ON THE FOUR-COLOUR CONJECTURE

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Introduction and summary

FOR the purposes of the four-colour conjecture it is convenient to consider only undirected graphs without loops and without multiple edges. In this paper an *abstract graph* Γ is taken to mean a set $\mathscr{V}(\Gamma)$, whose elements are called the *vertices* of Γ, together with a set $\mathscr{E}(\Gamma)$, whose elements are unordered pairs of distinct vertices of Γ and are called the *edges* of Γ; and a *planar graph* Γ is taken to mean a polygonal line complex homeomorphically embedded in the Euclidean plane, the 0-simplexes being called the *vertices*—they form the set $\mathscr{V}(\Gamma)$—and the 1-simplexes being called the *edges*—they form the set $\mathscr{E}(\Gamma)$—in which no edge joins a vertex to itself and two vertices are joined by at most one edge; *an edge is not considered to include the two vertices which it joins.* To each planar graph there corresponds an abstract graph, but not conversely. The usual notation of set theory will be used. The term *subgraph* will be used in the sense that every graph is a subgraph of itself, as distinct from *proper subgraph.*† The axiom of choice will be assumed. In what follows, definitions and statements concerning *graphs* apply both to abstract and to planar graphs.

DEFINITION 1. A graph will be called *κ-colourable* if its vertices can be partitioned into κ or fewer (colour) classes in such a way that no two vertices belonging to the same class are joined by an edge.

The four-colour conjecture is equivalent to the statement that **every planar graph is 4-colourable** (cf. (**11**), or (**10**) Lemma 1; and (**14**) 295). The present paper contains two results connected with the four-colour conjecture; one applies to all abstract graphs and the other to planar graphs.

DEFINITION 2. A graph with α vertices ($\alpha \geqslant 1$) in which every pair of distinct vertices is joined by an edge will be denoted by $\langle\alpha\rangle$, and a graph obtained from an $\langle\alpha\rangle, \alpha \geqslant 2$, by deleting a single edge (but not the end-vertices of this edge) will be denoted by $\langle\alpha -\rangle$. A graph obtained from an $\langle\alpha\rangle, \alpha \geqslant 2$, by the process of subdividing edges through the insertion of new vertices applied any number of times, will be denoted by $\langle\alpha \cup\rangle$; $\langle\alpha\rangle$ is not considered as a special case of $\langle\alpha \cup\rangle$.

† $\Gamma' \subseteq \Gamma$, where Γ is a graph, implies that Γ' is a graph such that $\mathscr{V}(\Gamma') \subseteq \mathscr{V}(\Gamma)$ and $\mathscr{E}(\Gamma') \subseteq \mathscr{E}(\Gamma)$. If $\mathscr{S} \subseteq \mathscr{V}(\Gamma) \cup \mathscr{E}(\Gamma)$ but $\mathscr{S}$ is not a graph, then $\mathscr{S}$ is not called a subgraph of Γ'.

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DEFINITION 3. If Γ is a graph and $\mathscr{S} \subseteq \mathscr{V}(\Gamma) \cup \mathscr{E}(\Gamma)$, then $\Gamma - \mathscr{S}$ will denote the graph obtained from Γ by deleting the set of vertices $\mathscr{S} \cap \mathscr{V}(\Gamma)$, together with all edges of Γ incident with at least one of these vertices and then deleting all those edges of $\mathscr{S} \cap \mathscr{E}(\Gamma)$ which still remain.

DEFINITION 4. If Γ is a graph and a and b are two distinct vertices of Γ then a graph obtained from $\Gamma - a - b$ by adding a new vertex not in Γ and joining it by edges to precisely all those vertices of $\Gamma - a - b$ which were joined in Γ to at least one of a, b will be called a *graph obtained from* Γ *by identifying the vertices* a *and* b. Γ *is homomorphic to* Γ', in symbols $\Gamma \succ \Gamma'$, will denote that Γ' is a subgraph of a graph which is isomorphic to the graph Γ or to a graph which can be obtained from Γ by the operation of identifying pairs of vertices joined by an edge applied any number of times. This relation is transitive. The operation of identifying pairs of adjacent vertices will be denoted by $\wp$.

DEFINITION 5. If Γ is a graph and a is a vertex, then the number of vertices of Γ to which a is joined by edges will be called the *valency* of a in Γ and denoted by $v(a, \Gamma)$.

DEFINITION 6. If Γ is a graph and Φ is a connected (Def. 11) subgraph of Γ such that every vertex of $\Gamma - \Phi$ is joined by an edge to at least one vertex of Φ then Φ will be called a *connected skeleton* of Γ.

DEFINITION 7. If Γ is a graph then the operation of deleting a set of vertices each of which have valency $\leqslant 4$ in Γ (together with every edge incident with at least one of the deleted vertices) will be denoted by $\wp_1$ and the operation of identifying pairs of vertices which are joined by an edge and at least one of which has valency $\leqslant 4$ in Γ will be denoted by $\wp_2$. If Γ is a planar graph then the operation of identifying pairs of vertices which are adjacent to the same region of the plane and are not joined by an edge will be denoted by $\wp_3$.

DEFINITION 8. A *member* of a graph is the same as a 'Glied'; see (**16**).

The two main results of this paper are:

THEOREM 1. *If* Γ *is an abstract graph and not* 4*-colourable, then either* $\Gamma \succ \langle 5 \rangle$, *or else*

$$(\forall a, b)\{a, b \in \mathscr{V}(\Gamma) . \rightarrow . \Gamma - a - b \succ \langle 5 - \rangle\} \quad \textit{and}$$

$$(\forall a, b, c)\{a, b, c \in \mathscr{V}(\Gamma) . \rightarrow . \Gamma - a - b - c \supseteq \langle 4 \cup \rangle\}.$$

THEOREM 2. *Let* Γ *denote any planar graph and let* Γ_0 *denote the graph (possibly empty and possibly identical with* Γ*) obtained from* Γ *by applying* $\wp_1$

until there remains a graph without vertices of valency $\leqslant 4$. Γ *is said to have the property* $\mathfrak{P}$ *if* $\Gamma_0 = \varnothing$ *or* Γ_0 *is 4-colourable or* $(\forall\Delta)\{\Delta$ *is a member of* $\Gamma_0 . \rightarrow . |\mathscr{V}(\Delta)| \leqslant 35 \vee |\mathscr{E}(\Delta)| \leqslant 101 \vee \Delta$ *contains at most one vertex whose valency in* Δ *exceeds* $6 \vee \Delta$ *has a connected skeleton such that all vertices of the skeleton with the exception of at most five have valency* $\leqslant 4$ *in* $\Delta \vee \Delta$ *is 4-colourable*$\}$.

If Γ *has the property* $\mathfrak{P}$, *or if* $(\forall\Delta)\{\Delta$ *is a member of* $\Gamma_0 . \rightarrow . \Delta$ *has the property* $\mathfrak{P} \vee$ *a graph having the property* $\mathfrak{P}$ *can be obtained from* Δ *by applying any or all of* $\wp_1, \wp_2$, *and* $\wp_3$ *in any suitable order and with repetitions allowed*$\}$, *then* Γ *is 4-colourable.*

Theorem 1 follows a line of approach indicated by two results due to K. F. Wagner: the four-colour theorem is equivalent to the statement that every graph which is not 4-colourable is homomorphic to a $\langle 5 \rangle$ ((**17**) 573); every graph which is not 4-colourable is homomorphic to a $\langle 5 - \rangle$ (**18**).

Theorem 2 establishes the truth of the four-colour conjecture for a new class of planar graphs, it incorporates the results of C. E. Winn that every planar graph with at most one vertex of valency $\geqslant 7$ is 4-colourable (**20**) and that every planar graph with fewer than 36 vertices is 4-colourable (**21**), and it furnishes a procedure for verifying that a given suitable planar graph is 4-colourable without having to find a 4-colouring for the whole of it.†

Notation and terminology

Graphs will be denoted by Greek capital letters, vertices by lower-case italic letters, and an edge incident with the vertices a and b by (a, b) and by (b, a); this edge is said to *join* a and b, and a and b are called the *end-vertices* of (a, b). An edge is not considered to include its end-vertices and so an edge by itself is not a graph, but an edge together with its two end-vertices is one. If Γ_1 and Γ_2 are graphs then $\Gamma_1 \cup \Gamma_2$ denotes the graph whose set of vertices is $\mathscr{V}(\Gamma_1) \cup \mathscr{V}(\Gamma_2)$ and whose set of edges is $\mathscr{E}(\Gamma_1) \cup \mathscr{E}(\Gamma_2)$, and analogously for $\Gamma_1 \cap \Gamma_2$. The graph Γ is called *finite* if $\mathscr{V}(\Gamma)$ is finite.

DEFINITION 9. $a \times b$ in Γ denotes that the vertices a and b are joined by an edge in the graph Γ; $a \circ b$ in Γ denotes that they are not.

DEFINITION 10. A graph with vertices $a_1, \ldots, a_\nu$ and edges $(a_1, a_2), \ldots, (a_{\nu-1}, a_\nu)$, where $\nu \geqslant 2$ and $a_1, \ldots, a_\nu$ are all distinct, is called a *path*; it is said to *join* a_1 and a_ν, and a_1 and a_ν are called its *end-vertices*. A graph with vertices $a_1, \ldots, a_\nu$ and edges $(a_1, a_2), \ldots, (a_{\nu-1}, a_\nu), (a_\nu, a_1)$, where $\nu \geqslant 3$ and $a_1, \ldots, a_\nu$ are all distinct, is called a *circuit*. ($\langle 2 \rangle$-s and $\langle 2 \cup \rangle$-s are paths and $\langle 3 \rangle$-s and $\langle 3 \cup \rangle$-s are circuits.) The *length* of a path (circuit) is the

† The procedure works immediately for the maps showing the counties of Britain, the states of the U.S.A., the cantons of Switzerland, and the départements of France.

number of edges in the path (circuit). If Γ_1 and Γ_2 are two mutually disjoint graphs and if Υ is a path with end-vertices a_1 and a_ν such that $\Upsilon \cap \Gamma_1 = a_1$ and $\Upsilon \cap \Gamma_2 = a_\nu$ then Υ is called a $(\Gamma_1)(\Gamma_2)$-*path*. In particular, Υ is an $(a_1)(a_\nu)$-path. If a and b are two vertices of the path Υ then the path contained in Υ which joins a and b is denoted by $\Upsilon[a,b]$ and by $\Upsilon[b,a]$; $\Upsilon[a,a] = a$.

DEFINITION 11. The graph Γ is called *connected* if $(\forall a,b)\{a,b \in \mathscr{V}(\Gamma)$ and $a \neq b . \rightarrow . \Gamma$ contains at least one $(a)(b)$-path$\}$; a graph which consists of a single vertex is connected. Γ is called *μ-fold connected* ($\mu \geqslant 1$) if $(\forall a,b)\{a,b \in \mathscr{V}(\Gamma)$ and $a \neq b . \rightarrow . \Gamma$ contains a set of μ or more distinct $(a)(b)$-paths, any two of which have nothing but a and b in common$\}$.

DEFINITION 12. A graph will be called *κ-chromatic* ($\kappa \geqslant 1$) if it is κ-colourable and not $(\kappa-1)$-colourable. A κ-chromatic graph will be called *critical* if it has no proper subgraph which is κ-chromatic; it will be called *contraction-critical* if it is connected and not homomorphic to any κ-chromatic graph other than itself.

DEFINITION 13. For $\alpha \geqslant 4$ any $\langle\alpha\rangle$ or any $\langle\alpha \cup\rangle$ contains exactly α vertices having valency $\alpha - 1$, these will be called its *branch-vertices*. A path contained in the $\langle\alpha\rangle$ or $\langle\alpha \cup\rangle$ which joins two branch-vertices and does not contain any of the remaining branch-vertices will be called a *side line* of the $\langle\alpha\rangle$ or $\langle\alpha \cup\rangle$.

DEFINITION 14. If $\mathscr{S} \subseteq \mathscr{V}(\Gamma)$ then $\Gamma - (\Gamma - \mathscr{S})$ is called the *subgraph of Γ generated by the vertices of $\mathscr{S}$* and is denoted by $\Gamma(\mathscr{S})$.

Proof of Theorem 1

1. *Preliminary results*

DEFINITION 15. If $\Gamma = \Delta$ or Δ is obtained from Γ by the operation $\mathscr{o}$ applied one or more times (see Definition 4) then we will say that Γ *can be contracted into* Δ. From this and Definition 4 it follows that $\Gamma \succ \Gamma'$ *if and only if Γ' is a subgraph of a graph into which Γ can be contracted.*

(1) *Γ can be contracted into Δ if and only if there exists a mapping m of $\mathscr{V}(\Gamma)$ onto $\mathscr{V}(\Delta)$ such that $(\forall d)\{d \in \mathscr{V}(\Delta) \rightarrow \Gamma(m^{-1}(d))$ is connected$\}$ and $(\forall e,f)\{e,f \in \mathscr{V}(\Delta)$ and $(e,f) \in \Delta . \leftrightarrow . \Gamma$ contains at least one $(m^{-1}(e))(m^{-1}(f))$-edge$\}$.*

(2) *$\Gamma \succ \Gamma'$ if and only if there exists a mapping m of a subset of $\mathscr{V}(\Gamma)$ onto $\mathscr{V}(\Gamma')$ such that $(\forall g)\{g \in \mathscr{V}(\Gamma') \rightarrow \Gamma(m^{-1}(g))$ is connected$\}$ and $(\forall h,i)\{h,i \in \mathscr{V}(\Gamma')$ and $(h,i) \in \Gamma' . \rightarrow . \Gamma$ contains at least one $(m^{-1}(h))(m^{-1}(i))$-edge$\}$.*

The proof is left to the reader.

DEFINITION 16. If there exists a mapping m of $\mathscr{V}(\Gamma)$ onto $\mathscr{V}(\Delta)$ with the properties described in (1) then Δ will be said to result from Γ *by contracting each of the subgraphs* $\Gamma(m^{-1}(d))$ *into a single vertex*. A mapping m of a subset of $\mathscr{V}(\Gamma)$ onto $\mathscr{V}(\Gamma')$ with the properties described in (2) will be called a *homomorphism* out of Γ onto Γ'.

(3) *If* $\Gamma \succ \Gamma'$ *then* $(\exists B)\{B \subseteq \Gamma$ *and* $B \succ \Gamma'$ *and* B *has at most as many connected components as* $\Gamma'\}$.

This follows from Definition 4.

(4) *If* $\Gamma \succ \Gamma'$ *and* Γ' *is finite then* $(\exists B)\{B \subseteq \Gamma$ *and* $B \succ \Gamma'$ *and* B *is finite*$\}$.

Proof. By (2) there exists a homomorphism m out of Γ onto Γ'. Let $g_1, \ldots, g_\gamma$ denote the vertices of Γ'. Corresponding to each edge (g_α, g_β) contained in Γ' select a vertex $f_{\alpha\beta}{}^{\alpha}$ in $m^{-1}(g_\alpha)$ and a vertex $f_{\beta\alpha}{}^{\beta}$ in $m^{-1}(g_\beta)$ such that $(f_{\alpha\beta}{}^{\alpha}, f_{\beta\alpha}{}^{\beta}) \in \Gamma$. In this way for each κ, $1 \leqslant \kappa \leqslant \gamma$, a subset $\mathscr{S}_\kappa$ of $m^{-1}(g_\kappa)$ is selected, where $|\mathscr{S}_\kappa| \leqslant v(g_\kappa, \Gamma')$. $\Gamma(m^{-1}(g_\kappa))$ is connected and therefore contains a finite connected subgraph Γ_κ such that $\mathscr{S}_\kappa \subseteq \mathscr{V}(\Gamma_\kappa)$. The subgraph of Γ generated by $\mathscr{V}(\Gamma_1) \cup \mathscr{V}(\Gamma_2) \cup \ldots \cup \mathscr{V}(\Gamma_\gamma)$ is finite and homomorphic to Γ'.

(5) *If* Γ *is finite and every vertex of* Γ, *with at most one exception, has valency* $\geqslant 3$, *then* Γ *contains a* $\langle 4 \rangle$ *or a* $\langle 4 \cup \rangle$ *as a subgraph* ((**7**) Satz 5).

(6) *If* $\Gamma \succ \langle 4 \rangle$ *then* Γ *contains a* $\langle 4 \rangle$ *or a* $\langle 4 \cup \rangle$ *as a subgraph.*

(I believe that this result is not new.) *Proof.* Suppose that (6) is false. Then there exist finite graphs for which it is false, by (4): let Ψ be one having a minimal number of vertices. It follows from the minimal property of Ψ that Ψ contains no vertex of valency < 3, and we have a contradiction to (5). Hence (6) is true.

(7) *If* Γ *is* 3*-fold connected then* $(\forall a, b)\{a, b \in \mathscr{V}(\Gamma) \rightarrow \Gamma$ *contains a* $\langle 4 \rangle$ *or a* $\langle 4 \cup \rangle$ *which contains* a *and* b *as branch-vertices*$\}$ ((**7**) Satz 8).

(8) *If a graph is contraction-critical then it is critical.*

The proof is left to the reader.

(9) *Every* κ*-chromatic graph,* κ *being finite, contains a critical* κ*-chromatic subgraph. Every critical* κ*-chromatic graph is finite and connected and each of its vertices has valency* $\geqslant \kappa - 1$, *and if* $\kappa \geqslant 3$ *then it is* 2*-fold connected* ((**5**) 43).

(10) *Every connected* κ*-chromatic graph,* κ *being finite, is either contraction-critical or contractible into a contraction-critical* κ*-chromatic graph.*

Proof. Let Γ denote a connected κ-chromatic graph and let Γ' denote a critical κ-chromatic subgraph of Γ ($\Gamma' = \Gamma$ possibly). Clearly Γ can be contracted into a graph Γ'' such that $\mathscr{V}(\Gamma'') = \mathscr{V}(\Gamma')$ and $\Gamma'' \supseteq \Gamma'$ ($\Gamma'' = \Gamma'$ possibly). Among the finite set of different κ-chromatic graphs into which the (finite connected) graph Γ'' can be contracted, let Γ''' be one having the least possible number of edges ($\Gamma''' = \Gamma''$ possibly). Obviously Γ''' is not homomorphic to any κ-chromatic graph with fewer vertices than Γ'''. Furthermore if (a, b) is any edge of Γ''' then $\Gamma''' - (a, b)$ is $(\kappa-1)$-colourable because otherwise the graph obtained from Γ''' by identifying a and b would not be $(\kappa-1)$-colourable, and this would contradict the definition of Γ'''. Hence Γ''' is contraction-critical.

(11) *If a critical κ-chromatic graph contains a $\langle\lambda\rangle$ as a subgraph, where $\lambda \leqslant \kappa-1$, then the graph can be contracted into a $\langle\lambda+1\rangle$* ((**6**) Theorem 1).

(12) *Every contraction-critical κ-chromatic graph with $\kappa \geqslant 5$ (except the $\langle 5\rangle$) is 5-fold connected* ((**8**) 128).

(13) *Every 4-chromatic graph contains a $\langle 4\rangle$ or a $\langle 4 \cup\rangle$ as a subgraph* ((**4**) 87).

This follows from (9) and (5).

2. *Results on connectivity*

Proposition 1. *If a 3-fold connected graph contains more than one vertex having valency >3, then the graph is homomorphic to a $\langle 5-\rangle$.*

Proof. It will first be shown that

(14) *If the 3-fold connected graph Γ contains as a subgraph a $\langle 4\rangle$ or a $\langle 4\cup\rangle$ to which not all vertices of Γ belong, then $\Gamma \succ \langle 5-\rangle$.*

Proof of (14). Let Λ denote a $\langle 4\rangle$ or a $\langle 4\cup\rangle$ and let x denote a vertex of Γ such that $\Lambda \subseteq \Gamma - x$. From the fact that Γ is 3-fold connected it follows, by an extension of Menger's Theorem (**9**), that Γ contains three $(x)(\Lambda)$-paths such that any two of them have only x in common. If the three end-vertices $\neq x$ of three such paths do not all belong to one and the same side line of Λ, then the union of Λ and the three paths is homomorphic to a $\langle 5-\rangle$. Therefore it only remains to consider the alternative that Γ has the property that

(i) $(\forall \Lambda, x)\{x \in \mathscr{V}(\Gamma)$ *and* $\Lambda = a\ \langle 4\rangle$ *or a* $\langle 4\cup\rangle$ *and* $\Lambda \subseteq \Gamma - x . \rightarrow .$ *one of the side lines of Λ has the property that it contains the end-vertex $\neq x$ of every $(x)(\Lambda)$-path contained in $\Gamma\}$.*

(i) will lead to a contradiction, however.

The side line of Λ on which all $(x)(\Lambda)$-paths contained in Γ end will be denoted by $\Upsilon_{\Lambda x}$, and the two branch-vertices of Λ which $\Upsilon_{\Lambda x}$ joins by $y_{\Lambda x}$ and $y'_{\Lambda x}$. Let $\Upsilon_1, \Upsilon_2, \Upsilon_3$ be three $(x)(\Lambda)$-paths contained in Γ and such that any two of them have only x in common, and let y_i denote $\Lambda \cap \Upsilon_i$, $i = 1, 2, 3$, the notation being chosen so that $y_{\Lambda x}, y_1, y_2, y_3, y'_{\Lambda x}$ occur in this order along $\Upsilon_{\Lambda x}$, where possibly $y_1 = y_{\Lambda x}$ and $y_3 = y'_{\Lambda x}$. Let $\delta(\Lambda, x, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ denote the sum of the lengths of $\Upsilon_{\Lambda x}[y_{\Lambda x}, y_1]$ and of $\Upsilon_{\Lambda x}[y'_{\Lambda x}, y_3]$. Let $\delta_{\Lambda x}$ denote the minimum value of $\delta(\Lambda, x, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ taken over all sets of three $(x)(\Lambda)$-paths contained in Γ, any two of the paths of the same set having only x in common. Let δ denote the minimum value of $\delta_{\Lambda x}$ taken over all Λ and x such that $x \in \mathscr{V}(\Gamma)$ and Λ is a $\langle 4 \rangle$ or a $\langle 4 \cup \rangle$ contained as a subgraph in $\Gamma - x$. Let Λ^* and x^* be such that $\delta_{\Lambda^* x^*} = \delta$ and let $\Upsilon_1{}^*, \Upsilon_2{}^*, \Upsilon_3{}^*$ be such that $\delta(\Lambda^*, x^*, \Upsilon_1{}^*, \Upsilon_2{}^*, \Upsilon_3{}^*) = \delta$, etc.

$\Gamma - y_1{}^* - y_3{}^*$ is connected because Γ is 3-fold connected; therefore $\Gamma - y_1{}^* - y_3{}^*$ contains at least one

$$(\Upsilon_{\Lambda^* x^*}[y_1{}^*, y_3{}^*] - y_1{}^* - y_3{}^*)\,(\Lambda^* - \Upsilon_{\Lambda^* x^*}[y_1{}^*, y_3{}^*])\text{-path}.$$

Let Υ denote such a path, let z denote the end-vertex of Υ belonging to $\Upsilon_{\Lambda^* x^*}[y_1{}^*, y_3{}^*] - y_1{}^* - y_3{}^*$, and let z' denote the other end-vertex of Υ.

(ii) $$(\Upsilon - z) \cap (\Upsilon_1{}^* \cup \Upsilon_2{}^* \cup \Upsilon_3{}^*) = \emptyset.$$

Proof. Suppose that this is untrue, and let w denote the last vertex belonging to $\Upsilon \cap (\Upsilon_1{}^* \cup \Upsilon_2{}^* \cup \Upsilon_3{}^*)$ which is encountered when going along Υ from z to z'; $w \neq z$, but possibly $w = x^*$. Suppose that $w \in \Upsilon_i{}^*$, $1 \leqslant i \leqslant 3$. $\Upsilon_i{}^*[x^*, w] \cup \Upsilon[w, z']$ is an $(x^*)(\Lambda^*)$-path; therefore, by (i),

$$z' \in \Upsilon_{\Lambda^* x^*} - \Upsilon_{\Lambda^* x^*}[y_1{}^*, y_3{}^*].$$

It may be assumed without loss of generality that $z' \in \Upsilon_{\Lambda^* x^*}[y_{\Lambda^* x^*}, y_1{}^*] - y_1{}^*$. Then it follows from the minimal property of $\delta_{\Lambda^* x^*}$ that $w \neq x^*$ and that $i = 3$. But in this case

$$\delta(\Lambda^*, w, \Upsilon[w, z'], \Upsilon_3{}^*[w, x^*] \cup \Upsilon_2{}^*, \Upsilon_3{}^*[w, y_3{}^*]) < \delta,$$

which is a contradiction. This proves (ii).

(iii) $$z' \in \Lambda^* - \Upsilon_{\Lambda^* x^*}.$$

Proof. Suppose that this is untrue. Then it may be assumed without loss of generality that $z' \in \Upsilon_{\Lambda^* x^*}[y_{\Lambda^* x^*}, y_1{}^*] - y_1{}^*$. Let Δ^* denote the $\langle 4 \cup \rangle$ obtained from Λ^* through replacing $\Upsilon_{\Lambda^* x^*}$ by

$$\Upsilon_{\Lambda^* x^*}[y_{\Lambda^* x^*}, z'] \cup \Upsilon \cup \Upsilon_{\Lambda^* x^*}[z, y'_{\Lambda^* x^*}].$$

It follows from (ii) that $\delta_{\Delta^* x^*} < \delta_{\Lambda^* x^*}$, and this contradicts the definition of $\delta_{\Lambda^* x^*}$. This proves (iii).

Let ∇^* denote the $\langle 4 \cup \rangle$ obtained from Λ^* through replacing $\Upsilon_{\Lambda^* x^*}[y_1{}^*, y_3{}^*]$ by $\Upsilon_1{}^* \cup \Upsilon_3{}^*$. It follows from (ii) and (iii) that $\Upsilon_{\Lambda^* x^*}[z, y_1{}^*]$, $\Upsilon_{\Lambda^* x^*}[z, y_3{}^*]$, and Υ are three $(z)(\nabla^*)$-paths of which any two have only z in common, and it follows from (iii) that their end-vertices $\neq z$ do not all belong to one and the same side line of ∇^*. This contradicts (i), and so (14) is proved.

Proof of Proposition 1. Suppose that the proposition is false, and let Γ denote a 3-fold connected graph not homomorphic to a $\langle 5-\rangle$ in which the vertices a and b $(a \neq b)$ have valency >3. It follows from (14) that Γ contains no $\langle 4\rangle$ as a subgraph and that

(iv) *every* $\langle 4\cup\rangle$ *contained in* Γ *as a subgraph includes all vertices of* Γ.

By (7), Γ contains at least one $\langle 4\cup\rangle$ of which a and b are branch-vertices, Λ say. Let c and d denote the other two branch-vertices of Λ and let $\Upsilon_{ab}, \Upsilon_{ac}$, etc. denote the side lines of Λ which join a and b, a and c, etc.

a is incident with an edge not belonging to Λ, (a,x) say, and b with an edge $(b,y)\notin\Lambda$; possibly $x=y$. $x\notin\Upsilon_{ab}\cup\Upsilon_{ac}\cup\Upsilon_{ad}$ and $y\notin\Upsilon_{ab}\cup\Upsilon_{bc}\cup\Upsilon_{bd}$ by (iv). Hence the union of $\Upsilon_{bc}\cup\Upsilon_{cd}\cup\Upsilon_{db}$ and (a,x) and any two of $\Upsilon_{ab}, \Upsilon_{ac}, \Upsilon_{ad}$ is a $\langle 4\cup\rangle$ contained in Γ, and therefore, by (iv), $\Upsilon_{ab}, \Upsilon_{ac}, \Upsilon_{ad}$ contain only one edge each; similarly $\Upsilon_{bc}, \Upsilon_{bd}$ contain only one edge each. Hence $x,y\in\Upsilon_{cd}-c-d$. It follows that $\Gamma\succ\langle 5-\rangle$. This contradiction proves Proposition 1.

COROLLARY TO PROPOSITION 1. *If a new edge is added to a 3-fold connected graph (without creating a double edge) then the resulting graph is homomorphic to a* $\langle 5-\rangle$.

PROPOSITION 2. *If* Γ *is a 5-fold connected graph then either* $\Gamma\succ\langle 5\rangle$ *or* $(\forall x,y)\{x,y\in\mathscr{V}(\Gamma)\rightarrow\Gamma-x-y\succ\langle 5-\rangle\}$.

Proof. Suppose on the contrary that $\Gamma\not\succ\langle 5\rangle$ and $x,y\in\mathscr{V}(\Gamma)$ and $\Gamma-x-y\not\succ\langle 5-\rangle$. This will lead to a contradiction.

Let Γ' denote $\Gamma-x-y$. Γ' is 3-fold connected; hence by Proposition 1 Γ' contains at most one vertex having valency >3 in Γ'. But every vertex of Γ has valency $\geqslant 5$ in Γ because Γ is 5-fold connected. Therefore every vertex belonging to Γ', with at most one exception, is joined to x and to y in Γ. Not every vertex of Γ' is joined to x in Γ, and not every vertex of Γ' is joined to y in Γ, because Γ' contains a $\langle 4\rangle$ or a $\langle 4\cup\rangle$ as subgraph by (7). Hence Γ' contains just one vertex, z say, which is joined neither to x nor to y in Γ, and all the other vertices of Γ' are joined both to x and to y in Γ.

$\Gamma'-z$ is 2-fold connected and therefore contains at least one circuit. Let Ξ denote a circuit contained in $\Gamma'-z$. Ξ contains at least five vertices and edges, because otherwise $\Gamma-\Xi$ would be connected and would therefore contain an $(x)(y)$-path, Υ say, and the union of Ξ, Υ, and the $(x)(\Xi)$ and $(y)(\Xi)$-edges contained in Γ would be homomorphic to a $\langle 5\rangle$.

From the fact that Γ' is 3-fold connected it follows, by an extension of Menger's Theorem (**9**), that Γ' contains three $(z)(\Xi)$-paths of which any two have only z in common, Υ_1, Υ_2, and Υ_3 say. Let $w_i = \Xi \cap \Upsilon_i$ for $i = 1, 2, 3$.

Contract Υ_i into a path with only one edge joining z and w_i for $i = 1, 2, 3$. Then contract Ξ into a pentagonal circuit (5-circuit) Ξ^* in such a way that the vertices w_1, w_2, w_3 remain distinct; $\Xi^* = \Xi$ possibly. Among the three vertices w_1, w_2, w_3 two can be found which are not joined by an edge belonging to Ξ^*, w_i and w_j say. Identify z with w_i. The graph so obtained from Γ' contains the pentagonal circuit Ξ^* and the edge (w_i, w_j). Re-label the vertices of Ξ^* as $a_1, \ldots, a_5$ in cyclic order around Ξ^* in such a way that $w_i = a_1$ and $w_j = a_3$.

The graph which results from Γ by this process contains $\Xi^*, x, y, (a_1, a_3)$, and the edges (x, a_i) and (y, a_i) for $i = 1, \ldots, 5$. Identify x with a_4 and y with a_5. The resulting graph contains a $\langle 5 \rangle$, i.e. $\Gamma \succ \langle 5 \rangle$. This contradiction proves Proposition 2.

3. *Deduction of Theorem* 1

1. If a graph is not 4-colourable then, by (9), it contains a critical 5-chromatic graph as a subgraph. It is therefore sufficient to prove Theorem 1 on the assumption that Γ is 5-chromatic and critical.

By (10), either Γ is contraction-critical or it can be contracted into a contraction-critical 5-chromatic graph. Let m denote a homomorphism of Γ onto the contraction-critical 5-chromatic graph Γ', where $\Gamma' = \Gamma$ possibly. Let $m(a), m(b), m(c)$ be denoted by a', b', c' respectively; possibly $a' = a$, $b' = b$, or $c' = c$, and a', b', c' are not necessarily distinct. Clearly $\Gamma - a - b \succ \Gamma' - a' - b'$ and $\Gamma - a - b - c \succ \Gamma' - a' - b' - c'$.

By (12), Γ' is 5-fold connected or a $\langle 5 \rangle$. By Proposition 2 it follows that $\Gamma' \succ \langle 5 \rangle$ or $\Gamma' - a' - b' \succ \langle 5- \rangle$. Consequently, since the homomorphism relation is transitive, $\Gamma \succ \langle 5 \rangle$ or $\Gamma - a - b \succ \langle 5- \rangle$.

2. Suppose that $\Gamma' \not\succ \langle 5 \rangle$ and $\Gamma' - a' - b' - c'$ does not contain a $\langle 4 \cup \rangle$ as a subgraph. This will lead to a contradiction.

By (9) and (11) it follows from this assumption that Γ' does not contain a $\langle 4 \rangle$ as a subgraph, and it follows further that at least two of a', b', c' are joined by an edge in Γ', because otherwise $\Gamma' - a' - b' - c'$ would be 4-chromatic and consequently would contain a $\langle 4 \cup \rangle$ by (13). Suppose that $(a', b') \in \Gamma'$.

$\Gamma' - a' - b' - c'$ is 2-fold connected. $\Gamma' - a' - b' - c'$ contains at least two vertices of valency 2, otherwise it would contain a $\langle 4 \cup \rangle$ by (5). Let x and y denote two vertices having valency 2 in $\Gamma' - a' - b' - c'$. x and y

are joined to a', b', and c' because $v(x, \Gamma') = v(y, \Gamma') = 5$. It follows that $(x, y) \notin \Gamma'$, otherwise x, y, a', b' would generate a $\langle 4 \rangle$ in Γ'.

$\Gamma' - a' - b' - c'$ contains at least one circuit to which x and y both belong, because $\Gamma' - a' - b' - c'$ is 2-fold connected. Let Ξ denote such a circuit having the greatest possible number of vertices and edges, and let Υ_1 and Υ_2 denote the two connected components of $\Xi - x - y$; $\Upsilon_1 \neq \emptyset$ and $\Upsilon_2 \neq \emptyset$ because $(x, y) \notin \Gamma'$.

Every vertex of Ξ other than x and y has valency $\geqslant 3$ in $\Gamma' - a' - b' - c'$. For if $z \in \mathscr{V}(\Xi)$ and $z \neq x$ and $z \neq y$ and $v(z, \Gamma' - a' - b' - c') = 2$, then z is joined to a', b', and c' because $v(z, \Gamma') = 5$, so that if Ξ is contracted into a $\langle 3 \rangle$ with vertices x, y, z, then the graph so obtained from Γ' contains a $\langle 5 \rangle$ whose vertices are a', b', x, y, z, and this contradicts the hypothesis that $\Gamma' \not\succ \langle 5 \rangle$.

Definition 17. A path which joins two vertices of Ξ not joined by an edge belonging to Ξ, and which has nothing except its two end-vertices in common with Ξ, will be called a *chord* of Ξ.

Every vertex of Ξ other than x and y is the end-vertex of a chord of Ξ contained in $\Gamma' - a' - b' - c'$. For let u denote a vertex of Ξ other than x and y. $v(u, \Gamma' - a' - b' - c') \geqslant 3$. Let t denote a vertex of $\Gamma' - a' - b' - c'$ such that $(u, t) \in \Gamma' - a' - b' - c'$ and $(u, t) \notin \Xi$. If $t \in \Xi$, then u, t, and (u, t) together constitute a chord of Ξ. If $t \notin \Xi$ then $\Gamma' - a' - b' - c' - u$, being connected since Γ' is 5-fold connected, contains a $(t)(\Xi - u)$-path, and it follows from the maximal property of Ξ that the union of such a path and u and (u, t) is a chord of Ξ contained in $\Gamma' - a' - b' - c'$.

Every chord of Ξ contained in $\Gamma' - a' - b' - c'$ joins a vertex of Υ_1 to a vertex of Υ_2. For suppose on the contrary that such a chord joins two vertices of, e.g., Υ_1. Among all such chords let Υ be one which cuts off a portion of Υ_1 containing a minimum number of vertices. Let w denote a vertex of Υ_1 situated between the two end-vertices of Υ. $\Gamma' - a' - b' - c'$ contains a chord of Ξ, Υ' say, having w as end-vertex. It follows directly from the maximal property of Ξ and the minimal property of Υ that $\Upsilon \cap \Upsilon' = \emptyset$ and that $\Xi \cap \Upsilon$ separates $\Xi \cap \Upsilon'$ on Ξ. Hence $\Xi \cup \Upsilon \cup \Upsilon'$ is a $\langle 4 \cup \rangle$ contained in $\Gamma' - a' - b' - c'$, which is contrary to hypothesis.

Let Υ^* denote a chord of Ξ contained in $\Gamma' - a' - b' - c'$. $\Gamma' - x - y - c'$ is 2-fold connected, and consequently, by an extension of Menger's Theorem (**9**), it contains an $(a')(\Xi - x - y)$-path $\Upsilon_{a'}$ and a $(b')(\Xi - x - y)$-path $\Upsilon_{b'}$ such that $\Upsilon_{a'} \cap \Upsilon_{b'} = \emptyset$. Contract $\Upsilon_{a'} - a'$ and $\Upsilon_{b'} - b'$ into a single vertex each, identify c' with x, and contract $\Upsilon_1 \cup \Upsilon_2 \cup \Upsilon^*$ into the single vertex d'. The graph so obtained from Γ' contains a $\langle 5 \rangle$ with vertices a', b', d', x, y, so that $\Gamma' \succ \langle 5 \rangle$, contrary to hypothesis.

Hence either $\Gamma' \succ \langle 5 \rangle$ or $\Gamma' - a' - b' - c' \supseteq$ a $\langle 4 \cup \rangle$. Consequently, since the homomorphism relation is transitive, either $\Gamma \succ \langle 5 \rangle$ or $\Gamma - a - b - c \succ \langle 4 \rangle$. Theorem 1 now follows by (6) and (11).

A consequence of Theorem 1

THEOREM 3. *If a graph is not 4-colourable and has a connected skeleton with fewer than 4 vertices then it can be contracted into a* $\langle 5 \rangle$.

Proof. Let Γ denote the graph and Φ the skeleton. Γ is necessarily connected. By Theorem 1, $\Gamma \succ \langle 5 \rangle$ or $\Gamma - \Phi \supseteq \langle 4 \cup \rangle$. If $\Gamma - \Phi \supseteq \langle 4 \cup \rangle$ then the graph obtained from Γ by contracting Φ into a single vertex, and a $\langle 4 \cup \rangle$ contained in $\Gamma - \Phi$ into a $\langle 4 \rangle$, contains a $\langle 5 \rangle$; and if $\Gamma \succ \langle 5 \rangle$ then Γ, being connected, can be contracted into a $\langle 5 \rangle$.

Proof of Theorem 2

1. *Preliminary results*

(15) *If* Γ *is a finite graph and* $|\mathscr{E}(\Gamma)| \geqslant 2|\mathscr{V}(\Gamma)| - 2$ *then* $\Gamma \supseteq \langle 4 \rangle$ *or* $\Gamma \supseteq \langle 4 \cup \rangle$ ((**7**) Satz 6).

(16) *If a graph is not* κ*-colourable then at least one of its members is not* κ*-colourable.*

This follows from the definition of member (**16**).

(17) *The operations* o_1, o_2, o_3 *applied to 5-chromatic planar graphs give 5-chromatic planar graphs.*

Proof. For o_1 the proof is essentially contained in A. B. Kempe's attempt to prove the 4-colour conjecture (**15**): Let Γ denote a 5-chromatic planar graph and x a vertex of valency $\leqslant 4$ in Γ. If $v(x, \Gamma) \leqslant 3$ then $\Gamma - x$ is obviously not 4-colourable. Suppose that $v(x, \Gamma) = 4$ and $\Gamma - x$ is 4-colourable. Let $(x, a), (x, b), (x, c), (x, d)$ denote the edges radiating from x in this cyclic order. Kempe proves that if $\Gamma - x$ is 4-colourable then $\Gamma - x$ can be coloured with four colours in such a way that two of a, b, c, d have the same colour. It follows that Γ is 4-colourable, contrary to hypothesis. The reader can reconstruct the proof from (**15**), or alternatively from (**1**).

The truth of (17) for o_2 follows immediately from its truth for o_1.

(17) is true for o_3, because the result of the operation is a planar graph, and if it were 4-colourable then the original graph would be 4-colourable.

2. *Planar graphs which triangulate the whole plane*

DEFINITION 18. A planar graph is said to *triangulate the whole plane* if exactly three vertices and three edges constituting a $\langle 3 \rangle$ are adjacent to each region of the plane.

DEFINITION 19. If Γ is a graph, Γ' is a connected component of Γ (possibly $\Gamma' = \Gamma$), $\mathscr{S} \subset \mathscr{V}(\Gamma')$, and $\Gamma' - \mathscr{S}$ is disconnected, then $\mathscr{S}$ will be called a *cut-set* of Γ. If no proper subset of $\mathscr{S}$ is a cut-set of Γ then $\mathscr{S}$ will be called a *minimal cut-set* of Γ.

(18) *Every planar graph which triangulates the whole plane, except a* $\langle 3\rangle$, *is* 3-*fold connected. If* Γ *is such a graph and* $\mathscr{S}$ *is a finite minimal cut-set of* Γ *then* $\Gamma(\mathscr{S})$ *is a circuit and* $\Gamma-\mathscr{S}$ *has exactly two connected components.*

The proof is left to the reader.

3. *Irreducible* 5-*chromatic planar graphs*

DEFINITION 20. A 5-chromatic planar graph will be called an *irreducible graph* if it is not possible to obtain a 5-chromatic planar graph from it by applying the operations $\mathscr{o}$ and $\mathscr{o}_3$.

It follows from the definition and from (8), (9), (11), and the 5-colour theorem that

(19) *Every connected* 5-*chromatic planar graph is either irreducible or an irreducible graph can be obtained from it by the operations* $\mathscr{o}$ *and* $\mathscr{o}_3$. *Every irreducible graph is critical and contraction-critical and contains no* $\langle 4\rangle$ *as subgraph.*

The following are some of the properties of irreducible graphs:

(20) *Every irreducible graph triangulates the whole plane.*

The proof is left to the reader.

(21) *Every irreducible graph is* 5-*fold connected.*

This follows from (12). P. J. Heawood proved a special case of this (**13**).

(21′) *Every vertex of an irreducible graph has valency* $\geqslant 5$.

(22) *If* Γ *is an irreducible graph and* $\mathscr{S}$ *is a cut-set of* Γ *containing five vertices then one of the connected components of* $\Gamma-\mathscr{S}$ *is a single vertex* ((**3**) 120).

DEFINITION 21. If Γ is a graph and $a \in \mathscr{V}(\Gamma)$ then the subgraph of Γ generated by those vertices which are joined to a will be called the *inner entourage* of a in Γ and denoted by $E_i(a)$. The subgraph of Γ generated by those vertices $\neq a$ which are joined to at least one vertex of $E_i(a)$ will be called the *outer entourage* of a in Γ and denoted by $E_o(a)$. $\Gamma(a \cup E_i(a) \cup E_o(a))$ will be called the *entourage* of a in Γ and denoted by $E(a)$.

From (18), (20), (21), and (22) it follows ((**3**) 123) that

(23) *If* a *is any vertex of an irreducible graph then*

(A) $E_i(a)$ *and* $E_o(a)$ *are circuits,*

(B) *every vertex of* $E_i(a)$ *is joined to two or more consecutive vertices of* $E_o(a)$, *and every vertex of* $E_o(a)$ *is joined either to exactly one vertex or to exactly two consecutive vertices of* $E_i(a)$,

(C) *exactly* $|\mathscr{V}(E_i(a))|$ *vertices of* $E_o(a)$ *are each joined to two consecutive vertices of* $E_i(a)$.

A typical $E(a)$ is drawn in Fig. 1.

(24) *An irreducible graph does not contain a vertex of valency* 5 *joined to three vertices of valency* 5 *and one vertex of valency* 6 ((**12**) 230).

(25) *If* a *is a vertex of an irreducible graph having valency* 5, 6, 7, *or* $\nu > 7$, *then no* 3, 3, 4, *or* $\nu - 1$ *consecutive vertices of* $E_i(a)$ *have valency* 5, *respectively.*

For $v(a, \Gamma) = 5, 6, 7, \nu > 7$ see ((**3**) 126), ((**12**) 229), (**19**), ((**12**) 233) respectively.

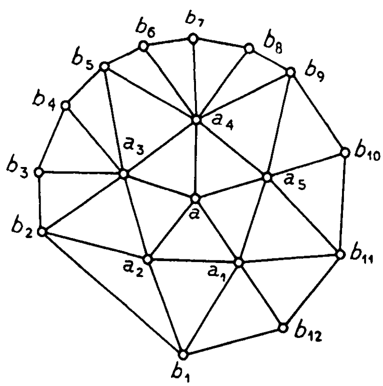

FIG. 1

(26) *In an irreducible graph every vertex of valency* $\leqslant 6$ *is joined to at least one vertex of valency* $\geqslant 7$ ((**20**) 515), *and every vertex of valency* $\geqslant 7$ *is joined to at least one vertex of valency* $\geqslant 7$ *or to at least three vertices of valency* 6 ((**19**) 260).

(27) *If* Γ *is an irreducible graph then*

$$|\mathscr{V}(\Gamma)| \geqslant 36 \quad \textit{and} \quad |\mathscr{E}(\Gamma)| \geqslant (17/6)|\mathscr{V}(\Gamma)|.$$

See (**21**). The last inequality but one on p. 408 of (**21**) is equivalent to $|\mathscr{E}(\Gamma)| \geqslant (17/6)|\mathscr{V}(\Gamma)|$, and this combined with Euler's inequality $|\mathscr{E}(\Gamma)| \leqslant 3|\mathscr{V}(\Gamma)| - 6$ gives $|\mathscr{V}(\Gamma)| \geqslant 36$. Hence $|\mathscr{E}(\Gamma)| \geqslant 102$.

Note. In the papers of P. Franklin and C. E. Winn referred to in connexion with (24)–(27) the equivalents of (24)–(27) are established for those 5-chromatic maps in the plane which contain the least possible number of regions; these maps are called irreducible maps by these authors. But in fact only the property of being irreducible under the equivalents of ρ and ρ_3 for maps is used in their proofs, not the absolute minimality of the number of countries, so that (24)–(27) are true for the irreducible graphs of Definition 19.

4. *Properties of the entourages in irreducible graphs*

In this section Γ will always denote an irreducible graph.

(28) $(\forall a)\{a \in \mathscr{V}(\Gamma) \rightarrow |\mathscr{V}(E_o(a))| \geqslant 9$ *and at least three vertices of* $E_o(a)$ *have valency* 3 *in* $E(a)\}$.

This follows from (24), (25), and (26).

(29) $(\forall a)\{a \in \mathscr{V}(\Gamma) \to |\mathscr{V}(\Gamma - E(a))| \geqslant 3\}$.

This follows from (21), (23), and (28).

(30) $(\forall a)\{a \in \mathscr{V}(\Gamma) \to$ *out of any three consecutive vertices of* $E_o(a)$ *at least one is joined to two or more vertices of* $\Gamma - E(a)\}$.

Proof. Suppose that this is false. Then the notation can be chosen so that the vertices of $E_o(a)$ are $b_1, \ldots, b_\beta$ consecutively around $E_o(a)$, and $v(b_j, \Gamma - E(a)) = 1$ for $j = 1, 2, 3$. By (21′) and (23B), $v(b_j, \Gamma) = 5$ and b_j is joined to two consecutive vertices of $E_i(a)$ for $j = 1, 2, 3$. The notation can be chosen so that the vertices of $E_i(a)$ are $a_1, \ldots, a_\alpha$ consecutively around $E_i(a)$ and $a_1 \times b_1, a_\alpha \times b_1, a_1 \times b_2, a_2 \times b_2, a_2 \times b_3$, and $a_3 \times b_3$. It follows that $v(a_1, \Gamma) = v(a_2, \Gamma) = 5$. Then we have a contradiction to (25) because three consecutive vertices of $E_i(a_1)$ have valency 5 in Γ. This proves (30).

(31) $(\forall a)\{a \in \mathscr{V}(\Gamma) \to$ *no two vertices of* $E_o(a)$ *have the property that every vertex of* $\Gamma - E(a)$ *is joined to at least one of them*$\}$.

Proof. Suppose that this is false. The notation can then be chosen so that the vertices of $E_o(a)$ are $b_1, \ldots, b_\beta$ consecutively around $E_o(a)$ ($\beta \geqslant 9$ by (28)), and

(*) *All vertices of* $\Gamma - E(a)$ *are adjacent to* b_1 *or to* b_τ, *and* $\tau \geqslant 6$.

Let c denote that vertex of $\Gamma - E(a)$ which is joined to b_1 and b_2, and d that vertex of $\Gamma - E(a)$ which is joined to $b_{\tau-1}$ and b_τ.

(i) $c \neq d$ *and* $c \circ b_{\tau-1}$ *and* $d \circ b_2$ *in* Γ.

For if, e.g., $c \times b_{\tau-1}$ in Γ, then by (*), $v(b_j, \Gamma - E(a)) = 1$ for $j = 2, \ldots, \tau - 2$ contrary to (30).

(ii) *It may be assumed without loss of generality that* $v(b_2, \Gamma - E(a) - c) = 0$.

For if this is not the case then b_2 is joined to one or more vertices of $E_i(b_\tau) \cap (\Gamma - E(a))$ and hence, by (23A) and (*), $v(b_{\tau-1}, \Gamma - E(a) - d) = 0$.

From (ii), (20), and (23A) it follows that

(iii) $c \times b_3$ *in* Γ *and* $v(b_2, \Gamma) = 5$.

From (ii), (21′), and (23) it follows that b_2 is joined to two consecutive vertices of $E_i(a)$. Let the notation be chosen so that the vertices of $E_i(a)$ are $a_1, \ldots, a_\alpha$ consecutively around $E_i(a)$ and $a_1 \times b_1$, $a_1 \times b_2$, $a_2 \times b_2$, and $a_2 \times b_3$.

The two alternatives $v(b_3, \Gamma - E(a)) = 1$ and $v(b_3, \Gamma - E(a)) \geqslant 2$ will now be considered in turn.

1. Suppose that $v(b_3, \Gamma - E(a)) = 1$. Then $c \times b_4$, $a_3 \times b_3$, and $a_3 \times b_4$ in Γ and $v(a_2, \Gamma) = v(b_3, \Gamma) = 5$ by (20), (21′), and (23). $v(b_4, \Gamma - E(a)) \geqslant 2$ by (30).

b_4 is joined to no vertex of $E_i(b_1) \cap (\Gamma - E(a))$ other than c, for otherwise $v(c, \Gamma) = 5$ by (22) and (21′), and so three consecutive vertices of $E_i(b_3)$, namely c, b_2, and a_2, have valency 5 in Γ, which contradicts (25). Hence by (*) b_4 is joined to at least one vertex of $E_i(b_\tau) \cap (\Gamma - E(a))$.

$v(a_3, \Gamma) \geqslant 6$, for otherwise three consecutive vertices of $E_i(b_3)$, namely b_2, a_2, and a_3, have valency 5 in Γ, which contradicts (25). Hence $a_3 \times b_5$ and $v(b_4, E(a)) = 3$.

$\tau \geqslant 7$. For if $\tau = 6$ then b_4 is joined to d and to no other vertex of $E_i(b_6)$ and so by (*), $v(b_4, \Gamma) = v(b_5, \Gamma) = 5$ and $a_4 \times b_5$; consequently $v(a_3, \Gamma) = 6$ and four consecutive vertices of $E_i(a_3)$, namely a_2, b_3, b_4, b_5, have valency 5 in Γ, which contradicts (25).

By (23) $v(b_{\tau-1}, \Gamma - E(a) - d) = v(b_{\tau-2}, \Gamma - E(a) - d) = 0$ because b_4 is joined to at least one vertex of $E_i(b_\tau) \cap (\Gamma - E(a))$; hence $d \times b_{\tau-2}$ and $d \times b_{\tau-3}$ in Γ by (20).

$b_{\tau-3}$ is joined to no vertex of $E_i(b_\tau) \cap (\Gamma - E(a))$ other than d for the same reason that b_4 is joined to no vertex of $E_i(b_1) \cap (\Gamma - E(a))$ other than c.

If $\tau = 7$ then $b_{\tau-3} = b_4$ and so $v(b_4, \Gamma) = 5$, and further $a_4 \times b_5$, so $v(a_3, \Gamma) = 6$. But four consecutive vertices of $E_i(a_3)$, namely a_2, b_3, b_4, and b_5, have valency 5 in Γ, and (25) is contradicted. If $\tau > 7$ then $b_{\tau-3} \neq b_4$ and so $v(b_j, \Gamma - E(a)) = 1$ for $j = \tau-3, \tau-2, \tau-1$; consequently (30) is contradicted. So the assumption that $v(b_3, \Gamma - E(a)) = 1$ leads to a contradiction.

2. Suppose that $v(b_3, \Gamma - E(a)) \geqslant 2$. $v(b_3, E_i(b_1) - b_2 - c) = 0$ by (21); hence b_3 is joined to one or more vertices of $E_i(b_\tau) \cap (\Gamma - E(a))$ by (*). It follows by (21) that $v(b_{\tau-1}, \Gamma - E(a) - d) = v(b_{\tau-2}, \Gamma - E(a) - d) = 0$, and so $d \times b_{\tau-2}$ and $d \times b_{\tau-3}$ by (20).

$b_{\tau-3}$ is joined to no vertex of $E_i(b_\tau) \cap (\Gamma - E(a))$ other than d for the same reason that it follows from $v(b_3, \Gamma - E(a)) = 1$ that b_4 is joined to no vertex of $E_i(b_1) \cap (\Gamma - E(a))$ other than c. $\tau \geqslant 7$ because if $\tau = 6$ then $b_{\tau-3} = b_3$ and we have essentially the same situation as in 1 with $\tau = 6$, which leads to a contradiction. So $b_{\tau-3} \neq b_3$ and therefore $v(b_j, \Gamma - E(a)) = 1$ for $j = \tau-3$, $\tau-2$, $\tau-1$, and (30) is contradicted. (31) has now been proved.

5. *Hexagonal circuits in irreducible graphs*

The propositions (32)–(38) here can also be deduced from the contents of the investigation of 6-rings in minimal 5-chromatic maps made by A. Bernhart (**2**). But § 5 can be read independently of (**2**). Ξ will in § 5 always denote a 6-circuit contained as a subgraph in the irreducible graph Γ, and the vertices of Ξ will be denoted by $x_1, \ldots, x_6$ consecutively around Ξ. Θ_i and Θ_o will denote the subgraphs of Γ generated by the vertices situated inside and outside Ξ, respectively.

(32) *Θ_i is empty or connected and Θ_o is empty or connected.*

This follows from (21).

(33) *For $j = i, o, |\mathscr{V}(\Theta_j)| \geqslant 3 . \rightarrow . (\forall a)\{a \in \mathscr{V}(\Theta_j) \rightarrow v(a, \Xi) \leqslant 3\}$.*

This follows from (21) and (22).

In the rest of § 5 *it will be assumed that $\Theta_i \neq \emptyset$ and $\Theta_o \neq \emptyset$.*

(34) *For $j = i, o$, $|\mathscr{V}(\Theta_j)| \geqslant 2 . \rightarrow . v(x_k, \Theta_j) \geqslant 1$ for $k = 1, \dots, 6$.*

Proof. If e.g. $|\mathscr{V}(\Theta_i)| \geqslant 2$ and $\Theta_o \neq \emptyset$ then, by (21) and (22), no two vertices of Ξ are joined by an edge situated inside Ξ, and hence by (20) $v(x_k, \Theta_i) \geqslant 1$ for $k = 1, \dots, 6$.

(35) *$(\forall a)\{a \in \mathscr{V}(\Gamma - \Xi)$ and $v(a, \Xi) \geqslant 2 . \rightarrow .$ the vertices of Ξ to which a is joined are consecutive around Ξ and around $E_i(a)\}$.*

Proof. Suppose that $a \in \mathscr{V}(\Theta_i)$ and $v(a, \Xi) \geqslant 2$, but the vertices of Ξ to which a is joined are not consecutive around Ξ. Then $\Theta_i - a \neq \emptyset$ since $v(a, \Gamma) \geqslant 5$ by (21′). Let the notation be chosen either so that $a \times x_1$, $a \circ x_2$, and $a \times x_3$ or so that $a \times x_1$, $a \circ x_2$, $a \circ x_3$, and $a \times x_4$.

Suppose that $a \times x_1$, $a \circ x_2$, and $a \times x_3$. Then, by (21), no vertex of Γ is situated inside the 4-circuit x_1, x_2, x_3, a, and hence, by (20), $x_1 \times x_3$ and (x_1, x_3) is situated inside this 4-circuit. It follows that all vertices of Θ_i are situated inside the 5-circuit x_1, x_3, x_4, x_5, x_6, while the vertices of Θ_o and x_2 are situated outside it. $|\mathscr{V}(\Theta_i)| \geqslant 2$ and $\Theta_0 \neq \emptyset$, so (22) is contradicted.

Suppose that $a \times x_1$, $a \circ x_2$, $a \circ x_3$, and $a \times x_4$. This will lead to a contradiction also. By (20) and (22) either exactly one vertex of Γ is situated inside the 5-circuit x_1, x_2, x_3, x_4, a, or it may be assumed without loss of generality that $x_1 \times x_3$ and $x_1 \times x_4$ and $(x_1, x_3), (x_1, x_4)$ are situated inside this 5-circuit. If the vertex b is situated inside the 5-circuit x_1, x_2, x_3, x_4, a, then b is joined to all vertices of the 5-circuit by (21), and so a is the only vertex of Γ situated inside the 5-circuit x_1, b, x_4, x_5, x_6 by (22), and consequently $a \times x_5$ and $a \times x_6$ by (21). But this contradicts the initial assumption made about a. If $x_1 \times x_4$ and (x_1, x_4) is situated inside the 5-circuit x_1, x_2, x_3, x_4, a, then a is situated inside the 4-circuit x_1, x_4, x_5, x_6, and x_2 is situated outside it, and so (21) is contradicted.

Hence if $a \in \mathscr{V}(\Theta_i)$ and $v(a, \Xi) \geqslant 2$ then the vertices of Ξ to which a is joined are consecutive around Ξ. Suppose that $a \times x_1, \dots, a \times x_\xi$, where $2 \leqslant \xi \leqslant 6$. By (21) no vertex of Γ is situated inside the triangulated $(\xi + 1)$-circuit $a, x_1, \dots, x_\xi$, and so $x_1, \dots, x_\xi$ are consecutive around $E_i(a)$. (35) is therefore proved for the vertices of $(\Gamma - \Xi) \cap \Theta_i$. For the vertices of $(\Gamma - \Xi) \cap \Theta_0$ it can be proved similarly.

(36) *Neither* Θ_i *nor* Θ_o *has a cut-vertex.*

Proof. If a is a cut-vertex of Θ_i or of Θ_o then $v(a, \Xi) \geqslant 2$ by (20), but the vertices of Ξ to which a is joined are not consecutive around Ξ by (21) and so (35) is contradicted.

(37) *For* $j = i, o$, $|\mathscr{V}(\Theta_j)| \geqslant 3 \rightarrow \Theta_j$ *contains as a subgraph a unique circuit* Ξ_j *such that every vertex of* Ξ_j *is joined to at least one vertex of* Ξ *and every vertex of* $\Theta_j - \Xi_j$ *is separated from* Ξ *by* Ξ_j.

Proof. If $|\mathscr{V}(\Theta_i)| \geqslant 3$ then Θ_i contains at least one circuit as a subgraph by (36). Let Ξ_i denote a circuit with maximal area such that $\Xi_i \subseteq \Theta_i$. It follows from (20) and (36) that Ξ_i has the required properties. If $|\mathscr{V}(\Theta_0)| \geqslant 3$ then Θ_o contains at least one circuit by (36), and if Ξ_0 is a circuit with minimal area such that $\Xi_o \subseteq \Theta_o$ then, by (20) and (36), Ξ_o has the required properties.

(38) *For* $j = i, o$, *the number of* $(\Xi)(\Theta_j)$*-edges contained in* Γ *is at most* $|\mathscr{V}(\Xi_j)| + 6$.

This follows from (35) and (37).

LEMMA 1. *For* $j = i, o$, $|\mathscr{V}(\Theta_j)| \geqslant 4 \rightarrow \Theta_j - \Xi_j$ *contains a connected component with at least three vertices.*

Proof. The truth of Lemma 1 with $j = o$ follows from its truth with $j = i$ because the plane can be deformed so that Ξ changes into a circle whose centre does not belong to Γ and the (deformed) plane can be inverted with respect to Ξ. Lemma 1 with $j = i$ will now be proved by *reductio ad absurdum.*

Let it be assumed that Lemma 1 with $j = i$ is not always true, and in what follows let Ξ denote a 6-circuit of minimal area among those 6-circuits of Γ for which it is false. At least one such 6-circuit with least area exists, because Γ is finite.

(i) $\Theta_i - \Xi_i \neq \emptyset$ *and* $|\mathscr{V}(\Xi_i)| \geqslant 5$.

For suppose on the contrary that $\Theta_i - \Xi_i = \emptyset$. Then $\Theta_i \not\supseteq \langle 4 \cup \rangle$ by (37), and $\Theta_i \not\supseteq \langle 4 \rangle$ by (21); therefore $|\mathscr{E}(\Theta_i)| \leqslant 2|\mathscr{V}(\Theta_i)| - 3$ by (15). On the other hand, all but at most three vertices of Ξ_i are joined to fewer than three vertices of Ξ by (35), from which it follows by (21′), (35), and (25) that at least one vertex of Θ_i has valency $\geqslant 6$ in Γ. Hence, by (21′) and (38), $2|\mathscr{E}(\Theta_i)| \geqslant 5|\mathscr{V}(\Theta_i)| + 1 - (|\mathscr{V}(\Theta_i)| + 6)$, and we have a contradiction. So $\Theta_i - \Xi_i \neq \emptyset$. From (21) and (37) it follows that $|\mathscr{V}(\Xi_i)| \geqslant 5$.

(ii) $(\forall y)\{y \in \mathscr{V}(\Xi_i) \rightarrow v(y, \Xi) \leqslant 2\}$.

For if this is false then in view of (33) and (35) it may be assumed without loss of generality that $y \in \mathscr{V}(\Xi_i)$ and $y \times x_1$, $y \times x_2$, $y \times x_3$. Let Ξ' denote the 6-circuit $x_1, y, x_3, x_4, x_5, x_6$, and let Θ_i' denote the subgraph of Γ generated by the vertices situated inside Ξ', etc. $|\mathscr{V}(\Theta_i')| \geqslant 4$ by (i) and Ξ' has smaller area than Ξ; therefore $\Theta_i' - \Xi_i'$ contains a connected component with more than two vertices. But $\Theta_i' - \Xi_i' \subseteq \Theta_i - \Xi_i$ and so we have a contradiction.

(iii) $(\forall y)\{y \in \mathscr{V}(\Xi_i) \rightarrow v(y, \Theta_i) \geqslant 3\}$.

This follows from (ii) and (21′).

(iv) *No edge of* Γ *joins two non-consecutive vertices of* Ξ_i.

Suppose that this is false. Then the vertices of Ξ_i can be labelled $y_1, \ldots, y_\xi$ consecutively around Ξ_i so that $y_1 \times y_\rho$ in Γ and $(y_1, y_\rho) \notin \Xi_i$. (y_1, y_ρ) is situated inside Ξ_i because Ξ_i is the circuit of greatest area contained in Θ_i. Consequently by (iii) and (21), $5 \leqslant \rho \leqslant \xi - 3$. It follows from (22) that the notation can be chosen so that $x_1 \times y_1$ and $x_4 \times y_\rho$ and $y_2, \ldots, y_{\rho-1}$ are situated inside the 6-circuit $x_1, y_1, y_\rho, x_4, x_3, x_2$. Let this 6-circuit be denoted by Ξ''. $|\mathscr{V}(\Theta_i'')| \geqslant 4$ because $\rho \geqslant 5$ and because it follows from (iii) that at least one vertex of Γ is situated inside the circuit $y_1, y_2, \ldots, y_\rho$. $\Theta_i'' - \Xi_i''$ contains a connected component with more than two vertices because Ξ'' has smaller area than Ξ. But $\Theta_i'' - \Xi_i'' \subset \Theta_i - \Xi_i$ and so we have a contradiction.

(v) $\Theta_i - \Xi_i$ *is connected.*

For suppose that $\Theta_i - \Xi_i$ is disconnected, let Ψ denote a connected component of $\Theta_i - \Xi_i$, and let $y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_\psi}$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_\psi$, denote those vertices of Ξ_i which are joined to one or more vertices of Ψ. $\psi \geqslant 5$ by (21). $y_{\alpha_1}, \ldots, y_{\alpha_\psi}$ determine ψ arcs on Ξ_i. Those of the arcs which contain an interior vertex joined to one or more vertices of $\Theta_i - \Xi_i - \Psi$ contain at least five vertices by (21). Suppose, as we may, that y_{α_1} and y_{α_2} are the two end-vertices of such an arc. Then $y_{\alpha_1} \times y_{\alpha_2}$ in Γ by (20), but $(y_{\alpha_1}, y_{\alpha_2}) \notin \Xi_i$. So (iv) is contradicted.

$|\mathscr{V}(\Theta_i - \Xi_i)| > 1$. For if $\Theta_i - \Xi_i$ consists of the single vertex a then, by (20) and (iv), $\Xi_i = E_i(a)$. It follows that $\mathscr{V}(E_0(a)) \subseteq \mathscr{V}(\Xi)$, and so (28) is contradicted.

$|\mathscr{V}(\Theta_i - \Xi_i)| > 2$.

Suppose that this is not true. Then it follows that $\Theta_i - \Xi_i = \langle 2 \rangle$. Let $\mathscr{V}(\Theta_i - \Xi_i) = \{a, b\}$. Then by (ii) and (iv), $(\forall y)\{y \in \mathscr{V}(\Xi_i) \rightarrow v(y, \Gamma) \leqslant 6\}$. Hence if $v(a, \Gamma)$ and $v(b, \Gamma)$ were $\leqslant 6$, neither a nor b would be joined to a vertex of valency $\geqslant 7$, so that by (26) it may be assumed that $v(a, \Gamma) \geqslant 7$. In view of (20) and (iv) the notation can be chosen so that

$y_1, \ldots, y_\xi$ denote the vertices of Ξ_i consecutively around Ξ_i and $a \times y_1$, $a \times y_2, \ldots, a \times y_\alpha$, $b \times y_\alpha$, $b \times y_{\alpha+1}, \ldots, b \times y_\xi$, $b \times y_1$, where $8 \leqslant \alpha + 2 \leqslant \xi$. By (21′), (ii), and (iv), $v(y_j, \Xi) = 2$ and $v(y_j, \Gamma) = 5$ for $j = 2, \ldots, \alpha - 1, \alpha + 1, \ldots, \xi$. From this and (38) it follows that $\xi = 8$ and $\alpha = 6$; consequently by (38), $v(y_1, \Gamma) = v(y_\alpha, \Gamma) = 5$. Hence $v(y_j, \Gamma) = 5$ for $j = 1, \ldots, \xi$, and so (25) is contradicted. Therefore $|\mathscr{V}(\Theta_i - \Xi_i)| \geqslant 3$. But this and (v) together contradict the initial assumption that $\Theta_i - \Xi_i$ contains no connected component with more than two vertices, and Lemma 1 is proved.

Note. It can be shown by a further analysis that for $j = i, o$, $|\mathscr{V}(\Theta_j)| \geqslant 4 \rightarrow \Theta_j - \Xi_j$ contains a connected component with at least four vertices.

LEMMA 2. *For* $j = i, o$, $|\mathscr{V}(\Theta_j)| \geqslant 4 \rightarrow (\forall t)\{t \in \mathscr{V}(\Theta_j) \rightarrow \Theta_j - t \supseteq \langle 4 \cup \rangle\}$.

Proof. The truth of Lemma 2 for $j = o$ follows from its truth for $j = i$ as in the case of Lemma 1. The truth of Lemma 2 for $j = i$ will be proved by *reductio ad absurdum.* Let it be assumed false and in what follows let Ξ denote a 6-circuit with minimal area among those 6-circuits of Γ for which it is false, and let t denote a vertex of Θ_i such that $\Theta_i - t \not\supseteq \langle 4 \cup \rangle$.

(i) $(\forall y)\{y \in \mathscr{V}(\Xi_i) \rightarrow v(y, \Xi) \leqslant 2\}$.

For if not, we may assume that $y \in \mathscr{V}(\Xi_i)$ and $y \times x_1$, $y \times x_2$, $y \times x_3$ by (35). Let Ξ' denote the 6-circuit $x_1, y, x_3, x_4, x_5, x_6$. $\Theta_i' = \Theta_i - y$. $|\mathscr{V}(\Theta_i')| \geqslant 4$ because it follows from Lemma 1 and (22) that $|\mathscr{V}(\Xi_i)| \geqslant 6$. Ξ' has smaller area than Ξ, hence $(\forall u)\{u \in \mathscr{V}(\Theta_i') \rightarrow \Theta_i' - u \supseteq \langle 4 \cup \rangle\}$. This clearly contradicts the assumption that $\Theta_i - t \not\supseteq \langle 4 \cup \rangle$.

(ii) $(\forall y)\{y \in \mathscr{V}(\Xi_i) \rightarrow v(y, \Theta_i) \geqslant 3\}$.

This follows from (i) and (21′).

(iii) *No edge of* Γ *joins two non-consecutive vertices of* Ξ_i.

This can be proved in the same way as (iv) in the proof of Lemma 1.

(iv) $t \in \Xi_i$.

For suppose on the contrary that $t \in \Theta_i - \Xi_i$. $\Theta_i - \Xi_i - t \neq \emptyset$ by Lemma 1. Let $u \in \mathscr{V}(\Theta_i - \Xi_i - t)$. $\Gamma - t$ is 4-fold connected and $\Gamma - t \supset \Xi_i$; it follows from this by an extension of Menger's Theorem (**9**) that $\Theta_i - t$ contains four $(u)(\Xi_i)$-paths any two of which have only u in common. Therefore $\Theta_i - t \supset \langle 4 \cup \rangle$ contrary to hypothesis.

(v) $(\forall v)\{v \in \mathscr{V}(\Theta_i - \Xi_i) \rightarrow t \times v\}$.

For if $v \in \mathscr{V}(\Theta_i - \Xi_i)$ and $t \circ v$ then the subgraph of Γ generated by $v \cup E_i(v)$, which contains a $\langle 4 \cup \rangle$ as a subgraph, is a subgraph of $\Theta_i - t$.

(vi) *The notation may be chosen so that the vertices of* Ξ_i *are* $y_1, \ldots, y_\xi$ *consecutively around* Ξ_i, $t = y_1$, *and the vertices of* $\Theta_i - \Xi_i$ *are* $z_1, \ldots, z_\omega$, *where* $\omega \geqslant 3$ *and* $y_2, z_1, \ldots, z_\omega, y_\xi$ *are consecutive vertices of* $E_i(y_1)$.

This follows from Lemma 1, (iv), and (v).

(vii) $\xi \geqslant 7$, *and* $v(y_j, \Xi) = 2$ *and* $v(y_j, \Gamma) = 5$ *for* $j = 2, 3, \xi-1, \xi$.

This follows from (23), (i), (iii), and (vi).

(viii) $v(z_1, \Gamma) \geqslant 6$ *or* $v(z_\omega, \Gamma) \geqslant 6$.

For suppose that on the contrary $v(z_1, \Gamma) = v(z_\omega, \Gamma) = 5$. Then $y_3 \times z_1$, $y_4 \times z_1$, $y_4 \times z_2$, $y_{\xi-1} \times z_\omega$, $y_{\xi-2} \times z_\omega$, $y_{\xi-2} \times z_{\omega-1}$, and $v(y_4, \Gamma) = v(y_{\xi-2}, \Gamma) = 6$ and $v(y_4, \Xi) = v(y_{\xi-2}, \Xi) = 2$ by (23), (25), (i), (vi), and (vii). Since $v(y_4, \Gamma) = 6$ it follows from (vii) and (25) that $v(z_2, \Gamma) \geqslant 6$, and consequently by (i), (iii), (21), and (23), $v(y_5, \Xi) = 2$ and $\xi \geqslant 8$. Thus $v(y_j, \Xi) = 2$ for $j = 2, 3, 4, 5, \xi-2, \xi-1, \xi$, and $\xi - 2 > 5$. This together with (37) contradicts (38), and so (viii) is proved.

Assume without loss of generality that $v(z_1, \Gamma) \geqslant 6$.

$$z_1 \times y_j \quad \text{for} \quad j = 1, 2, \ldots, v(z_1, \Gamma) - 1$$

and hence by (21), (23), and (i), $v(y_j, \Gamma) = 5$ for $j = 2, 3, \ldots, v(z_1, \Gamma) - 2$. From this and (25) it follows that $v(z_1, \Gamma) \geqslant 8$. Therefore by (21), (23), and (i) and (vii),

$$v(y_j, \Xi) = 2 \quad \text{for} \quad j = 2, 3, \ldots, v(z_1, \Gamma) - 2, \xi - 1, \xi$$

and

$$\xi - 1 > v(z_1, \Gamma) - 2 \geqslant 6.$$

Hence (38) is contradicted. Lemma 2 is therefore proved.

LEMMA 3. *If* $\{j, k\} = \{i, o\}$ *then* $|\mathscr{V}(\Theta_j)| \geqslant 4 \rightarrow$

$(\forall p, q, r)\{p \in \mathscr{V}(\Xi)$ *and* $q, r \in \mathscr{V}(\Theta_j) . \rightarrow . \Gamma - \Theta_k - p - q - r \supseteq \langle 4 \cup \rangle\}$.

Proof. The truth of Lemma 3 for $j = o$ and $k = i$ follows from its truth for $j = i$ and $k = o$ as in the case of Lemma 1. Lemma 3 with $j = i$ and $k = o$ will be proved by *reductio ad absurdum*. Let it be assumed not generally true and in what follows let Ξ denote a 6-circuit of minimal area, contained in Γ as a subgraph, for which Lemma 3 with $j = i$ and $k = o$ is false, and let p, q, r be such that $p \in \mathscr{V}(\Xi)$, $q, r \in \mathscr{V}(\Theta_i)$ and $\Gamma - \Theta_0 - p - q - r \not\supseteq \langle 4 \cup \rangle$. *Then every vertex of* Θ_i *is joined to at least one of* p, q, r.

(i) $(\forall y)\{y \in \mathscr{V}(\Xi_i) \rightarrow v(y, \Xi) \leqslant 2\}$.

For if this is false we may assume that $y \in \mathscr{V}(\Xi_i)$ and $y \times x_1, y \times x_2, y \times x_3$ by (33) and (35). Let Ξ' denote the 6-circuit $x_1, y, x_3, x_4, x_5, x_6$. $|\mathscr{V}(\Theta_i')| \geqslant 4$ because $\Theta_i' = \Theta_i - y$ and because it follows from Lemma 1 and (22) that $|\mathscr{V}(\Xi_i)| \geqslant 6$. $q, r \in \mathscr{V}(\Theta_i')$ by Lemma 2. Hence, by the minimal property of Ξ,

$$\Gamma - \Theta_0' - p - q - r \supseteq \langle 4 \cup \rangle.$$

But this is a contradiction because

$\Gamma - \Theta_0' - p - q - r = \Gamma - \Theta_0 - y - p - q - r$ and $\Gamma - \Theta_0 - p - q - r \not\supseteq \langle 4 \cup \rangle$.

(ii) $(\forall y)\{y \in \mathscr{V}(\Xi_i) \rightarrow v(y, \Theta_i) \geqslant 3\}$.

(iii) $v(x_h, \Xi_i) \geqslant 2$ *for* $h = 1, \ldots, 6$.

(ii) and (iii) follow from (i) by (20) and (21′).

(iv) *No edge of* Γ *joins two non-consecutive vertices of* Ξ_i.

Suppose that this is false. Then, as in the proof of (iv) in the proof of Lemma 1, the vertices of Ξ_i can be labelled $y_1, \ldots, y_\xi$ consecutively around Ξ_i so that $y_1 \times y_\rho$ and $(y_1, y_\rho) \notin \Xi_i$, (y_1, y_ρ) is situated inside Ξ_i, $5 \leqslant \rho \leqslant \xi - 3$, $x_1 \times y_1$ and $x_4 \times y_\rho$, and at least four vertices of Γ are situated inside each of the 6-circuits $x_1, y_1, y_\rho, x_4, x_3, x_2$ and $x_1, y_1, y_\rho, x_4, x_5, x_6$. Hence $\Theta_i - q - r \supset \langle 4 \cup \rangle$ by Lemma 2, which contradicts the hypothesis that $\Gamma - \Theta_0 - p - q - r \not\supseteq \langle 4 \cup \rangle$. This proves (iv).

The notation may be chosen so that $p = x_1$ and $y_1, \ldots, y_\xi$ denote the vertices of Ξ_i consecutively around Ξ_i and $x_1 \times y_j$ for $j = 1, \ldots, \mu$, and $x_2 \times y_1$, $x_2 \times y_\xi$, and $x_6 \times y_\mu$, $x_6 \times y_{\mu+1}$. $2 \leqslant \mu \leqslant \xi - 4$ by (iii).

(v) $x_1 \times q$ *or* $x_1 \times r$.

For suppose that $x_1 \circ q$ and $x_1 \circ r$. Let Ξ^* denote the circuit

$$x_2, y_1, y_2, \ldots, y_\mu, x_6, x_5, x_4, x_3.$$

$\Xi^* \subset \Gamma - \Theta_0 - p - q - r$ because $x_1 \circ q$ and $x_1 \circ r$. At least four vertices of Γ are situated inside Ξ^* because $\mu \leqslant \xi - 4$; let u denote a vertex of $\Gamma - q - r$ which is situated inside Ξ^*. $\Gamma - p - q$ is 3-fold connected by (21), and it follows (**9**) that $\Gamma - \Theta_0 - p - q - r$ contains three $(u)(\Xi^*)$-paths any two of which have only u in common. Therefore $\Gamma - \Theta_0 - p - q - r \supset \langle 4 \cup \rangle$ contrary to hypothesis.

(vi) $x_1 \circ q$ *or* $x_1 \circ r$.

For suppose that $x_1 \times q$ and $x_1 \times r$. Then $\{q, r\} \subseteq \{y_1, \ldots, y_\mu\}$. Hence, by (iv), $p \circ y_j$, $q_0 \circ y_j$, and $r \circ y_j$, if $\mu + 2 \leqslant j \leqslant \xi - 1$. Therefore

$$\Gamma - \Theta_0 - p - q - r \supset \Gamma(y_j \cup E_i(y_j)) \supset \langle 4 \cup \rangle,$$

contrary to hypothesis. This proves (vi).

Let the notation be chosen so that $x_1 \times q$, $x_1 \circ r$, $q = y_\nu$, and $2 \leqslant \nu \leqslant \mu$.

(vii) $q \times r$.

For suppose that $q \circ r$. Let

$$\Sigma = (\Theta_i \cap E_i(y_\nu)) \cup (\Xi_i - y_\nu - y_{\mu+1} - y_{\mu+2} - \cdots - y_\xi) \cup (\Xi - x_1) \cup (x_2, y_1) \cup (x_6, y_\mu) \qquad \text{if} \quad 2 \leqslant \nu < \mu$$

and let

$$\Sigma = (\Theta_i \cap E_i(y_\mu)) \cup (\Xi_i - y_\mu - y_{\mu+2} - y_{\mu+3} - \dots - y_\xi) \cup (\Xi - x_1) \cup (x_2, y_1) \cup (x_6, y_{\mu+1}) \qquad \text{if} \quad \nu = \mu.$$

Σ is a circuit, p and q are situated outside Σ and r is situated inside Σ (because $q \circ r$ by hypothesis). Consequently $\Sigma \subset \Gamma - \Theta_0 - p - q - r$. $y_{\mu+2}, y_{\mu+3}, \dots, y_\xi$ are situated inside Σ. Let s denote a vertex of $\Gamma - r$ which is situated inside Σ. $\Gamma - r$ is 4-fold connected, and it follows (**9**) that $\Gamma - \Theta_0 - p - q - r$ contains four $(s)(\Sigma)$-paths any two of which have only s in common. So $\Gamma - \Theta_0 - p - q - r \supset \langle 4 \cup \rangle$ contrary to hypothesis. This proves (vii).

$p \circ y_\xi$ and $q \circ y_\xi$ because of the way in which the notation has been chosen and by (iv). Therefore $r \times y_\xi$. So, by (vii), Γ contains the 5-circuit $x_1, y_\nu(=q), r, y_\xi, x_2$. By (22), at most one vertex of Γ is situated inside this 5-circuit, so $q = y_2$. $\mu \leqslant 3$, because if $\mu > 2$ then $p \circ y_{\mu+1}$ and $q \circ y_{\mu+1}$ (by (iv)), and so $r \times y_{\mu+1}$; therefore Γ contains the 5-circuit $x_1, y_2, r, y_{\mu+1}, x_6$, and by (22) at most one vertex of Γ is situated inside this 5-circuit.

By (iv) and (vii), and since $\Gamma - \Theta_0 - p - q - r \not\supseteq \langle 4 \cup \rangle, r \times y_2, r \times y_4, r \times y_5, \dots, r \times y_\xi$. Therefore, by (iv), (vii), (21), and since $2 \leqslant \mu \leqslant \xi - 4$, $\Theta_i - \Xi_i = r$. This contradicts Lemma 1, and Lemma 3 is proved.

From Lemmas 2 and 3 there follows

Proposition 3. *If Γ is an irreducible graph which contains a 6-circuit such that at least four vertices of Γ are situated inside it and at least four outside it, then*

$$(\forall v_1, v_2, v_3, v_4, v_5)\{v_1, \dots, v_5 \in \mathscr{V}(\Gamma) \rightarrow \Gamma - v_1 - \dots - v_5 \supset \langle 4 \cup \rangle\}.$$

6. *Proof that no irreducible graph contains a connected skeleton with fewer than six vertices*

Proposition 4. *If Γ is an irreducible graph then*

$$(\forall \Omega)\{\Omega \subset \Gamma \text{ and } \Omega \text{ connected and } |\mathscr{V}(\Omega)| \leqslant 5 . \rightarrow . \Gamma - \Omega \supseteq \langle 4 \cup \rangle\}.$$

Proof. By *reductio ad absurdum*. Suppose that Ω is a connected subgraph of Γ with fewer than six vertices and $\Gamma - \Omega \not\supseteq \langle 4 \cup \rangle$.

(i) $(\forall x)\{x \in \mathscr{V}(\Gamma - \Omega) \rightarrow v(x, \Omega) \geqslant 1\}$.

For if $x \in \mathscr{V}(\Gamma - \Omega)$ and $v(x, \Omega) = 0$ then $\Gamma - \Omega \supset \Gamma(x \cup E_i(x))$, but $\Gamma(x \cup E_i(x)) \supset \langle 4 \cup \rangle$ and so the hypothesis that $\Gamma - \Omega \not\supseteq \langle 4 \cup \rangle$ is contradicted.

(ii) $(\forall x)\{x \in \mathscr{V}(\Gamma - \Omega)$ *and* $|\mathscr{V}(\Omega \cap E_i(x)| = 1 . \rightarrow . |\mathscr{V}(\Omega \cap E_o(x))| \geqslant 2\}$.

For $(\forall y, z)\{y \in \mathscr{V}(E_i(x))$ and $z \in \mathscr{V}(E_o(x)) . \rightarrow . E(x) - y - z \supset \langle 4 \cup \rangle\}$, as the reader can easily verify.

$\Gamma - \Omega$ contains at least two vertices of valency $\leqslant 2$ by (5). Let a denote such a vertex. $v(a, \Omega) \geqslant 3$ by (21′).

(iii) $|\mathscr{V}(\Omega)| = 5$, $v(a,\Omega) = 3$, *and* $v(a,\Gamma) = 5$.

Proof. $\Gamma-\Omega \not\supseteq \Gamma-E_i(a)-a$ because of (23), (29), and (i). It follows that $v(a,\Omega)<|\mathscr{V}(\Omega)|$. If $v(a,\Omega) = |\mathscr{V}(\Omega)|-1$ then $\Omega\cap E_o(a)$ consists of a single vertex and $\Omega\subset E(a)$ because $\Gamma-\Omega\not\supseteq\Gamma-E_i(a)-a$ and because Ω is connected; this and (31) together contradict (i), and consequently $v(a,\Omega)\leqslant|\mathscr{V}(\Omega)|-2$. But $|\mathscr{V}(\Omega)|\leqslant 5$ and $v(a,\Omega)\geqslant 3$; therefore $|\mathscr{V}(\Omega)| = 5$ and $v(a,\Omega) = 3$. It follows that $v(a,\Gamma) = 5$ because $v(a,\Gamma-\Omega)\leqslant 2$.

(iv) Ω *contains exactly one vertex of* $E_o(a)$ *and exactly one vertex of* $\Gamma-E(a)$ *and these two vertices are joined by an edge in* Γ.

Proof. $\Omega\cap E_o(a)\neq\emptyset$ because $|\mathscr{V}(\Omega)| = 5$, $v(a,\Omega) = 3$, and Ω is connected. $\Omega\not\subset E(a)$ because if $\Omega\subset E(a)$ then $|\mathscr{V}(\Omega\cap E_o(a))| = 2$, which together with (31) contradicts (i). So $|\mathscr{V}(\Omega\cap E_o(a))| = 1$. This and $|\mathscr{V}(\Omega\cap E_i(a))| = 3$ and $|\mathscr{V}(\Omega)| = 5$, and Ω connected, together imply (iv).

(v) *The three vertices of* $\Omega\cap E_i(a)$ *are consecutive around* $E_i(a)$.

This follows from (23), (iii), (iv), and the fact that Ω is connected.

Let the notation be chosen so that the vertices of $E_i(a)$ are $a_1,\ldots,a_5$ consecutively around $E_i(a)$ and the vertices of $E_o(a)$ are $b_1,\ldots,b_\beta$ consecutively around $E_o(a)$ and $a_1\times b_1$, $a_2\times b_1$, $a_2\times b_2$, and $\{a_3,a_4,a_5\}\subset\Omega$. Let b and c denote $\Omega\cap E_o(a)$ and $\Omega\cap(\Gamma-E(a))$ respectively.

(vi) $b\times c$, $b\times a_3\vee b\times a_4\vee b\times a_5$, $b\neq b_1$.

For Ω is connected.

There are two alternatives: $c\circ b_1$ or $c\times b_1$; these will now be considered in turn.

1. Suppose that $c\circ b_1$. Then $b\times b_1$ by (i) and (vi). Therefore $b = b_2$ or $b = b_{\beta-1}$. Let the notation be chosen so that $b = b_2$. Then $b_2\times a_3$ and $v(a_2,\Gamma) = 5$ by (23) and (vi).

$v(b_1,\Gamma) = 5$. For suppose on the contrary that $v(b_1,\Gamma)\geqslant 6$. Then $v(b_1,\Gamma-E(a))\geqslant 2$ by (23). Let $x,y\in\mathscr{V}(\Gamma-E(a))$ be such that b_2,x,y in this order are consecutive vertices of $E_i(b_1)$. By (i) and (23), $y\times c$. Therefore the 4-circuit y,b_1,b_2,c separates a and x, and so (21) is contradicted. Hence $v(b_1,\Gamma) = 5$. $v(b_1,\Gamma) = v(a_2,\Gamma) = v(a,\Gamma) = 5\rightarrow v(a_1,\Gamma)\geqslant 7$ by (25). $v(a_1,\Gamma)\geqslant 7\rightarrow b_{\beta-1}\circ a_5$ by (23). Therefore $b_{\beta-1}\times c$ by (i) and (23). Hence the 5-circuit $b_{\beta-1},a_1,a_2,b_2,c$ separates b_1 and b_β from a, a_3, a_4, and a_5, and so (22) is contradicted.

2. Suppose that $c\times b_1$. Then $c\notin E_o(a_4)$ because if $c\times b_i$ and $b_i\times a_4$ for some i, then, since $\beta\geqslant 9$ by (28), at least one of the 6-circuits b_1,c,b_i,a_4,a_3,a_2; b_1,c,b_i,a_4,a,a_1; b_1,c,b_i,a_4,a_5,a_1 would have four or more vertices on each side of it, so that, by Proposition 3, $\Gamma-\Omega\supset\langle 4\cup\rangle$, which is contrary to

hypothesis. Therefore $b \circ a_4$, so by (vi), $b \times a_3$ or $b \times a_5$. It may be assumed that $b \times a_3$.

$b = b_2$ or $b = b_3$, $v(a_2, \Gamma) = 5$, $b_2 \times a_3$, and $b_2 \times c$.

For put $b = b_\gamma$. If $\gamma = 2$ then the statement follows from (23) and (vi). If $\gamma > 2$ then the 5-circuit $c, b_1, a_2, a_3, b_\gamma$ separates $b_2, \ldots, b_{\gamma-1}$ from a, a_1, a_4, and a_5; therefore, by (22) and (20), $\gamma = 3$ and $b_2 \times c$ and $b_2 \times a_3$, and consequently also $v(a_2, \Gamma) = 5$.

$v(b_1, \Gamma) = 5$.

Suppose on the contrary that $v(b_1, \Gamma) \geqslant 6$. Let f denote a vertex of $(\Gamma - E(a) - c) \cap E_i(b_1)$. $f \circ b_2$ because if $f \times b_2$ then the 3-circuit f, b_1, b_2 separates a from c. $f \notin E_o(a_3)$ because if $f \times b_\kappa$ and $b_\kappa \times a_3$ and $\kappa \geqslant 3$ then the 5-circuit $f, b_1, a_2, a_3, b_\kappa$ separates b_2 and c from a, a_1, a_4, and a_5 in contradiction to (22). Hence $f \circ b$. Therefore $f \times c$ by (i), and $\Omega \cap E_i(f) = c$. b_2, c, f, in this order, are therefore consecutive vertices of $E_i(b_1)$. Consequently $v(b_1, \Gamma - E(a)) = 2$, $v(b_1, \Gamma) = 6$, and $f \times b_\beta$. It follows from Proposition 3 that $a_4 \notin E_0(f)$. Since $a_3, a_4 \notin E_0(f)$ and $\Omega \cap E_i(f) = c$ it follows from (ii) that $a_5 \in E_0(f)$. It follows that $v(a_1, \Gamma) = 5$, because if $v(a_1, \Gamma) \geqslant 6$ then let δ be such that $f \times b_\delta$ and $b_\delta \times a_5$: $\delta < \beta$ because $b_\beta \circ a_5$ since $v(a_1, \Gamma) \geqslant 6$; therefore $b_\delta \notin E_i(a_1)$ because $v(f, E_i(a_1)) \leqslant 2$ by (23); therefore $\delta \leqslant \beta - 2$ and the 5-circuit $f, b_1, a_1, a_5, b_\delta$ separates $b_{\beta-1}$ and b_β from a, a_2, a_3, and a_4 in contradiction to (22). By (25), $v(a_1, \Gamma) = v(a_2, \Gamma) = 5$ and $v(b_1, \Gamma) = 6 . \rightarrow . v(b_\beta, \Gamma) \geqslant 6$, so $v(b_\beta, \Gamma - E(a)) \geqslant 2$. Let g denote a vertex of $E_i(b_\beta) \cap (\Gamma - E(a) - f)$. $g \neq c$ because c, f, b_β, in this order, are consecutive vertices of $E_i(b_1)$. $g \circ c$ because if $g \times c$ then the 4-circuit g, b_β, b_1, c separates a from f. $g \circ b$ by Proposition 3, since $b \times a_3$. Hence $v(g, \Omega) = 0$ and (i) is contradicted. Therefore $v(b_1, \Gamma) = 5$.

By (25), $v(a, \Gamma) = v(a_2, \Gamma) = v(b_1, \Gamma) = 5 \rightarrow v(a_1, \Gamma) \geqslant 7$. Consequently $b_\beta \circ a_5$ and $b_{\beta-1} \circ a_5$ by (23). $b_{\beta-1} \circ c$ because $v(b_{\beta-1}, E_i(b_1)) < 3$ by (23). $b_{\beta-1} \circ b$ by (23). Hence $v(b_{\beta-1}, \Omega) = 0$ and (i) is contradicted.

So each of the two alternatives $c \circ b_1$ and $c \times b_1$ leads to a contradiction. Proposition 4 is thereby proved.

It follows from Proposition 4 that

(39) *No irreducible graph contains a connected skeleton with fewer than six vertices.*

7. *The proof of Theorem 2 completed*

(40) *If a graph Γ has a connected skeleton Σ then any graph obtained from Γ by identifying vertices has a connected skeleton containing not more than $|\mathscr{V}(\Sigma)|$ vertices.*

The proof is left to the reader.

(41) *If a planar graph has a connected skeleton all of whose vertices, with at most five exceptions, have valency* $\leqslant 4$ *in the graph, then the graph is 4-colourable.*

Proof. By *reductio ad absurdum.* Suppose that (41) is false, let Γ denote a 5-chromatic planar graph with a skeleton which satisfies the conditions of (41), and let Φ denote the subgraph of Γ generated by the vertices of the skeleton. Let $\mathscr{F}$ denote the set of those vertices of Φ whose valency in Γ exceeds 4, and let $\mathscr{F}' = \mathscr{V}(\Phi) - \mathscr{F}$.

(i) $\mathscr{F} \neq \varnothing$.

For if $\mathscr{F} = \varnothing$ then $\Gamma - \Phi$ is 5-chromatic by (17), but on the other hand $\Gamma - \Phi$ is 3-colourable by (13) because $\Gamma - \Phi \not\supseteq \langle 4 \rangle$ and $\Gamma - \Phi \not\supseteq \langle 4 \cup \rangle$ since Φ' is connected and Γ is a planar graph. Hence $\mathscr{F} \neq \varnothing$.

(ii) *If* $|\mathscr{V}(\Phi)| > 5$ *then a 5-chromatic planar graph* Γ^* *with a connected skeleton* Φ^* *having fewer than six vertices can be obtained from* Γ *by the operation* $\mathscr{o}$.

Proof. $\mathscr{F}' \neq \varnothing$. Corresponding to each connected component Φ_j of $\Phi - \mathscr{F}$ choose a vertex $f_j \in \mathscr{F}$ which is joined to at least one vertex of Φ_j in Γ (f_j exists because Φ is connected), and let Ψ_j denote $\Phi - (\Phi - f_j - \Phi_j)$. Ψ_j is connected. Let $\bigcup_j \Psi_j = \Psi$. Let Γ^* and Φ^* denote planar graphs obtained from Γ and Φ respectively by contracting each connected component of Ψ into a single vertex. Γ^* is 5-chromatic because $\Gamma^* \supseteq \Gamma - \mathscr{F}'$, and $\Gamma - \mathscr{F}'$ is 5-chromatic by (17). Φ^* is connected and a skeleton of Γ^*. Finally, $|\mathscr{V}(\Phi^*)| = |\mathscr{F}| \leqslant 5$. This proves (ii).

If $|\mathscr{V}(\Phi)| \leqslant 5$ then put $\Gamma^* = \Gamma$ and $\Phi^* = \Phi$.

By (39), Γ^* is not irreducible. Therefore, by (19), an irreducible graph Γ^{**} can be obtained from Γ^* by applying the operations $\mathscr{o}$ and $\mathscr{o}_3$. By (40), Γ^{**} has a connected skeleton with fewer than six vertices, and so (39) is contradicted. This contradiction proves (41).

From (17), (16), (27), (41), and the theorem of C. E. Winn that every planar map containing at most one region having more than six neighbours is 4-colourable (**20**), it follows that if a planar graph has the property $\mathfrak{P}$ described in Theorem 2 then the graph is 4-colourable. Theorem 2 now follows from (16) and (17).

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