UNIVERSITY
OF TRENTO

**DEPARTMENT OF INFORMATION AND COMMUNICATION TECHNOLOGY**

38050 Povo – Trento (Italy), Via Sommarive 14
http://www.dit.unitn.it

# EFFICIENCY OF ADC LINEARITY ESTIMATORS

Paolo Carbone, Emilia Nunzi, Dario Petri

February 2004

Technical Report # DIT-04-043

# Efficiency of ADC Linearity Estimators

P.Carbone[1], E.Nunzi[1], D.Petri[2]

[1]DIEI – Università degli Studi di Perugia, via G.Duranti 93 – 06125 Perugia, Italy.
Phone: +39 075 5853629, Fax: +39 075 5853654, Email: carbone@diei.unipg.it
[2] DIT – Università degli Studi di Trento, Via Sommarive, 14 – 38050 Trento, Italy
Phone: +39 0461 883902, Fax: +39 0461 882093, Email: petri@dit.unitn.it.

***Abstract*** ***– The paper presents an analysis of the statistical efficiency of the sinewave histogram test used for estimating the unknown transition levels of an analog-to-digital converter. Accordingly, at first a closed–form determination of the Cramér–Rao bound is derived under the assumption of a noiseless stimulus signal. Both unbiased and biased versions of the bound are described in order to account for the eventual bias introduced by commonly employed estimators. Then, additive Gaussian noise is assumed and comments are made about its effects on the maximum achievable accuracy.***

***Keywords*** ***– Cramér–Rao bound, sinewave histogram test, statistical efficiency.***

## I. INTRODUCTION

The assurance and improvement of quality levels characterizing the performance of analog-to-digital converters (ADCs), require the careful selection of post-production device testing methods. In fact, considerations mainly based on economical reasons demand the tests to be quick, effective, easily reproducible, and simple. In practice, both static and dynamic information on the ADC behavior is usually needed, as well as figures showing its performance both in the time and frequency domains. It is widely recognized that, among other parameters, the ADC integral (INL) and differential (DNL) nonlinearities represent quantities of paramount importance for the description of the tested device quality under both static and dynamic conditions. In order to measure the ADC linearity parameters the histogram test method is usually applied. This technique is based on the use of a signal source exciting the ADC under test and on the evaluation of the histogram of the device output codes. Accordingly, the a priori knowledge about the amplitude distribution of the converter input signal is employed. The linearity parameters are finally estimated by processing the ADC output code histogram. The most frequently used device stimuli are the sinusoidal and the ramp ones [1], even though Gaussian noise has been recently proposed as an alternative approach [2].

Whichever the excitation signal type, if the information carried by the device output sequence is effectively processed, the time needed to obtain prescribed levels of estimation accuracy is minimized. Since the test costs are directly related to the test process duration, efficiently processing data helps in keeping test expenses at a minimum.

The goal of this paper is twofold. At first, the theoretically achievable accuracy of sinewave histogram testing (SHT) is investigated through the evaluation of the Cramér–Rao bound (CRB). Then, an analysis is carried out to determine how efficiently the commonly adopted data processing techniques exploit the information provided by the ADC data, assuming both noiseless and noisy stimulus signals.

## II. EFFICIENCY OF THE SINEWAVE HISTOGRAM TEST

### *A. The Sinewave Histogram Test*

A brief description of the fundamentals of the SHT is given in the following. The converter under test is stimulated through a highly linear sinewave generator. Thus, a suitable model of the single record input data set is:

$$x[n] \triangleq d - A\cos\left(2\pi\frac{D}{M}n + \phi\right), \quad n = 0, 1, \ldots, M-1, \tag{1}$$

with $A$, $\phi$ and $d$ as the sinewave amplitude, initial record phase and offset respectively, while $D/M$ represents the ratio between sinewave frequency and converter sampling rate. Since the phase difference between sinewave and ADC sampling sequence is usually not controlled and varies at random between separate data records, $\phi$ can be considered as a random variable uniformly distributed in $[0, 2\pi)$.

The purpose of SHT is that of estimating the $k$-th transition level, $T_k$, where $k = 1, \ldots, N-1$, $N \triangleq 2^b$ and $b$ is the number of ADC bits. Once each $T_k$ has been evaluated, estimates of the converter INLs and DNLs easily follow. In order to minimize the total number of samples required for a given estimation accuracy, the coherence condition should be met. This happens when $D$ and $M$ are mutually prime integer numbers. Other directions on how to choose appropriate values for $A$, $d$, $M$ and the number $R$ of data records are provided in [1], [3], [4]. Finally, the transition levels are estimated by means of

$$\hat{T}_k \triangleq d - A\cos\left(\widehat{\psi}_k\right), \qquad k = 1, \ldots, N-1, \tag{2}$$

where $\widehat{\psi}_k$ represents the cumulative histogram corresponding to the $k$-th converter output code normalized by $(M \cdot R)/\pi$. It can be shown that [3], [5]:

$$\begin{aligned}\text{bias}[\hat{T}_k] &\triangleq E[\hat{T}_k - T_k] \\ &= -2A\sin\left[\frac{\pi}{2M}(2n_k+\alpha_k)\right]\sin\left(\frac{\pi}{2M}\alpha_k\right)(1-\alpha_k) \\ &\quad +2A\sin\left[\frac{\pi}{2M}(2n_k+\alpha_k+1)\right]\sin\left[\frac{\pi}{2M}(1-\alpha_k)\right]\alpha_k \end{aligned} \quad (3)$$

and

$$\text{var}[\hat{T}_k] = A^2\alpha_k(1-\alpha_k)\left[\cos\left(\frac{\pi}{M}n_k\right) - \cos\left(\frac{\pi}{M}(n_k+1)\right)\right]^2 \quad (4)$$

where $E[\cdot]$ is the expectation operator,

$$\begin{aligned} n_k &\triangleq \left\lfloor \frac{2\psi_k}{\Delta\phi}\right\rfloor, \quad \alpha_k \triangleq \left\langle \frac{2\psi_k}{\Delta\phi}\right\rangle, \\ \psi_k &\triangleq \arccos\left(-\frac{\overline{T}_k}{A}\right), \quad \Delta\phi \triangleq \frac{2\pi}{M}, \end{aligned} \quad (5)$$

with $\overline{T}_k \triangleq T_k - d$ and where $A \geq \overline{T}_k$ is assumed. In (5), $\lfloor x \rfloor$ and $\langle x \rangle$ represent the largest integer lower than or equal to $x$ and the fractional part of $x$, respectively. By employing trigonometric equivalencies and by observing that $0 \leq \alpha_k(1-\alpha_k) \leq 1/4$, from (3) and (4) follows that

$$|\text{bias}[\hat{T}_k]| \leq \frac{A\pi^2}{8M^2}, \qquad \text{var}[\hat{T}_k] \leq \frac{\pi^2}{4M^2}(A^2 - \overline{T}_k^2). \quad (6)$$

Thus, (2) is an asymptotically unbiased estimator of $T_k$, whose bias is negligible for all values of $M$ employed in practice.

### *B. The Cramér–Rao bound for the SHT*

It is well known that, under certain regularity conditions, for a given sample probability distribution and size, the *mean square error* of any estimator of an unknown parameter can not be made arbitrarily small. In other terms, there is a lower bound on the amount of information that can be extracted from the available data set about the parameter to be estimated. This limit is called the Cramér–Rao bound [6], [7]. In particular, since the adoption of unbiased estimators represents a common engineering practice, it follows that the estimator *variance* can not be made lower than a limit depending entirely on the sample size and data joint distribution function. The application of such theoretical principles to the accuracy analysis of the SHT applied to a single–bit ADC, allows the following inequality to be written for any unbiased estimator $\widetilde{T}_0$ of $T_0$:

$$\text{var}[\widetilde{T}_0] \triangleq E\{(\widetilde{T}_0 - T_0)^2\} \geq CRB, \quad (7)$$

where the CRB has been obtained through the likelihood function derived in App. A, as follows (App. B):

$$CRB = \frac{\pi^2}{M^2}\alpha_0(1-\alpha_0)(A^2 - \overline{T}_0^2). \quad (8)$$

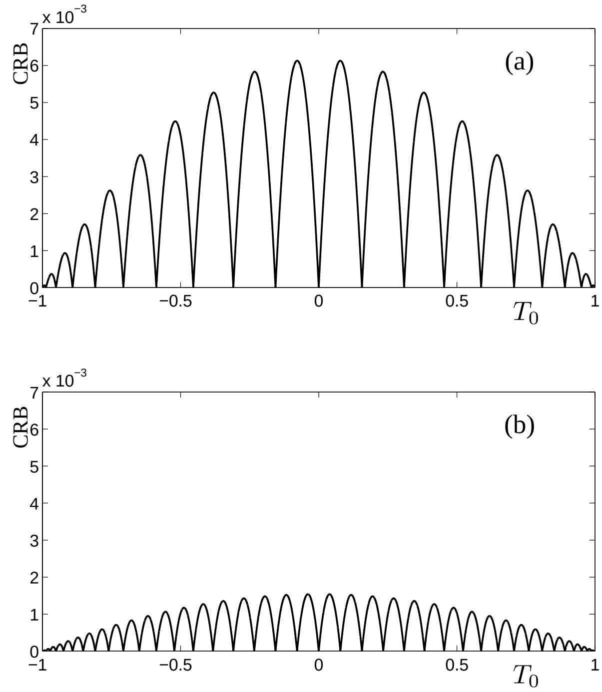


Figure 1. Cramér–Rao bound as a function of the unknown transition level using single record sinewave with random initial record phase: $A = 1$, $d = 0$; (a) $M = 20$; (b) $M = 40$.

Moreover, from (8) it follows that

$$\max_{T_0} CRB = \frac{A^2\pi^2}{4M^2}. \quad (9)$$

In order to appreciate its behavior, (8) has been plotted in Fig. 1 as a function of $T_0$, by assuming $A = 1$, $d = 0$ and $M = 20, 40$. Notice that the Cramér–Rao bound vanishes when $\alpha_0 = 0$, that is when $\hat{T}_0$ and $\widehat{\psi}_0$ are no longer random variables but become deterministic functions of the input data sequence [5]. Notice also that $M$ sets the number of lobes appearing in Fig. 1.

In order to compare the accuracy of (2) with the corresponding theoretical limit, it is necessary to include the contribution of estimator bias. Accordingly, for any biased estimator $\widetilde{T}_0$ of $T_0$ the following inequality applies:

$$E\{(\widetilde{T}_0 - T_0)^2\} = \text{bias}^2[\widetilde{T}_0] + \text{var}[\widetilde{T}_0] \geq CRB_b, \quad (10)$$

where $CRB_b$ is the biased SHT Cramér–Rao bound which includes the contribution of (3) (App. B):

$$CRB_b \triangleq A^2\alpha_0(1-\alpha_0)\left[\cos\left(\frac{\pi}{M}n_0\right) - \cos\left(\frac{\pi}{M}(n_0+1)\right)\right]^2. \quad (11)$$

Notice that (11) differs appreciably from (8) only for small values of $M$. Observe also that, the comparison of (4) with (11) evidences that $CRB_b$ is equal to var$[\hat{T}_0]$. Thus, since bias$[\hat{T}_0] \neq 0$ when $\alpha_0 \neq 0$, it follows that the equal sign in

(10) never applies. Consequently, the statistical efficiency of (2) can be written as:

$$\eta \triangleq \frac{CRB_b}{\text{bias}^2[\hat{T}_0] + \text{var}[\hat{T}_0]} = \frac{1}{1 + \dfrac{\text{bias}^2[\hat{T}_0]}{CRB_b}}, \qquad (12)$$

which converges rapidly to 1, and uniformly with respect to $T_0$, for all values of $M$ used in practice.

Consider now the unbiased CRB related to the estimation of ADC transition levels corresponding to a multibit quantizer. Output data related to the device under test can be regarded as data coming from a set of converters each characterized by one of the transition levels pertaining to the original device. Thus, the CRB corresponding to the estimation of more than a single transition level can not be larger than (8). Furthermore, the numerical evaluation of such bound under various settings of both test parameters and using several values of $N$, has confirmed that (8) appears also to apply to the multibit quantizer case. Consequently, on the basis of (8) and (12), the currently adopted estimator (2) is an almost optimum one, under the specified system and signal hypotheses.

When additive noise corrupts (1), an additional source of uncertainty must be taken into consideration. Usually, even under these testing conditions (2) still applies. This results in estimator bias and variance whose approximate expressions have been given in [3]. Because of the noise, it is expected that the corresponding estimation accuracy be worse than that theoretically achievable through (8). In fact, by following a reasoning similar to that described in App. A and B, a numerical evaluation of the CRB has been carried out. The CRB that applies to the unbiased estimator of a single transition level is graphed in Fig. 2. Data have been derived by assuming random initial record phase, zero–mean Gaussian noise with standard deviation $\sigma = 0.05, 0.12, 0.2$, $A = 1$, $d = 0$ and $M = 11$. For comparison, also (8) has been reported in the same figure. Data shown in Fig. 2 confirms the overall increase of the CRB due to the additive noise. However, for particular values of $T_0$, the addition of noise with low enough variance, seems to allow slightly more accurate estimates than those achievable assuming a noiseless sinewave. Moreover, numerical simulations and theoretical results published in [3] confirm that (2), is almost efficient even in this case, once the associated bias has been neglected.

Notice that when multiple data records are collected, each obtained by independently setting the initial record phase, (8) scales by a factor $1/R$. Moreover, by assuming $M = 1$ and $R > 1$, the presented results apply also when sinewave random sampling is adopted, that is when the input sinewave is sampled at random.

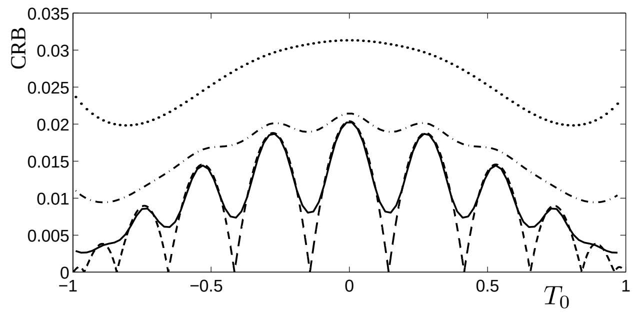


Figure 2. Cramér–Rao bound of an unknown ADC transition level. Single record sinewave is assumed with random initial record phase and zero-mean Gaussian additive noise with $A = 1$, $d = 0$, $M = 11$ and $\sigma = 0.05$ (solid line), $\sigma = 0.12$ (dash–dotted line), $\sigma = 0.2$ (dotted line). Expression (8) is graphed for comparison, using a dashed line.

## III. CONCLUSION

In this paper, it is analyzed the efficiency of the code density test in exploiting the information provided by the converter output when the device is properly stimulated using a sinusoidal signal generator. A closed form relationship is given for the Cramér–Rao bound related to the estimation of converter transition levels under the assumption of sinewave coherent sampling and random initial record phase. By using this result, it is shown that the ADC sinewave histogram test is an asymptotically efficient estimator of converter linearity. Moreover, the contribution of additive noise is taken into consideration. At this regard, results are presented on the increase in minimum attainable variance when estimating the converter transition levels.

## APPENDIX A: DERIVATION OF THE ADC OUTPUT LIKELIHOOD FUNCTION

Assume (1) as the noiseless converter input signal. Define

$$\begin{aligned} P\{\mathbf{Y} = Y; \underline{T}\} &\triangleq P\{\mathbf{y}_0 = y_0, \mathbf{y}_1 = y_1, \ldots, \mathbf{y}_N = y_N; \underline{T}\} \\ &= P\{y_0, y_1, \ldots, y_N; \underline{T}\} \end{aligned} \qquad (A.1)$$

where $\underline{T}$ is the set of unknown transition levels, that is the parameters to be estimated, and

$$\mathbf{Y} \triangleq [\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N], \quad Y \triangleq [y_0, y_1, \ldots, y_N], \quad N \geq 1 \qquad (A.2)$$

with $\mathbf{y}_k$ as the random variable representing the number of occurrences of the $k$-th output code when $M$ samples are acquired, and $y_k$ as the value taken by $\mathbf{y}_k$. Such variable is based on the definition of the ADC input–output characteristic,

$$y(x) = \begin{cases} y_0 & x < T_0 \\ y_k & T_{k-1} \leq x < T_k, \quad k = 1, \ldots, N-1 \\ y_N & x \geq T_{N-1} \end{cases} \qquad (A.3)$$

assuming $x$ as the converter input quantity. By using the properties of conditional probabilities, it can be proved that

$$P\{\mathbf{Y} = Y; \underline{T}\} = P_C\{\mathbf{C} = C; \underline{T}\} \qquad (A.4)$$

where $\mathbf{C} \triangleq [\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_N]$, $\quad C \triangleq [c_0, c_1, \ldots, c_N]$, $N \geq 1$ and $\mathbf{c}_i \triangleq \sum_{j=0}^{i} \mathbf{y}_j$. Observe that $\mathbf{c}_N = M$ must hold true. Thus,

$$P\{y_0, y_1, \ldots, y_N; \underline{T}\} = P_C\{c_0, c_1, \ldots, c_{N-1}, M; \underline{T}\} \quad \text{(A.5)}$$

It can be shown that $\mathbf{c}_k$ is a function of $T_k$ and $\overline{\phi} \triangleq \langle \phi / \Delta\phi \rangle$ as follows:

$$\mathbf{c}_k \triangleq c(\overline{\phi}, T_k) = \begin{cases} \left.\begin{array}{ll} n_k & 0 \leq \overline{\phi} < \frac{1}{2} - \frac{\alpha_k}{2} \\ n_k + 1 & \frac{1}{2} - \frac{\alpha_k}{2} \leq \overline{\phi} < \frac{1}{2} + \frac{\alpha_k}{2} \\ n_k & \overline{\phi} \geq \frac{1}{2} + \frac{\alpha_k}{2} \end{array}\right\} n_k \text{ odd} \\ \\ \left.\begin{array}{ll} n_k + 1 & 0 \leq \overline{\phi} < \frac{\alpha_k}{2} \\ n_k & \frac{\alpha_k}{2} \leq \overline{\phi} < 1 - \frac{\alpha_k}{2} \\ n_k + 1 & \overline{\phi} \geq 1 - \frac{\alpha_k}{2} \end{array}\right\} n_k \text{ even} \end{cases} \quad \text{(A.6)}$$

Moreover, when $\overline{\phi}$ is uniformly distributed in $[0, 1)$, (A.5) becomes the measure $\mu(\mathcal{A})$, in the Lebesgue sense, of the set

$$\mathcal{A} \triangleq \bigcap_{k=0}^{N-1} I_k(c_k) \quad \text{(A.7)}$$

where the sets $I_k(\cdot)$ are defined as follows:

$$I_k(c_k) \triangleq \begin{cases} \left.\begin{array}{ll} \left[0, \frac{1}{2} - \frac{\alpha_k}{2}\right) \cup \left[\frac{1}{2} + \frac{\alpha_k}{2}, 1\right) & c_k = n_k \\ \left[\frac{1}{2} - \frac{\alpha_k}{2}, \frac{1}{2} + \frac{\alpha_k}{2}\right) & c_k = n_k + 1 \end{array}\right\} n_k \text{ odd} \\ \\ \left.\begin{array}{ll} \left[\frac{\alpha_k}{2}, 1 - \frac{\alpha_k}{2}\right) & c_k = n_k \\ \left[0, \frac{\alpha_k}{2}\right) \cup \left[1 - \frac{\alpha_k}{2}, 1\right) & c_k = n_k + 1 \end{array}\right\} n_k \text{ even} \end{cases} \quad \text{(A.8)}$$

Thus, $P\{\mathbf{Y} = Y; \underline{T}\} = \mu(\mathcal{A})$.

## APPENDIX B: DERIVATION OF THE SINGLE LEVEL CRB

The CRB on the variance of an estimator $\hat{t}$ of a scalar differentiable function $t_{\underline{T}}$ of $\underline{T}$ exhibiting bias $b_{\underline{T}} = E\{\hat{t}\} - t_{\underline{T}}$ is given by:

$$CRB = [\nabla t_{\underline{T}} + \nabla b_{\underline{T}}]^T \underline{F}^{-1} [\nabla t_{\underline{T}} + \nabla b_{\underline{T}}], \quad \text{(B.1)}$$

where $\nabla f \triangleq [\partial f / \partial T_0, \partial f / \partial T_1, \ldots, \partial f / \partial T_{N-1}]^T$ is the gradient function and

$$\underline{F} \triangleq E\{[\nabla \log P\{\mathbf{Y} = Y; \underline{T}\}][\nabla \log P\{\mathbf{Y} = Y; \underline{T}\}]^T\} \quad \text{(B.2)}$$

denotes the $N \times N$ Fisher information matrix, which is assumed to be non–singular. Since $\mathbf{Y}$ is a discrete random vector, the element $F_{r,c}$ in row $r$ and column $c$ in such a matrix has the following representation:

$$F_{r,c} \triangleq \sum_{Y \in \mathcal{Y}} P\{\mathbf{Y} = Y; \underline{T}\} \frac{\partial \log P\{\mathbf{Y} = Y; \underline{T}\}}{\partial T_r} \times \frac{\partial \log P\{\mathbf{Y} = Y; \underline{T}\}}{\partial T_c}, \quad \text{(B.3)}$$

where $\mathcal{Y}$ represents the set of admissible values for $\mathbf{Y}$. By using (A.4) it follows that

$$F_{r,c} = \sum_{C \in \mathcal{C}} \frac{1}{P_C\{\mathbf{C} = C; \underline{T}\}} \frac{\partial P_C\{\mathbf{C} = C; \underline{T}\}}{\partial T_r} \times \frac{\partial P_C\{\mathbf{C} = C; \underline{T}\}}{\partial T_c}, \quad \text{(B.4)}$$

where $\mathcal{C}$ represents the set of admissible values for $\mathbf{C}$. Notice that (B.3) and (B.4) apply only when the values of the model parameters allow $\partial P_C\{\mathbf{C} = C; \underline{T}\} / \partial \underline{T}$ to exist.

Under the assumption of a single transition level, e.g. a single-bit quantizer for which $\underline{T} = T_0$, two outcomes are possible and the joint probabilities become:

$$\begin{aligned} P\{y_0, y_1; \underline{T}\} &= P\{y_0, M - y_0; \underline{T}\} = P_C\{c_0, M; \underline{T}\} \\ &= \mu(I_0(c_0)) = \begin{cases} p_0 \triangleq 1 - \alpha_0 & c_0 = n_0 \\ \\ p_1 \triangleq \alpha_0 & c_0 = n_0 + 1 \end{cases} . \end{aligned} \quad \text{(B.5)}$$

Moreover, Fisher matrix becomes the following scalar:

$$\begin{aligned} \underline{F} &= E\left[\left(\frac{\partial \log P_C\{c_0, M; T_0\}}{\partial T_0}\right)^2\right] \\ &= \frac{1}{p_0}\left(\frac{\partial p_0}{\partial T_0}\right)^2 + \frac{1}{p_1}\left(\frac{\partial p_1}{\partial T_0}\right)^2 . \end{aligned} \quad \text{(B.6)}$$

When $\alpha_0 \neq 0$, we obtain:

$$\left(\frac{\partial p_0}{\partial T_0}\right)^2 = \left(\frac{\partial p_1}{\partial T_0}\right)^2 = \frac{M^2}{\pi^2 (A^2 - T_0^2)}. \quad \text{(B.7)}$$

Thus, from (B.1) and (B.6) it follows that $\underline{F} = M^2 / (\pi^2 \alpha_0 (1 - \alpha_0)(A^2 - T_0^2))$. Consequently, when $b_{\underline{T}} = 0$, (8) results. Conversely, by differentiating (3) with respect to $T_k$, $k = 0$, and by using (B.1), we obtain (11).

## References


[1] IEEE STD 1241 – Draft, *Standard for Terminology and Test Methods for Analog-to-Digital Converters*, March 2000.

[2] R.C.Martins, A.M.Serra, "Automated ADC Characterization Using the Histogram Test Stimulated by Gaussian Noise," *IEEE Trans. Instr. Meas.*, vol.48, n.2, pp. 471–474, Apr. 1999.

[3] J.Blair, "Histogram Measurement of ADC Nonlinearities using Sine Waves," *IEEE Trans. Instr. Meas.*, vol.43, n.3, pp. 373–383, June 1994.

[4] P.Carbone, G.Chiorboli, "ADC Sinewave Histogram Testing with Quasi-Coherent Sampling," Proc. $17$–$th$ *IEEE Instr. Meas. Tech. Conf.*, Baltimore, pp. 108–113, 4–6 June, 2000.

[5] P.Carbone, D.Petri, "Design of ADC Sinewave Histogram Test," *Computer Standards & Interfaces*, vol. 22, no. 4, pp. 239–244, Sept. 2000.

[6] H.Cramér, *Mathematical Methods of Statistics*, Princeton University Press, $19^{th}$ Printing, 1999.

[7] G.Ivchenko, Yu. Medvedev, *Mathematical Statistics*, MIR Publishers Moscow, 1990.