How Harmful the Paradox Can Be in the Braess/Cohen-Kelly-Jeffries Networks

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Abstract— **Consider networks in Wardrop equilibria, *i.e.,* situations where each user in a network strives to optimize its own cost noncooperatively but has only an infinitesimal impact on other users. In computer networking, some shortest-path routing protocols that reflect link queueing delays may bring about situations close to Wardrop equilibria. The Braess paradox is a famous example of paradoxical cases where adding capacity to a network degrades the costs for all users. This paper investigates, in particular, networks generalized from what were studied by Cohen, Kelly, and Jeffries in comparison with the networks of the same topology as the original Braess network. The measure of cost degradation considered is the ratio of the cost for each user of a network after adding capacity (a link) to that before adding capacity. It has been shown that a value of the measure is less than 2 for every general Braess network. The results of this paper show that the measure of paradoxical cost degradation is also less than 2 for all of the networks considered in this paper.**

I. Introduction

THERE exist networks that consist of a finite number of links or facilities and at which threads or flows of users arrive from infinite sources. For example, communication networks have flows of packets or calls to pass through communication links, distributed computer systems have continuing arrivals of transactions or jobs to be processed by computers, transportation flow networks have incoming threads of vehicles to drive through roads, *etc.*

It would be anticipated that users would be benefited by adding capacity to a network. This is not always the case, however, as first exemplified in the Braess paradox [1]. The Braess paradox shows that adding capacity (a link) to a network may sometimes degrade the costs for all users in a Wardrop equilibrium. A Wardrop equilibrium is attained in the situation where each user chooses a path of the minimum cost, the choice of a single user has only a negligible impact on the cost of each link, and the equilibrium cost of each used path is identical, which is not greater than the costs of unused paths (see *e.g.,* [2]). In computer networking, some dynamic shortest-path routing protocols that reflect link queueing delays may bring about situations close to Wardrop equilibria (see, *e.g.,* [3–6]). The present paper considers networks in Wardrop equilibria.

The Braess paradox attracted attention of many researchers including Nobel laureate Paul Samuelson [7], and a lot of work has been accumulated. Many papers related to the Braess paradox have been published (*e.g.,* [8–13]), including a paper [14], published in a scientific journal, *Nature*, that discusses similar phenomena in mechanical and electrical networks. Examples where a paradox similar to Braess's appear in a Nash equilibrium have been found for networks of a topology similar to the Braess one [15, 16] and for a network of another topology [17] for which no paradox occurs in Wardrop equilibria. It seems that most studies have not emphasized the estimation of how harmful the paradox can be, *i.e.,* the worst ratio of paradoxical cost degradation by adding capacity.

The present paper addresses the estimation of the worst ratio of paradoxical cost degradation by adding capacity. This paper investigates, in particular, networks generalized from the networks studied by Cohen, Kelly, and Jeffries in comparison with the networks of the same topology as the original Braess network. The measure of cost degradation considered is the ratio of the cost for each user of the network after adding capacity (a link) to that before adding capacity. The Braess paradox shows that there exists a network for which the measure is greater than one. It has been shown that a value of the measure is less than 2 for every general Braess network [18]. The results of this paper show that the measure of paradoxical cost degradation is also less than 2 for all of the networks considered in this paper. This is in contrast to KAKH networks [17] where the measure of paradoxical cost degradation can increase without bound in Nash equilibria [19].

The description of the model investigated is given in Section II, and Section III presents results obtained on networks generalized from the networks studied by Cohen and Jeffries [13] and from the Cohen-Kelly network [11], and show that the measure of paradoxical cost degradation is also less than 2 for all of them considered, just as the measure is less than 2 for general Braess networks. Section IV concludes this paper.

II. The Model and Assumptions

This paper considers networks that have nonlinear link cost functions. The cost of a path is the sum of the cost of each link in the path. Each user chooses a path of the minimum cost. The choice of a single user has only a negligible impact on the cost of each link. The situation where no user can reduce its cost by unilaterally choosing another path is called a *Wardrop equilibrium* (see *e.g.,* [2]). Therefore, in a Wardrop equilibrium, the cost of each used path is identical and is not greater than the costs of unused paths. In this paper, networks in Wardrop equilibria are considered. The uniqueness of the flow of each path at a Wardrop equilibrium in each network considered in this paper is clear (as to uniqueness of routing decisions, see *e.g.,* [20]).

A. General Braess networks

The Braess network considered (Fig. 1) consists of four nodes: one origin (numbered 0), two relay nodes (1 and 2), and one destination (3). Before adding capacity (a link), the network has two paths, 0-1-3 and 0-2-3, each of which contains two links, the origin to one relay (0-1 or 0-2) and the relay to the destination (1-3 or 2-3), respectively. After adding capacity, i.e., a one-way link connecting two relays (1-2), the network

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has three paths including the new path (0-1-2-3) connecting the origin, one relay, the other relay, and the destination. Each user flows through one of the paths.

Denote by C_o and C_c, respectively, the costs of the paths that are used before and after adding link 1-2. All the costs of used paths must be equal, which are not greater than the cost of unused paths. The costs of paths used before and after adding a link may be different from each other. Denote by k the ratio of cost degradation by adding a link, and thus, $k = C_c/C_o$. In this paper, k is considered the measure of cost degradation by adding link 1-2. $k > 1$ means paradoxical cost degradation. $k < 1$ means cost improvement that is naturally expected when a link is added to the network. (In the original Braess network, the cost of each link is a linear function of the amount of flow through the link, and the measure of paradoxical cost degradation is $k = C_c/C_o = 1.1084\ldots$)

Denote the amounts of the flow through paths 0-1-3 and 0-2-3, respectively, by x and y before adding link 1-2. Denote the amounts of the flow through paths 0-1-3, 0-2-3, and 0-1-2-3, respectively, by u, v, and w after adding link 1-2. Denote by X the total flow (of users), and thus

$$x + y = u + v + w = X. \tag{1}$$

Consider a number of types of networks reflecting the degree of specialization in this subsection. Note, however, that many of them are used only in the intermediate stages of the entire flow of proofs that leads to main results.

• [Type-G0] or [**General Braess network**] (Fig. 1) The cost

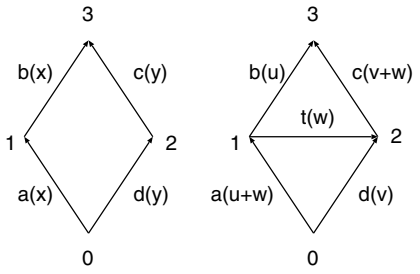

Fig. 1. General Braess network. (Type-G0). **Left**: The network before link 1-2 is added. **Right**: The network after link 1-2 is added. Similarly for the subsequent pairs of figures except for Figures 4 and 5.

of links 0-1, 1-3, 0-2, and 2-3 are, respectively, $a(x)$, $b(x)$, $d(y)$, and $c(y)$ before adding link 1-2. The cost of links 0-1, 1-3, 0-2, 2-3, and 1-2 are, respectively, $a(u+w)$, $b(u)$, $d(v)$, $c(v+w)$, and $t(w)$ after adding the link. a and c are continuous and (strictly) increasing functions. b, d, and t are continuous and non-decreasing functions. If all of x, y, u, v, and w are positive, the following relations hold.

$$C_o = a(x) + b(x) = c(y) + d(y), \tag{2}$$
$$C_c = a(u+w) + b(u) = c(v+w) + d(v) \tag{3}$$
$$= a(u+w) + t(w) + c(v+w). \tag{4}$$

If either $x = 0$ or $y = 0$, the Braess paradox never occurs since adding link 1-2 cannot increase the cost of the path used before adding the link. Thus, in the case where the paradox occurs, *i.e.,* $k > 1$, the following relations hold.

$$0 < x < X, 0 < y < X. \tag{5}$$

• [Type-G1] A subset of Type-G0 networks for which the cost of link 1-2 is independent of the flow of the link, i.e., $t(\cdot) = t$.

• [Type-G2] A subset of Type-G1 networks for which the costs of links 1-3 and 0-2 are independent of the link flow, and, respectively, $b(\cdot) = b$ and $d(\cdot) = d$. If all of x, y, u, v, and w are positive, the following relations hold.

$$C_o = a(x) + b = c(y) + d, \tag{6}$$
$$C_c = a(u+w) + b = c(v+w) + d = b + d - t$$
$$= a(u+w) + t + c(v+w). \tag{7}$$

• [Type-G3] or [**Reduced Braess network**] (Fig. 2) A subset of

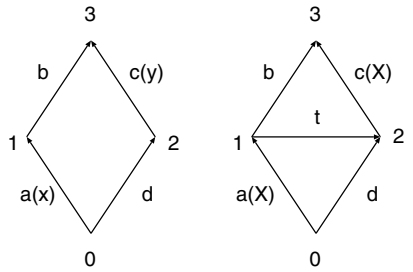

Fig. 2. Reduced Braess (Type-G3) network.

Type-G2 networks for which the following relations hold after the link 1-2 is added:

$$a(X) + t = d \text{ and } c(X) + t = b. \tag{8}$$

Therefore, by noting that $a(\cdot)$ and $c(\cdot)$ are (strictly) increasing, it follows that $u = v = 0$ and $w = X$. Thus,

$$C_c = a(X) + t + c(X) = b + d - t$$
$$= a(X) + b = c(X) + d. \tag{9}$$

In Type-G3 networks, it holds that $0 < x < X$ and $0 < y < X$ (relations (5)). Therefore,

$$C_o = a(x) + b = c(y) + d. \tag{10}$$

Since the cost of any unused path cannot be less than the cost of used paths, the above (relations (5)) is easily seen as follows. Assume, for example, that $x = X$. Then, $y = 0$, and $C_o \le c(0) + d < c(X) + d = C_c = a(X) + b = C_o$, which is a contradiction.

• [Type-G4] or [**Symmetric reduced Braess network**] (Fig. 3) A subset of Type-G3 networks for which $a = c$, $b = d$, and thus $x = y = X/2$ hold.

B. Cohen-Jeffries networks

Cohen and Jeffries [13] considered networks that contain multiple Braess networks, that is, networks that have multiple origin-destination pairs. Consider a general Cohen-Jeffries network that consists of N Braess networks, each of which is a kind

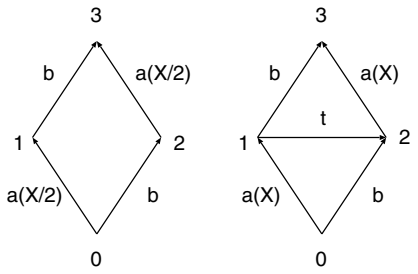

Fig. 3. Symmetric Braess (Type-G4) network.

of general Braess network. These Braess networks are numbered, $1, 2, \cdots, N$, and line in a row from the left to the right (see Figs. 4 and 5). Braess network i in the Cohen-Jeffries network considered in this paper consists of six nodes: one origin (denoted by 0_i), four relay nodes ($1\mathrm{B}_i$, $1\mathrm{T}_i$, $2\mathrm{B}_i$ and $2\mathrm{T}_i$), and one destination (3_i). Before adding capacity (a link), the network has two paths, $0_i - 1\mathrm{B}_i - 1\mathrm{T}_i - 3_i$ and $0_i - 2\mathrm{B}_i - 2\mathrm{T}_i - 3_i$, each of which contains three links, the origin to one relay ($0_i - 1\mathrm{B}_i$ or $0_i - 2\mathrm{B}_i$), one relay to another relay ($1\mathrm{B}_i - 1\mathrm{T}_i$ or $2\mathrm{B}_i - 2\mathrm{T}_i$), and the relay to the destination ($1\mathrm{T}_i - 3_i$ or $2\mathrm{T}_i - 3_i$), respectively. Link $1\mathrm{B}_{i+1} - 1\mathrm{T}_{i+1}$ is identical with link $2\mathrm{B}_i - 2\mathrm{T}_i$. Thus, Braess network i, $i \neq 1, N$, is embedded in such a way that it shares a left-hand-side link $1\mathrm{B}_i - 1\mathrm{T}_i$ with network $i-1$ and a right-hand-side link $2\mathrm{B}_i - 2\mathrm{T}_i$ with network $i+1$.

After adding capacity, i.e., a one-way link connecting two relays ($1\mathrm{B}_i - 2\mathrm{T}_i$), the network has three paths including the new path ($0_i - 1\mathrm{B}_i - 2\mathrm{T}_i - 3_i$) connecting the origin, one relay, another relay, and the destination. Each user flows through one of the paths.

The O-D pair (also numbered i) of Braess network i has incoming total flow X_i. Before adding the link $1\mathrm{B}_i - 2\mathrm{T}_i$, paths $0_i - 1\mathrm{B}_i - 1\mathrm{T}_i - 3_i$ and $0_i - 2\mathrm{B}_i - 2\mathrm{T}_i - 3_i$ have flows x_i and y_i, respectively. After adding the link $1\mathrm{B}_i - 2\mathrm{T}_i$, paths $0_i - 1\mathrm{B}_i - 1\mathrm{T}_i - 3_i$, $0_i - 2\mathrm{B}_i - 2\mathrm{T}_i - 3_i$, and $0_i - 1\mathrm{B}_i - 2\mathrm{T}_i - 3_i$ have flows u_i, v_i, and w_i, respectively.

The costs of links $0_i - 1\mathrm{B}_i$ and $2\mathrm{T}_i - 3_i$ are increasing functions $a_i(\cdot)$ and $c_i(\cdot)$ of the flow of the link, respectively. The costs of links $1\mathrm{B}_i - 1\mathrm{T}_i$, $1\mathrm{T}_i - 3_i$, $0_i - 2\mathrm{B}_i$, and $2\mathrm{B}_i - 2\mathrm{T}_i$ are non-decreasing functions $m_i(\cdot)$, $b_i(\cdot)$, $d_i(\cdot)$, and $m_{i+1}(\cdot)$ of the flow of the link, respectively.

Consider the measure k_i of paradoxical cost degradation for each Braess network i embedded within a general Cohen-Jeffries network as follows:

$$k_i = \frac{C_c^i}{C_o^i}.$$

Two possibile ways of adding capacity (a link) to each Braess network embedded in general Cohen-Jeffries network, are considered. One is that links are added sequentially one at a time (Type-C0). The other is that all the links are added at the same time (Type-C1).

• [Type-C0] or [**General Cohen-Jeffries network with links added one at a time**] (Fig. 4) Denote by $\lambda_o^{m_i}$ and $\lambda_c^{m_i}$ the flows through link m_i of the O-D pair next left to network i before and after the link is added to network i. Denote by $\lambda_o^{m_{i+1}}$ and $\lambda_c^{m_{i+1}}$ the

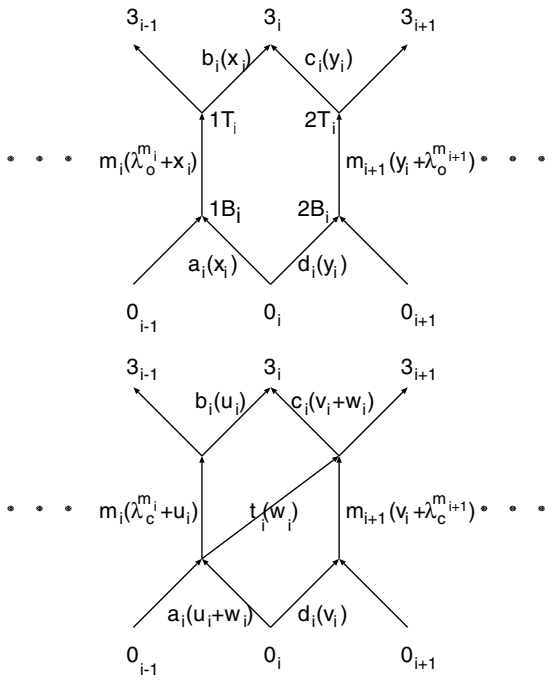

Fig. 4. General Cohen-Jeffries (Type-C0) network with links added one at a time. **Top**: The network before link $1\mathrm{B}_i - 2\mathrm{T}_i$ is added. **Bottom**: The network after link $1\mathrm{B}_i - 2\mathrm{T}_i$ is added.

flows through link m_{i+1} of the O-D pair next right to network i before and after the link is added to network i.

If all of x_i, y_i, u_i, v_i, and w_i are positive, the following relations hold.

$$\begin{aligned} C_o^i &= a_i(x_i) + b_i(x_i) + m_i(x_i + \lambda_o^{m_i}) \\ &= c_i(y_i) + d_i(y_i) + m_{i+1}(y_i + \lambda_o^{m_{i+1}}), \qquad (11) \\ C_c^i &= a_i(u_i{+}w_i) + b_i(u_i) + m_i(u_i + \lambda_c^{m_i}) \\ &= c_i(v{+}w_i) + d(v_i) + m_{i+1}(v_i + \lambda_c^{m_{i+1}}), \\ &= a(u_i{+}w_i) + t_i(w_i) + c_i(v_i{+}w_i). \qquad (12) \end{aligned}$$

• [Type-C1] or [**General Cohen-Jeffries network with all links added at the same time**] (Fig. 5) If all of x_i, y_i, u_i, v_i, and w_i are positive, the following relations hold.

$$\begin{aligned} C_o^i &= a_i(x_i) + b_i(x_i) + m_i(x_i + y_{i-1}) \\ &= c_i(y_i) + d_i(y_i) + m_{i+1}(y_i + x_{i+1}), \qquad (13) \\ C_c^i &= a_i(u_i{+}w_i) + b_i(u_i) + m_i(u_i + v_{i-1}) \\ &= c_i(v_i{+}w_i) + d(v_i) + m_{i+1}(v_i + u_{i+1}), \\ &= a(u_i{+}w_i) + t_i(w_i) + c_i(v_i{+}w_i). \qquad (14) \end{aligned}$$

C. Cohen-Kelly networks

Cohen and Kelly [11] considered a Braess network that has a type of hyperbolic functions expressing exponential single-server queueing delays and constants expressing simple delays for link costs. In fact, networks that are discussed by Cohen and Jeffries [13] consist of networks of this type.

• [Type-N0] or [**General Cohen-Kelly network**] (Fig. 6) A subset of general Braess (Type-G0) networks for which each

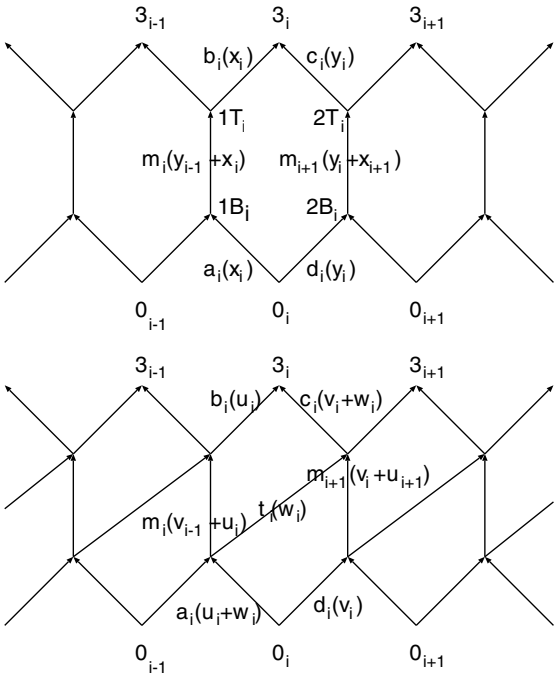

Fig. 5. General Cohen-Jeffries (Type-C1) network with all links added at the same time. **Top**: The network before links $1B_i - 2T_i$ for all i are added. **Bottom**: The network after links $1B_i - 2T_i$ for all i are added.

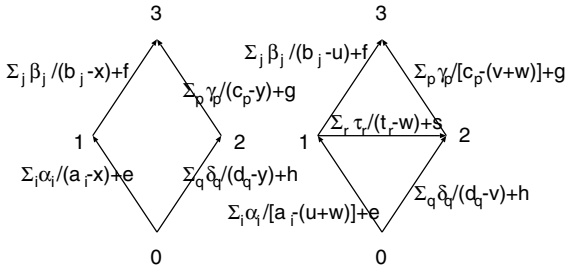

Fig. 6. General Cohen-Kelly (Type-N0) network.

link cost is a type of hyperbolic function and/or constant or a cascade connection of them [11]. The costs of links 0-1, 1-3, 2-3, 0-2, and 1-2 are expressed by $\sum_i \alpha_i/(a_i - \lambda_{01}) + e$, $\sum_j \beta_j/(b_j - \lambda_{13}) + f$, $\sum_p \gamma_p/(c_p - \lambda_{23}) + g$, $\sum_q \delta_q/(d_q - \lambda_{02}) + h$, and $\sum_r \tau_r/(t_r - \lambda_{12}) + s$, respectively, where λ_{ij} denotes the flow of link i-j. Each link cost is defined for $a_i > \lambda_{01}$, $b_j > \lambda_{13}$, $c_k > \lambda_{23}$, $d_l > \lambda_{02}$, and $t_r > \lambda_{12}$, and otherwise infinite.

• [Type-N1] (Fig. 7) Networks that are both Type-G3 and Type-N0, and for which the costs of links 0-1 and 2-3 are certain nonlinear (a type of hyperbolic) functions and are, respectively, $a(x) = \sum_i \alpha_i/(a_i - x) + e$ and $c(y) = \sum_p \gamma_p/(c_p - y) + g$ for

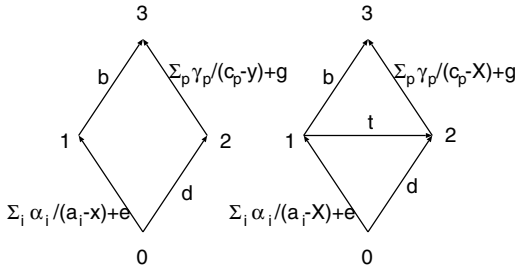

Fig. 7. Type-N1 network.

$0 \le x < a$ and $0 \le y < c$. Otherwise, both are infinite.

$$\begin{aligned} C_o &= \sum_i \frac{\alpha_i}{a_i - x} + e + b \\ &= \sum_p \frac{\gamma_p}{c_p - y} + g + d, \end{aligned} \tag{15}$$

$$\begin{aligned} C_c &= \sum_i \frac{\alpha_i}{a_i - X} + e + b = \sum_p \frac{\gamma_p}{c_p - X} + g + d \\ &= \sum_i \frac{\alpha_i}{a_i - X} + e + t + \sum_p \frac{\gamma_p}{c_p - X} + g \\ &= b + d - t. \end{aligned} \tag{16}$$

• [Type-N2] (Fig. 8) A subset of Type-N1 networks for which

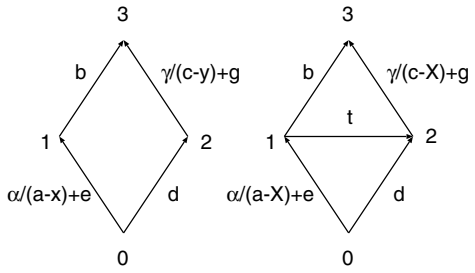

Fig. 8. Type-N2 network.

the costs of links 0-1 and 2-3 are, respectively, $a(x) = \alpha/(a - x) + e$ and $c(y) = \gamma_p/(c_p - y) + g$ for $0 \le x < a$ and $0 \le y < c$. Otherwise, both are infinite. As in Type-G3 networks, both x and y are positive and less than X, and the following relations hold.

$$C_o = \frac{\alpha}{a - x} + e + b = \frac{\gamma}{c - y} + g + d, \tag{17}$$

$$\begin{aligned} C_c &= \frac{\alpha}{a - X} + e + b = \frac{\gamma}{c - X} + g + d = b + d - t \\ &= \frac{\alpha}{a - X} + e + t + \frac{\gamma}{c - X} + g. \end{aligned} \tag{18}$$

• [Type-N3] or [**Reduced Cohen-Kelly network**] (Fig. 9) A subset of Type-N1 networks for which $e = g = 0$. As in Type-G3 networks, both x and y are positive and less than X, and the

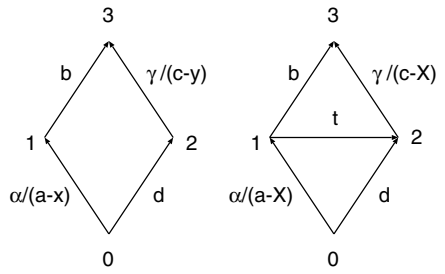

Fig. 9. Reduced Cohen-Kelly (Type-N3) network.

following relations hold.

$$C_o = \frac{\alpha}{a-x} + b = \frac{\gamma}{c-y} + d, \tag{19}$$

$$C_c = \frac{\alpha}{a-X} + b = \frac{\gamma}{c-X} + d = b + d - t$$
$$= \frac{\alpha}{a-X} + t + \frac{\gamma}{c-X}. \tag{20}$$

Cohen and Kelly [11] considered a network of this type for which $\alpha = \gamma = 1$, $a = c = \phi$, $b = d = 2$, $t = 1$, and $X = 2\lambda$, which is actually symmetric while some equalities in (20) may not hold. They showed that $C_c = 3$ and $C_o = 1/(\phi - \lambda) + 2 < 3$ assuming that $2\lambda > \phi - 1 > \lambda > 0$, which is a Braess paradox.

• [Type-N4] or [**Symmetric reduced Cohen-Kelly network**] (Fig. 10) A subset of Cohen-Kelly (Type-N3) networks for

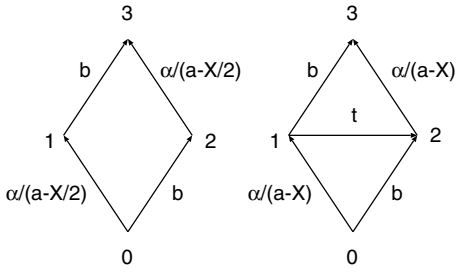

Fig. 10. Symmetric reduced Cohen-Kelly (Type-N4) network.

which $a = c$, $b = d$, and thus $x = y = X/2$, that is, the networks are symmetric.

III. The Results

A. General Braess networks

The properties stated in this subsection are mostly given in [18], but since the results of this paper are based on these properties, the outlines of the proofs are also given here. The most important property is given in Proposition 3, and all lemmas and most propositions in this subsection are required step-by-step in finally establishing this Proposition 3. k^X denotes the value of k for a type-X network where X is either of G0, G1, *etc.*

Proposition 1: The Braess paradox, *i.e.,* paradoxical cost degradation, always occurs for reduced Braess (Type-G3) networks. ($k^{G3} > 1$)

Proof: It is proved by noting

$$k^{G3} = \frac{C_c}{C_o} = \frac{a(X) + c(X) + t}{a(x) + c(X) + t} = \frac{a(X) + c(X) + t}{c(y) + a(X) + t} \tag{21}$$
$$> 1.$$

■

Lemma 1: The value, k^{G3}, of the measure of paradoxical cost degradation for every reduced Braess (Type-G3) network is less than 2. ($k^{G3} < 2$)

Proof: From (8) and (10),

$$a(x) + c(X) = c(y) + a(X). \tag{22}$$

Then, this lemma can be proved by noting that

$$k^{G3} = \frac{a(X) + c(X) + t}{a(x) + c(X) + t} = \frac{a(x) + 2c(X) - c(y) + t}{a(x) + c(X) + t}$$
$$< \frac{2c(X) - c(y)}{c(X)} < 2.$$

■

Lemma 2: For any Type-G1 network, it holds that $x \geq u$, $y \geq v$, $x \leq u + w$, and $y \leq v + w$.

Proof: Although this can be proved independently, this happens to be a special case ($N = 1$) of Lemma 7 given in the next subsection III-B. ■

Lemma 3: For any Type-G0 network, there exists a Type-G1 network that has the same measure k as the Type-G0 network has.

Proof: C_o is independent of the cost of link 1-2. Thus even if the cost of link 1-2 is replaced by a new constant cost $t = t(w)$, the value of measure k does not change. ■

Lemma 4: For any Type-G1 network, there exists a Type-G2 network that has a value of measure k that is not less than that of the Type-G1 network.

Proof: 1) It has been shown in Lemma 2 that $x \geq u$, $y \geq v$, $x \leq u + w$, and $y \leq v + w$.

2) Note that if the link costs $b(\cdot)$ and $d(\cdot)$ are replaced by constants b (= $b(u)$) and d (= $d(v)$), respectively, the equilibrium flows of the network with link 1-2 are the same as u, v, and w before this replacement.

Then, make a Type-G2 network by replacing the link costs $b(\cdot)$ and $d(\cdot)$ by constants b and d, respectively. Let the flows of paths 0-1-3 and 0-2-3 be $\hat{x}$ and $\hat{y}$ before adding link 1-2 to the Type-G2 network, respectively. Note that

$\hat{x} + \hat{y} = x + y = X$. Therefore, either $\hat{x} \leq x$ or $\hat{y} \leq y$ holds. Then,

$a(\hat{x}) \leq a(x)$ or $c(\hat{y}) \leq c(y)$.

3) Denote by $\hat{C_o}$ the value of C_o of the Type-G2 network. By noting the above 2) and by recalling that $b = b(u) \leq b(x)$ and $d = d(v) \leq d(y)$ it holds that

$$\hat{C_o} = a(\hat{x}) + b = c(\hat{y}) + d$$
$$\leq a(x) + b(x) = c(y) + d(y) = C_o.$$

4) Recall that the value of C_c is the same for both the Type-G1 and Type-G2 networks. It has thus been shown that the measure

k of the Type-G2 network is not less than that of the Type-G1 network. ■

Lemma 5: In the Type-G2 network, C_o is increasing in b and d.

Proof: This can be proved by contradiction. Suppose C_o would not increase even if b increases. Then, since $C_o = c(y)+d$, y should not increase, which should imply that $x = X - y$ should not decrease. Thus $C_o = a(x) + b$ should increase because x should not decrease and b increases, which should be a contradiction. Similarly for d. Therefore, C_o is increasing in b and d. ■

Lemma 6: For any Type-G2 network, there exists a Type-G3 network that has a value of measure k that is not less than that of the Type-G2 network.

Proof: Clearly, it is sufficient to show the case where the original Type-G2 network has the paradox, *i.e.,* $k^{G2} > 1$, since it has been already shown by Proposition 1 that Type-G3 networks have the paradox, *i.e.,* $k^{G3} > 1$.

The sketch of a proof is given as follows:

1) Show that if a Type-G2 network has $b > c(X) + t$, there exists a new Type-G2 network with $b = c(X) + t$ for which the value, k^{G2}_{new}, of k is not less than the value, k^{G2}, of the original Type-G2 network with $b > c(X) + t$. Similarly for Type-G2 networks with $d > a(X) + t$. Therefore, for every Type-G2 network, there exists a new Type-G2 network with $b \le c(X) + t$ and $d \le a(X) + t$ for which the value, k^{G2}_{new}, of k is not less than the value, k^{G2}, of the original Type-G2 network. ($k^{G2} \le k^{G2}_{new}$)

2) Show that if a Type-G2 network has $b \le c(X) + t$ and $d \le a(X) + t$, there exists a Type-G3 network, *i.e.,* with $b = c(X) + t$ and $d = a(X) + t$, for which the value, k^{G3}, of k is not less than the value, k^{G2}_{new}, of the Type-G2 network. ($k^{G2}_{new} \le k^{G2}$)

3) Give the procedure of obtaining, from a given Type-G2 network, a Type-G3 network for which the value, k^{G3}, of k is not less than the value, k^{G2}, of the given Type-G2 network, on the basis of steps 1) and 2). ($k^{G2} \le k^{G2}_{new} \le k^{G3}$) ■

From Lemmas 3, 4, and 6, the following proposition is derived.

Proposition 2: For any general Braess (Type-G0) network, there exists a reduced Braess (Type-G3) network that has a value of measure k that is not less than that of the general Braess network.

Proposition 3: The measure of paradoxical cost degradation of general Braess (Type-G0) networks is less than 2.

Proof: Follows from Proposition 2 and Lemma 1. ■

B. General Cohen-Jeffries networks

The following properties hold on general networks in Wardrop equilibria under the condition that every link cost function is non-decreasing and dependent only on the flow through the link.

[Property I] If one link in the network is improved while the rest remain unchanged, the load on the link cannot decrease and the incurred link cost cannot increase. If one link in the network is degraded while the rest remain unchanged, the load on the link cannot increase and the incurred link cost cannot decrease.

Proof: This property is given as Corollary 3.1 of Dafermos and Nagurney [10] with a stronger assumption on the monotonicity of link cost functions than what can be used in this paper. A proof to this property can be derived by modifying the proof of Dafermos and Nagurney [10] so that the assumption may be used in this paper. ■

[Property II] If only the total flow associated with a particular O-D pair is increased, while the rest remain the same, then the cost of the paths associated with the O-D pair can never decrease.

Proof: This can be easily seen from the Corollary 4.1 of [10]. ■

[Property I'] Consider the case where one link is changed (improved or degraded) while the rest remain unchanged. If the cost of the link decreases because of the change, the load on the link cannot decrease. If the cost of the link increases because of the change, the load on the link cannot increase.

Proof: From Property I, it is easily seen that, if the cost of a link decreases (and increases) due to the change of the link while the rest remain unchanged, then it must hold that the link is improved (and degraded). Then this property can be proved by applying Property I. ■

This subsection discusses general Cohen and Jeffries networks [13] that contain multiple Braess networks, that is, networks that have multiple origin-destination pairs.

Lemma 7: Consider adding a link to Braess network i embedded within an arbitrary general Cohen-Jeffries network while the other parts of the entire network remain unchanged (Type-C0). For the embedded Braess network i, it holds that $x_i \ge u_i$ and $y_i \ge v_i$, and thus $x_i \le u_i + w_i$, and $y_i \le v_i + w_i$.

Proof: Consider an arbitrary Braess network i. It can be shown that $x_i \ge u_i$ by contradiction as follows: Assume $x_i < u_i$. Since $x_i + y_i = X_i = u_i + v_i + w_i$, then $x_i < u_i + w_i$, $y_i > v_i + w_i$, and $y_i > v_i$.

By adding the link to Braess network i, for the network consisting of sub-networks $1, 2, \cdots, i-1$, only the cost function of link $1\mathrm{B}_i - 1\mathrm{T}_i$ is degraded since $x_i < u_i$ by assumption, and the rest remain unchanged. Thus by applying Property I to the network, the incurred link cost cannot decrease, that is

$m_i(x_i + \lambda_o^{m_i}) \le m_i(u_i + \lambda_c^{m_i})$, from which, due to the monotonicity of the cost function $m_i(\cdot)$, follows,

$x_i + \lambda_o^{m_i} \le u_i + \lambda_c^{m_i}$.

Also, for the network consisting of sub-networks $i+1, i+2, \cdots, N$, only the cost function of link $2\mathrm{B}_{i+1} - 2\mathrm{T}_{i+1}$ is improved by adding the link to Braess network i, since $y_i > v_i$ as derived by assumption, and the rest remain unchanged. Thus by noting Property I, similarly as above,

$y_i + \lambda_o^{m_{i+1}} \ge v_i + \lambda_c^{m_{i+1}}$.

Then, due to the monotonicity of link cost functions $a_i(\cdot), b_i(\cdot), m_i(\cdot), c_i(\cdot), d_i(\cdot)$, and $m_{i+1}(\cdot)$,

$C_o^i = a_i(x_i) + b_i(x_i) + m_i(x_i + \lambda_o^{m_i}) < a_i(u_i + w_i) + b_i(u_i) + m_i(u_i + \lambda_c^{m_i}) = C_c^i = c_i(v_i + w_i) + d_i(v_i) + m_{i+1}(v_i + \lambda_c^{m_{i+1}}) < c_i(y_i) + d_i(y_i) + m_{i+1}(y_i + \lambda_o^{m_{i+1}}) = C_o^i$, which is impossible. Therefore $x_i \ge u_i$ and thus $y_i \le v_i + w_i$. By noting Property I as was used above, $\lambda_o^{m_i} \le \lambda_c^{m_i}$. Similarly $y_i \ge v_i$ and thus $x_i \le u_i + w_i$. ■

Proposition 4: Consider adding a link to Braess network i embedded within an arbitrary general Cohen-Jeffries network while the other parts of the entire network remain unchanged (Type-C0). There exists an isolated Braess network that has the same link cost functions a_i, b_i, c_i, d_i, t_i, $\hat{m}_i$ and $\hat{m}_{i+1}$ each of which has the same value as the corresponding link cost function of the embedded network i (both before and after adding the link). It can be considered that all these functions depend only on the flow through the link. a_i and b_i are increasing and c_i, d_i, t_i, $\hat{m}_i$ and $\hat{m}_{i+1}$ are non-decreasing, and this isolated network i is regarded as a general Braess (Type-G0) network.

Proof: It needs to say nothing about a_i, b_i, c_i, d_i, and t_i. As to $\hat{m}_i$ and $\hat{m}_{i+1}$, consider a sub-network that consists of embedded Braess networks $1, 2, \cdots, i-1$ as one network and consider the effect of the flow λ through link $1\mathrm{B}_i - 1\mathrm{T}_i$ from the O-D pair i. Then by Property I, the cost function m_i of link $1\mathrm{B}_i - 1\mathrm{T}_i$ of the embedded network i can be regarded as a non-decreasing function of the flow λ by the O-D pair i. Thus, $\hat{m}_i(\lambda) = m_i(\lambda + \lambda^{m_i})$ is non-decreasing in λ where λ is the flow through link $1\mathrm{B}_i - 1\mathrm{T}_i$ by O-D pair i (or isolated network i) and λ^{m_i} denotes the flow by O-D pair $i-1$.

Similarly for the link cost functions m_{i+1}, and $\hat{m}_{i+1}(\lambda) = m_i(\lambda + \lambda^{m_{i+1}})$ is non-decreasing in λ where λ is the flow through link $2\mathrm{B}_i - 2\mathrm{T}_i$ by O-D pair i (or isolated network i) and $\lambda^{m_{i+1}}$ denotes the flow by O-D pair $i+1$. ■

Then, from Propositions 3 and 4, the following property is derived:

Theorem 1: Consider adding a link to a Braess network i embedded within an arbitrary general Cohen-Jeffries network while the other parts of the entire network remain unchanged (Type-C0). The measure of paradoxical cost degradation for the embedded Braess network is less than 2.

Lemma 8: Consider adding a link to each Braess network embedded within an arbitrary general Cohen-Jeffries network at the same time (Type-C1). Assume that Braess paradox occurs for all the embedded networks. For the embedded Braess network i, it holds that $x_i \geq u_i$ and $y_i \geq v_i$, and thus $x_i \leq u_i + w_i$, and $y_i \leq v_i + w_i$ for all i.

Proof: Consider an arbitrary Braess network i. It can be shown that $x_i \geq u_i$ for arbitrary i by contradiction as follows:

(i) Suppose that $x_i < u_i$ holds for $i < N$. Then it must hold that $x_{i+1} < u_{i+1}$. This is shown as follows: Since $x_i + y_i = X_i = u_i + v_i + w_i$, then $x_i < u_i + w_i$, $y_i > v_i + w_i$, and $y_i > v_i$. If $x_{i+1} \geq u_{i+1}$ then $C_c^i = c_i(v_i + w_i) + d_i(v_i) + m_{i+1}(v_i + u_{i+1}) < c_i(y_i) + d_i(y_i) + m_{i+1}(y_i + x_{i+1}) = C_o^i$, which is against the assumption $C_o^i < C_c^i$. Therefore it must hold that $x_{i+1} < u_{i+1}$.

(ii) Assume $x_i < u_i$ holds for $i = N$. This leads to contradiction as follows: Since $x_i + y_i = X_i = u_i + v_i + w_i$, then $x_i < u_i + w_i$, $y_i > v_i + w_i$, and $y_i > v_i$ for $i = N$. Thus, $C_c^i = c_i(v_i + w_i) + d_i(v_i) + m_{i+1}(v_i) < c_i(y_i) + d_i(y_i) + m_{i+1}(y_i) = C_o^i$, which is against the assumption $C_o^i < C_c^i$.

Therefore, if it is assumed that $x_i < u_i$ for some i, it is led to a contradiction. Thus, $x_i \geq u_i$ for arbitrary i. Similarly, $y_i \geq v_i$ for arbitrary i. ■

Proposition 5: Consider adding links to all Braess networks embedded within an arbitrary general Cohen-Jeffries network at the same time (Type-C1). Assume that the Braess paradox occurs for all the embedded Braess networks. Then, for each of the Braess networks embedded, there exists an isolated Braess network that has a value of measure k that is not less than that of the Braess network embedded.

Proof: From the above lemma, $x_i \geq u_i$ and thus $y_i \leq v_i + w_i$. Similarly $y_i \geq v_i$ and thus $x_i \leq u_i + w_i$ for arbitrary i.

1) Consider an arbitrary isolated Braess network i that has link cost functions $m_i(\lambda + v_{i-1})$ and $m_{i+1}(\lambda' + u_{i+1})$ where λ and λ' denote flows by O-D pair i. Clearly the isolated network i has the same value of C_c^i as the embedded network i.

2) On the other hand, note from the above lemma that $y_{i-1} \geq v_{i-1}$ and $x_{i+1} \geq u_{i+1}$. Consider an arbitrary isolated Braess network i' that has link cost functions $m_i(\lambda + y_{i-1})$ and $m_{i+1}(\lambda' + x_{i+1})$ where λ and λ' denote flows by O-D pair i. Clearly the isolated network i' has the same value of C_o^i as the embedded network i.

3) Note from the above lemma that $y_{i-1} \geq v_{i-1}$ and $x_{i+1} \geq u_{i+1}$. Thus, the isolated network i has the cost functions of links $1\mathrm{B}_i - 1\mathrm{T}_i$ and $2\mathrm{B}_i - 2\mathrm{T}_i$ both improved over those of the isolated network i'. Then, it is shown that the value of C_o^i for the isolated network i is not greater than the value of $C_o^{i'}$ for the isolated network i' and the embedded sub-network i, by contradiction as follows.

Suppose $C_o^{i'} < C_o^i$. Denote the values of x_i and y_i for the isolated network i by x_i^{iso} and y_i^{iso}, respectively. Note that the isolated network i' has values x_i and y_i before adding the link. Due to the improvement and monotonicity of link cost functions and assumption $C_o^{i'} < C_o^i$, it must hold that $x_i^{iso} < x_i$ and $y_i^{iso} < y_i$, which is impossible since $x_i + y_i = X_i = x_i^{iso} + y_i^{iso}$. Therefore, $C_o^{i'} \geq C_o^i$. Thus, before adding the link, the isolated network i has the value of C_o^i that is not greater than that of the embedded Braess network i.

4) It is noted in 1) that, after adding the link, the isolated network i has the same value of C_c^i as the last value of C_c^i shown above. On the other hand, it is noted in 3) that, before adding the link, the isolated network i has the value of C_o^i that is not greater than that of the embedded Braess network i. Therefore the measure $k = C_c^i / C_o^i$ of paradoxical cost degradation for the embedded network is not greater than that for the isolated network. ■

Then, from Proposition 3 and 5, the following property is derived:

Theorem 2: Consider adding links to all Braess networks embedded within an arbitrary general Cohen-Jeffries network at the same time (Type-C1). Assume that the Braess paradox occurs for all the embedded Braess networks. The measure of paradoxical cost degradation for every embedded Braess network is less than 2.

C. Cohen-Kelly networks

This subsection discusses Cohen-Kelly networks, which have a type of hyperbolic functions and/or constants for their link costs [11]. From Proposition 2, the following corollary is derived:

Corollary 1: For any general Cohen-Kelly (Type-N0) network, there exists a Type-N1 network that has a value of mea-

sure k that is not less than that of the general Cohen-Kelly network.

Lemma 9: For any Type-N1 network, there exists a Type-N2 network that has a value of measure k that is not less than that of the Type-N1 network.

Proof: 1) Consider a Type-N1 network. Recall that $a(x) = \sum_i \alpha_i/(a_i - x) + e$ and $c(y) = \sum_p \gamma_p/(c_p - y) + g$.

Let $\grave{a} = \min_i a_i$ and $\grave{c} = \min_p c_p$. Then define $\check{\alpha}_i$ and $\check{\gamma}_p$ such that

$$\frac{\check{\alpha}_i}{\grave{a} - X} = \frac{\alpha_i}{a_i - X} \text{ and } \frac{\check{\gamma}_p}{\grave{c} - X} = \frac{\gamma_p}{c_p - X}. \tag{23}$$

Thus, since $x, y < X$, by noting that $\grave{a} \le a_i$ and $\grave{c} \le c_p$, for arbitrary i and p,

$$\frac{a_i - x}{\grave{a} - x} \le \frac{a_i - X}{\grave{a} - X}, \text{ and } \frac{c_i - y}{\grave{c} - y} \le \frac{c_p - X}{\grave{c} - X}, \tag{24}$$

Then, clearly, combining (23) and (24),

$$\frac{\alpha_i}{a_i - x} \ge \frac{\check{\alpha}_i}{\grave{a} - x} \text{ and } \frac{\gamma_p}{c_p - y} \ge \frac{\check{\gamma}_p}{\grave{c} - y}. \tag{25}$$

2) Define α and γ such that $\alpha = \sum_i \check{\alpha}_i$ and $\gamma = \sum_p \check{\gamma}_p$, respectively.

Then make a Type-N2 network for which $a(\lambda_{01}) = \sum_i \alpha_i/(a_i - \lambda_{01}) + e$ and $c(\lambda_{23}) = \sum_p \gamma_p/(c_p - \lambda_{23}) + g$ of the Type-N1 network are replaced by $\check{a}(\lambda_{01}) = \alpha/(\grave{a} - \lambda_{01}) + e$ and $\check{c}(\lambda_{23}) = \gamma/(\grave{c} - \lambda_{23}) + g$ in the Type-N2 network, respectively. Then, by noting the above definitions and inequalities,

$$a(X) = \check{a}(X), c(X) = \check{c}(X), a(x) \ge \check{a}(x), c(y) \ge \check{c}(y). \tag{26}$$

After adding the link 1-2, this Type-N2 network has the same value of C_c as the Type-N1 network. On the other hand, before adding link 1-2, the flows λ_{01} and λ_{23} of the Type-N1 network may be different from the flows λ_{01} and λ_{23} of the Type-N2 network that are denoted by $\check{x}$ and $\check{y}$, respectively.

3) It can be shown by contradiction that both of $a(x) \ge \check{a}(\check{x})$ and $c(y) \ge \check{c}(\check{y})$ hold, as follows. Note that $a(x) = \sum_i \alpha_i/(a_i - x) + e$ and $c(y) = \sum_p \gamma_p/(c_p - y) + g$ in the Type-N1 network are replaced by $\check{a}(\check{x}) = \sum_i \check{\alpha}_i/(\grave{a} - \check{x}) + e$ and $\check{c}(\check{y}) = \sum_p \check{\gamma}_p/(\grave{c} - \check{y}) + g$. Then, since the cost of every path used in a network is identical, $a(x) - \check{a}(\check{x}) = c(y) - \check{c}(\check{y})$. Therefore, either both of $a(x) \ge \check{a}(\check{x})$ and $c(y) \ge \check{c}(\check{y})$ hold, or both of $a(x) < \check{a}(\check{x})$ and $c(y) < \check{c}(\check{y})$ hold. Assume that both of $a(x) < \check{a}(\check{x})$ and $c(y) < \check{c}(\check{y})$ hold. From (26), $\check{a}(x) < \check{a}(\check{x})$ and $\check{c}(y) < \check{c}(\check{y})$, which contradict each other since $\check{a}$ and $\check{c}$ are (strictly) increasing and $x + y = \check{x} + \check{y} = X$.

4) Since $a(x) \ge \check{a}(\check{x})$ and $c(y) \ge \check{c}(\check{y})$ hold, before adding link 1-2, the cost C_o of the Type-N1 network is not less than the cost $\check{C}_o$ of the Type-N2 network while the costs, C_c and $\check{C}_c$, of both networks are identical after adding link 1-2. ∎

Lemma 10: For any Type-N2 network, there exists a reduced Cohen-Kelly (Type-N3) network that has a value of measure k that is not less than that of the Type-N2 network.

Proof: Consider the case where $g > 0$ in a Type-N2 network. Note that $b \ge g$ since, from the definition of Type-N2 networks, $b = \gamma/(c - X) + g + t$ holds in the network with link 1-2. Then consider a Type-N3 network that has, as the costs of links 1-3 and 2-3, $b - g$ and $\gamma/(c - X)$ replacing b and $\gamma/(c - X) + g$ of the Type-N2 network. It is clear that the link flows of the network before and after adding link 1-2 remain the same in both the Type-N2 and Type-N3 networks.

By noting $x \le X$, however, it holds that

$$k^{Type-N3} = \frac{a(X) + e + b - g}{a(x) + e + b - g} \ge \frac{aX + e + b}{ax + e + b} = k^{Type-N2}.$$

Thus the newType-N3 network has a value of measure k not less than the old Type-N2 network. Similar arguments hold for the links 0-1 and 0-2. ∎

From Corollary 1 and Lemmas 9 and 10, the following property is derived.

Theorem 3: For any general Cohen-Kelly network, there exists a reduced Cohen-Kelly (Type-N3) network that has a value of measure k that is not less than that of the general Cohen-Kelly network.

Remark 1: This property implies that a complication of Cohen-Kelly networks would add nothing in making the ratio of paradoxical cost degradation worse than that of a simple Type-N3 network of the original Cohen-Kelly type [11].

Theorem 4: The worst value, 2, of the measure k of paradoxical cost degradation due to adding link 1-2 in general Braess networks is asymptotically reached in symmetric reduced Cohen-Kelly (Type-N4) networks.

Proof: Consider a symmetric reduced Cohen-Kelly network with $t = 0$. By noting that $b = \alpha/(a - X)$,

$$k = \frac{C_c}{C_o} = \frac{\dfrac{2\alpha}{a - X}}{\dfrac{\alpha}{a - X/2} + \dfrac{\alpha}{a - X}} = \frac{2}{1 + \dfrac{a - X}{a - X/2}}.$$

It can be easily seen, therefore, that, in the symmetric reduced Cohen-Kelly network, the measure of paradoxical cost degradation k is less than but close to 2 as X approaches a with $t = 0$. ∎

Remark 2: The above implies that the worst value, 2, is asymptotically reached in symmetric reduced Cohen-Kelly (Type-N4) networks with asymptotically infinite link costs.

IV. Concluding Remarks

The present paper has investigated networks generalized from what were studied by Cohen, Kelly, and Jeffries in comparison with the networks of the same topology as the original Braess network. The measure of cost degradation considered is the ratio of the cost for each user of the network after adding capacity (a link) to that before adding capacity.

On one hand, all of the Braess-type networks in Wardrop equilibria have the degradation ratio less than 2. The worst value, 2, cannot be surpassed even by networks with asymptotically infinite link costs. (In computer networking, some shortest path routing protocols may bring about situations close to Wardrop equilibria. Thus, the above-mentioned situations regarding paradoxical cost degradation by adding capacity may

appear in computer networking.) On the other hand, as to networks in Nash equilibria, there exist cases of Braess-like paradoxical cost degradation the measure of which can increase without bounds.

Therefore, it seems important to continue research on this type of paradoxes.

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