Ann. Sci. Math. Québec **23** (1999), no. 1, 97–107.

σ-SMOOTHNESS CRITERIA FOR LATTICE MEASURES

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RÉSUMÉ. Soit X un ensemble quelconque et $\mathcal{L}$ un treillis de parties de X tel que $\emptyset, X \in \mathcal{L}$. Dans ce travail, nous étendons des caractérisations de $\sigma-$continuité pour des mesures $\mathcal{L}-$régulières $\mu \in M_R(\mathcal{L})$ au cas général $\mu \in M(\mathcal{L})$, en considérant différentes mesures extérieures associées à μ ainsi que la mesure induite $\underline{\mu}$ sur un espace de Wallman.

ABSTRACT. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X such that $\emptyset, X \in \mathcal{L}$. In this paper we extend $\sigma-$smoothness characterizations of $\mathcal{L}-$regular measures $\mu \in M_R(\mathcal{L})$ to the general case of $\mu \in M(\mathcal{L})$, by considering different outer measures associated with μ and the induced measure $\underline{\mu}$ on a Wallman space.

1. Introduction. Let X be an arbitrary non-empty set and $\mathfrak{L}$ a lattice of subsets of X such that $\emptyset, X \in \mathfrak{L}$. $A(\mathfrak{L})$ is the algebra generated by $\mathfrak{L}$ and $M(\mathfrak{L})$ denotes those non-negative, finite, finitely additive measures on $A(\mathfrak{L})$. $M_\sigma(\mathfrak{L})$ denotes those measures $\mu \in M(\mathfrak{L})$ which are $\sigma-$smooth on $\mathfrak{L}$, $M^\sigma(\mathfrak{L})$ denotes $\mu \in M(\mathfrak{L})$ which are $\sigma-$smooth on $A(\mathfrak{L})$ hence countably additive. $M_R(\mathfrak{L})$ denotes those $\mu \in M(\mathfrak{L})$ which are $\mathfrak{L}-$regular and $M_R^\sigma(\mathfrak{L}) = M_R(\mathfrak{L}) \cap M_\sigma(\mathfrak{L})$. Clearly, if $\mu \in M_R^\sigma(\mathfrak{L})$ then $\mu \in M^\sigma(\mathfrak{L})$.

Associated with a $\mu \in M(\mathfrak{L})$, there is a finitely subadditive outer measure μ' (see below for definitions) whose properties, especially pertaining to measurability, have been investigated. For a measure $\mu \in M_\sigma(\mathfrak{L})$, the elements of $M(\mathfrak{L})$ which are $\sigma-$smooth on $\mathfrak{L}$, we associate an outer measure μ''. If μ is also $\mathfrak{L}-$regular then μ'' coincides with the usual induced outer measure μ^*. The more general case is investigated here.

In [2,3], conditions were given for a $\mu \in M_R(\mathfrak{L})$ to belong to $M_R^\sigma(\mathfrak{L})$ in terms of an induced measure $\underline{\mu}$ on an associated Wallman space (see below for definitions). In this paper we wish to extend this result by starting more generally with a $\mu \in M(\mathfrak{L})$, and giving conditions in terms of an induced measure $\underline{\mu}$ on a Wallman space for $\mu \in M_\sigma(\mathfrak{L})$ and conditions for $\mu \in M^\sigma(\mathfrak{L})$. This will involve considering other measures associated with μ and $\underline{\mu}$ than the customary Caratheodory outer measure. We give the necessary background material in the next section.

We note that the notation and the terminology is consistent with that of [1,2,3,10] and that further relates matters on μ' and μ'' and properties can be found in [4,6,7,8,9].

Reçu le 25 janvier 1998 et, sous forme définitive, le 22 mai 1998.

2. Background and notations. We begin with some standard background material for the reader's convenience. Let X be an arbitrary non-empty abstract set and $\mathfrak{L}$ a lattice of subsets of X. It is assumed that $\emptyset, X \in \mathfrak{L}$. $A(\mathfrak{L})$ denotes the algebra generated by $\mathfrak{L}$ and $\sigma(\mathfrak{L})$ the $\sigma-$algebra generated by $\mathfrak{L}$.

Definition 1. The lattice is called *normal* if for any $L_1, L_2 \in \mathfrak{L}$ with $L_1 \cap L_2 = \emptyset$ there exist $L_3, L_4 \in \mathfrak{L}$ with $L_1 \subset L_3', L_2 \subset L_4'$ and $L_3' \cap L_4' = \emptyset$ (where prime denotes complement); *disjunctive* if for $x \in X$ and $L_1 \in \mathfrak{L}$ such that $x \notin L_1$, there exists an $L_2 \in \mathfrak{L}$ with $x \in L_2$ and $L_1 \cap L_2 = \emptyset$; *compact* if for any collection $\{L_\alpha\}$ of sets of $\mathfrak{L}$ with $\cap L_\alpha = \emptyset$, there exists a finite subcollection with empty intersection; *countably compact* if for any countable collection $\{L_\alpha\}$ of sets of $\mathfrak{L}$ with $\cap L_\alpha = \emptyset$, there exists a finite collection with empty intersection; *countably paracompact* if for any sequence $\{A_n \in \mathfrak{L}\}$, $A_n \downarrow \emptyset$, there exists sequences $\{L_n \in \mathfrak{L}\}$, such that $A_n \subseteq L_n'$ and $L_n' \downarrow \emptyset$; *delta lattice* ($\delta$*-lattice*) if $\mathfrak{L}$ is under countable intersertions.

Given two lattices $\mathfrak{L}_1, \mathfrak{L}_2$ of subsets of X such that $\mathfrak{L}_1 \subset \mathfrak{L}_2$, $\mathfrak{L}_1$ *separates* $\mathfrak{L}_2$ if $A, B \in \mathfrak{L}_2$ and $A \cap B = \emptyset$ imply there exist $C, D \in L_1$ such that $A \subset C, B \subset D$ and $C \cap D = \emptyset$.

We give now some measure terminology. $M(\mathfrak{L})$ denotes the set of finite valued, non-negative finitely additive measures on $A(\mathfrak{L})$.

Definition 2. A measure $\mu \in M(\mathfrak{L})$ is called:

σ*-smooth on* $\mathfrak{L}$ if for all sequences $\{L_n\}$ of sets of $\mathfrak{L}$ with $L_n \downarrow \emptyset$, $\mu(L_n) \to 0$.
σ*-smooth on* $A(\mathfrak{L})$ if for all sequences $\{A_n\}$ of sets of $A(\mathfrak{L})$ with $A_n \downarrow \emptyset$, $\mu(A_n) \to \emptyset$, i.e. countably additive.
$\mathfrak{L}$*-regular* if for any $A \in A(\mathfrak{L})$, $\mu(A) = \sup\{\mu(L) \mid L \subset A,\ L \in \mathfrak{L}\}$.

We denote by $M_R(\mathfrak{L})$ the set of $\mathfrak{L}$-regular measures of $M(\mathfrak{L})$; $M_\sigma(\mathfrak{L})$ the set of σ-smooth measures on $\mathfrak{L}$ of $M(\mathfrak{L})$; $M^\sigma(\mathfrak{L})$ the set of σ-smooth measures on $A(\mathfrak{L})$ of $M(\mathfrak{L})$; $M_R^\sigma(\mathfrak{L})$ the set of $\mathfrak{L}$-regular measures of $M^\sigma(\mathfrak{L})$.

In addition , $I(\mathfrak{L})$, $I_R(\mathfrak{L})$, $I_\sigma(\mathfrak{L})$, $I^\sigma(\mathfrak{L})$ and $I_R^\sigma(\mathfrak{L})$ are the subsets of the corresponding M's which consist of the non-trivial zero-one valued measures.

Definition 3. A measure $\mu \in M(\mathfrak{L})$ is *strongly* σ*-smooth on* $\mathfrak{L}$ or $\mu \in M(\sigma, \mathfrak{L})$ if, and only if, for any sequence $\{L_n \in \mathfrak{L}\}$ $L_n \downarrow L$ where $L \in \mathfrak{L}$, then $\mu(L) = \inf_n \mu(L_n) = \lim_n \mu(L_n)$. Correspondingly, for $\mu \in I(\mathfrak{L})$ we have $I(\sigma, \mathfrak{L})$, the set of strongly σ-smooth zero-one valued measures on $\mathfrak{L}$.

3. Finitely subadditive outer measures and Wallman spaces. In this section we define finitely subadditive outer measures and (cover) regular measures in contrast to an ordinary outer measure which is countably subadditive.

Associated with $\mu \in M(\mathfrak{L})$ and $\mu \in M_\sigma(\mathfrak{L})$ are the outer measures μ' and μ'' which have been investigated in [7]. We first review some of the basic properties of these outer measures and then develop new results.

Definition 4. A non-negative measure μ defined on $\mathcal{P}(X)$, the class of all subsets of X, is a finitely subadditive outer measure if:

a) μ nondecreasing ;

b) $\mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu(E_i)$, for any $E_1, E_2, \ldots, E_n \subset X$;

c) $\mu(\emptyset) = 0$.

Definition 5. Let ν be a finitely subadditive outer measure. We say that a set E is a measurable with respect to ν if for any $A \subset X : \nu(A) = \nu(A \cap E) + \nu(A \cap E')$. Let $\mathbf{S}_\nu$ be the set of ν-measurable sets, with ν a finitely subadditive outer measure. ν is called *(cover) regular* if for any $S \subset X$, there exists $E \in \mathbf{S}_\nu$ such that $S \subset E$ and $\nu(S) = \nu(E)$. It is easy to prove (see [5]) that if ν is a finitely subadditive outer measure outer (cover) regular measure and $\nu(X)$ is finite, then $E \in \mathbf{S}_\nu$ if, and only if, $\nu(X) = \nu(E) + \nu(E')$.

Definition 6. Let $\mu \in M(\mathfrak{L})$ and define

$$\mu'(E) = \inf \sum_{i=1}^{n} \mu(L_i'), E \subset \bigcup_{i=1}^{n} L_i', L_i \in \mathfrak{L}, E \subset X \ .$$

The definition is equivalent to:

$$\mu'(E) = \inf \mu(L'), E \subset L', L \in \mathfrak{L} \ .$$

Clearly, μ' is a finitely subadditive outer measure outer measure and $E \in \mathbf{S}_{\mu''}$ if, and only if,

$$\mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E') \text{ for all } A \in \mathfrak{L}.$$

(See [7]).

Definition 7. Let $\mu \in M_\sigma(\mathfrak{L})$ and define

$$\mu''(E) = \inf \sum_{i=1}^{\infty} \mu(L_i'), E \subset \bigcup_{i=1}^{\infty} L_i', L_i \in \mathfrak{L}, E \subset X \ .$$

Clearly, μ'' is a countably subadditive outer measure and $E \subset \mathbf{S}_{\mu''}$ if, and only if,

$$\mu''(A') \geq \mu''(A' \cap E) + \mu''(A' \cap E') \text{ for all } A \in \mathfrak{L} \text{ (see [7])} \ .$$

Clearly, for $\mu \in I(\mathfrak{L})$ (or $I_\sigma(\mathfrak{L})$) μ' and μ'' are regular outer measures. Furthermore, if $\mu \in I_\sigma(\mathfrak{L})$ then:

(a) $\mu \leq \mu'' \leq \mu'(\mathfrak{L})$ i.e. $\mu(L) \leq \mu''(L) \leq \mu'(L)$ for all $L \in \mathfrak{L}$;
(b) $\mu'' \leq \mu' = \mu(\mathfrak{L}')$;
(c) If $\mu \in I_R^\sigma(\mathfrak{L})$ then $\mu = \mu'' = \mu'(\mathfrak{L})$ and $\mu'' = \mu' = \mu(\mathfrak{L})$.

These results extend readily to the general case of $\mu \in M_\sigma(\mathfrak{L})$.

Theorem 1. *Let $\mu \in M_\sigma(\mathfrak{L})$. Then:*

(a) $\mu''(X) = \mu(X)$ *;*
(b) $\mu \leq \mu''(\mathfrak{L})$.

Proof. (a) Clearly $\mu''(X) \leq \mu(X)$; now suppose $\mu''(X) < \mu(X)$. Then, there exist $L_i \in \mathfrak{L}$, such that $X \subset \bigcup_{i=1}^{\infty} L_i'$ and $\sum_{i=1}^{\infty} \mu(L_i') < \mu(X)$. Hence $\mu(X) > \sum_{i=1}^{\infty} \mu(L_i') = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(L_i') \geq \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} L_i'\right)$ with $\bigcup_{i=1}^{n} L_i' \in \mathfrak{L}'$ and $\bigcup_{i=1}^{n} L_i \uparrow \bigcup_{i=1}^{\infty} L_i' = X$. Since $\mu \in M_\sigma(\mathfrak{L})$, $\lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} L_i'\right) = \mu(X)$, contradiction.

(b) $\mu''(L) \geq \mu''(X) - \mu''(L') = \mu(X) - \mu''(L') \geq \mu(X) - \mu'(L') = \mu(L)$ since $\mu' = \mu(\mathfrak{L}')$.

Definition 8. Let $\mu \in M(\mathfrak{L})$ and define $\mu_i(E) = \sup\{\mu(L), L \subset E, L \in \mathfrak{L}, E \subset X\}$. The following statements can be found in [7].

(a) $\mu(X) = \mu_i(L) + \mu'(L'), L \in \mathfrak{L}$;
(b) $\mu(X) = \mu_i(L') + \mu'(L), L \in \mathfrak{L}$;
(c) $E \in \mathbf{S}_{\mu''}$ if, and only if, $\mu_i(E) = \mu'(E), E \subset X$;
(d) If $\mathfrak{L}$ is normal then μ_i is finitely additive on $\mathfrak{L}'$.

Definition 9. Let $\mu \in M_\sigma(\mathfrak{L})$ and define for $E \subset X : \mu_j(E) = \mu(X) - \mu''(E')$.

Theorem 2. *Let $\mu \in M_\sigma(\mathfrak{L})$ Then :*

(a) $\mu_i \leq \mu_j \leq \mu'' \leq \mu'$;
(b) *If, in addition μ'' is a (cover) regular outer measure, then $E \in \mathbf{S}_{\mu''}$ if, and only if, $\mu_j(E) = \mu''(E)$.*

Proof. (a) Clearly, the relation $\mu'' \leq \mu'$ always holds. Now consider $\mu(X) - \mu_i(E) = \mu(X) - \sup\{\mu(L), L \subset E, L \in \mathfrak{L}\} = \inf\{\mu(X) - \mu(L), E' \subset L', L \in \mathfrak{L}\} = \inf\{\mu(L'), E' \subset L', L \in \mathfrak{L}\} = \mu'(E')$. Hence $\mu_i(E) = \mu(X) - \mu'(E')$ and since $\mu_j(E) = \mu(X) - \mu''(E')$ it follows that $\mu_i \leq \mu_j$. Finally, $\mu_j(E) = \mu''(X) - \mu''(E')$ since μ'' is an outer mesure.

(b) Since μ'' is a regular outer measure, $E \in \mathbf{S}_{\mu''}$ if, and only if, $\mu''(X) = \mu''(E) + \mu''(E')$. Suppose $E \in \mathbf{S}_{\mu''}$. Since $\mu \in M_\sigma(\mathfrak{L})$, $\mu(X) = \mu''(X) = \mu''(E) + \mu''(E')$, hence $\mu_j(E) = \mu''(E)$. Conversely, assume $\mu_j(E) = \mu''(E)$. We have $\mu(X) = \mu_j(E) + \mu''(E') = \mu''(E) + \mu''(E')$, hence $\mu''(X) = \mu''(E) + \mu(E')$.

Definition 10. Let $E \subset X$ and $\mu \in M(\mathfrak{L})$ and define

$$\mu^*(E) = \inf \sum_{i=1}^{\infty} \mu(A_i), E \subset \bigcup_{i=1}^{\infty} A_i, A_i \in \mathbf{A}(\mathfrak{L}) \ .$$

Definition 11. Let $E \subset X$ and $\mu \in M_R(\mathfrak{L})$ and define

$$\mu^\bullet(E) = \inf \mu(A), E \subset A, A \in \mathbf{A}(\mathfrak{L}) \ .$$

The following theorem is generally well-known (see [4]) and we state it without proof.

Theorem 3. *Let $\mu \in M_\sigma(\mathfrak{L})$. Then $\mu' = \mu''(\mathfrak{L})$ if*

(a) *$\mathfrak{L}$ is countably compact; or*
(b) *$\mathfrak{L}$ is δ and normal; or*
(c) *$\mathfrak{L}$ is normal and countably paracompact.*

Theorem 4. *Let $\mathfrak{L}_1 \subset \mathfrak{L}_2$ and suppose that $\mathfrak{L}_1$ separates $\mathfrak{L}_2$ and that $\delta(\mathfrak{L}_1) \subset \mathfrak{L}_2$. Let $\mu \in M_\sigma(\mathfrak{L}_1)$. If $\mu' = \mu''(\mathfrak{L}_1)$, then $\mu' = \mu''(\mathfrak{L}_2)$.*

Proof. $\mu'' \leq \mu'$, hence in particular $\mu'' \leq \mu'(\mathfrak{L}_2)$. Supppose there exists $L_2 \in \mathfrak{L}_2$ such that $\mu''(L_2) < \mu'(L_2)$. Then there exists $A_n \in \mathfrak{L}_1$ such that $L_2 \subset \bigcup_{n=1}^{\infty} A'_n$ and $\sum_{n=1}^{\infty} \mu(A'_n) < \mu'(L_2)$. Since $\mathfrak{L}_1$ separates $\mathfrak{L}_2$, there exist $B, C \in \mathfrak{L}_1$ such that $L_2 \subset B \subset C' \subset \bigcup_{n=1}^{\infty} A'_n$. Then $\mu'(L_2) \leq \mu'(B) = \mu''(B) \leq \mu''\left(\bigcup_{n=1}^{\infty} A'_n\right) \leq \sum_{n=1}^{\infty} \mu''(A'_n) \leq \sum_{n=1}^{\infty} \mu(A'_n) < \mu'(L_2)$. This is a contradiction. Therefore $\mu' = \mu''(\mathfrak{L}_2)$.

Related matters can be found in [6].

Next, we briefly summarize some facts (see[7]) about the Wallman space $I_R(\mathfrak{L})$ and then proceed to consider some relations between the measures and the induced measures on the Wallman spaces, and also relations between their associated outer measures. For $x \in X$, denote by μ_x the measure concentrated at x:

$$\mu_x(A) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A; \end{cases}$$

for $A \in \mathbf{A}(\mathfrak{L})$. Then $\mu_x \in I_R(\mathfrak{L})$ if, and only if, $\mathfrak{L}$ is disjunctive. If $\mathfrak{L}$ is separating and disjunctive and if X is given the topology $t\mathfrak{L}$ of closed sets and if $I_R(\mathfrak{L})$ is given the Wallman topology, then the map $x \to \mu_x$ is a homeomorphism of X into $I_R(\mathfrak{L})$. Thus we may identify X with its image. The Wallman topology is obtained by taking all

$$W(L) = \{\mu \in I_R(\mathfrak{L}) \mid \mu(L) = 1\}, L \in \mathfrak{L}$$

as a base for the closed sets and $I_R(\mathfrak{L})$ is called the general Wallman space associated with X and $\mathfrak{L}$. The lattice $\mathfrak{W}(\mathfrak{L}) = \{W(L) \mid L \in \mathfrak{L}\}$ is a base for the closed sets $t\mathfrak{W}(\mathfrak{L})$. If $A \in \mathbf{A}(\mathfrak{L})$, then $W(A) = \{\mu \in I_R(\mathfrak{L}) \mid \mu(A) = 1\}$ and the following statements are true:

(a) $W(A \cup B) = W(A) \cup W(B)$;
(b) $W(A \cap B) = W(A) \cap W(B)$;
(c) $W(A') = W(A)'$;
(d) $B \subset A$ if, and only if, $W(B) \subset W(A)$;
(e) $\mathbf{A}(\mathfrak{W}(\mathfrak{L})) = \mathfrak{W}(\mathbf{A}(\mathfrak{L}))$.

Theorem 5. *Let $\mu \in M(\mathfrak{L})$ and consider the induced measure $\underline{\mu} \in M(\mathfrak{W}(\mathfrak{L}))$ where $\underline{\mu}(W(A)) = \mu(A)$. We have:*

(a) $\underline{\mu} \in M_\sigma(\mathfrak{W}(\mathfrak{L}))$;
(b) *If $\mu \in M(\sigma, \mathfrak{L})$, the set of strongly σ-smooth measures on $\mathfrak{L}$, then $\underline{\mu} \in M(\sigma, \mathfrak{W}(\mathfrak{L}))$, the set of strongly σ-smooth measures on $\mathfrak{W}(\mathfrak{L})$;*
(c) *If $\mu \in M^\sigma(\mathfrak{L})$, then $\underline{\mu} \in M^\sigma(\mathfrak{W}(\mathfrak{L}))$.*

Proof. We only prove (b), the proofs of (a) and (c) are clear. Consider $W(L_n) \downarrow W(L)$, with $L_n, L \in \mathfrak{L}$. But then $W(L_n) \cap X \downarrow W(L) \cap X$ or $L_n \downarrow L$. Then $\lim \underline{\mu}(W(L_n)) = \lim \mu(L_n) = \mu(L) = \underline{\mu}(W(L))$ since $\mu \in M(\sigma, \mathfrak{L})$. Hence $\underline{\mu} \in M(\sigma, \mathfrak{W}(\mathfrak{L}))$.

Theorem 6. *Let $\mu \in M(\mathfrak{L})$. Then $\underline{\mu}' = \underline{\mu}''$ on $t\mathfrak{W}(\mathfrak{L})$.*

Proof. $\mathfrak{W}(\mathfrak{L})$ compact implies $\mathfrak{W}(\mathfrak{L})$ countably compact and by Theorem 1(a) it follows $\underline{\mu}' = \underline{\mu}''(\mathfrak{W}(\mathfrak{L}))$. But $\mathfrak{W}(\mathfrak{L})$ separates $t\mathfrak{W}(\mathfrak{L})$ and then by Theorems 2 and 3(a) we obtain that $\underline{\mu}' = \underline{\mu}''(t\mathfrak{W}(\mathfrak{L}))$.

Theorem 7.

(a) *Let $E \subset X$ and $\mu \in M(\mathfrak{L})$. Then the following are true:*
(a.1) $\mu'(E) = \underline{\mu}'(E)$;
(a.2) $\mu''(E) = \underline{\mu}''(E)$;
(a.3) $\mu^*(E) = \underline{\mu}^*(E)$;
(b) *Let $L \in \mathfrak{L}$ and $\mu \in M(\mathfrak{L})$. Then $\mu''(L) = \underline{\mu}''(L) \leq \underline{\mu}''(W(L)) = \underline{\mu}'(W(L)) = \mu'(L)$.*
(c) *Let $L \in \mathfrak{L}$. Then $L \in \mathbf{S}_{\mu'}$ if, and only if, $W(L) \in \mathbf{S}_{\underline{\mu}'}$.*

Proof. (a) $\mu'(E) = \inf\{\mu(L'), E \subset L', L \in \mathfrak{L}\} = \inf\{\underline{\mu}(W(L)'), E \subset W(L)', L \in \mathfrak{L}\} = \underline{\mu}'(E)$, since $E \subset L'$ implies $E \subset W(L)'$, and conversely if $E \subset W(L)', L \in \mathfrak{L}$, then intersecting on both sides with X, we get $E \subset L'$. Similar proofs for μ'' and μ^*.

(b) $\underline{\mu}''(W(L)) = \inf\{\sum_{i=1}^{\infty} \underline{\mu}(W(L_i)'), W(L) \subset \bigcup_{i=1}^{\infty} W(L_i)\} \geq$
$\inf\{\sum_{i=1}^{\infty} \mu(L_i'), L \subset \bigcup_{i=1}^{\infty} L_i'\} = \mu''(L)$, since $\underline{\mu}(W(L_i)') = \mu(L_i')$ and $W(L) \subset \bigcup_{i=1}^{\infty} < W(L_i')$ implies $W(L) \cap X \subset \bigcup_{i=1}^{\infty} W(L_i') \cap X$, so $L \subset \bigcup_{i=1}^{\infty} L_i'$. (c) $L \in \mathbf{S}_{\mu'}$ if, and only if, $\mu'(L) = \mu(L)$ if, and only if, $\underline{\mu}'(W(L)) = \underline{\mu}(W(L))$ if, and only if, $W(L) \in \mathbf{S}_{\underline{\mu}'}$ (see [7] for more details on $\mathbf{S}_{\mu'}$).

Theorem 8.

(a) *If $E \in \mathbf{S}_{\underline{\mu}''}$ and μ'' regular, then $E \cap X \in \mathbf{S}_{\mu''}$.*
(b) *If $E \in \mathbf{S}_{\underline{\mu}'}$, then $E \cap X \in \mathbf{S}_{\mu'}$.*

Proof. (a) $E \in \mathbf{S}_{\underline{\mu}''}$ implies $\underline{\mu}''(X) = \underline{\mu}''(E \cap X) + \underline{\mu}''(E' \cap X)$. But $E \cap X \subset X$ and therefore $\underline{\mu}''(E \cap X) = \mu''(E \cap X)$. Similarily $\underline{\mu}(E' \cap X) = \mu''(E' \cap X)$ and $\underline{\mu}''(X) = \mu''(X)$. Hence $\mu''(X) = \mu''(E \cap X) + \mu''(E' \cap X)$ i.e. $E \cap X \in \mathbf{S}_{\mu''}$ since μ'' is regular.

(b) The proof is clear

4. Smoothness characterizations.

We wish to extend the work done in [2,3]. In particular, we want to extend the following theorem proved in [2]:

Theorem 9. *Let X be an arbitrary non-empty set and $\mathfrak{L}$ a lattice of subsets of X such that $\emptyset, X \in \mathfrak{L}$ and where $\mathfrak{L}$ is separating and disjunctive. If $\mu \in M_R(\mathfrak{L})$, then $\mu \in M_R^{\sigma}(\mathfrak{L})$ if, and only if, $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$.*

Here we want to consider the case where $\mu \in M(\mathfrak{L})$ only, and give conditions analogous to the above for μ to belong to $M_{\sigma}(\mathfrak{L})$ or $M^{\sigma}(\mathfrak{L})$.

Theorem 10.

(a) *Let $\mu \in M_{\sigma}(\mathfrak{L})$. Then $\underline{\mu}''(X) = \underline{\mu}(I_R(\mathfrak{L}))$ and consequently $\underline{\mu}''(I_R^{\sigma}(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$.*

(b) *If* $\mu \in M(\mathfrak{L})$ *and* $\underline{\mu}''(X) = \underline{\mu}(I_R(\mathfrak{L}))$ *or if just* $\underline{\mu}''(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$ *and if* $\underline{\mu}''$ *is a regular outer measure, then* $\mu \in M_\sigma(\mathfrak{L})$.

Proof. (a) By Theorem 5(a), for $E \subset X$ we have $\mu'(E) = \underline{\mu}'(E)$ and $\mu''(E) = \underline{\mu}''(E)$. Thus, since $\mu \in M_\sigma(\mathfrak{L}) : \mu(X) = \mu''(X) = \underline{\mu}''(X) \leq \underline{\mu}'(X) = \mu(X)$. Hence $\underline{\mu}''(X) = \mu(X) = \underline{\mu}(W(X)) = \underline{\mu}(I_R(\mathfrak{L}))$. But $X \subset I_R^\sigma(\mathfrak{L}) \subset I_R(\mathfrak{L})$ and then $\underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}''(X) \leq \underline{\mu}(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$ since $\underline{\mu}'' \in M_\sigma(\mathfrak{W}(\mathfrak{L}))$. It follows that $\underline{\mu}''(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$.

(b) Let $L_n \in \mathfrak{L}, L_n \downarrow \emptyset$. Then $W(L_n) \downarrow$ and $\bigcap_{n=1}^\infty W(L_n) \subset I_R(\mathfrak{L}) - X$. Hence $X \subset \bigcup_{n=1}^\infty W(L_n)'$. Now, since $\underline{\mu} \in M_\sigma(\mathfrak{W}(\mathfrak{L}))$:

$$\underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}''(I_R(\mathfrak{L})) \geq \underline{\mu}''(\bigcup_{n=1}^\infty W(L_n)') \geq \underline{\mu}''(X) = \underline{\mu}(I_R(\mathfrak{L}))\ .$$

Using the regularity of $\underline{\mu}''$ we get:

$\mu(X) = \underline{\mu}(W(X)) = \underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}''(\bigcup_{n=1}^\infty W(L_n)') = \lim \underline{\mu}''(W(L_n)') \leq \lim \underline{\mu}'(W(L_n)') = \lim \underline{\mu}(W(L_n)') = \lim \mu(L_n')$. So, $\mu(X) \leq \lim \mu(L_n')$. But $L_n' \uparrow X$ and clearly $\lim \mu(L_n') \leq \mu(X)$. It follows $\mu(X) = \lim \mu(L_n')$ and since $\lim \mu(L_n) = 0, \mu \in M_\sigma(\mathfrak{L})$. In case $\underline{\mu}''(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$, take $L_n \in \mathfrak{L}, L_n \downarrow \emptyset$ and get $W(L_n) \downarrow$ and $\bigcap_{n=1}^\infty W(L_n) \subset I_R(\mathfrak{L}) - I_R^\sigma(\mathfrak{L})$, hence $I_R^\sigma(\mathfrak{L}) \subset \bigcup_{n=1}^\infty W(L_n)'$.

Theorem 11.

(a) *If* $\mu \in M^\sigma(\mathfrak{L})$, *then* $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$ *and consequently* $\underline{\mu}^*(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$.

(b) *If* $\mu \in M(\mathfrak{L})$ *and if* $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$ *or if just* $\underline{\mu}^*(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$ *and if* $\underline{\mu}^*$ *is regular, then* $\mu \in M^\sigma(\mathfrak{L})$.

Proof. (a) If $\mu \in M^\sigma(\mathfrak{L})$ then $\mu^*(X) = \mu(X)$ and then $\underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}(W(X)) = \mu(X) = \mu^*(X) = \underline{\mu}^*(X)$. Consider now $X \subset I_R^\sigma(\mathfrak{L}) \subset I_R(\mathfrak{L})$. We have: $\underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}^*(X) \leq \underline{\mu}^*(I_R^\sigma(\mathfrak{L})) \leq \underline{\mu}(I_R(\mathfrak{L}))$ since $\underline{\mu} \in M^\sigma(\mathfrak{W}(\mathfrak{L}))$. Hence $\underline{\mu}^*(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$.

(b) Let $A_n \in \mathbf{A}(\mathfrak{L}), A_n \downarrow \emptyset$. then $W(A_n) \downarrow$ and $\bigcap_{n=1}^\infty W(A_n) \subset I_R(\mathfrak{L}) - X$, hence $X \subset \bigcup_{n=1}^\infty W(A_n)'$. If $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$, then $\mu(X) = \underline{\mu}(W(X)) = \underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}^*(X) \leq \underline{\mu}^*(\bigcup_{n=1}^\infty W(A_n)')$. By using the regularity of $\underline{\mu}^*$ as in the proof of Theorem 2 we obtain that $\mu \in M^\sigma(\mathfrak{L})$. Similar proof for the case of $\underline{\mu}^*(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$.

Corollary 1. *Let* $\mu \in I(\mathfrak{L})$. *Then :*

(a) $\mu \in I_\sigma(\mathfrak{L})$ *if, and only if,* $\underline{\mu}''(X) = \underline{\mu}(I_R(\mathfrak{L}))$*;*

(b) $\mu \in I^\sigma(\mathfrak{L})$ *if, and only if,* $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$.

In terms of measurability we have the following criterion:

Theorem 12.

(a) *Let* $\mu \in M(\mathfrak{L})$. *If* $X \in \mathbf{S}_{\underline{\mu}'}$, *then* $X \in \mathbf{S}_{\underline{\mu}''}$ *and if* $\underline{\mu}''$ *is a regular outer measure then* $\mu \in M_\sigma(\mathfrak{L})$.

(b) *Let* $\mu \in M_R(\mathfrak{L})$. *If* $X \in \mathbf{S}_{\underline{\mu}\bullet}$, *then* $X \in \mathbf{S}_{\underline{\mu}^*}$ *and* $\mu \in M_R^\sigma(\mathfrak{L})$.

Proof. (a) Since $X \in \mathbf{S}_{\underline{\mu}'}$, $\underline{\mu}'(I_R(\mathfrak{L})) = \mu'(X) = \underline{\mu}'(I_R(\mathfrak{L}) - X)$, and since $\underline{\mu}'(I_R(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}(W(X)) = \mu(X)$ and $\underline{\mu}'(X) = \mu'(X) = \mu(X)$, we obtain $\underline{\mu}'(I_R(\mathfrak{L} - X) = 0$. But $\underline{\mu}'' \leq \underline{\mu}'$, hence $\underline{\mu}''(I_R(\mathfrak{L}) - X) = 0$ then $I_R(\mathfrak{L}) - X \in \mathbf{S}_{\underline{\mu}''}$ and also $X \in \mathbf{S}_{\underline{\mu}''}$. Finally, $\underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}''(X) + \underline{\mu}''(I_R(\mathfrak{L}) - X) = \underline{\mu}''(X)$. Since $\underline{\mu}''$ is regular, by Theorem 2 it follows that $\mu \in M_\sigma(\mathfrak{L})$.

(b) Since $\mathfrak{W}(\mathfrak{L})$ compact, $\underline{\mu} \in M_R(\mathfrak{W}(\mathfrak{L})) \cap M_\sigma(\mathfrak{W}(\mathfrak{L})) = M_R^\sigma(\mathfrak{W}(\mathfrak{L}))$. Hence $\underline{\mu}^*$ is regular (see [5]). Now $X \in \mathbf{S}_{\underline{\mu}^\bullet}$, implies $\underline{\mu}^\bullet(I_R(\mathfrak{L})) = \underline{\mu}^\bullet(X) + \underline{\mu}^\bullet(I_R(\mathfrak{L}) - X)$. But $\underline{\mu}^\bullet(I_R(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L})) = \mu(X)$ and $\underline{\mu}^\bullet(X) = \mu(X)$, hence $\underline{\mu}^\bullet((I_R\mathfrak{L}) - X) = 0$. But $\underline{\mu}* \leq \underline{\mu}^\bullet$, hence $\underline{\mu}^*(I_R(\mathfrak{L}) - X) = 0$, which implies $I_R(\mathfrak{L}) - X \in \mathbf{S}_{\underline{\mu}^*}$ and $X \in \mathbf{S}_{\underline{\mu}^*}$. Now, since $\underline{\mu} \in M_R^\sigma(\mathfrak{W}(\mathfrak{L}))$: $\underline{\mu}(I_R(\mathfrak{L})) = \underline{\mu}^*(I_R(\mathfrak{L})) = \underline{\mu}^*(X) + \underline{\mu}^*(I_R(\mathfrak{L}) - X)$, hence $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$. Since $\underline{\mu}^*$ is regular, by Theorem 3 we obtain that $\mu \in M^\sigma(\mathfrak{L})$. Hence $\mu \in M_R^\sigma(\mathfrak{L})$.

Résumé substantiel en français.

Introduction.

Soit X un ensemble non vide arbitraire et $\mathfrak{L}$ un treillis de parties de X tel que $\emptyset, X \in \mathfrak{L}$. Désignons par $\mathbf{A}(\mathfrak{L})$ l'algèbre engendrée par $M(\mathfrak{L})$ l'espace des mesures non négatives finies, finiment additives sur $\mathbf{A}(\mathfrak{L})$, par $M_\sigma(\mathfrak{L})$ l'espace des mesures $\mu \in M(\mathfrak{L})$ qui sont σ-continues sur $\mathbf{A}(\mathfrak{L})$, donc dénombrablement additives. Désignons de plus par $M_R(\mathfrak{L})$ l'espace des $\mu \in M(\mathfrak{L})$ qui sont $\mathfrak{L}$-régulières et par $M_R^\sigma(\mathfrak{L}) = M_R(\mathfrak{L}) \cap M_\sigma(\mathfrak{L})$. Clairement, $\mu \in M_R^\sigma(\mathfrak{L})$ entraîne $\mu \in M^\sigma(\mathfrak{L})$.

Pour toute mesure $\mu \in M(\mathfrak{L})$, il existe une mesure extérieure finiment sous-additive associée, μ', dont les propriétés ont été étudiées, spécialement celles qui sont reliées à la mesurabilité. À toute mesure $\mu \in M_\sigma(\mathfrak{L})$, on associe une mesure extérieure μ''. Si μ est aussi $\mathfrak{L}$-régulière, alors μ'' coincide avec la mesure extérieure habituelle μ^*. Des conditions assurant qu'une $\mu \in M_R(\mathfrak{L})$ appartienne à $M_R^\sigma(\mathfrak{L})$ ont été données dans [2,3] en faisant appel à une mesure induite $\underline{\mu}$ sur un espace de Wallman associé. Dans le présent travail, nous étendons ce résultat à une mesure $\mu \in M(\mathfrak{L})$ en donnant des conditions en termes d'une mesure induite $\underline{\mu}$ sur un espace de Wallman pour $\mu \in M_\sigma(\mathfrak{L})$ ainsi que des conditions sur $M^\sigma(\mathfrak{L})$.

Rappels et notations.

Cette section contient les rappels classiques nécessaires à la bonne compréhension du texte.

Mesures extérieures finiment additives et espaces de Wallman.

Définition 1. Une mesure non négative μ est une *mesure extérieure subadditive outer measure* si

(a) μ est non décroissante;
(b) $\mu(\cup_{i=1}^n E_i) \leq \sum_{i=1}^n \mu(E_i)$ pour tous $E_1, E_2, \ldots, E_n \subset X$.
(c) $\mu(\emptyset) = 0$.

Définition 2. Soit ν une mesure extérieure subadditive outer measure. On dit qu'un ensemble E est *ν-mesurable* si, pour toute partie $A \subset X$, on a $\nu(A) = \nu(A \cap E) + \nu(A \cap E')$. Désignons par $\mathbf{S}_\nu$ l'espace des ensembles ν-mesurables; ν est dite *régulière*

(au sens des recouvrements) si, pour tout $S \subset X$, il existe un $E \in \mathbf{S}_\nu$ tel que $S \subset E$ et $\nu(S) = \nu(E)$.

Définition 3. Étant donné $\mu \in M(\mathfrak{L})$, on définit $\mu'(E) = \inf \sum_{i=1}^{n} \mu(L_i'), E \subset \cup_{i=1}^{n} L_i', L_i \in \mathfrak{L}, E \subset X$

Définition 4. Étant donné $\mu \in M_\sigma(\mathfrak{L})$, on définit $\mu''(E) = \inf \sum_{i=1}^{n} \mu(L_i'), E \subset \cup_{i=1}^{n} L_i', L_i \in \mathfrak{L}, E \subset X$

Pour $\mu \in I(\mathfrak{L})$ (ou $I_\sigma(\mathfrak{L})$), μ' et μ'' sont clairement des mesures extérieures régulières. De plus, si $\mu \in I_\sigma(\mathfrak{L})$ alors

(a) $\mu \leq \mu'' \leq \mu'(\mathfrak{L})$, c'est-à-dire $\mu(L) \leq \mu''(L) \leq \mu'(L)$ pour tout $L \in \mathfrak{L}$;
(b) $\mu'' \leq \mu' = \mu(\mathfrak{L}')$;
(c) Si $\mu \in I_R^\sigma(\mathfrak{L})$, alors $\mu = \mu'' = \mu'(\mathfrak{L})$ et $\mu'' = \mu' = \mu(\mathfrak{L}')$.

Théorème 1. *Soit $\mu \in M_\sigma(\mathfrak{L})$. Alors*

(a) $\mu''(X) = \mu(X)$*;*
(b) $\mu \leq \mu''(\mathfrak{L})$.

Définition 5. Étant donné $\mu \in M(\mathfrak{L})$, on définit $\mu_i(E) = \sup\{\mu(L), L \subset E, L \in \mathfrak{L}, E \subset X\}$

Définition 6. Étant donné $\mu \in M_\sigma(\mathfrak{L})$ et $E \subset X$, on définit $\mu_j(E) = \mu(X) - \mu''(E')$.

Théorème 2. *Soit $\mu \in M_\sigma(\mathfrak{L})$. Alors*

(a) $\mu_i \leq \mu_j \leq \mu'' \leq \mu'$*;*
(b) *Si, de plus, μ'' est une mesure extérieure régulière (au sens des recouvrements), alors $E \in \mathbf{S}_{\mu''}$ ssi $\mu_j(E) = \mu''(E)$.*

Définition 7. Étant donné $E \subset X$ et $\mu \in M(\mathfrak{L})$, on définit

$$\mu^*(E) = \inf \sum_{i=1}^{\infty} \mu(A_i), E \subset \bigcup_{i=1}^{\infty} A_i, A_i \in \mathbf{A}(\mathfrak{L}) \ .$$

Définition 8. Étant donné $E \subset X$ et $\mu \in M_R(\mathfrak{L})$, on définit

$$\mu^\bullet(E) = \inf \mu(A), E \subset A, A \in \mathbf{A}(\mathfrak{L}) \ .$$

Théorème 3. *Soit $\mu \in M_\sigma(\mathfrak{L})$. Alors $\mu' = \mu''(\mathfrak{L})$ si*

(a) *$\mathfrak{L}$ est dénombrablement compact; ou*
(b) *$\mathfrak{L}$ normal et est un δ-treillis; ou*
(c) *$\mathfrak{L}$ est normal et dénombrablement paracompact.*

Théorème 4. *Soit $\mathfrak{L}_1 \subset \mathfrak{L}_2$ et supposons que $\mathfrak{L}_1$ sépare $\mathfrak{L}_2$ et que $\delta(\mathfrak{L}_1) \subset \mathfrak{L}_2$. Soit $\mu \in M_\sigma(\mathfrak{L}_1)$. Si $\mu' = \mu''(\mathfrak{L}_1)$, alors $\mu' = \mu''(\mathfrak{L}_2)$.*

La topologie de Wallman est obtenue en prenant $W(L) = \{\mu \in I_R(\mathfrak{L}) \mid \mu(L) = 1\}$, $L \in \mathfrak{L}$, comme base pour les ensembles fermés et $I_R(\mathfrak{L})$ est appelé l'espace général de Wallman associé à X et $\mathfrak{L}$. Le treillis $\mathfrak{W}(\mathfrak{L}) = \{W(L) \mid L \in \mathfrak{L}\}$ est une base des fermés $t\mathfrak{W}(\mathfrak{L})$.

Théroème 5. *Soit $\mu \in M(\mathfrak{L})$ et considérons la mesure induite $\underline{\mu} \in M(\mathfrak{W}(\mathfrak{L}))$ où $\underline{\mu}(W(A)) = \mu(A)$. On a*

(a) $\underline{\mu} \in M_\sigma(\mathfrak{W}(\mathfrak{L}))$
(b) *Si $\mu \in M(\sigma, \mathfrak{L})$,l'ensemble des mesures σ-continues sur $\mathfrak{L}$, alors $\underline{\mu} \in M((\sigma, \mathfrak{W}(\mathfrak{L}))$, l'ensemble des mesures σ-continues sur $\mathfrak{W}(\mathfrak{L})$.*
(c) *Si $\mu \in M^\sigma(\mathfrak{L})$, alors $\underline{\mu} \in M^\sigma(\mathfrak{W}(\mathfrak{L}))$.*

Théorème 6. *Soit $\mu \in M(\mathfrak{L})$. Alors $\underline{\mu'} = \underline{\mu''}$ sur $t\mathfrak{W}(\mathfrak{L})$.*

Théorème 7.

(a) *Soit $E \subset X$ et $\mu \in M(\mathfrak{L})$. Alors on a:*
(a.1) $\mu'(E) = \underline{\mu}'(E)$
(a.2) $\mu''(E) = \underline{\mu}''(E)$
(a.3) $\mu^*(E) = \underline{\mu}^*(E)$
(b) *Soit $L \in \mathfrak{L}$ et $\mu \in M(\mathfrak{L})$. Alors $\mu''(L) = \underline{\mu}''(L) \leq \underline{\mu}''(W(L)) = \underline{\mu}'(W(L)) = \mu'(L)$.*
(c) *Soit $L \in \mathfrak{L}$. Alors $L \in \mathbf{S}_{\mu'}$ ssi $W(L) \in \mathbf{S}_{\underline{\mu}'}$.*

Théorème 8.

(a) *Si $E \in \mathbf{S}_{\underline{\mu}''}$ et μ'' regulière, alors $E \cap X \in \mathbf{S}_{\mu''}$.*
(b) *Si $E \in \mathbf{S}_{\underline{\mu}''}$, alors $E \cap X \in \mathbf{S}_{\mu'}$.*

4. Caractérisations de la σ-continuité.

Théorème 9. *Soit X un ensemble non vide arbitraire et $\mathfrak{L}$ un treillis de parties de X séparant et disjonctif. Si $\mu \in M_R(\mathfrak{L})$, alors $\mu \in M_R^\sigma(\mathfrak{L})$ ssi $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$.*

Théorème 10.

(a) *Soit $\mu \in M_\sigma(\mathfrak{L})$. Alors $\underline{\mu}''(X) = \underline{\mu}^*(I_R(\mathfrak{L}))$ et, par conséquent, $\underline{\mu}''(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$.*
(b) *Si $\mu \in M(\mathfrak{L})$ et $\underline{\mu}''(X) = \underline{\mu}(I_R(\mathfrak{L}))$ ou si seulement $\underline{\mu}''(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$ et si $\underline{\mu}''$ est une mesure régulière, alors $\mu \in M_\sigma(\mathfrak{L})$.*

Théorème 11.

(a) *Soit $\mu \in M^\sigma(\mathfrak{L})$. Alors $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$ et, par conséquent, $\underline{\mu}^*(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$.*
(b) *Si $\mu \in M(\mathfrak{L})$ et si $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$ ou si seulement $\underline{\mu}^*(I_R^\sigma(\mathfrak{L})) = \underline{\mu}(I_R(\mathfrak{L}))$ et si $\underline{\mu}^*$ est régulière, alors $\mu \in M^\sigma(\mathfrak{L})$.*

Corollaire.

(a) *Soit $\mu \in I(\mathfrak{L})$. Alors $\mu \in I_\sigma(\mathfrak{L})$ ssi $\underline{\mu}''(X) = \underline{\mu}(I_R(\mathfrak{L}))$;*
(b) *Soit $\mu \in I(\mathfrak{L})$. Alors $\mu \in I^\sigma(\mathfrak{L})$ ssi $\underline{\mu}^*(X) = \underline{\mu}(I_R(\mathfrak{L}))$.*

Théorème 12.

(a) *Soit $\mu \in M(\mathfrak{L})$. Si $X \in \mathbf{S}_{\underline{\mu}'}$, alors $X \in \mathbf{S}_{\underline{\mu}''}$ et si $\underline{\mu}''$ est une mesure extérieure régulière, alors $\mu \in M_\sigma(\mathfrak{L})$.*
(b) *Soit $\mu \in M_R(\mathfrak{L})$. Si $X \in \mathbf{S}_{\underline{\mu}^\bullet}$. Alors $X \in \mathbf{S}_{\underline{\mu}^*}$ and $\mu \in M_R^\sigma(\mathfrak{L})$.*

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