) = p(z)$ for $z \in E$ and $y \in Y$ is a Serre fibration such that for each $z \in E$ the map $y \mapsto (z, y)$ is a weak homotopy equivalence from Y to $q^{-1}(p(z))$. Then there is a natural DGC quasi-isomorphism $B(C_*(E); C_*(G); C_*(Y)) \xrightarrow{\simeq} C_*(E \times_G Y)$. The proof is the same as above interpreting the general Eilenberg-Moore formula ([12] Theorem 3.9) $H_*(E \times_G Y) \cong \mathrm{Tor}^{C_*(G)}(C_*(E), C_*(Y))$ at the chain level.

Proposition 3.8 *([9] 6.7) Let $f : E \to B$ be a continuous map between path connected spaces and F_f its homotopy fiber then there is a natural DGC quasi-isomorphism $B(\Bbbk; C_*(\Omega B); C_*(F_f)) \xrightarrow{\simeq} C_*(E)$.*

Remark 3.9 This natural quasi-isomorphism is the identity on $C_*(E)$ for the map $E \to *$.

Proof. The Moore path space fibration $PB \twoheadrightarrow B$ with PB being the Moore paths that begin at the basepoint, is a left ΩB-fibration. So, by pull back, we obtain a left ΩB-fibration $p_0 : F_f \twoheadrightarrow E$. We apply Theorem 3.5 to p_0. QED

Proposition 3.10 *Let $f : E \to B$ be a continuous pointed map with E, B and F_f path connected. Then $C_*(F_f)$ is naturally weakly DGC equivalent to $B(C_*(\Omega B); C_*(\Omega E))$.*

Proof. We have the morphism of topological monoids $\Omega f : \Omega E \to \Omega B$. So ΩB is a right ΩE-space and $C_*(\Omega f) : C_*(\Omega E) \to C_*(\Omega B)$ is a DGH morphism. Consider the previous map $p_0 : F_f \twoheadrightarrow E$. Let $\tilde{F}_{p_0}$ denote its homotopy fiber where the holonomy acts on the right. There is a natural morphism of right ΩE-spaces $\eta : \tilde{F}_{p_0} \xrightarrow{\simeq} \Omega B$ which is a homotopy equivalence and is the identity if $E = *$ ([10] I.2.(c)). By Proposition 3.8 applied to p_0, we have the chain of natural DGC quasi-isomorphisms

$$C_*(F_f) \xleftarrow{\simeq} B(C_*(\tilde{F}_{p_0}); C_*(\Omega E)) \xrightarrow[B(\eta;id)]{\simeq} B(C_*(\Omega B); C_*(\Omega E)).$$

QED

Remark 3.11 This chain of natural quasi-isomorphisms is just the identity on $C_*(\Omega B)$ for the map $* \to B$. So by naturality, we have the commutative diagram of DGC's

$$\begin{array}{ccc}
C_*(\Omega B) & \longrightarrow & B(C_*(\Omega B); C_*(\Omega E)) \\
{\scriptstyle C_*(\partial)}\downarrow & \searrow & \uparrow{\scriptstyle \simeq\, B(\eta; id)} \\
C_*(F_f) & \overset{\simeq}{\longleftarrow} & B(C_*(\tilde{F_{p0}}); C_*(\Omega E))
\end{array}$$

where $\partial : \Omega B \hookrightarrow F_f$ is the inclusion.

4 Homotopy of augmented chain complexes and of augmented DGA's and lifting lemmas

We recall the notion of homotopy of augmented chain complexes and of augmented DGA's using cylinders since our proof will rely heavily on it.

To develop homotopy theory using cylinders in a category, a good framework is to have a structure of cofibration category where all objects are fibrant.

Definition 4.1 (Compare [4] I.1.1) A *cofibration category where all objects are fibrant* is a category **C** with two classes of morphisms called *cofibrations* (denoted by $\rightarrowtail$) and *weak equivalences* (denoted by $\overset{\simeq}{\to}$), subject to axioms C1, C2, C3 and C4. The axioms in question are:

(C1) *Composition axiom*: The isomorphisms in **C** are weak equivalences and also cofibrations. For two maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

if any two of f, g and $g \circ f$ are weak equivalences, then so is the third. The composite of cofibrations is a cofibration.

(C2) *Push out axiom*: For a cofibration $i : B \rightarrowtail A$ and map $f : B \to Y$

there exists the push out in $\mathbf{C}$

$$\begin{array}{ccc} B & \xrightarrow{\;f\;} & Y \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \bar{i}} \\ A & \xrightarrow[\;\bar{f}\;]{} & A \cup_B Y \end{array}$$

and $\bar{i}$ is a cofibration. Moreover:

(a) if f is a weak equivalence, so is $\bar{f}$,

(b) if i is a weak equivalence, so is $\bar{i}$.

(C3) *Factorization axiom*: For a map $f : B \to Y$ in $\mathbf{C}$ there exists a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\qquad f \qquad} & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle g,\ \simeq} & \\ & A & \end{array}$$

where i is a cofibration and g is a weak equivalence.

(C4) *All objects are fibrant*: Each cofibration which is also a weak equivalence $i : R \overset{\simeq}{\rightarrowtail} Q$ in $\mathbf{C}$ admits a retraction $r : R \to Q$, $r \circ i = 1$.

The axiom C4 can be replaced in Definition 4.1 by the Property:

Property 4.2 ([4] II.1.11=II.2.11a)) Given a commutative diagram of unbroken arrows

$$\begin{array}{ccc} B & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p}\ \simeq \\ A & \xrightarrow[\;g\;]{} & Y \end{array}$$

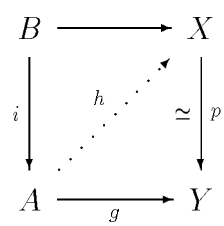
where i is a cofibration and p is a weak equivalence then there is a map h for which the upper triangle commutes.

Property 4.3 (i) ([4] II.6.4) The category of augmented chain complexes is a cofibration category where all objects are fibrant and where cofibrations are injections and weak equivalences, quasi-isomorphisms.

(ii) ([4] II.7.10) The category of augmented DGA's is a cofibration category where all objects are fibrant and where cofibrations are free extensions and weak equivalences, quasi-isomorphisms.

Remark 4.4 In a cofibration category where all objects are fibrant, Property 4.2 is the key to obtain the basic properties of the notion of homotopy. In the particular case of augmented chain complexes and of augmented DGA's (Property 4.3), we can use the following lifting lemma instead of Property 4.2.

Property 4.5 Given a commutative diagram of unbroken arrows



where i is a cofibration and p is both a surjection and a weak equivalence then the dotted arrow h exists such that both triangles commute.

We can now define the notion of homotopy in the category of augmented chain complexes and in the category of augmented DGA's.

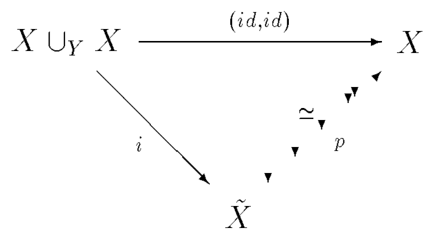
In this section, the morphism $Y \to X$ is going to be either

(i) a cofibration and we define *homotopy relative* Y *or under* Y and follow [4] II§2 in the case of a cofibration category where all objects are fibrant,

(ii) or just the unit of X, $\Bbbk \to X$ and we define absolute homotopy for augmented DGA's following [9] §3.

Remark 4.6 In a cofibration category $\mathbf{C}$ with an initial object Φ, homotopy relative Φ is called *absolute homotopy* and we call an object X *cofibrant* if $\Phi \rightarrowtail X$ is a cofibration. Following case (i), absolute homotopy is defined only when X is cofibrant. In the category of augmented chain complexes, all objects are cofibrant and therefore by case (i), absolute homotopy is defined for every complex X. In the category of augmented DGA's, only free DGA's are cofibrant but case (ii) define absolute homotopy even when X is a not a free DGA.

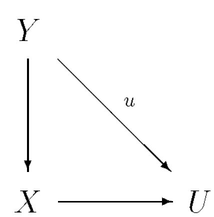
Definition 4.7 ([19] 1.1) An object denoted $\tilde{X}$ is a *left homotopy object* on $Y \to X$ if there is a factorization of the folding map



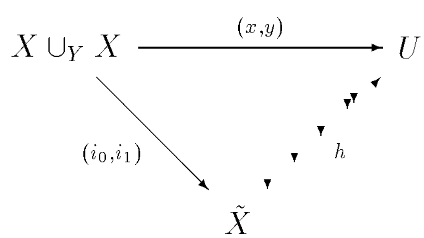
Remark 4.8 Let i_0 (respectively i_1) be the composite of the first (respectively second) inclusion $X \to X \cup_Y X$ with i. Then by universal property, $i = (i_0, i_1)$ and we use this last notation.

Definition 4.9 A *cylinder* on $Y \to X$, denoted $I_Y X$ is a left homotopy object on $Y \to X$ such that (i_0, i_1) is a cofibration. If the category has an initial object Φ, IX will stand for a cylinder on $\Phi \to X$ instead of $I_\Phi X$.

Let $u : Y \to U$ be a fixed morphism. Let $x, y : X \to U$ be two morphisms such that for each of them the following diagram commutes:



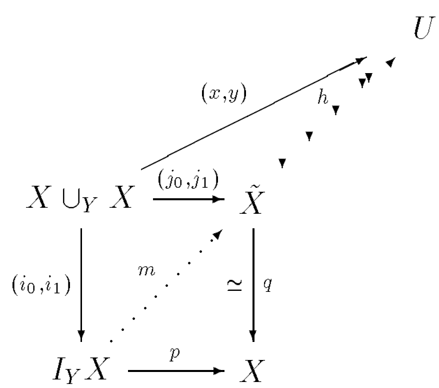
Definition 4.10 The morphisms x and y are *homotopic* for the left homotopy object $\tilde{X}$ if there is a commutative diagram



We call h a *homotopy* from x to y, and denoted it $h : x \approx y$.

Property 4.11 If we fix a cylinder $I_Y X$, then for any homotopy $h : x \approx y$ starting from a left homotopy object $\tilde{X}$, there exists a homotopy $h' : x \approx y$ starting from $I_Y X$. In particular, all cylinders define the same notion of homotopy between morphisms.

Proof. By the lifting lemma (Property 4.5), we obtain a morphism $m : I_Y X \to \tilde{X}$ such that the following diagram commutes

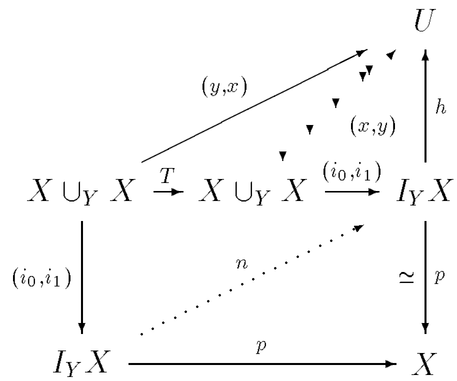


and we set $h' = h \circ m$. QED

Property 4.12 The homotopy relation defined with a cylinder is an equivalence relation.

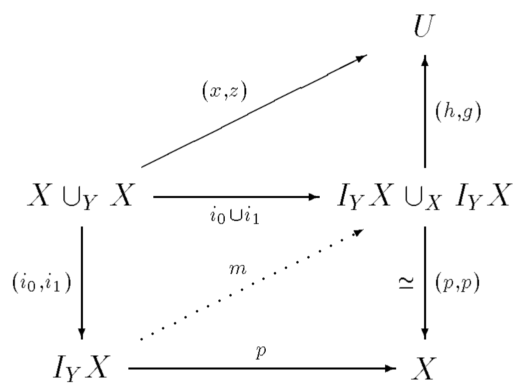
Definition 4.13 (i) The homotopy $x \circ p : x \approx x$ is called the *trivial homotopy* and is denoted 0.

(ii) Let $h : x \approx y$ be a homotopy. By the lifting lemma (Property 4.5), we obtain a morphism $n : I_Y X \to I_Y X$ such that the following diagram commutes



Here T is the interchange map of the two factors. The homotopy $h \circ n : y \approx x$ is called a *negative* of the homotopy h and is denoted $-h$.

(iii) Let $h : x \approx y$ and $g : y \approx z$ be two homotopies for the same cylinder $I_Y X$. The push out of two cylinders is a left homotopy object. So again as in Property 4.11, we can apply the lifting lemma (Property 4.5) to the diagram

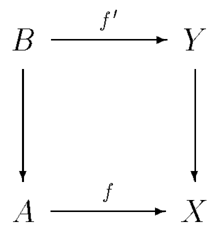


The homotopy $(h, g) \circ m : x \approx z$, is called the *sum* of the homotopies and is denoted $h + g$.

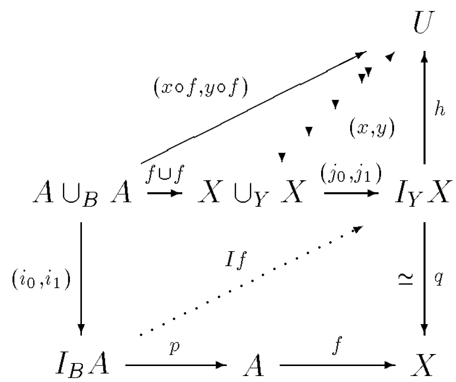
Property 4.14 The notion of homotopy is stable by composition.

Proof. • Let $g : U \to V$ be a morphism and $h : x \approx y$ be a homotopy. Then $g \circ h : g \circ x \approx g \circ y$ is a homotopy.

- Let



be any commutative diagram with $B \rightarrowtail A$ a cofibration or the unit of A. Then by the lifting lemma (Property 4.5), we obtain a morphism $If : I_B A \to I_Y X$ such that the following diagram commutes



So $h \circ If : x \circ f \approx y \circ f$ is the desired homotopy. $\boxed{QED}$

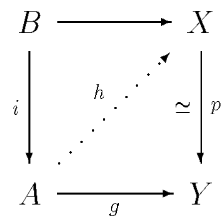
Definition 4.15 We denote by $If : I_B A \to I_Y X$ any morphism from a cylinder on $B \to A$ to a cylinder on $Y \to X$ such that the preceding diagram commutes.

Remark 4.16 In the category of augmented DGA when X is a free DGA, there is a canonical cylinder IX called the Baues-Lemaire cylinder and a canonical map $If : IA \to IX$ ([4] I.7.15). For this cylinder, a given homotopy $h : x \approx y$ has a canonical negative $-h$ and the sum of two homotopies is canonically defined ([4] II.17.3).

For the Baues-Lemaire cylinder, any homotopy h from x to y corresponds uniquely to an (x, y)-derivation H ([9] 3.5, [4] I.7.12). Of course, the canonical negative of the homotopy h corresponds to the (y, x)-derivation $-H$. And the composite of the homotopy h and of the canonical map If, $h \circ If$ corresponds to the $(x \circ f, y \circ f)$-derivation $H \circ f$. Warning, the sum $H + G$ of an (x, y)-derivation H and an (y, z)-derivation G is not in general an (x, z)-derivation.

In section 6, we will use for DGA's two lifting lemmas other than Property 4.5, the first of which refines Property 4.2.

Property 4.17 ([4] II.1.11=II.2.11a)) Given a commutative diagram of unbroken arrows

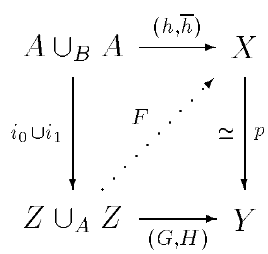


where i is a cofibration and p is a weak equivalence then

(i) there is a map h for which the upper triangle commutes and for which $p \circ h$ is homotopic to g relative to B, and

(ii) this map h is unique up to homotopy relative to B.

Proof. We recall just the proof of (ii). Let h and $\overline{h}$ be two maps satisfying (i) and let H and G be homotopies relatively to B for a cylinder Z from $p \circ h$ to g and from g to $p \circ \overline{h}$ respectively. We apply Property 4.2 to the commutative diagram



Now F is a homotopy from h to $\overline{h}$ for the cylinder $Z \cup_A Z$. QED

Property 4.18 ([9] 3.6) Consider the following diagram, that commutes up to a homotopy H:

$$\begin{array}{ccc} TV & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p}\ \simeq \\ TW & \underset{g}{\longrightarrow} & Y \end{array}$$

where TV and TW are free DGA's, i is a cofibration and p is a weak equivalence. Then there exists a map h for which the upper triangle commutes and such that $p \circ h$ is homotopic to g. The homotopy G from $p \circ h$ to g can be chosen such that $G \circ Ii = H$ (G extends H).

5 Bar construction and homotopies

After reviewing Félix-Halperin-Thomas diagonal on the bar construction and the notion of homotopy defined with cylinders, we prove in this section the key lemma from which derives all our theorems. This lemma is a homotopic version of Corollary 3.2. First, we need a "functoriality up to homotopy" of the bar construction provided by the Property.

Property 5.1 Let

$$\begin{array}{ccc} A & \xrightarrow{h:\varphi\approx\varphi'} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[h':\Psi\approx\Psi']{} & M' \end{array}$$

be a "diagram" of chain augmented DGA's where $h : IA \to A'$ and $h' : IM \to M'$ are homotopies, and where $\Psi \circ f = g \circ \varphi$, and $\Psi' \circ f = g \circ \varphi'$. Consider one of the morphisms $If : IA \to IM$ (Definition 4.15). If $h' \circ If = g \circ h$ (*naturality of the homotopies*) then the morphisms of augmented chain complexes $B(\Psi; \varphi)$ and $B(\Psi'; \varphi')$ are homotopic.

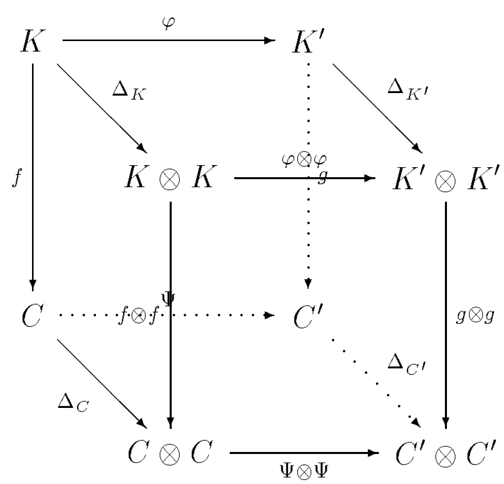
Proof. Since the bar construction is a functor preserving quasi-isomorphisms from the category of pairs of chain augmented DGA's to the category of augmented chain complexes ([9] 4.3(iii)), $B(IM; IA)$ is a left homotopy object on $0 \rightarrowtail B(M; A)$ in the category of augmented chain complexes. So $B(h'; h) : B(\Psi; \varphi) \approx B(\Psi'; \varphi')$ is a homotopy. $\boxed{QED}$

Lemma 5.2 *(i) Let K (respectively C) be a strictly counitary chain HAH, coassociative up to a homotopy h_{assocK} (respectively h_{assocC}): $(\Delta \otimes 1) \circ \Delta \approx (1 \otimes \Delta) \circ \Delta$. Let $f : K \to C$ be a morphism of augmented DGA's such that $\Delta_C f = (f \otimes f)\Delta_K$ and $h_{assocC} I f = (f \otimes f \otimes f) h_{assocK}$ (f commutes with the diagonals and the homotopies of coassociativity). Then $B(C; K)$ with the diagonal*

$$B(C; K) \xrightarrow{B(\Delta_C; \Delta_K)} B(C \otimes C; K \otimes K) \xrightarrow{AW} B(C; K) \otimes B(C; K)$$

is a strictly counitary coalgebra up to homotopy.

(ii) Consider the following cube of augmented chain DGA's



where all the faces commute exactly except the top and the bottom ones. Suppose that the top face commutes up to a homotopy h_{top} : $(\varphi \otimes \varphi)\Delta_K \approx \Delta_{K'}\varphi$ and the bottom face commutes up to a homotopy $h_{bottom} : (\Psi \otimes \Psi)\Delta_C \approx \Delta_{C'}\Psi$ such that $h_{bottom} I f = (g \otimes g) h_{top}$. Then the morphism of augmented chain complexes $B(\Psi; \varphi) : B(C; K) \to B(C'; K')$ commutes with the diagonals up to homotopy.

Proof. the same as the proof of Corollary 3.2, with Property 5.1 replacing the functoriality of the bar construction. QED

Remark 5.3 The results of this section remain true if we replace the bar construction by any functor B preserving quasi-isomorphisms from the category of pairs of chain augmented DGA's to the category of augmented chain complexes equipped with a natural, associative morphism of augmented complexes

$$AW : B(P \otimes Q; A \otimes B) \to B(P; A) \otimes B(Q; B).$$

In particular, Lemma 5.2 is valid for the functor $B(M; A) = M \otimes_A \Bbbk$ if $f : K \to C$ is a free extension and K is a free DGA (The extra hypothesis is needed to preserve quasi-isomorphisms.).

Remark 5.4 There is a generalization of Lemma 5.2(ii) to homotopy commutative cubes. In [20], we define diagonals on $B(C; K)$ and $B(C'; K')$ and a morphism of augmented chain complexes from $B(C; K)$ to $B(C'; K')$ commuting with the diagonals up to homotopy provided that the homotopies in each face of the cube satisfies a compatibility condition.

6 HAH structure on free models

Let X be a graded vector space. We denote a free DGA (TX, ∂) simply by TX except when the differential ∂ can be specified. In particular, a free DGA with zero differential is still denoted by $(TX, 0)$.

Definition 6.1 ([19] D.28) An *explicit HAH* is a free DGA TX equipped with a morphism of augmented DGA's $\Delta : TX \to TX \otimes TX$ such that $(\varepsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \varepsilon) \circ \Delta$, a homotopy $h_{assoc} : (\Delta \otimes 1) \circ \Delta \approx (1 \otimes \Delta) \circ \Delta$ and a homotopy $h_{com} : \Delta \approx \tau\Delta$. Note that if $(TX, \Delta, h_{assoc}, h_{com})$ is an *explicit HAH* then (TX, Δ) is a strictly counitary HAH, coassociative and cocommutative up to homotopy. Let $(TX, \Delta_{TX}, h_{assocTX}, h_{comTX})$ and $(TY, \Delta_{TY}, h_{assocTY}, h_{comTY})$ be two explicit HAH's. Let $f : TX \to TY$ be an augmented DGA morphism. Then f is a *morphism of explicit HAH's* if $f(X) \subset Y$, $\Delta_{TY} f = (f \otimes f)\Delta_{TX}$, $h_{assocTY} If = (f \otimes f \otimes f)h_{assocTX}$ and $h_{comTY} If = (f \otimes f)h_{comTX}$.

Theorem 6.2 *Let $f : E \to B$ be a map between path connected pointed topological spaces with a path connected homotopy fiber F. We consider a* *commutative diagram of augmented chain algebras as follows:*

$$\begin{array}{ccc} TX & \xrightarrow[\Theta_X]{\simeq} & C_*(\Omega E) \\ {\scriptstyle m(f)}\downarrow & & \downarrow{\scriptstyle C_*(\Omega f)} \\ TY & \xrightarrow[\Theta_Y]{\simeq} & C_*(\Omega B) \end{array}$$

where TX, TY are free DGA's and $m(f) : TX \rightarrowtail TY$ is a free extension. Then

1. *TX (respectively TY) can be endowed with an explicit HAH structure such that Θ_X (respectively Θ_Y) commutes with the diagonals up to a homotopy h_{Θ_X} (respectively h_{Θ_Y}) and such that $m(f)$ is a morphism of explicit HAH's and h_{Θ_Y} extends $(C_*(\Omega f) \otimes C_*(\Omega f))h_{\Theta_X}$.*

2. *$B(\Theta_Y; \Theta_X) : B(TY; TX) \overset{\simeq}{\rightrightarrows} B(C_*(\Omega B); C_*(\Omega E))$ is a morphism of coalgebras up to homotopy.*

3. *The homology of the coalgebra up to homotopy $TY \otimes_{TX} \Bbbk$ is isomorphic to $H_*(F)$ as coalgebras.*

Remark 6.3 • The isomorphism of graded coalgebras between $H_*(TY \otimes_{TX} \Bbbk)$ and $H_*(F)$ fits into the commutative diagram of graded coalgebras:

$$\begin{array}{ccc} H_*(TY) & \xrightarrow[H_*(\Theta_Y)]{\cong} & H_*(\Omega B) \\ {\scriptstyle H_*(q)}\downarrow & & \downarrow{\scriptstyle H_*(\partial)} \\ H_*(TY \otimes_{TX} \Bbbk) & \xrightarrow{\cong} & H_*(F) \end{array}$$

where $\partial : \Omega B \hookrightarrow F$ is the inclusion and $q : TY \twoheadrightarrow TY \otimes_{TX} \Bbbk$ the quotient map.

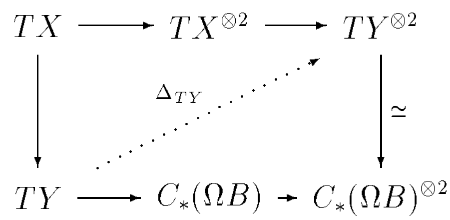
• The quasi DGC $TY \otimes_{TX} \Bbbk$ can be made explicit using Example 2.2.

Remark 6.4 • The exact commutativity of the diagram in Theorem 6.2 is not important. If the diagram commutes only up to homotopy, since $m(f)$ is a cofibration, by Property 4.18, we can replace Θ_Y by another Θ_Y which is homotopic to it, so that now the diagram strictly commutes.

• But it is important that $m(f)$ is a cofibration. We will show it in Remark 7.5. Indeed, the general idea for the proof of 1 is to control the homotopies using the homotopy extension property of cofibrations.

Proof of Theorem 6.2

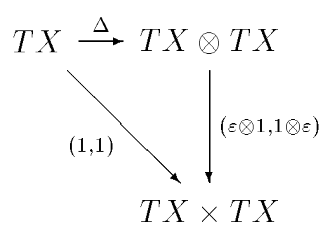
1. By Property 4.17(i), we put a diagonal on TX, Δ_{TX}, such that Θ_X commutes with the diagonals up to a homotopy h_{Θ_X}. The diagram of unbroken arrows



commutes, with homotopy $C_*(\Omega f)^{\otimes 2} h_{\Theta_X}$. By Property 4.18, there exists a diagonal on TY, Δ_{TY}, satisfying

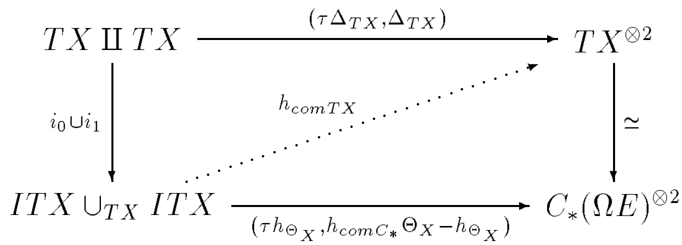
$$(*) \quad \begin{cases} \Delta_{TY} \text{ extends the diagonal on } TX \text{ such that there exists a} \\ \text{homotopy } h_{\Theta_Y} \text{ between } (\Theta_Y \otimes \Theta_Y)\Delta_{TY} \text{ and } \Delta_{C_*(\Omega B)}\Theta_Y \\ \text{extending } C_*(\Omega f)^{\otimes 2} h_{\Theta_X}. \end{cases}$$

We can assume that, both the diagonal of TX and the diagonal of TY are counitary. Let's give a sketch of proof of that: Since $C_*(\Omega E)$ has a counitary diagonal, by Property 4.17(ii), Δ_{TX} is counitary up to a homotopy h_{unitTX}. That is, the diagram

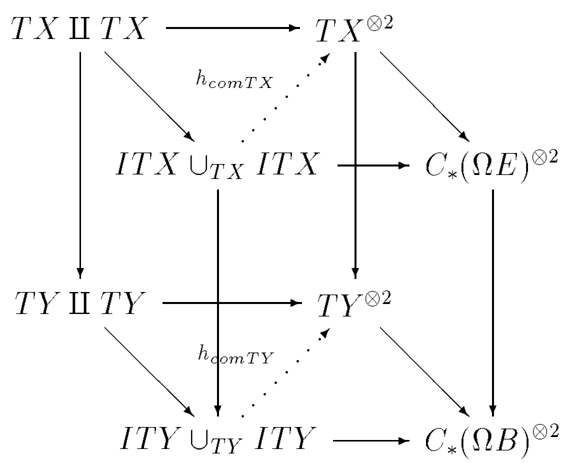


commutes up to the homotopy h_{unitTX}. Furthermore, Δ_{TY} is counitary up to a homotopy h_{unitTY} extending h_{unitTX}. We can change the diagonal of TX up to homotopy to get a counitary one [2, Lemma 5.4 i)]. Moreover, since h_{unitTY} extends h_{unitTX}, we can change up to homotopy the diagonal of TY to get a counitary one such that the condition $(*)$ is still satisfied with the new counitary diagonals.

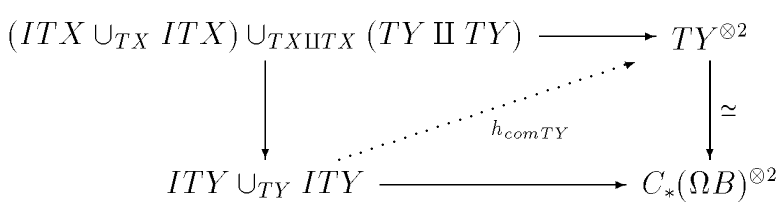
We give now a detailed proof that Δ_{TX} is cocommutative up to a homotopy h_{comTX} and that Δ_{TY} is cocommutative up to a homotopy h_{comTY} extending h_{comTX}: Since the diagonal on $C_*(\Omega E)$ is cocommutative up to a homotopy h_{comC_*}, by Property 4.17(ii), Δ_{TX} is cocommutative up to a homotopy h_{comTX}. More precisely (Proof of Property 4.17(ii)), h_{comTX} is given by Property 4.2 in the diagram:



where ITX is the Baues-Lemaire cylinder (Remark 4.16). Now, since the homotopy of cocommutativity of $C_*(\Omega B)$ is natural ([2] (23)) and the sums and negatives of homotopies are canonically defined (Remark 4.16), the following cube of unbroken arrows is commutative:



The homotopy of cocommutativity of TY, h_{comTY} is given by applying Property 4.2 to the commutative diagram



A similar proof shows that Δ_{TX} is coassociative up to a homotopy $h_{assocTX}$ and that Δ_{TY} is coassociative up to a homotopy $h_{assocTY}$ extending $h_{assocTX}$.

2. Now, by Lemma 5.2, the augmented chain complexes quasi-isomorphism

$$B(\Theta_Y;\Theta_X): B(TY;TX) \stackrel{\simeq}{\to} B(C_*\Omega B; C_*\Omega E)$$

commutes with the diagonals up to homotopy. Since $m(f)$ is a morphism of explicit HAH's, this diagonal on $B(TY;TX)$ is counitary exactly and is coassociative up to homotopy.

3. By Property 2.4(i) and Property 3.3, the augmented chain complexes quasi-isomorphism

$$B(TY;TX) \stackrel{\simeq}{\to} TY \otimes_{TX} \Bbbk$$

commutes exactly with the diagonals. By Remark 5.3, $TY \otimes_{TX} \Bbbk$ is a strictly counitary coalgebra up to homotopy, coassociative up to homotopy and cocommutative up to homotopy. By Proposition 3.10, $C_*(F)$ is weakly DGC equivalent to $B(C_*\Omega B; C_*\Omega E)$. So now by 2, the coalgebra $H_*(TY \otimes_{TX} \Bbbk)$ is isomorphic to $H_*(F)$. *QED*

7 The fiber of a suspended map

Lemma 7.1 *Let X be a path connected space. Then there is a natural DGH quasi-isomorphism $T\overline{C_*(X)} \stackrel{\simeq}{\to} C_*(\Omega\Sigma X)$.*

Proof. The adjunction map ad induces a morphism of coaugmented DGC's $C_*(ad): C_*(X) \to C_*(\Omega\Sigma X)$. By universal property of the tensor algebra on the complex $\overline{C_*(X)}$, denoted $T\overline{C_*(X)}$, $C_*(ad)$ extends to a natural DGH morphism. By Bott-Samelson Theorem ([16] appendix 2 Theorem 1.4), it is a quasi-isomorphism, since the functors H and T commute. *QED*

Lemma 7.2 *Let $f: E \to B$ be a continuous map between path connected spaces. Then $C_*(F_{\Sigma f})$ is naturally weakly DGC equivalent to $B(T\overline{C_*(B)}; T\overline{C_*(E)})$.*

Proof. It is a direct consequence of Lemma 7.1, Proposition 3.10 and Corollary 3.2. *QED*

Theorem 7.3 *Let $f: E \to B$ be a continuous map between path connected spaces such that $H_*(f)$ is injective. Then the graded coalgebra $TH_+(B)\otimes_{TH_+(E)} \Bbbk$ is isomorphic to $H_*(F_{\Sigma f})$.*

Remark 6.3 holds here too.
Proof of Theorem 7.3 Since $H_*(f)$ is injective, we can apply Theorem 6.2 and Lemma 7.2 to the homotopy commutative diagram of DGA's:

$$\begin{array}{ccc} (TH_+(E),0) & \xrightarrow{\simeq} & T\overline{C_*(E)} \\ {\scriptstyle TH_+(f)}\downarrow & & \downarrow{\scriptstyle T\overline{C_*(f)}} \\ (TH_+(B),0) & \xrightarrow{\simeq} & T\overline{C_*(B)} \end{array} \qquad (7.4)$$

Since the horizontal arrows induce the identity in homology, the diagonals on $TH_+(E)$ and $TH_+(B)$ must be obtained by tensorization of the diagonals of $H_+(E)$ and $H_+(B)$. QED

Remark 7.5 If $H_*(f)$ is not injective, Theorem 7.3 is not true in general: the algebra $H^*(F)$ does not depend only on $H_*(f)$. Indeed, since $TH_+(f)$ is not a free extension, we cannot apply Theorem 6.2 to the diagram 7.4.

For an example over $\mathbb{F}_p$, we can take a map f from S^{2p-1} to $\mathbb{CP}^{p-1}$. Let y_2 be a generator of $H^2(F_{\Sigma f})$. If f is the Hopf map, there is a map $\psi : \mathbb{CP}^p \to F_{\Sigma f}$ such that $H^2(\psi)$ is an isomorphism. So $y_2^p \neq 0$. If f is the constant map then $y_2^p = 0$.

Remark 7.6 When $H_*(B)$ is of finite type and $H_1(f)$ is an isomorphism, the isomorphism given by Theorem 7.3 can be proved using a spectral sequence argument. Recall first that by the Bott-Samelson Theorem, the adjunction maps ad induce an isomorphism of graded coalgebras between $TH_+(B) \otimes_{TH_+(E)} \Bbbk$ and $H_*(\Omega\Sigma B) \otimes_{H_*(\Omega\Sigma E)} \Bbbk$. The inclusion $\partial : \Omega\Sigma B \to F_{\Sigma f}$ is up to a homotopy equivalence a right $\Omega\Sigma E$-fibration (Proof of Proposition 3.10). So $\ker H_*(\partial)$ contains the left ideal generated by $\mathrm{Im} H_+(\Omega\Sigma f)$ and by Property 2.1, $H_*(\partial)$ induces a morphism of graded coalgebras

$$\overline{H_*(\partial)} : H_*(\Omega\Sigma B) \otimes_{H_*(\Omega\Sigma E)} \Bbbk \to H_*(F_{\Sigma f}).$$

Since $H_*(\Omega\Sigma f)$ is injective, the Serre spectral sequence applied to ∂ collapses at the E_2-term, $H_*(\partial)$ is surjective and $\ker H_*(\partial)$ is isomorphic to $H_*(F_{\Sigma f}) \otimes H_+(\Omega\Sigma E)$. Using again the Bott-Samelson Theorem, $\ker H_*(\partial)$ is the left ideal generated by the image of $H_+(E) \stackrel{H_+(f)}{\longrightarrow} H_+(B) \stackrel{H_+(ad)}{\longrightarrow} H_+(\Omega\Sigma B)$. So $\overline{H_*(\partial)}$ is an isomorphism.

8 Homotopy cofibers for CDGA's

In this section, we develop the notions of cofibration, cofiber, homotopy cofiber and homotopy push out in the category of augmented CDGA's. We give an example of homotopy cofiber crucial for the proof of Theorem 9.2 and we notice that the weak CDGA equivalence class of homotopy cofibers of a CDGA morphism is preserved if one changes the CDGA morphism up to quasi-isomorphisms.

Definition 8.1 Let $i : A \to C$ be a morphism of augmented CDGA's. Consider the A-module structure on C induced by i. If C is an A-semifree module, we say that i is a *cofibration* (in the category of augmented CDGA's) and we denote $i : A \rightarrowtail C$. The *cofiber* of a cofibration $i : A \rightarrowtail C$ is the augmented CDGA $\mathbb{k} \otimes_A C$.

Property 8.2 The category of augmented CDGA's where the cofibrations are the morphisms as defined above and where the weak equivalences are the quasi-isomorphisms, satisfies axioms $C1$, $C2$ and $C3$ of Definition 4.1 (but not C4!).

Remark 8.3 If we restrict our definition of cofibrations to morphisms being relative Sullivan models, the category of augmented $\mathbb{Q}$-CDGA's forms a cofibration category where all objects are fibrant ([4] I.§8). However, over a field of characteristic p, the category of augmented CDGA's is still not a cofibration category.

The topological notions of homotopy push out and homotopy cofibers can be defined more generally in any category with a final object satisfying axiom $C1$, $C2$ and $C3$ ([6], chapter 4).

Using Property 8.2, we develop now the notion of homotopy push out and of homotopy cofiber in the category of augmented CDGA's:

Let $f : A \to B$, $g : A \to C$ be two morphisms of augmented CDGA's. Consider two factorizations $f = p \circ i$, $g = q \circ j$ where $i : A \rightarrowtail D$, $j : A \rightarrowtail E$ are cofibrations and p, q quasi-isomorphisms. By Property 8.2, we can construct the commutative diagram of augmented CDGA's:

$$
\begin{array}{ccccc}
A & \xrightarrow{i} & D & \xrightarrow[p]{\simeq} & B \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \bar{j}} & & \downarrow \\
E & \xrightarrow{\bar{i}} & D \otimes_A E & \xrightarrow{\simeq} & B \otimes_A E \\
\simeq \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \simeq} & & \\
C & \longrightarrow & D \otimes_A C & &
\end{array}
\qquad (8.4)
$$

All the rectangles appearing in this diagram are push outs, $\bar{i}$ and $\bar{j}$ are cofibrations. We have a chain of quasi-isomorphisms of augmented CDGA's

$$D \otimes_A C \xleftarrow{\simeq} D \otimes_A E \xrightarrow{\simeq} B \otimes_A E.$$

In particular, the augmented CDGA's $D \otimes_A C$, $D \otimes_A E$ and $B \otimes_A E$ are weakly CDGA equivalent and their weak CDGA equivalence class is independent of the factorization chosen of f and g.

Definition 8.5 The augmented CDGA's $D \otimes_A C$, $D \otimes_A E$ and $B \otimes_A E$ obtained by considering various factorizations of f and g as above are called *homotopy push outs* of f and g. All the homotopy push outs of f and g are weakly CDGA equivalent. The *homotopy cofibers* of f are the homotopy push outs of f and of the augmentation on A, $\varepsilon : A \twoheadrightarrow \Bbbk$.

Example 8.6 Let $f : E \to B$ a chain Lie algebras morphism. If B is positively graded and of finite type then $C^*(UB; E) = ((UB)^\vee \otimes (?\, sE)^\vee, d_1 + d_2)$ equipped with the tensor product algebra structure becomes a CDGA which is a homotopy cofiber of $C^*(f) : C^*(B) \to C^*(E)$.
Proof. By [9] 6.10, $C^*(UB; B)$ is an acyclic CDGA. Since B is of finite type, $C^*(UB; B)$ is $C^*(B)$-semifree. By the universal property of push out, there is a CDGA morphism

$$C^*(E) \otimes_{C^*(B)} C^*(UB; B) \xrightarrow{\cong} C^*(UB; E)$$

which is an isomorphism since B is of finite type. So we get the commutative diagram of augmented CDGA's

$$\begin{array}{ccccc} & & C^*(B) & \xrightarrow{C^*(f)} & C^*(E) \\ & \swarrow & \downarrow & & \downarrow \\ \Bbbk & \xleftarrow{\simeq} & C^*(UB;B) & \longrightarrow & C^*(UB;E) \end{array}$$

where the square is a push out and where $C^*(B) \to C^*(UB;B)$ is a cofibration. Therefore, $C^*(UB;E)$ is a homotopy push out of $C^*(f)$ and of the augmentation of $C^*(B)$. QED

Proposition 8.7 *(particular case of [6] 4.13) Suppose given a commutative diagram of augmented CDGA's*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \simeq & & \downarrow \simeq \\ A' & \xrightarrow{f'} & B' \end{array}$$

where the vertical arrows are quasi-isomorphisms. Consider two factorizations $f = \Phi \circ i$, $f' = \Phi' \circ i'$ where $i : A \rightarrowtail C$ is a morphism of augmented CDGA's such that C is an A-semifree module, $i' : A' \rightarrowtail C'$ is a morphism of augmented CDGA's such that C' is an A'-semifree module and $\Phi : C \overset{\simeq}{\twoheadrightarrow} B$, $\Phi' : C' \overset{\simeq}{\twoheadrightarrow} B'$ are quasi-isomorphisms of augmented CDGA's. Then the cofibers $\Bbbk \otimes_A C$ and $\Bbbk \otimes_{A'} C'$ are weakly CDGA equivalent.

Proof. By Property 8.2, we have the commutative diagram of augmented CDGA's

$$\begin{array}{ccccc} A & \xrightarrow{i} & C & \underset{\simeq}{\xrightarrow{\Phi}} & B \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ A' & \xrightarrow{\bar{i}} & A' \otimes_A C & \underset{\simeq}{\overset{\bar{\Phi}}{\cdots\rightarrow}} & B' \end{array}$$

where $\bar{i}$ is a cofibration and $f' = \overline{\Phi} \circ \bar{i}$. So the CDGA $\Bbbk \otimes_{A'} A' \otimes_A C$ is a homotopy cofiber of f' as the CDGA $\Bbbk \otimes_{A'} C'$. Therefore they are weakly CDGA equivalent. QED

9 The fiber of the model in the Anick range

Let $r \geq 1$ be a fixed integer. p is going to be the least noninvertible prime (or $+\infty$) in $\Bbbk$. We suppose now $p \neq 2$.

Definition 9.1 [14] A topological space X is (r,p)-*mild* or in the *Anick range* if it is r-connected and its homology is concentrated in degrees $\leq rp$ and of finite type.

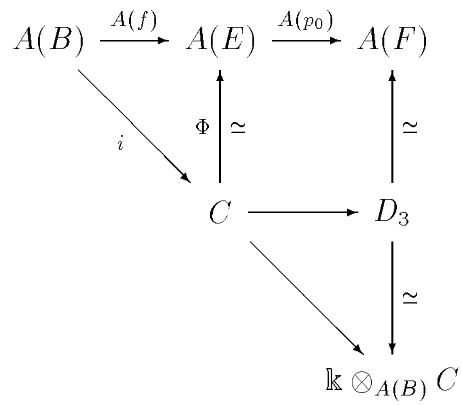
Theorem 9.2 *Let $f : E \to B$ be a continuous map between two topological spaces both (r,p)-mild with $H_{rp}(f)$ injective. Consider the homotopy fiber F and the fibration $p_0 : F \twoheadrightarrow E$. Then there are two morphisms of augmented CDGA's, denoted $A(f) : A(B) \to A(E)$ and $A(p_0) : A(E) \to A(F)$ such that*

1. *there is a commutative diagram of cochain complexes*

$$\begin{array}{ccccc}
C^*(B) & \xrightarrow{C^*(f)} & C^*(E) & \xrightarrow{C^*(p_0)} & C^*(F) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
D_1(B) & \longrightarrow & D_1(E) & \longrightarrow & D_1(F) \\
\uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
D_2(B) & \longrightarrow & D_2(E) & \longrightarrow & D_2(F) \\
\downarrow \simeq & & \downarrow \simeq & & \Theta \downarrow \simeq \\
A(B) & \xrightarrow{A(f)} & A(E) & \xrightarrow{A(p_0)} & A(F)
\end{array}$$

where all the vertical maps are quasi-isomorphisms and where all the maps are DGA morphisms except $\Theta : D_2(F) \xrightarrow{\simeq} A(F)$ who induces a morphism of graded algebras only in homology.

2. *for any factorization $A(f) = \Phi \circ i$ where $i : A(B) \rightarrowtail C$ is a morphism of augmented CDGA's such that C is an $A(B)$-semifree module and where $\Phi : C \xrightarrow{\simeq} A(E)$ is a quasi-isomorphism of augmented CDGA's, we have a commutative diagram of augmented CDGA's*



In particular, the cohomology algebra of the homotopy fiber of f, $H^*(F)$, is isomorphic to the cohomology of the homotopy cofiber of $A(f)$, $H^*(\mathbb{k} \otimes_{A(B)} C)$.

Remark 9.3 Over $\mathbb{Q}$, the functor A_{PL} due to Sullivan [22] is such that the two CDGA morphisms $A_{PL}(f) : A_{PL}(B) \to A_{PL}(E)$ and $A_{PL}(p_0) : A_{PL}(E) \to A_{PL}(F)$ verifies 1 and 2: by Corollary 10.10 of [10], for any topological space X, there are natural quasi-isomorphisms of cochain algebras

$$C^*(X) \xrightarrow{\simeq} D(X) \xleftarrow{\simeq} A_{PL}(X)$$

and by the Grivel-Thomas-Halperin theorem "the fiber of a model is a model of the fiber" ([10], 15.5), $\mathbb{k} \otimes_{A(B)} C$ is weakly CDGA-equivalent to $A_{PL}(F)$.

Proof. By naturality of Proposition 3.10 with respect to continuous maps, we have a commutative diagram of DGC's

$$\begin{array}{ccccc}
C_*(F) & \xrightarrow{C_*(p_0)} & C_*(E) & \xrightarrow{C_*(f)} & C_*(B) \\
\uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
G(F) & \longrightarrow & G(E) & \longrightarrow & G(B) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
B(C_*(\Omega B); C_*(\Omega E)) & \longrightarrow & BC_*(\Omega E) & \xrightarrow{BC_*(\Omega f)} & BC_*(\Omega B)
\end{array} \qquad (9.4)$$

There is a commutative diagram of augmented DGA's

$$\begin{array}{ccccc}
TX & \xrightarrow{\simeq} & \Omega C_*(E) & \overset{\simeq}{\rightarrow} & C_*(\Omega E) \\
\downarrow m(f) & & \downarrow \Omega C_*(f) & & \downarrow C_*(\Omega f) \\
TY & \xrightarrow{\simeq} & \Omega C_*(B) & \overset{\simeq}{\rightarrow} & C_*(\Omega B)
\end{array}$$

where Ω denotes the cobar construction ([8] Theorem I), TX is a minimal free DGA and $m(f) : TX \rightarrowtail TY$ is a minimal free extension. Since the indecomposables functor Q preserves quasi-isomorphism between free DGA's ([5] 1.5),

$$X \cong s^{-1}H_+(E) \quad \text{and} \quad Y \cong s^{-1}H_+(E) \oplus s^{-1}\text{coker}H_+(f) \oplus \ker H_+(f).$$

So X and Y are graded vector spaces of finite type concentrated in degree $\geq r$ and $\leq rp-1$. Denote by Θ_X the composite $TX \overset{\simeq}{\rightarrow} \Omega C_*(E) \overset{\simeq}{\rightarrow} C_*(\Omega E)$ and by Θ_Y the composite $TY \overset{\simeq}{\rightarrow} \Omega C_*(B) \overset{\simeq}{\rightarrow} C_*(\Omega B)$. By Theorem 6.2, $m(f) : TX \rightarrowtail TY$ is a morphism of explicit HAH's and $B(\Theta_Y;\Theta_X) : B(TY;TX) \overset{\simeq}{\rightarrow} B(C_*(\Omega E); C_*(\Omega B))$ is a morphism of coalgebras up to homotopy.

By the naturality of Anick's Theorem ([19] D.29 and D.21, see also the proof of Theorem 8.5(g)[2]), there exists a DGL morphism $L(f) : L(E) \rightarrow$

$L(B)$ and a commutative diagram of DGA's

$$\begin{array}{ccc} UL(E) & \xrightarrow[\varphi]{\cong} & TX \\ {\scriptstyle UL(f)}\downarrow & & \downarrow{\scriptstyle m(f)} \\ UL(B) & \xrightarrow[\Psi]{\cong} & TY \end{array}$$

where φ and Ψ are two DGA isomorphisms equipped with two DGA homotopies

$$h_{top} : (\varphi \otimes \varphi)\Delta_{UL(E)} \approx \Delta_{TX}\varphi \quad \text{and} \quad h_{bottom} : (\Psi \otimes \Psi)\Delta_{UL(B)} \approx \Delta_{TY}\Psi$$

$$\text{such that} \quad h_{bottom} I(UL(f)) = (m(f) \otimes m(f))h_{top}$$

(*the horizontal arrows commute with the diagonals up to natural homotopies*). By Lemma 5.2(ii), the isomorphism $B(\Psi;\varphi) : B(UL(B);UL(E)) \overset{\cong}{\to} B(TY;TX)$ commutes up to chain homotopy with the diagonals. We give $C_*(UL(B);L(E))$ the tensor product coalgebra structure of $UL(B) \otimes ?\, sL(E)$. The injection $C_*(UL(B);L(E)) \overset{\simeq}{\to} B(UL(B);UL(E))$ is a DGC quasi-isomorphism ([9] 6.11). By functoriality of the bar construction and the Cartan-Chevalley-Eilenberg complex with coefficients, finally we get the commutative diagram of coalgebras up to homotopy

$$\begin{array}{ccccc} B(C_*(\Omega B);C_*(\Omega E)) & \longrightarrow & BC_*(\Omega E) & \xrightarrow{BC_*(\Omega f)} & BC_*(\Omega B) \\ {\scriptstyle B(\Theta_Y;\Theta_X)}\uparrow\simeq & & {\scriptstyle B(\Theta_X)}\uparrow\simeq & & {\scriptstyle B(\Theta_Y)}\uparrow\simeq \\ B(TY;TX) & \longrightarrow & B(TX) & \xrightarrow{B(m(f))} & B(TY) \\ {\scriptstyle B(\Psi;\varphi)}\uparrow\cong & & {\scriptstyle B(\varphi)}\uparrow\cong & & {\scriptstyle B(\Psi)}\uparrow\cong \\ B(UL(B);UL(E)) & \longrightarrow & B(UL(E)) & \xrightarrow{B(UL(f))} & B(UL(B)) \\ \uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq \\ C_*(UL(B);L(E)) & \longrightarrow & C_*L(E) & \xrightarrow{C_*L(f)} & C_*L(B) \end{array} \qquad (9.5)$$

where all the coalgebras up to homotopy are counitary and coassociative exactly except $B(TY;TX)$, all the morphisms commute exactly with the diagonals except $B(\Theta_Y;\Theta_X)$ and $B(\Psi;\varphi)$, and where all the vertical maps are quasi-isomorphisms. Define $A(f)$ to be $C^*L(f): C^*L(B)\to C^*L(E)$ and $A(p_0)$ to be the inclusion $C^*L(E)\hookrightarrow C^*(UL(B);L(E)$. By dualizing diagram 9.4 and diagram 9.5, we obtain the diagram of 1.

By its definition, the CDGA $\Bbbk\otimes_{A(B)} C$ is a homotopy cofiber of $A(f)$ (Definition 8.5). The CDGA $A(F):=C^*(UL(B);L(E))$ is also a homotopy cofiber of $A(f):=C^*(L(f))$ (Example 8.6). So $A(F)$ is weakly CDGA-equivalent to $\Bbbk\otimes_{A(B)}C$. More precisely diagram 8.4 in the proof of Definition 8.5 gives the diagram of 2 with $D_3=C^*(UL(B);L(B))\otimes_{C^*L(B)} C$. QED

To construct a factorization of $A(f)$ is quite difficult. As in the rational case, we would rather construct a factorization of a model of $A(f)$:

Corollary 9.6 • *Let $A(f): A(B)\to A(E)$ be a CDGA morphism as in Theorem 9.2. Let ΛY be a Sullivan model of $A(B)$, ΛX a Sullivan model of $A(E)$. Then there is an acyclic CDGA U and a commutative diagram of CDGA's*

$$\begin{array}{ccccc} & & \Lambda Y & \xrightarrow{\simeq} & A(B) \\ & \swarrow^{\Psi} & \downarrow & & \downarrow{\scriptstyle A(f)} \\ \Lambda X & \xleftarrow{\simeq} & \Lambda X\otimes U & \xrightarrow{\simeq} & A(E) \end{array}$$

• *Let $\Lambda Y\rightarrowtail C\overset{\simeq}{\twoheadrightarrow}\Lambda X$ be a factorization of $\Psi:\Lambda Y\to\Lambda X$ such that C is a ΛY-semifree module. Then the algebra $H^*(F)$ is isomorphic to $H^*(\Bbbk\otimes_{\Lambda Y}C)$. (This isomorphism identifies in homology $C^*(p_0): C^*(E)\to C^*(F)$ and the quotient map $C\twoheadrightarrow\Bbbk\otimes_{\Lambda Y}C$.)*

Proof. The first part of this Corollary is just Proposition 7.7 and Remark 7.8 of [14]. The second part is Proposition 8.7 and Theorem 9.2. QED

As in the rational case, we can take a factorization of Ψ with relative Sullivan models. But mod p, since the p^{th} power of an element of even degree is always a cycle, our relative Sullivan model will have infinitely many generators. We'd rather use a free divided powers algebra ?V where for $v\in V_{even}$, $v^p=0$. But now arises the problem of constructing CDGA morphisms from a free divided power algebra to any CDGA where the p^{th} powers are not zero. We give now an effective construction of a factorization of Ψ with divided powers algebras. Over $\mathbb{Q}$, this factorization will be just a factorization of Ψ through a minimal relative Sullivan model.

Lemma 9.7 *Let $\Psi : (\Lambda Y, d) \to (\Lambda X, d)$ be a CDGA morphism between two minimal Sullivan models such that X and Y are concentrated in degree ≥ 2. Then there is an explicit factorization of Ψ:*

$$(\Lambda Y, d) \overset{i}{\rightarrowtail} (\Lambda Y \otimes \Lambda coker\varphi \otimes ?\, sker\varphi, D) \underset{p}{\overset{\simeq}{\twoheadrightarrow}} (\Lambda X, d)$$

where

- *φ is the composite $Y \hookrightarrow \Lambda Y \overset{\Psi}{\to} \Lambda X \twoheadrightarrow X$ and D is a ?-derivation,*
- *i is a minimal inclusion of augmented CDGA's such that $(\Lambda Y \otimes \Lambda coker\varphi \otimes ?\, sker\varphi, D)$ is $(\Lambda Y, d)$-semifree, and*
- *p is a surjective CDGA quasi-isomorphism vanishing on $?\, sker\varphi$.*

Proof. We proceed by induction on the degree $n \in \mathbb{N}^*$. Suppose we have constructed the factorization:

$$\left(\Lambda(Y^{\leq n})\right), d \rightarrowtail \left(\Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes ?\, s(\mathrm{ker}\varphi^{\leq n}), D\right) \underset{p_n}{\overset{\simeq}{\twoheadrightarrow}} \left(\Lambda(X^{\leq n}), d\right)$$

Let $w \in \mathrm{coker}\varphi^{n+1}$. Define $p_{n+1}(w)$ in obvious way. $dp_{n+1}(w)$ is a cycle of $\Lambda X^{\leq n}$. Since p_n is a surjective quasi-isomorphism, there is a cycle $z \in \Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes ?\, s(\mathrm{ker}\varphi^{\leq n})$ such that $p_n(z) = dp_{n+1}(w)$. Define $Dw = z$.

Let $v \in \mathrm{ker}\varphi^{n+1}$. Since p_{n+1} is a surjective morphism of graded algebras, there is $u \in \Lambda^{\geq 2}(Y^{\leq n} \oplus \mathrm{coker}\varphi^{\leq n})$ such that $p_{n+1}(v+u) = 0$. Since $D(v+u)$ is a cycle of $\Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes ?\, s(\mathrm{ker}\varphi^{\leq n})$ and p_n is a surjective quasi-isomorphism, there is $\gamma \in \Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes ?\, s(\mathrm{ker}\varphi^{\leq n})$ such that $p_n(\gamma) = 0$ and $D\gamma = D(v+u)$. Define $Dsv = v + u - \gamma$.

Now we have the commutative diagram of CDGA's:

$$\begin{array}{ccc}
\Lambda(Y^{\leq n})\otimes\Lambda(\mathrm{coker}\varphi^{\leq n})\otimes ?\, s(\ker\varphi^{\leq n}), D & \xrightarrow[\simeq]{p_n} & \Lambda(X^{\leq n}), d \\
\downarrow & & \downarrow \\
\Lambda(Y^{\leq n+1})\otimes\Lambda(\mathrm{coker}\varphi^{\leq n+1})\otimes ?\, s(\ker\varphi^{\leq n+1}), D & \xrightarrow{p_{n+1}} & \Lambda(X^{\leq n+1}), d \\
\downarrow & & \downarrow \\
\Lambda(Y^{n+1})\otimes\Lambda(\mathrm{coker}\varphi^{n+1})\otimes ?\, s(\ker\varphi^{n+1}), \overline{D} & \xrightarrow[\simeq]{\overline{p_{n+1}}} & \Lambda(X^{n+1}), 0
\end{array}$$

Since p_n and $\overline{p_{n+1}}$ are quasi-isomorphisms, by comparison of the E_2-term of the algebraic Serre spectral sequence associated to each column, p_{n+1} is a quasi-isomorphism. *QED*

Example 9.8 Let $f : S^2 \hookrightarrow \mathbb{CP}^n$ be the inclusion of CW-complexes with $n \geq 2$. Applying Corollary 9.6, ψ is $(\Lambda(x_2, y_{2n+1}), d) \to (\Lambda(x_2, z_3), d)$ with $dy_{2n+1} = x_2^{n+1}$ and $dz_3 = x_2^2$. By Lemma 9.7, ψ factorises through the CDGA $(\Lambda(x_2, y_{2n+1}, z_3) \otimes ?\, sy_{2n+1}, D)$ with $Dz_3 = x_2^2$ and $Dsy_{2n+1} = y_{2n+1} - z_3 x_2^{n-1}$. So $H^*(F) \cong \Lambda z_3 \otimes ?\, sy_{2n+1}$ for $p \geq 2n$.

Remark 9.9 The hypotheses of the Theorem 9.2 are necessary:

• B must be (r,p)-mild. Indeed even for a path fibration $\Omega X \hookrightarrow PX \twoheadrightarrow X$, a commutative model of X does not determine the cohomology algebra of the loop space. $\Sigma\mathbb{CP}^p$ and $S^3 \vee .. \vee S^{2p+1}$ just not $(2,p)$-mild, have a same commutative model but the cohomology algebras of their loop spaces are not isomorphic.

• E and B both (r,p)-mild is not enough: $H_{rp}(f)$ must also be injective. Take the same example as in Remark 7.5: the suspension of the Hopf map $\Sigma f : \Sigma S^{2p-1} \to \Sigma\mathbb{CP}^{p-1}$.

Remark 9.10 Over $\mathbb{Q}$, replacing A by A_{PL}, the Grivel-Thomas-Halperin theorem implies that the CDGA $\Bbbk \otimes_{A(B)} C$ is weakly DGA equivalent to $C^*(F)$ (Remark 9.3). But over a field of characteristic p, we can't improve Theorem 9.2, by $\Bbbk \otimes_{A(B)} C \sim C^*(F)$ as DGA's. Indeed, let X be the $2p+3$ skeleton of a $K(\mathbb{Z}, 4)$, X is $(3,p)$-mild and $C^*(\Omega X)$ is not weakly DGA equivalent to a CDGA.

Proof. A consequence of Milnor is that there exist two CW-complexes denoted Y and $K(\mathbb{Z},3)$ with the same $2p+2$ skeleton, respectively homotopic to ΩX and $\Omega K(\mathbb{Z},4)$. The two morphisms of topological monoids

$$\Omega(Y^{(2p+2)}) \to \Omega Y \quad \text{and} \quad \Omega(K(\mathbb{Z},3)^{(2p+2)}) \to \Omega K(\mathbb{Z},3)$$

induce in homology two algebra morphisms which are isomorphisms in degree $\leq 2p$. Since $H_*(\Omega K(\mathbb{Z},3)) \cong ?\,\alpha_2$ as algebras, ΩY is 1-connected, $H_2(\Omega Y) = \mathbb{F}_p\alpha_2$ and $\alpha_2^p = 0$. Suppose $C^*(Y)$ is weakly DGA equivalent to a commutative chain algebra A. We can suppose that A is of finite type. The Quillen construction [10, §22 e) and §23 a)] on the coalgebra $A^\vee$ is a DGL $\mathcal{L}_A$ equipped with a DGA quasi-isomorphism $U\mathcal{L}_A := \Omega(A^\vee) \overset{\simeq}{\twoheadrightarrow} C_*(\Omega Y)$. The homology of an universal enveloping algebra of a DGL, $U\mathcal{L}_A$ is an universal enveloping algebra of a Lie algebra, UE ([14] 8.3). So $H_*(\Omega Y)$ admits by the Poincaré-Birkoff-Witt Theorem a basis containing $\alpha_2^p \neq 0$. *QED*

10 Divided powers algebras

The key to the proof of Theorem 9.2 is to apply Anick's Theorem ([2] 5.6). One of the goal of Anick for developing this theorem was to prove a result suggested by McGibbon and Wilkerson "If X is a finite simply-connected CW-complex then for large primes, p^{th} powers vanish in $\tilde{H}^*(\Omega X;\mathbb{F}_p)$." ([17], p. 699). Anick proved precisely that "If X is (r,p)-mild then p^{th} powers vanish in $\tilde{H}^*(\Omega X;\mathbb{F}_p)$." ([2] 9.1). After Anick, Halperin proved in [14] (Theorem 8.3 and Poincaré-Birkoff-Witt Theorem) that in fact:

Corollary 10.1 *[14] If X is (r,p)-mild then the algebra $H^*(\Omega X)$ is isomorphic to $?\,sV$ where ΛV is a minimal Sullivan model of $A(X)$.*

Proof. Apply Corollary 9.6 to $* \to X$ and see that the homotopy cofiber of $(\Lambda V, d) \twoheadrightarrow (\mathbb{k}, 0)$ given by Lemma 9.7, $(\mathbb{k},0) \otimes_{(\Lambda V,d)} (\Lambda V \otimes ?\,sV, D)$ has a null differential ([14] 2.6). *QED*

Actually, we can show now that Anick's result on p^{th} powers and Halperin's result on a divided powers algebra structure remain valid if we consider the fiber of any fibration in the Anick range instead of just the loop fibration. But before we need the notion of an admissible CGDA and of a ?-admissible CGDA.

Definition 10.2 A CDGA (respectively ?-algebra) A is *admissible* (respectively ?*-admissible*) if there is a surjective CDGA morphism (respectively ?-morphism) $C \twoheadrightarrow A$ with C acyclic.

Property 10.3 ([15] II.2.6) Let $f : A \to B$ a CDGA morphism (respectively ?-morphism). If f is surjective and A is admissible (respectively ?-admissible) then so is B.

Proposition 10.4 *([15] II.2.7)*

(i) If $f : A \to B$ is a CGDA morphism with B admissible then we have the commutative diagram of CDGA's

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \searrow & \uparrow \simeq \\ A \otimes ?V' & \xleftarrow{\ \simeq\ } & A \otimes \Lambda V \end{array}$$

where $A \rightarrowtail A \otimes \Lambda V$ is a relative Sullivan model and $A \rightarrowtail A \otimes ?V'$ is a ?-free extension.

(ii) In particular, if B is any admissible CDGA, there are CDGA quasi-isomorphisms

$$?V' \xleftarrow{\simeq} \Lambda V \xrightarrow{\simeq} B$$

where $?V'$ is a ?-algebra.

The essential role of ?-admissible algebras is that

Property 10.5 ([3] 1.3) If A is a ?-admissible algebra then $H(A)$ is a ?-algebra (not true if A was only a ?-algebra!).

Lemma 10.6 *Let A be a cochain commutative algebra. Assume that for some $r \geq 1$, A satisfies $A = \mathbb{k} \oplus \{A^i\}_{i \geq r}$.*

(i) If $H^i(A) = 0$, $i \geq rp$, then A is admissible.

(ii) If A is a ?-algebra and $H^i(A) = 0, i \geq rp+p-1$, then A is ?-admissible.

Proof. (i) This lemma is just a slight improvement from Lemma 7.6 [14] and the proof is the same: For each $a \in \overline{A}^{odd}$, construct an obvious CDGA morphism σ_a from the acyclic CDGA $\Lambda(\Bbbk a \oplus \Bbbk da)$ to A. For each $a \in \overline{A}^{even}$, the cohomology class of lowest degree in $\Lambda(\Bbbk a \oplus \Bbbk da)$ is represented by a^p. Extend this CDGA to an acyclic Sullivan model of the form $\Lambda(\Bbbk a \oplus \Bbbk da \oplus V)$ where V is a graded vector space concentrated in degree $\geq rp - 1$. Construct a CDGA morphism $\sigma_a : \Lambda(\Bbbk a \oplus \Bbbk da \oplus V) \to A$. Now $\otimes_{a \in A^+} \sigma_a$ is a surjective morphism from an acyclic CDGA to A.

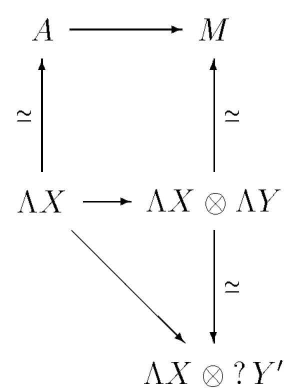
(ii) For each $a \in \overline{A}^{odd}$, the cohomology class of lowest degree in the ?-algebra $?(\Bbbk a \oplus \Bbbk da)$ is represented by $\gamma^{p-1}(da)a$. After replacing Λ by ?, the proof is the same as in (i). QED

Lemma 10.7 *Let A and M be two cochain commutative algebras concentrated in degrees $\geq r+1$. Consider a CDGA morphism $A \to M$. If $H^{\geq rp+p}(A) = H^{\geq rp+p-1}(M) = 0$ then $Tor^A(M, \Bbbk)$ is a divided powers algebra.*

Proof. By Lemma 10.6 (i), A and M are admissible. By Proposition 10.4 (ii), there are CDGA quasi-isomorphisms

$$? X' \xleftarrow{\simeq} \Lambda X \xrightarrow{\simeq} A$$

where X and X' are concentrated in degrees $\geq r+1$. By Proposition 10.4 (i), we get the commutative diagram of CDGA's



where Y and Y' are concentrated in degrees $\geq r$. By push-out, we have the commutative diagram of CDGA's

$$\begin{array}{ccc} \Lambda X & \longrightarrow & \Lambda X \otimes ?Y' \\ \simeq \downarrow & & \downarrow \simeq \\ ?X' & \longrightarrow & ?X' \otimes ?Y' \end{array}$$

where $\Lambda X \otimes ?Y' \xrightarrow{\simeq} ?X' \otimes ?Y'$ is a CDGA quasi-isomorphism ([9] 2.3(i)) since $\Lambda X \otimes ?Y'$ is ΛX-semifree (Property 2.4(iii)). Since push-outs preserve $?$-free extension, $?X' \rightarrowtail ?X' \otimes ?Y'$ is a $?$-free extension. So $?X' \otimes ?Y'$ is $?X'$-semifree, and by Proposition 8.7, the cohomology algebra of the cofiber $?Y'$ is $\mathrm{Tor}^A(M, \mathbb{k})$. Now since $?X'$ is a $?$-algebra, so is $?X' \otimes ?Y'$. Since $?X' \otimes ?Y'$ is concentrated in degrees $\geq r$ and its cohomology is null in degrees $\geq rp + p - 1$, by Lemma 10.6(ii), $?X' \otimes ?Y'$ is $?$-admissible. Since $?X' \otimes ?Y' \twoheadrightarrow ?Y'$ is a surjective $?$-morphism, by Property 10.3, $?Y'$ is a $?$-admissible. So by Property 10.5, $H(?Y')$ is a $?$-algebra. *QED*

Theorem 10.8 *Let p be an odd prime and let $f : E \twoheadrightarrow B$ be a fibration of fiber F such that E and B are both (r, p)-mild with $H_{rp}(f)$ injective. Then the cohomology algebra $H^*(F; \mathbb{F}_p)$ is a (not necessarily free!) divided powers algebra. In particular, p^{th} powers vanish in the reduced cohomology $\tilde{H}^*(F; \mathbb{F}_p)$.*

Proof. By Theorem 9.2, $H^*(F; \mathbb{k}) \cong \mathrm{Tor}^{A(B)}(A(E)$. Since $A(B)$ and $A(E)$ are concentrated in degrees $\geq r + 1$ and their cohomology is null in degrees $\geq rp$, by Lemma 10.7, $\mathrm{Tor}^{A(B)}(A(E), \mathbb{k})$ is a divided powers algebra. *QED*

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