made a very substantial contribution in spatial kinematic analysis. It is hoped that the authors' work will be extended to deal with problems of kinematic synthesis. For both analysis and synthesis, it would be useful if, say, Fortran subroutines were worked out to allow dealing with dual quantities by a single symbol in computer programming. Since dual vectors have six components and dual quaternions have eight components, such "dual mode" of computation would afford considerable economy and reduce the risk of error in programming and would facilitate the application of the authors' methods to new problems in analysis and synthesis. Furthermore, since spherical and planar mechanisms are included in the authors' spatial theory as special cases, programs written in such "dual mode" would be universally applicable in all three areas.

Authors' Closure

The authors wish to thank the discussers for their illuminating comments and generous remarks. Professor Beggs' remark on the concept of the instant screw axis in spatial motion—that it is analogous to the concept of the instant center of a body in plane motion (extension of Aronhold-Kennedy theorem)—is interesting. Incidentally, the authors believe that, using dual-number quaternion algebra and the concept of the instant screw axis, one may determine analytically the velocity of any point on the floating link of a spatial four-link mechanism.

Professor Beggs' comments on terminology in mechanisms are well taken. The authors share Professor Beggs' hope that kinematicians will continue to work out a set of standard terminology to facilitate communication for the benefit of all, in line with the recent efforts of Professors Artobolevskii, Denavit, Goodman, Hartenberg, and others.

The authors would like to take this opportunity to acknowledge their debt to Professor Bottema for suggesting the form $Q(KQ)$ for the left-hand side of equation (25); this form was adopted in the paper as published in the Journal. As to equation (40), the authors agree with Professor Bottema that it might well have been written in the more elegant form $Q_{34}Q_{23}Q_{12}Q_{41} = 1$. The expression given in the paper, however, has some computational advantages.

On the somewhat paradoxical situation—rotation corresponding to force and linear velocity to moment—it may be clarified if we consider two points, O and P, on a moving body whose angular velocity is $\boldsymbol{\omega}$. We may transfer the linear velocity at O to P as follows:

$$\mathbf{V}_P = \mathbf{V}_0 + \boldsymbol{\omega} \times \overline{OP} = \mathbf{V}_0 + \overline{PO} \times \boldsymbol{\omega}$$

If force $\mathbf{F}$ and moment IM_P are applied at point P, the moment about O is given by

$$\mathbf{M}_0 = \mathbf{M}_P + \overline{OP} \times \mathbf{F}$$

Comparing the two equations, we note that moment corresponds to linear velocity and force to angular velocity. In the paper, angular and linear velocities are represented in dual vector form; so are force and moment. Hence, the terms dual velocity and dual force. More detailed treatment of the terms can be found in references [5] and [20] of the paper.

And now to Professor Bottema's three specific questions:

1 The statement that the theory of spatial four-link mechanism will not attain the elegance of its planar counterpart is very true. The authors also agree that, in principle, a satisfactory theory could be developed for a spatial four-link mechanism with four identical turn-slides and two inputs (rotation and sliding). In fact, when its sliding input velocity is zero, such a mechanism is the spatial four-link under study in this paper. The most symmetrical expressions derived for four-link spatial mechanism seem to be those of F. M. Dimentberg, reference [10] of the paper.

2 The authors believe that the tools developed in the paper may be used for the study of the path of a point on the floating link; such a point may be specified as a point on the line which is perpendicular to $\hat{\mathbf{a}}_{23}$ and subtends a constant dual angle with either one of the two floating axes, $\hat{\mathbf{s}}_2$ or $\hat{\mathbf{s}}_3$.

3 In principle, the analysis developed in this paper can be applied to the case where one of the joints is a spherical joint and the others are a turn-slide and two turning pairs; for a spherical joint can be replaced by three turning pairs while a turning pair is but a special case of the turn-slide.

Over the past ten years, the matrix methods developed by Professors J. Denavit and R. S. Hartenberg have become a fundamental contribution to the analysis of spatial mechanisms in general. The investigation presented in this paper, as acknowledged by the authors in the opening paragraph, owes much to their work.

A spatial four-link mechanism with one turning pair and three turn-slides is selected by the authors for study in depth, because of its generality. The authors believe that as the screw operator derived from dual-number quaternions seems to be well adapted for the description of screw motion, and, since the motion of a rigid body in space in general can be considered as a series of screw motions about instant axes, this tool may possess good possibilities for the description of the motion of a link in space, for example, the floating link of a spatial four-link mechanism.

The point raised by Professors Denavit and Hartenberg—that, in the case of a closed spatial chain of seven turning pairs, dual-number quaternion may not lead to analytical solutions—seems well taken. The authors also agree with their comment that, although algebraic properties of many operators are somewhat similar, practice shows some are better suited than others to investigate particular situations. It is the authors' hope that quaternion methods may find a suitable companion place to the matrix methods already in existence. The authors are grateful to Professor George N. Sandor for his valuable suggestion that FORTRAN subroutines should be worked out to allow dealing with dual quantities by a single symbol in computer programming. It is intended to carry this out and, hopefully, in the not-too-distant future.

Dynamics of Synchronous-Precessing Turborotors With Particular Reference to Balancing. Part 1—Theoretical Foundations[1]

E. J. GUNTER, JR.[2] The concept of indifferent equilibrium was formulated by Rankine [1][3] in 1869 in the first recorded article on rotor dynamics and has persisted to this date. Rankine examined the equilibrium of a frictionless, uniform shaft and concluded that the motion is stable below the first critical speed, neutral or in "indifferent" equilibrium at the critical speed, and unstable above this speed. The problem of "indifferent equilibrium" was not resolved until 1919 when H. H Jeffcott [2] analyzed the whirling of a single-mass unbalanced flexible rotor with viscous damping (see Fig. 2 of the paper). His analysis revealed that synchronous forward precession is the only possible steady-state motion and the rotor deflection becomes unbounded only if the friction coefficient approaches zero. Jeffcott's analysis shows that viscous damping increases the critical speed. Examination of equation (14) with both P_1 and P_2 equal to zero indicates that turbulence or damping proportional to the velocity squared will not change the critical speed.

In this case, the rotor amplitude factor is given by

$$\frac{\delta}{e} = \frac{1}{\left(\mu_f^2 + \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^2\right)^{1/2}} \tag{1}$$

[1] By T. M. Tang and P. R. Trumpler, published in the March, 1964, issue of the Journal of Applied Mechanics, vol. 31, Trans. ASME, vol. 86, pp. 115–122.

[2] Senior Research Engineer, The Franklin Institute, Laboratories for Research and Development, Philadelphia, Pa.

[3] Numbers in brackets indicate References at end of Discussion.

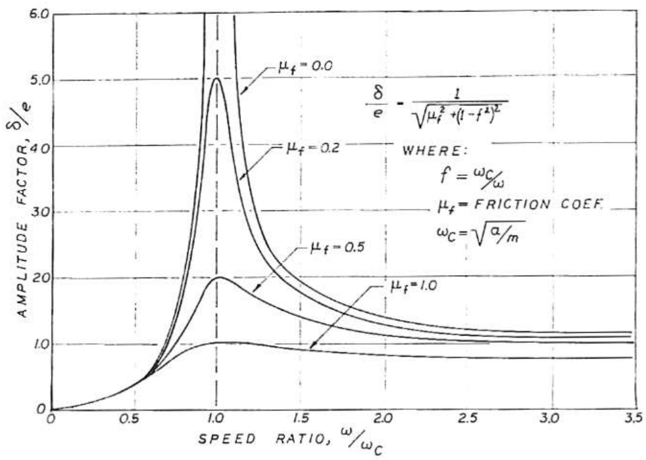

Fig. 1 Rotor amplitude response for various damping values

and the rotor phase angle is

$$\beta = \tan^{-1}\left[\frac{\mu_f}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}\right] \tag{2}$$

$\omega_c{}^2 = \dfrac{\alpha}{m}$ = undamped lateral frequency of rotor

Fig. 1 of this discussion is a plot of the rotor amplification factor versus speed for various values of the damping parameter μ_f. From the chart it is seen that high amplitudes are encountered at the rotor critical speed only if the damping coefficient μ_f is much less than unity.

The assumption of simple-support boundary conditions as a basis for rotor critical speeds is obviously inadequate since the bearings and foundation flexibility will considerably alter the behavior. Numerous investigators, such as Smith [3], Linn and Prohl [4], Koenig, and Lund and Sternlicht [5], have considered the effects of the bearings on the rotor critical speeds and the bearing attenuation. In each of these approaches, the bearings are treated as linear springs and dashpots. The problem of determining the system critical speeds is reduced to the problem of finding the natural lateral frequencies of an equivalent beam on damped elastic supports.

A fluid film bearing cannot be represented adequately by a single spring and damping coefficient. For small perturbations from an equilibrium position, the bearing characteristics may be approximated by 8 film coefficients—4 damping factors and 4 film stiffness terms. For a symmetric bearing, this further reduces to 4—2 damping and 2 spring coefficients. The most common practice is to neglect the normal or cross-coupling coefficient Cd (see equation (32)) in order to reduce the bearing to a one-dimensional representation. Lund and Sternlicht have attempted to improve upon this assumption by lumping the cross-coupling coefficients with the principle spring rates, but the manner in which this is done is nebulous and the results are not general.

The normal film coefficient has a considerable influence not only on the system critical speeds but also on the forces transmitted through the bearings. As an illustration, using the approach outlined by the authors, it is possible to write the equations of motion of the single-mass symmetric rotor including the bearings and foundation flexibility. This represents a system of seven degrees of freedom as compared to the original three.

From equation (36) it is seen that the bearing force-deflection relationship (neglecting damping) can be expressed as

$$\mathbf{F}_{br} = -K_{br}\boldsymbol{\delta}_j \tag{3}$$

where

K_{br} = complex bearing stiffness coefficient

$= \omega[C_s{}^* + iC_d{}^*] = K_s{}^* + iK_d{}^*$

$\boldsymbol{\delta}_j$ = journal displacement $= \boldsymbol{\delta}_b + \boldsymbol{\delta}_c$

If the elastic forces acting on the rotor center are similar in form to equation (3), then

$$\mathbf{F}_s = -\alpha\boldsymbol{\delta}_r \tag{4}$$

Let

$$\boldsymbol{\delta} = \text{total rotor deflection} = \boldsymbol{\delta}_r + \boldsymbol{\delta}_j \tag{5}$$

Combining [3, 4, 5]

$$\mathbf{F}_s = -K_s\boldsymbol{\delta} \tag{6}$$

where

K_s = total system complex stiffness coefficient

$$= \frac{\alpha\, K_{br}}{\alpha + K_{br}} = K_{sr} + iK_{si} \tag{7}$$

and

$$\left.\begin{aligned} K_{sr} &= \frac{\alpha[(\omega C_s{}^*)^2 + (\omega C_d{}^*)^2 + \alpha\omega C_s{}^*]}{(\alpha + \omega C_s{}^*)^2 + (\omega C_d{}^*)^2} \\ K_{si} &= \frac{\alpha^2\omega C_d{}^*}{(\alpha + \omega C_s{}^*)^2 + (\omega C_d{}^*)^2} \end{aligned}\right\} \tag{8}$$

If rotor damping similar to equation (10) and unbalance are considered, then the vectorial equation of motion of the rotor is given by

$$M\ddot{\boldsymbol{\delta}} + M\mu_f\omega\boldsymbol{\delta} + K_s\boldsymbol{\delta} = -M\mathbf{e} \tag{9}$$

(Note that total time derivatives are used in the above equation.)

If the disk is revolving with constant angular velocity, then

$$\mathbf{e} = e[e^{i\omega t}]$$

and

$$\mathbf{e} = -e\omega^2[e^{i\omega t}] \tag{10}$$

Equation (7) becomes

$$\ddot{\boldsymbol{\delta}} + \mu_f\omega\dot{\boldsymbol{\delta}} + \frac{K_s}{M}\,\boldsymbol{\delta} = (e\omega^2)e^{i\omega t} \tag{11}$$

If the motion of the system is assumed to be forward synchronous precession (rotor motion may be nonsynchronous precessive, which is associated with self-excited whirling), then a particular steady-state solution of the following form may be assumed:

$$\boldsymbol{\delta} = \boldsymbol{\delta}_0 e^{i\omega t} \tag{12}$$

Solving for $\boldsymbol{\delta}_0$, the complex rotor amplitude

$$\boldsymbol{\delta}_0 = \frac{e}{1 - \dfrac{K_s}{M\omega^2} + i\mu_f} \tag{13}$$

$$\boldsymbol{\delta}_0 = \frac{e}{1 - \dfrac{K_{sr}}{M\omega^2} + i\left(\mu_f + \dfrac{K_{si}}{M\omega^2}\right)} \tag{14}$$

The complex rotor amplitude

$$\boldsymbol{\delta}_0 = A - iB \tag{15}$$

where

$$\left.\begin{aligned} A &= \frac{e\left(1 - \dfrac{K_{sr}}{M\omega^2}\right)}{\left(1 - \dfrac{K_{sr}}{M\omega^2}\right)^2 + \left(\mu_f + \dfrac{K_{si}}{M\omega^2}\right)^2} \\ B &= \frac{e\left(\mu_f + \dfrac{K_{si}}{M\omega^2}\right)}{\left(1 - \dfrac{K_{sr}}{M\omega^2}\right)^2 + \left(\mu_f + \dfrac{K_{si}}{M\omega^2}\right)^2} \end{aligned}\right\} \tag{16}$$

The vectorial rotor deflection $\boldsymbol{\delta}$ may be expressed in terms of real quantities by the introduction of the rotor phase angle β.

$$\boldsymbol{\delta} = R_e e^{i(\omega t - \beta)} \tag{17}$$

Re = total rotor deflection

$$= (A^2 + B^2)^{1/2} = \left[\left(1 - \frac{K_{sr}}{M\omega^2}\right)^2 + \left(\mu_f + \frac{K_{si}}{M\omega^2}\right)^2\right]^{1/2} \tag{18}$$

$$\beta = \tan^{-1}\left(\frac{\mu_f + \dfrac{K_{si}}{M\omega^2}}{\dfrac{K_{sr}}{M\omega^2} - 1}\right) \tag{19}$$

When $\omega = \omega_{cs}$, the system critical speed, the displacement vector $\boldsymbol{\delta}_0$ is lagging the eccentricity vector $\mathbf{e}$ by 90 deg. This is given by the conditions that

$$A = 0 \tag{20}$$

and

$$\beta = 90 \text{ deg}$$

This is only possible if

$$1 - \frac{K_{sr}}{M\omega^2} = 0 \tag{21}$$

Hence, the system critical speed is given by

$$\omega_{cs} = \left(\frac{K_{sr}}{M}\right)^{1/2} = \left(\frac{\alpha[K_s^{*2} + K_d^{*2} + \alpha K_s^*]}{M[(\alpha + K_s^*)^2 + (K_d^*)^2]}\right)^{1/2} \tag{22}$$

Let

$$R = \frac{\alpha}{K_s^*}$$

and

$$\tan\phi = \frac{K_d^*}{K_s^*}$$

$$\frac{\omega_{cs}}{\omega_{cR}} = \left(\frac{1 + R + \tan^2\phi}{(1 + R)^2 + \tan^2\phi}\right)^{1/2} = N \tag{23}$$

The foregoing expression represents the ratio of the system critical speed to the critical speed of the rotor on simple or fixed supports. The ratio N is always less than 1.

Fig. 2 represents a plot of equation (23) for various values of N. From the graph it is seen that the bearing attitude angle has a pronounced effect on the system critical speed, particularly at high values of ϕ. Previous rotor dynamic programs have neglected the normal bearing coefficients K_d^* in the calculation of critical speeds. This approximation is justified only if the effective bearing attitude angle is low or less than 20 deg.

For the case of a rigid rotor ($\alpha \to \infty$):

Equation (22) reduces to

$$\omega_{cs} = \left(\frac{K_s^*}{M}\right)^{1/2} \tag{24}$$

where

$$K_s = \frac{K_s + \dfrac{1}{K_b}(K_s^2 + K_d^2)}{\left(1 + \dfrac{K_s}{K_b}\right)^2 + \left(\dfrac{K_d}{K_b}\right)^2}$$

(See equation (38) of text.)

For the case of a rigid foundation ($K_b \to \infty$), the rotor resonance frequency reduces to a function of only the radial oil film stiffness coefficient K_s.

The rotor displacement at the critical speed is given by

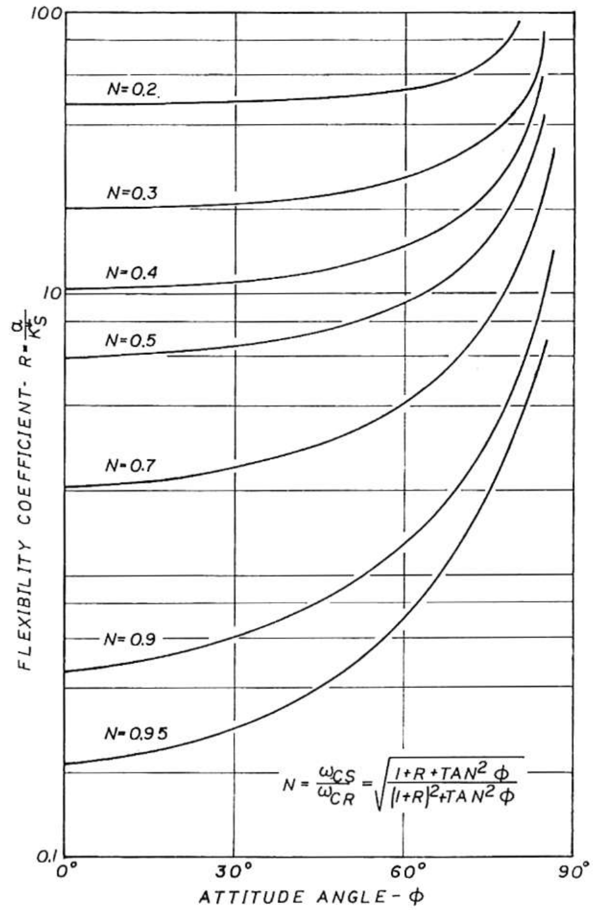

Fig. 2 The effects of bearing attitude angle and foundation flexibility on rotor critical speed

$$\left.\frac{\delta}{e}\right|_{\omega=\omega_{cs}} = \frac{B}{e} = \frac{K_{sr}}{K_{si} + \mu_f K_{sr}} \tag{25}$$

If the external damping coefficient $\mu_f = 0$, then the rotor amplitude at the critical speed is unbounded for the case of the simple-support rotor. When oil film bearings are introduced, the deflection is limited due to the influence of the out-of-phase bearing coefficient. Equation (25) reduces to

$$\left.\frac{\delta}{e}\right|_{\omega=\omega_{cs}} = \frac{1 + R + \tan^2\phi}{R\tan\phi} \tag{26}$$

Thus, it is seen that the rotor amplification factor at the critical speed is a function of the bearing attitude angle. Attempts to increase the stability of a rotor, with respect to self-excited whirling by reducing the bearing attitude angle, will result in higher amplitude ratios when passing through the rotor critical speeds.

It is felt that, in general, the method outlined by the authors represents a more accurate approach to the problem of determining the rotor critical speeds and also to the problem of calculating the forces transmitted to the bearings.

It would be of interest to know:

1 Have the authors applied their approach to the calculation of an actual turborotor system?

2 If so, in what manner did they prescribe the rotor unbalance distribution?

3 Are any problems of computer convergence encountered when trying to iterate to the proper bearing boundary conditions?

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1 A. W. Rankine, "On the Centrifugal Force of Rotating Shafts," *Engineer*, London, vol. 27, p. 249.

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3 D. M. Smith, "The Motion of a Rotor Carried by a Flexible Shaft in Flexible Bearings," *Proceedings of the Royal Society*, London, Series A, vol. 142, pp. 92–118.
4 F. C. Linn and M. A. Prohl, "The Effect of Flexibility of Support Upon the Critical Speeds of High-Speed Rotors," *Trans. SNAME*, vol. 59, 1951, pp. 536–553.
5 J. W. Lund and B. Sternlicht, "Rotor-Bearing Dynamics With Emphasis on Attenuation," *Journal of Basic Engineering*, TRANS. ASME, Series D, vol. 84, 1962, p. 491.

Authors' Closure

Mr. Gunter presents a significant discussion of the single-mass rotor including bearing and support stiffness factors. The results may be looked upon as an asymptotic case of the general multimass rotor treated by the authors. The advantage in carefully analyzing the simple systems lies in the understanding one gains about the way in which the system parameters may be expected to enter the solution of the realistically complex system. Sometimes these expectations are misleading, and therein lies the weakness of the simple model. The authors hope that their comparatively elaborate model will prove to be the basis of fruitful analytical work at a level beyond that of the single-mass system.

In answer to the specific questions:

1 Yes. With the digital computer, the speed-response spectra (Fig. 1) of numerous industrial turbomachines have been predicted.

2 The unbalance of each rotor mass is a system parameter. Since a turborotor is assembled without regard to the location of the mass centers, the predicted vibratory performance must be statistical. The computer is programmed to obtain a number of speed-response spectra based on various random-number eccentricity sets.

3 There were convergence problems in the early programs, but it appears that moderate attention to the problem has eliminated the difficulty.

A Note on the Transient Axisymmetric Thermoelastic Problem for the Solid Sphere[1]

G. SONNERMANN.[2] In this paper, the numerical result of Fig. 1 shows very clearly that the thermal penetration into the center of the sphere lags the outside surface thermal transient. Consequently, the concept of utilizing only elementary theory, i.e., ignoring geometry effects, should be feasible.[3, 4]

With the temperature distribution of Warren

$$T(1, u, \tau) = \Sigma B_n(1 - e^{-\beta_n \tau})P_n(u)$$

approximated at $\theta = 0$ ($u = \cos\theta = 1$) by the $n = 0$ term

$$T(1, 1, \tau) = B_0(1 - e^{-\beta_0 \tau})$$

the temperature distribution becomes for small nondimensional times

$$T(1, 1, \tau) \approx B_0\beta_0\tau \qquad (1)$$

[1] By W. E. Warren, published in the June, 1964, issue of the JOURNAL OF APPLIED MECHANICS, vol. 31, TRANS. ASME, vol. 86, Series E, pp. 348–350.

[2] Senior Technical Assistant to Program Manager, Guidance and Control, Corporate Systems Center, United Aircraft Corporation, Farmington, Conn.

[3] G. W. Reichard, Jr., "Temperature Distributions and Thermal Stresses in Finite Hollow Cylinders," MS thesis, University of Pittsburgh, Mechanical Engineering, 1964.

[4] M. A. Biot, "New Methods in Heat Flow Analysis with Application to Flight Structures," *Journal of the Aeronautical Sciences*, vol. 24, 1957, pp. 857–873.

If one uses for the thermal stress

$$\sigma = E\alpha\Delta T = 2(1 + \nu)G\alpha\Delta T \qquad (2)$$

and then substitutes the values of Warren for B_0 and β_0, equation (1) yields

$$T(1, 1, \tau) = 92.2T_0\tau \qquad (3)$$

Also

$$\bar{\sigma} = \sigma/G\alpha T_0 = 2(1 + \nu)92.2\tau \qquad (4)$$

The peak stress should occur at the time when the outside temperature reaches T_0, i.e., $T = T_0$, which is at $\tau \simeq 0.011$. This ignores the penetration of the thermal front into the sphere and consequently will give a higher thermal stress than one should expect in practice. Substitution of $\tau = 0.011$ into equation 4 yields, with $\nu = 0.3$

$$\bar{\sigma} = 2.64 \qquad (5)$$

This value is to be compared to the peak of about 2.2 at the $\theta = 0$ of Warren. The stress computed in this note is 20 percent greater than Warren's value. Although the technique does not yield the details of the thermal-stress distribution, it does bound the value with sufficient accuracy for many practical problems. It has been applied to transient thermal stresses in cylinders.[3] The main attribute of the method is rapid evaluation without the need to conduct either a heat-conduction analysis or a thermal-stress analysis.

Author's Closure

The remarks of Dr. Sonnemann pointing out the conservative nature of an elementary theory for bounding the thermal stresses in a solid sphere should be of practical interest. It should be mentioned that Dr. Sonnemann's equation (1) is only valid for $\beta_0\tau \ll 1$, but is subsequently evaluated at $\beta_0\tau = 1.83$ to arrive at (5). Fortunately, the actual temperature distribution employed has nothing to do with arriving at (5) since (1), (3), (4) merely produce max ΔT = max $T = T_0$ from which (2) renders directly $\bar{\sigma} = 2(1 + \nu) = 2.60$. Equation (5) yields 2.60 rather than 2.64 if three significant figures are carried in τ.

The axisymmetric nature of this problem would appear to make the two-dimensional stress analog of (2) more physically realistic than (2). This presents

$$\bar{\sigma} = 2\left(\frac{1 + \nu}{1 - \nu}\right) = 3.71$$

which is considerably more conservative than (5).

Longitudinal Impact on Viscoplastic Rods—Linear Stress-Strain Rate Law[1]

P. C. CHOU.[2] The authors have studied the longitudinal impact on rods using a Bingham type of constitutive equation. Their study will certainly be helpful in the evaluation of the strain-rate effect under hypervelocity impact. In evaluating their efforts, the writer would like to offer the following comments:

1 Viscoplastic theory involving only shear has been applied recently to the perforation of a plate due to hypervelocity impact [1, 2].[3] The governing equations and initial and boundary conditions are similar to those used in this paper. The solutions in [1, 2] involve Bessel functions, rather than error functions, because of the axisymmetric nature of the shear problem.

[1] By T. C. T. Ting and P. S. Symonds, published in the June, 1964, issue of the JOURNAL OF APPLIED MECHANICS, vol. 31, TRANS. ASME, vol. 86, Series E, pp. 199–207.

[2] Professor of Aeronautical Engineering, Drexel Institute of Technology, Philadelphia, Pa. Mem. ASME.

[3] Numbers in brackets designate References at end of Discussion.