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CHARACTER TABLES OF PARABOLIC SUBGROUPS OF STEINBERG'S TRIALITY GROUPS

FRANK HIMSTEDT

ABSTRACT. We compute the conjugacy classes of elements and the character tables of the parabolic subgroups of Steinberg's triality groups ${}^3D_4(q)$, where q is a power of an odd prime. Our main tools are Clifford theory applied to the Levi decomposition of the parabolic subgroups and the decomposition of restrictions of the unipotent characters of ${}^3D_4(q)$. For many of the calculations we use the CHEVIE–package.

1. INTRODUCTION

Let ${}^3D_4(q)$ be Steinberg's simple triality group defined over a finite field with $q = p^n$ elements, where p is a prime number and n is a positive integer. In [17] N. Spaltenstein computed the values of the eight unipotent irreducible characters of ${}^3D_4(q)$, and in [4] D.I. Deriziotis and G.O. Michler determined the remaining irreducible characters of ${}^3D_4(q)$.

In this paper we construct the irreducible characters of the proper parabolic subgroups of ${}^3D_4(q)$ in the case $p > 2$. Some of our methods are similar to those used by H. Enomoto and H. Yamada in [6]. We start by computing the characters of a Borel subgroup B of ${}^3D_4(q)$. The conjugacy classes of B together with their fusions in ${}^3D_4(q)$ have already been determined by M. Geck in [8]. We construct all irreducible characters of B by inducing linear characters from suitable subgroups of B.

Then, we make up the character tables of two maximal parabolic subgroups P and Q containing B. First, we compute the conjugacy classes of the elements of P and Q using the fusions of the conjugacy classes of B in ${}^3D_4(q)$. Next, we construct the irreducible characters of P and Q using the following two methods.

First, we compute the orbits of P and Q on the irreducible characters of their unipotent radicals and determine the corresponding inertia subgroups. It turns out that both for P and for Q the number of these orbits does not depend on q. By inducing suitable characters of the inertia subgroups we obtain irreducible characters of P and Q. Secondly, we restrict the unipotent characters of ${}^3D_4(q)$ to these two maximal parabolic subgroups and split off irreducible constituents.

Since up to conjugacy B, P, Q and ${}^3D_4(q)$ are the only parabolic subgroups of ${}^3D_4(q)$, this paper completes the task of determining the character tables of the parabolic subgroups of ${}^3D_4(q)$ in the case $p > 2$.

The applications we have in mind of the character tables of B, P and Q are to the study of the modular representation theory of the triality groups. In [9], M. Geck has determined the decomposition numbers of ${}^3D_4(q)$ in non–defining characteristic up to a few numbers in the principal block. Using methods like T. Okuyama and

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K. Waki in [15], the character tables in this paper might be helpful in proving new results about these numbers. A first step in this direction is given in [11].

We have implemented the character tables of B, P and Q in the format of the Maple[3] part of the CHEVIE [10] package and have written Maple–programs based on CHEVIE for restricting and inducing class functions. The use of CHEVIE allows us to compute easily scalar products of class functions and provides good tests for the obtained character tables. For calculations with elements of ${}^3D_4(q)$ we use computer programs written by C. Köhler and the author in the language of the CHEVIE package of GAP[7]. These programs are based on the commutator relations and the relations describing the action of a maximally split torus and the Weyl group of ${}^3D_4(q)$ on the root subgroups given in Tables 2.2–2.4 in [9].

The conjugacy classes and character tables of the parabolic subgroups of ${}^3D_4(q)$ are summarized in tabular form in an appendix at the end of this paper.

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2. Notation and Group Theoretical Properties of ${}^3D_4(q)$

We choose the notation similar to that in M. Geck [9]. Let Φ be a root system of type D_4 in some Euclidean space V, with basis $\Delta = \{r_1, r_2, r_3, r_4\}$ of simple roots such that r_1, r_3 and r_4 are orthogonal to each other, and let L be a simple complex Lie algebra of type D_4. We fix a total ordering on Φ: Given $r = \sum_{i=1}^4 n_i r_i$, $s = \sum_{i=1}^4 m_i r_i$, $(r, s \in \Phi)$, we define $r < s$ if the first non–vanishing difference $m_i - n_i$ is positive. Those roots $r = \sum_{i=1}^4 n_i r_i \in \Phi$ with $n_i \geq 0$ for $i = 1, \ldots, 4$ are the positive roots. We choose a Chevalley basis $\{h_r | r \in \Delta\} \cup \{e_r | r \in \Phi\}$ of L in the same way as in [9]. In particular, we choose the sign of the integers N_{rs} in $[e_r, e_s] = N_{rs} e_{r+s}$ for all extraspecial pairs of roots $(r, s) \in \Phi \times \Phi$ in the same way as in [9], Table 2.1 (see also [1], Section 4.2).

Let q be a power of an odd prime p, $\mathbb{F}_q$ the field with q elements, $\mathbb{F}$ its algebraic closure and $\mathbf{G} = \langle x_r(t) | r \in \Phi, t \in \mathbb{F} \rangle$ the Chevalley group over the field $\mathbb{F}$ constructed from the Lie algebra L. Then $\mathbf{G}$ is a simple adjoint algebraic group, defined over $\mathbb{F}_q$. Let ρ be the isometry of Φ which is given by $r_1 \mapsto r_3$, $r_3 \mapsto r_4$, $r_4 \mapsto r_1$, $r_2 \mapsto r_2$ and linear extension. $\mathbf{G}$ has a graph automorphism (which is also denoted by ρ) sending $x_r(t)$ to $x_{\rho(r)}(t)$, and a field automorphism F_q sending $x_r(t)$ to $x_r(t^q)$ for all $r \in \Phi$ and $t \in \mathbb{F}$. Setting $F := \rho \circ F_q = F_q \circ \rho$ we get $\mathbf{G}^F = {}^3D_4(q)$.

Fix an F–stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ as in [9], normalizing all root subgroups $X_r = \{x_r(t) | t \in \mathbb{F}\}$ of $\mathbf{G}$. Each element $h \in \mathbf{T}$ can be written as a product $h = h(t_1, t_2, t_3, t_4) = \prod_{i=1}^4 h_{r_i}(t_i)$ with $t_i \in \mathbb{F} - \{0\}$ (see [1], Section 7.1). Then F acts on $\mathbf{T}$ by $F(h(t_1, t_2, t_3, t_4)) = h(t_4^q, t_2, t_1^q, t_3^q)$ and $\mathbf{T}^F$ consists precisely of the elements $h(t_1, t_2) := h(t_1, t_2, t_1^q, t_1^{q^2})$ with $t_1^{q^3-1} = t_2^{q-1} = 1$.

For $r \in \Phi$ let $r^1 := \frac{1}{3}(r + \rho(r) + \rho^2(r))$ and call $r, s \in \Phi$ equivalent, if $r^1 = s^1$. Each equivalence class consists either of one root fixed by ρ or of three roots cyclically permuted by ρ. Define $\alpha := \frac{1}{3}(r_1 + r_3 + r_4)$ and $\beta := r_2$. The set $\Phi^1 := \{r^1 | r \in \Phi\}$ is a root system of type G_2 (in the Euclidean subspace of V spanned by α and β) with basis $\{\alpha, \beta\}$ (α is the short root). For $r \in \Phi$, $t \in \mathbb{F}$ let

$$x_{r^1}(t) := \begin{cases} x_r(t) & \text{, if } \rho(r) = r\, ,\ t^q = t, \\ x_r(t)x_{\rho(r)}(t^q)x_{\rho^2(r)}(t^{q^2}) & \text{, if } \rho(r) \neq r\, ,\ t^{q^3} = t. \end{cases}$$

Then

$$X_{r^1} := \begin{cases} \{x_{r^1}(t) | t \in \mathbb{F}, t^q = t\} & \text{, if } \rho(r) = r, \\ \{x_{r^1}(t) | t \in \mathbb{F}, t^{q^3} = t\} & \text{, if } \rho(r) \neq r, \end{cases}$$

is the root subgroup of $\mathbf{G}^F$ corresponding to $r^1 \in \Phi^1$. For $r \in \Phi$, let n_r be the element in $\mathbf{N} := N_{\mathbf{G}}(\mathbf{T})$, given by [1], Lemma 6.4.4. Then $\mathbf{N}$ is generated by $\mathbf{T}$ and the elements n_r for $r \in \Delta$. We have a canonical homomorphism with kernel $\mathbf{T}$ from $\mathbf{N}$ onto $\mathbf{W}$, the Weyl group of the root system Φ, mapping n_r to the reflection w_r of V at the hyperplane orthogonal to r. This allows us to identify $\mathbf{W}$ with $\mathbf{N}/\mathbf{T}$. As $\mathbf{N}$ and $\mathbf{T}$ are F–stable, this homomorphism induces an action of F on $\mathbf{W}$. Define

$$n_{r^1} := \begin{cases} n_r & \text{, if } \rho(r) = r, \\ n_r n_{\rho(r)} n_{\rho^2(r)} & \text{, if } \rho(r) \neq r, \end{cases}$$

for $r \in \Phi$. Then $\mathbf{N}^F$ is generated by $\mathbf{T}^F$ and the elements n_{r^1} for $r \in \Phi$, and there is a canonical homomorphism with kernel $\mathbf{T}^F$ from $\mathbf{N}^F$ to $\mathbf{W}^1$, the Weyl group of the root system Φ^1. The action of $\mathbf{W}$ on $\mathbf{T}$ can be computed using [1], Theorem 7.2.2.

Table 2.1 *Action of* $\mathbf{W}$ *on* $\mathbf{T}$. *For* $t_i \in \mathbb{F}$, $t_i \neq 0$ *we have:*

$$\begin{aligned} n_{r_1} h(t_1,t_2,t_3,t_4) n_{r_1}^{-1} &= h(t_1^{-1}t_2, t_2, t_3, t_4)\ , \\ n_{r_2} h(t_1,t_2,t_3,t_4) n_{r_2}^{-1} &= h(t_1, t_1t_2^{-1}t_3t_4, t_3, t_4)\ , \\ n_{r_3} h(t_1,t_2,t_3,t_4) n_{r_3}^{-1} &= h(t_1, t_2, t_2t_3^{-1}, t_4)\ , \\ n_{r_4} h(t_1,t_2,t_3,t_4) n_{r_4}^{-1} &= h(t_1, t_2, t_3, t_2t_4^{-1})\ . \end{aligned}$$

We will use frequently the commutator relations and the relations describing the action of $\mathbf{T}^F$ and $\mathbf{W}^1$ on the root subgroups of ${}^3D_4(q)$ given in [9], Tables 2.2–2.4.

Define $\delta \in \{1, -1\} \subseteq \mathbb{Z}$ by $q \equiv \delta \mod 4$. For any field K, we denote its multiplicative group by $K^\times$. Following [14], we establish the following notation to parametrize conjugacy classes and characters: Let $\mathbb{Q}_{p'}$ be the additive group of all rational numbers with a denominator not divisible by p. We fix an ismorphism of groups $\varphi_1 : \mathbb{Q}_{p'}/\mathbb{Z} \to \mathbb{F}^\times$ as in [2], Proposition 3.1.3, and the embedding $\varphi_2 : \mathbb{Q}_{p'}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$, $\frac{r}{s} \mapsto e^{2\pi i r/s}$. (We describe the elements of $\mathbb{Q}_{p'}/\mathbb{Z}$ via representatives $\frac{r}{s}$ in $\mathbb{Q}_{p'}$ ($r, s \in \mathbb{Z}$, $p \not| s$)). Let ϕ_n denote the n–th cyclotomic polynomial in q, in particular: $\phi_1 = q-1$, $\phi_2 = q+1$, $\phi_3 = q^2+q+1$, $\phi_6 = q^2-q+1$, $\phi_{12} = q^4-q^2+1$.

Table 2.2 *Notation for generic roots of unity:*

$\mu \in \mathbb{Q}_{p'}/\mathbb{Z}$	$\varphi_1(\mu) \in \mathbb{F}^\times$	$\varphi_2(\mu) \in \mathbb{C}^\times$
$\frac{1}{q^n-1}$	$\tilde{\zeta}_n$	ζ_n
$\frac{1}{q^n+1}$	$\tilde{\xi}_n$	ξ_n
$\frac{1}{(q^n+1)(q-1)}$	$\tilde{\eta}_n$	η_n
$\frac{1}{(q^n-1)(q+1)}$	$\tilde{\mu}_n$	μ_n
$\frac{1}{\phi_n}$	$\tilde{\varphi}_n$	φ_n

For any finite group H let $(\cdot,\cdot)_H$ be the usual scalar product on the space of class functions of H and let $\mathrm{Irr}(H)$ be the set of (complex) irreducible characters of H. We denote the trivial character of H by $\mathbf{1}_H$ or $\mathbf{1}$. If χ is a character of a subgroup H_1 of H we write χ^H for the induced character, and if χ is a character of H we write χ_{H_1} for the restriction of χ to the subgroup H_1 of H.

3. The Conjugacy Classes of ${}^3D_4(q)$

The conjugacy classes of $\mathbf{G}^F = {}^3D_4(q)$ have been determined by D.I. Deriziotis and G.O. Michler in [4] (see also M. Geck [8]). To fix notation and because we will use these conjugacy classes when restricting characters from ${}^3D_4(q)$ to parabolic subgroups, we give a short description of these classes.

We call semisimple elements $s_1, s_2 \in \mathbf{G}^F$ equivalent if and only if their centralizers $C_{\mathbf{G}}(s_1)$ and $C_{\mathbf{G}}(s_2)$ are $\mathbf{G}^F$–conjugate. Let s_1, s_2 be elements which are equivalent to each other. After conjugation in $\mathbf{G}^F$ we can suppose $C_{\mathbf{G}}(s_1) = C_{\mathbf{G}}(s_2) =: C$. Then, s_1 and s_2 are contained in all maximal tori of C. Hence, we may always assume that two equivalent semisimple elements are contained in the same F–stable maximal torus of $\mathbf{G}$.

For each F–stable maximal torus $\tilde{\mathbf{T}}$ there is $g \in \mathbf{G}$ such that $\tilde{\mathbf{T}}^g = \mathbf{T}$ and $n := g^{-1}F(g) \in \mathbf{N} = N_{\mathbf{G}}(\mathbf{T})$. The action of F on $\tilde{\mathbf{T}}$ corresponds to the action of the map $(Fn^{-1}) : \mathbf{G} \to \mathbf{G}, x \mapsto {}^nF(x)$ on $\mathbf{T}$ (i.e. $F(\tilde{t})^g = {}^nF(\tilde{t}^g)$ for all $\tilde{t} \in \tilde{T}$) and this action is already determined by $w := n\mathbf{T} \in \mathbf{W} = N(\mathbf{T})/\mathbf{T}$, so that we can write $(Fw^{-1}) : \mathbf{T} \to \mathbf{T}$, $t \mapsto {}^wF(t)$. The $\mathbf{G}^F$–classes of F–stable maximal tori of $\mathbf{G}$ are parametrized by the F–conjugacy classes of $\mathbf{W}$ (cf. [2], Proposition 3.3.3). Thus, we can describe the $\mathbf{G}^F$–classes of maximal tori of $\mathbf{G}^F$ by the sets $T_j := \mathbf{T}^{(Fw_j^{-1})}$ of fixed points of (Fw_j^{-1}) on $\mathbf{T}$, where w_j runs over a set of representatives for the F–conjugacy classes of $\mathbf{W}$. We choose the representatives w_j, $j = 0, 1, \dots, 6$ as in [4], Table 1.1, except for w_1, which we replace by $w_1 := w_{r_2}$ for later use. (We may do this, because $w^{-1}w_{r_2}F(w) = w_{1+2}$ by CHEVIE with $w := w_{r_2}w_{r_1}w_{r_3}w_{r_2}w_{r_4}w_{r_2}w_{r_1}w_{r_3}$ and w_{1+2} as in [4].) The sets $T_j = \mathbf{T}^{(Fw_j^{-1})}$ are given by [4], Table 1.1, after replacing T_1 by

$$T_1 = \mathbf{T}^{(Fw_1^{-1})} = \{h(t^{q^2+q}, t^{q^2+q+1}, t^{q^3+q^2}, t^{q+1}) \mid t^{(q^3-1)(q+1)} = 1\} \cong \mathbb{Z}_{(q^3-1)(q+1)}$$

due to our choice of the representative w_1. To fix a parametrization of the semisimple conjugacy classes, a set of representatives is given in Table A.1 in the Appendix (see [4], Table 2.1 or [8], Tafel 3.6.3). Representatives for the unipotent classes of $\mathbf{G}^F$ are contained in [9], Table 3.1, and a classification of the mixed conjugacy classes of $\mathbf{G}^F$ is given in [4], Table 2.4. Representatives for the mixed classes can be obtained by computing the nontrivial unipotent conjugacy classes of the centralizers $C_{\mathbf{G}^F}(s)$ where s runs through a set of representatives for the nontrivial semisimple conjugacy classes of $\mathbf{G}^F$. We may assume $s \in \tilde{\mathbf{T}} = {}^g\mathbf{T}$ where $n_j = g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T})$ is a preimage of one of the w_j, $j = 0, \dots, 6$. Because of $(C_{\mathbf{G}}(s)^F)^g = C_{\mathbf{G}}(s')^{(Fn_j^{-1})}$ with $s' := s^g \in \mathbf{T}$ the finite groups $C_{\mathbf{G}}(s')^{(Fn_j^{-1})}$ and $C_{\mathbf{G}}(s)^F$ are $\mathbf{G}$–conjugate. Hence, it suffices to determine the unipotent conjugacy classes of $C_{\mathbf{G}}(s')^{(Fn_j^{-1})}$. Using the classification in [4], Table 2.4, representatives for the unipotent conjugacy classes of $C_{\mathbf{G}}(s')^{(Fn_j^{-1})}$, where s' runs through the representatives in Table A.1, are computed in [11], Abschnitt 3.1. This gives the list of representatives for all conjugacy classes of ${}^3D_4(q)$ in Table A.2 in the Appendix.

4. The Character Table of a Borel Subgroup B

Let $\mathbf{B}$ be the F–stable Borel subgroup $\mathbf{TU}$ of $\mathbf{G}$ where $\mathbf{U}$ is the product of all root subgroups of $\mathbf{G}$ to positive roots, and let $B := \mathbf{B}^F$ be the corresponding Borel subgroup of $\mathbf{G}^F$. The conjugacy classes of B and their fusions in $\mathbf{G}^F$ have been determined by M. Geck in [8], Tafeln 3.8.1 and 3.8.2, using the relations in [9], Tables 2.2–2.4. In this section, we compute the irreducible characters of B. We construct these characters by inducing linear characters from subgroups of B. All scalar products of characters in this section have been computed using CHEVIE.

For convenience and to fix notation we give a short description of the conjugacy classes of B. The group B is the semidirect product of $T = \mathbf{T}^F$ and the unipotent normal subgroup $U = \mathbf{U}^F = X_\alpha X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. The elements of T form a set of representatives for the semisimple conjugacy classes of B and we parametrize these classes according to Table A.3 in the Appendix. A list of all $q^4+2q^3+q^2+5q+9$ conjugacy classes of B is given in Table A.4 (cf. [8], Tafeln 3.8.1 and 3.8.2). The fusions of these classes in $\mathbf{G}^F$ are given by Tables A.4 and A.8.

The parameter sets I_1, I_2 in Table A.4 are defined as subsets of the set $S := \{(i,b) \mid i \in \{0,1\}, b \in \mathbb{F}_q^\times\} - \{(0,2),(0,-2)\}$ as follows: We call $(i,b),(i',b') \in S$ equivalent if and only if $i = i'$ and $b = \pm b'$. Let I be a set of representatives for the equivalence classes of this equivalence relation. Then, I_1 is defined as the set of all pairs $(i,b) \in I$ such that $x_\beta(1)x_{2\alpha+\beta}(\zeta^i)x_{3\alpha+\beta}(b)$ is $\mathbf{G}^F$–conjugate with $x_\beta(1)x_{2\alpha+\beta}(-1)$. The set I_2 is defined as $I_2 := I - I_1$.

The parameter set I_3 in Table A.4 is defined as follows: We call two elements $a', a'' \in I' := \{a' \in \mathbb{F} \mid (a')^{q^3} = a', (a')^q \neq a'\}$ equivalent if and only if there exist $t, y \in \mathbb{F}$ with $t^{q-1} = 1$, $y^q = y$ such that $a'' = t(a'+y)$. Then, I_3 is defined as a set of representatives for the equivalence classes of this equivalence relation on I'.

Now, we start to construct the irreducible characters of B. Let $\mathbb{F}_{q^3}$ be the unique subfield of $\mathbb{F}$ with exactly q^3 elements. For abbreviation, set $\tau := \tilde{\zeta}_3$ and $\pi := \tilde{\zeta}_1 = \tau^{q^2+q+1}$ such that τ and π are generators of $\mathbb{F}_{q^3}^\times$, $\mathbb{F}_q^\times$ respectively. Let $\mathrm{Tr} : \mathbb{F}_{q^3} \to \mathbb{F}_q$, $u \mapsto u^{q^2} + u^q + q$ be the trace function of the field extension $\mathbb{F}_{q^3}/\mathbb{F}_q$. Fix nontrivial linear characters $\phi', \phi'' : \mathbb{F}_{q^3} \to \mathbb{C}^\times$ of the additive group of $\mathbb{F}_{q^3}$, such that ϕ' restricts nontrivially on $\mathbb{F}_q$ and such that $\{r \in \mathbb{F}_{q^3} | \mathrm{Tr}(r) = 0\} \subseteq \ker(\phi'')$. Let ϕ be the restriction of ϕ' to $\mathbb{F}_q$. Then (see Problem (2.1) in [13]):

$$\sum_{t\in\mathbb{F}_q} \phi(t) = \sum_{t\in\mathbb{F}_{q^3}} \phi'(t) = \sum_{t\in\mathbb{F}_{q^3}} \phi''(t) = 0 \quad \text{and} \tag{1}$$

$$\sum_{t\in\mathbb{F}_q^\times} \phi(t) = \sum_{t\in\mathbb{F}_{q^3}^\times} \phi'(t) = \sum_{t\in\mathbb{F}_{q^3}^\times} \phi''(t) = -1. \tag{2}$$

For $\lambda \in \mathbb{F}_{q^3}$, $\nu \in \mathbb{F}_q \subseteq \mathbb{F}_{q^3}$ and

$$\begin{aligned} f &:= X^{q+1} + X^{q^2+q} + X^{q^2+1} + \nu \in \mathbb{F}_q[X], \\ g &:= X^{q^2+q+1} - (\lambda^q X + \lambda^{q^2} X^q + \lambda X^{q^2}) - \nu \in \mathbb{F}_{q^3}[X]. \end{aligned}$$

we set $L(\nu) := |\{x \in \mathbb{F}_{q^3} | f(x) = 0\}|$ and $M(\lambda,\nu) := |\{x \in \mathbb{F}_{q^3} | g(x) = 0\}|$. Then

$$L(\nu) = \begin{cases} q^2+q & \text{if } \nu \neq 0 \text{ is a square in } \mathbb{F}_q, \\ q^2-q & \text{if } \nu \neq 0 \text{ is not a square in } \mathbb{F}_q, \\ q^2 & \text{if } \nu = 0 \end{cases} \tag{3}$$

and

$$M(\lambda,\nu)=M(\lambda,-\nu)=\begin{cases}1 & \text{if } \lambda=\nu=0,\\ q^2+1 & \text{if } x_\beta(1)x_{2\alpha+\beta}(\lambda)x_{3\alpha+\beta}(\nu)\in c_{1,2}^G,\\ q^2+q+1 & \text{if } x_\beta(1)x_{2\alpha+\beta}(\lambda)x_{3\alpha+\beta}(\nu)\in c_{1,3}^G,\\ q^2-q+1 & \text{if } x_\beta(1)x_{2\alpha+\beta}(\lambda)x_{3\alpha+\beta}(\nu)\in c_{1,4}^G,\end{cases}\tag{4}$$

where $c_{1,j}^G$ denotes the conjugacy class $c_{1,j}$ of $G:=\mathbf{G}^F$ given in Table A.2 (for a proof of (3) and (4) see [8], p. 97). Let $U_P:=X_\beta X_{\alpha+\beta}X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$ and $U_Q:=X_\alpha X_{\alpha+\beta}X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$. Each element in B can be written uniquely as

$$h(t_1,t_2)x_\alpha(d_1)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6)$$

with $t_1,t_2,d_1,\dots,d_6\in\mathbb{F}$, $t_1^{q^3-1}=t_2^{q-1}=1$, $d_i^{q^3}=d_i$ for $i=1,3,4$ and $d_i^q=d_i$ for $i=2,5,6$. The irreducible characters of B can be constructed as follows:

For $k=0,\dots,q^3-2$, $l=0,\dots,q-2$, let ${}_B\chi_1(k,l)$ be the linear character of B defined by

$$h(\tau^i,\pi^j)x_\alpha(d_1)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6)\mapsto\zeta_3^{ik}\zeta_1^{jl}.$$

For $k=0,\dots,q-2$, define a linear character of $C_T(X_\alpha)U$ by

$$h(\pi^i,\pi^{2i})x_\alpha(d_1)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6)\mapsto\zeta_1^{ik}\phi'(d_1).$$

Inducing this character to B, we obtain ${}_B\chi_2(k)$. Similarly ${}_B\chi_3(k)$, $k=0,\dots,q^3-2$, is defined to be the induced character of the following linear character of $C_T(X_\beta)U$:

$$h(\tau^{2i},\pi^i)x_\alpha(d_1)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6)\mapsto\zeta_3^{ik}\phi(d_2).$$

Let ${}_B\chi_4$ be the character of B induced from the following linear character of U:

$$x_\alpha(d_1)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6)\mapsto\phi'(d_1+d_2).$$

Let ${}_B\chi_5(k)$, $k=0,\dots,q-2$, be the character of B induced from the following linear character of $C_T(X_{\alpha+\beta})U_Q$:

$$h(\pi^i,\pi^i)x_\alpha(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)\mapsto\zeta_1^{ik}\phi''(d_2).$$

Let ${}_B\chi_7(k)$, $k=0,\dots,q-2$, be the character of B induced from the following linear character of $C_T(X_{2\alpha+\beta})U_P$:

$$h(1,\pi^i)x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)\mapsto\zeta_1^{ik}\phi'(d_3).$$

Let ${}_B\chi_{12}(k)$, $k=0,\dots,q^3-2$, be the character of B induced from the following linear character of $C_T(X_{3\alpha+\beta})U_P$:

$$h(\tau^i,\pi^i)x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)\mapsto\zeta_3^{ik}\phi(d_4).$$

Let ${}_B\chi_{13}(k)$, $k=0,\dots,q^2+q$, be the character of B induced from the following linear character of $(C_T(X_\beta)\cap C_T(X_{3\alpha+\beta}))U_P$:

$$h(\tau^{(q-1)i},1)x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)\mapsto\phi_3^{ik}\phi(d_1+d_4).$$

Let ${}_B\chi_{17}(k)$, $k=0,\dots,q^3-2$, be the character of B induced from the following linear character of $C_T(X_{3\alpha+2\beta})X_\alpha X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$:

$$h(\tau^i,1)a_\alpha(d_1)x_{2\alpha+\beta}(d_2)x_{3\alpha+\beta}(d_3)x_{3\alpha+2\beta}(d_4)\mapsto\zeta_3^{ik}\phi(d_4).$$

It is not difficult to compute the values of the above characters using (1), (2), (3) and (4). We demonstrate how to determine the values of ${}_B\chi_{13}(k)$, $k=0,\dots,q^2+q$, on the conjugacy classes $c_{1,3}$, $c_{1,5}$, $c_{1,6}$, $c_{1,8}$, $c_{1,9}$, $c_{1,10}$, $c_{1,12}$, $c_{1,13}$, $c_{1,14}$, $c_{1,16}$

of B: The set $\{h_{i',j',s} := h(\tau^{i'},\pi^{j'})x_\alpha(s)\,|\,i',j' = 1,\dots,q-1\,,\,s\in\mathbb{F}_{q^3}\}$ is a set of representatives for the right cosets of $(C_T(X_\beta)\cap C_T(X_{3\alpha+\beta}))U_P$ in B and for all $d_4\in\mathbb{F}_{q^3}$, $d_5\in\mathbb{F}_q$ the element ${}^{h_{i',j',s}}(x_\beta(1)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5))$ is equal to

$$x_\beta(\pi^{2j'-i'})x_{\alpha+\beta}(\dots)x_{2\alpha+\beta}(\dots)x_{3\alpha+\beta}(\pi^{i'-j'}(d_5+\mathrm{Tr}(sd_4^q)-s^{q^2+q+1})))x_{3\alpha+2\beta}(\dots).$$

Hence by the definition of induced characters (see [13], p. 62), the value of ${}_B\chi_{13}(k)$ on a conjugacy class of B with representative $x_\beta(1)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)$ is

$$\sum_{s\in\mathbb{F}_{q^3}}\sum_{i',j'=1}^{q-1}\phi(\pi^{2j'-i'}+\pi^{i'-j'}(d_5-\mathrm{Tr}(sd_4^q)+s^{q^2+q+1})).$$

By (2) we have

$$\sum_{i',j'=1}^{q-1}\phi(\pi^{j'-i'}+\pi^{i'}(d_5-\mathrm{Tr}(sd_4^q)+s^{q^2+q+1}))=\left(\sum_{i'=1}^{q-1}\phi(\pi^{i'})\right)\cdot\left(\sum_{j'=1}^{q-1}\phi(\pi^{j'})\right)=1$$

for $d_5-\mathrm{Tr}(sd_4^q)+s^{q^2+q+1}\neq 0$ and

$$\sum_{i',j'=1}^{q-1}\phi(\pi^{j'-i'}+\pi^{i'}(d_5-\mathrm{Tr}(sd_4^q)+s^{q^2+q+1}))=\sum_{i',j'=1}^{q-1}\phi(\pi^{j'-i'})=-(q-1)$$

for $d_5-\mathrm{Tr}(sd_4^q)+s^{q^2+q+1}=0$. Hence the value of ${}_B\chi_{13}(k)$ on a conjugacy class of B with representative $x_\beta(1)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)$ is $q^3-qM(d_4,d_5)$. Using (4) this gives the values of ${}_B\chi_{13}(k)$ on the conjugacy classes $c_{1,3}$, $c_{1,6}$, $c_{1,8}$, $c_{1,9}$, $c_{1,12}$, $c_{1,13}$, $c_{1,16}$ of B. Analogously, by (3), the value of ${}_B\chi_{13}(k)$ on a conjugacy class of B with representative $x_{\alpha+\beta}(1)x_{3\alpha+\beta}(d_5)$ ($d_5\in\mathbb{F}_q$) is $q(q-1)(L(d_5)-q^2)$. This gives the values of ${}_B\chi_{13}(k)$ on the classes $c_{1,5}$, $c_{1,10}$, $c_{1,14}$.

Because of $({}_B\chi_j(k),{}_B\chi_j(k))_B=({}_B\chi_4,{}_B\chi_4)_B=1$ for $j=2,3,5,7,12,13,17$ and k as in Table A.5 the above characters are irreducible.

We proceed to construct further irreducible characters of B. Let $\tilde X_\alpha$ be the subgroup $\tilde X_\alpha := \{x_\alpha(s)|\mathrm{Tr}(s)=0\}$ of order q^2 of the root subgroup X_α. The linear characters

$$\varphi_\beta : x_\alpha(d_1)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6)\mapsto\beta(x_\alpha(d_1))\phi''(d_3)$$

with $\beta\in\mathrm{Irr}(\tilde X_\alpha)$ are the extensions of the linear character

$$\varphi : x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6)\mapsto\phi''(d_3)$$

of U_P to $\tilde X_\alpha U_P$. Hence, (6.17) in [13] implies $\varphi^{\tilde X_\alpha U_P}=\sum_{\beta\in\mathrm{Irr}(\tilde X_\alpha)}\varphi_\beta$. By the relations in Table 2.3 in [9], the centralizer $C_T(X_{\alpha+\beta})=\{h(t,t)|t\in\mathbb{F}_q^\times\}$ acts on $\{\varphi_\beta\,|\,\beta\in\mathrm{Irr}(\tilde X_\alpha)\}$ by conjugation with exactly $q+2$ orbits: the orbit of $\varphi_{\mathbf{1}}$ and $q+1$ orbits of size $q-1$. Let $\{\psi_0=\varphi_{\mathbf{1}},\psi_1,\dots,\psi_{q+1}\}$ be a set of representatives for these orbits. Let $H := C_T(X_{\alpha+\beta})\tilde X_\alpha U_P$. Then $\varphi^H=(\varphi^{\tilde X_\alpha U_P})^H=\psi_0^H+(q-1)\sum_{k=1}^{q+1}\psi_k^H$ and therefore $\varphi^B=\psi_0^B+(q-1)\sum_{k=1}^{q+1}\psi_k^B$. Let ${}_B\chi_6(k):=\psi_k^B$ for $k=1,\dots,q+1$. Using (1), (2) it is not difficult to compute the values of φ^B. Then we can compute $(\varphi^B,{}_B\chi_5(k))_B=1$ for $k=0,\dots,q-2$. Hence $\psi_0^B=\sum_{k=0}^{q-2}{}_B\chi_5(k)$. Thus, we can compute the values of $\sum_{k=1}^{q+1}{}_B\chi_6(k)$ via $\sum_{k=1}^{q+1}{}_B\chi_6(k)=\frac{1}{q-1}\left(\varphi^B-\sum_{k=0}^{q-2}{}_B\chi_5(k)\right)$. We can verify $(\sum_{k=1}^{q+1}{}_B\chi_6(k),\sum_{k=1}^{q+1}{}_B\chi_6(k))_B=q+1$. Hence, the ${}_B\chi_6(k)$'s are $q+1$ distinct irreducible characters of B.

Next, we construct the characters ${}_B\chi_8$, ${}_B\chi_9$, ${}_B\chi_{10}$, ${}_B\chi_{11}$. The linear characters $\varphi_k : x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5) \mapsto \phi'(\pi^k d_1 + d_3)$ of U_P $(k = 0, 1)$ are invariant under the action of the involution $h(1, -1)$ on $\mathrm{Irr}(U_P)$ and therefore extend to $\langle h(1, -1)\rangle U_P$. Hence, by (6.17) in [13], $(\varphi_0 + \varphi_1)^B = \varphi_0^B + \varphi_1^B$ is the sum of 4 characters of degree $\frac{1}{2}q^3(q^3 - 1)(q - 1)$. Let $\varphi := \varphi_0 + \varphi_1$. Using (1) and (2) it is straightforward to compute the values of φ^B and we can verify $(\varphi^B, \varphi^B)_B = 4$. Hence (because φ^B is a sum of four characters) φ^B is the sum of four different irreducible characters of degree $\frac{1}{2}q^3(q^3 - 1)(q - 1)$. The $q^4 + 3q - 1$ distinct irreducible characters ${}_B\chi_1, \ldots, {}_B\chi_7$ of B have the normal subgroup $X_{3\alpha+\beta}X_{3\alpha+2\beta} \trianglelefteq B$ in their kernel, and hence, we can identify these characters with irreducible characters of $\bar{B} := B/X_{3\alpha+\beta}X_{3\alpha+2\beta}$. Using the relations in [9], Tables 2.2 and 2.3, one can compute the conjugacy classes of $\bar{B}$ and the orders of the corresponding centralizers and one gets, that $\bar{B}$ has precisely $q^4 + 3q + 3$ conjugacy classes. Thus, the four constituents of $\varphi^B = \varphi_0^B + \varphi_1^B$ are the missing four irreducible characters of $\bar{B}$ (with the above identification).

By construction, the two constituents of φ_0^B both have the same values on the unipotent classes and the same is true for the two constituents of φ_1^B. Because we also know the values of $\varphi_0^B + \varphi_1^B$, there is just one unknown parameter in each column of the character table of $\bar{B}$ corresponding to the unipotent classes. This parameter can be determined by the column orthogonality relations of the character table of $\bar{B}$. This gives us the values of the four constituents ${}_B\chi_8$, ${}_B\chi_9$, ${}_B\chi_{10}$, ${}_B\chi_{11}$ of φ^B on all unipotent conjugacy classes of B. By construction, these constituents vanish on all non–unipotent conjugacy classes of B except for the classes $c_{4,j}$. It is straightforward to compute the values of these constituents on the class $c_{4,0}$. The values on $c_{4,1}, \ldots, c_{4,4}$ are determined by the conditions $({}_B\chi_n, \mathbf{1}_B)_B = ({}_B\chi_n, {}_B\chi_7(0))_B = 0$, $({}_B\chi_n, {}_B\chi_n)_B = 1$, $({}_B\chi_n, [\rho_1]_B)_B \in \mathbb{Z}$ where $n = 8, 9, 10, 11$ and $[\rho_1]_B$ is the restriction of the unipotent character $[\rho_1]$ of ${}^3D_4(q)$ to B (see [17], Table 2). This gives us the remaining values of ${}_B\chi_8$, ${}_B\chi_9$, ${}_B\chi_{10}$, ${}_B\chi_{11}$ in Table A.6.

Next, we construct the characters ${}_B\chi_{14}(k)$, ${}_B\chi_{15}(k)$. For $k, l = 0, 1$, let φ_{kl} be the linear character of $\langle h(-1, -1)\rangle U_P$ mapping

$$h(-1,-1)^i x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5) \mapsto (-1)^{li}\phi'(d_2 + \pi^k d_4).$$

Let ${}_B\chi_{14}(k)$ $(k = 0, 1)$ be the character of B induced from φ_{k0} and let ${}_B\chi_{15}(k)$ $(k = 0, 1)$ be the character of B induced from φ_{k1}. It is straightforward to compute the values of $\chi := \sum_{k=0}^{1} {}_B\chi_{14}(k)$ and $\psi := \sum_{k=0}^{1} {}_B\chi_{15}(k)$. Then, we can verify $(\chi, \chi)_B = (\psi, \psi)_B = 2$ and $(\chi, \psi)_B = (\chi, {}_B\chi_n)_B = (\psi, {}_B\chi_n)_B = 0$ for $n = 8, 9, 10, 11$. Thus, ${}_B\chi_{14}(k)$, ${}_B\chi_{15}(k)$ are four distinct irreducible characters of B.

Next, we construct the characters ${}_B\chi_{16}(k)$. For $k = 1, \ldots, q - 1$, let φ_k be the linear character of U_P mapping $x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)$ to:

$$\begin{cases} \phi'(\pi^k d_1 + d_2 + d_4) & \text{if } 1 \le k \le (q-1)/2, \\ \phi'(\pi^k d_1 + d_2 + \pi d_4) & \text{if } (q+1)/2 \le k \le q-1, \end{cases}$$

and let ${}_B\chi_{16}(k) := \varphi_k^B$ for $k = 1, \ldots, q - 1$. For $x \in \mathbb{F}_q$, we define the map $\varphi_x : U_P \to \mathbb{C}$ by

$$\varphi_x : x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5) \mapsto \sum_{m=0}^{1} \phi'(xd_1 + d_2 + \pi^m d_4).$$

Hence, each φ_x is the sum of two linear characters of U_P. Let $\varphi := \sum_{x\in\mathbb{F}_q}\varphi_x$. Using (1), (2) and (3) it is straightforward to compute the values of φ^B. As φ_x is conjugate to φ_{-x} under the action of $h(-1,-1)$ on $\mathrm{Irr}(U_P)$, we get $\varphi_x^B = \varphi_{-x}^B$. Hence, we can compute the values of $\psi := \sum_{k=1}^{q-1} {}_B\chi_{16}(k)$ from

$$\varphi^B = {}_B\chi_{14}(0) + {}_B\chi_{14}(1) + {}_B\chi_{15}(0) + {}_B\chi_{15}(1) + 2\psi.$$

We can verify $(\psi,\psi)_B = q-1$. Hence ${}_B\chi_{16}(k)$, $k = 1,\dots,q-1$, are $q-1$ different irreducible characters of B.

So far, we have constructed $q^4 + 2q^3 + q^2 + 5q + 5$ irreducible characters of B. Hence, there are still four irreducible characters of B missing. The linear characters

$$\varphi_k : x_\alpha(d_1)x_{2\alpha+\beta}(d_2)x_{3\alpha+\beta}(d_3)x_{3\alpha+2\beta}(d_4) \mapsto \phi'(\pi^k d_1 + d_4)$$

of $H := X_\alpha X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$ $(k = 0,1)$ are invariant under the action of the involution $h(-1,1)$ on $\mathrm{Irr}(H)$ and therefore extend to $\langle h(-1,1)\rangle H$. Hence, by (6.17) in [13], $(\varphi_0+\varphi_1)^B = \varphi_0^B + \varphi_1^B$ is the sum of 4 characters of degree $\frac{1}{2}q^4(q^3-1)(q-1)$. Let $\varphi := \varphi_0 + \varphi_1$. Using (1), (2) and (3) it is straightforward to compute the values of φ^B and we can verify $(\varphi^B,\varphi^B)_B = 4$. Hence φ^B is the sum of four different irreducible characters of degree $\frac{1}{2}q^4(q^3-1)(q-1)$ and these have to be the four missing irreducible characters. Their values can be computed analogously to those of ${}_B\chi_8$, ${}_B\chi_9$, ${}_B\chi_{10}$, ${}_B\chi_{11}$ using the orthogonality relations for the character table of B. So we have proved:

Theorem 4.1. *The character table of the Borel subgroup B is given by Tables A.5 and A.6 in the Appendix.* ∎

We would like to point out that we are not able to describe all values of the characters ${}_B\chi_6(k)$, ${}_B\chi_{14}(k)$, ${}_B\chi_{15}(k)$, ${}_B\chi_{16}(k)$ generically because an inspection of small values of q $(q = 13)$ shows, that the values of these characters on the unipotent conjugacy classes $c_{1,12}(i,b)$, $c_{1,16}(i,b)$ or $c_{1,17}(a')$ of B depend on i, b or a' and we do not have a generic description of these classes. For fixed q (not too large), there is no difficulty in computing the values of these characters on all conjugacy classes only using the above definition of these characters.

5. The Character Table of a Maximal Parabolic Subgroup P

Let $\mathbf{P} = \langle \mathbf{B}, n_{r_1}, n_{r_3}, n_{r_4}\rangle$ be the F–stable maximal parabolic subgroup of $\mathbf{G}$ corresponding to the subset $\{r_1, r_3, r_4\} \subseteq \Delta$ and $P := \mathbf{P}^F$ be the corresponding maximal parabolic subgroup of $\mathbf{G}^F = {}^3D_4(q)$. Then, P is generated by B and n_α and $|P| = q^{12}(q^6-1)(q-1)$. In this section, we compute the conjugacy classes and the character table of P.

P is the semidirect product of the Levi complement $L_P = \langle \mathbf{T}^F, X_\alpha, X_{-\alpha}\rangle$ and the unipotent radical $U_P := X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. To compute the conjugacy classes of P we proceed in two steps: In the first step, we determine the conjugacy classes of L_P, and in the second step we investigate the "splitting" of these classes when interpreting L_P as the factor group P/U_P.

The Levi complement $\mathbf{L}_P = \langle \mathbf{T}, X_{\pm r_1}, X_{\pm r_3}, X_{\pm r_4}\rangle$ of $\mathbf{P}$ is F–stable and the group $L_P = (\mathbf{L}_P)^F$ is a finite group of Lie type of order $|L_P| = q^3(q^6-1)(q-1)$. The Weyl group $\mathbf{W}_{\mathbf{L}_P} = \langle w_{r_1}, w_{r_3}, w_{r_4}\rangle$ of $\mathbf{L}_P$ is elementary abelian of order 8 and has exactly two F–conjugacy classes with representatives $w_0 = 1$ and $w_2 = w_{r_1}w_{r_3}w_{r_4}$ $(= -w_{r_0}$ in the notation of [4]). Thus, $\mathbf{L}_P$ has exactly two L_P–conjugacy classes of F–stable maximal tori, and we can choose representatives $\mathbf{T}_0 = \mathbf{T}$ and $\mathbf{T}_2$ such that the corresponding maximal tori $\mathbf{T}^F$ and $\mathbf{T}_2^F$ of L_P are given by $T_0 = T = \mathbf{T}^F$ and $T_2 = \mathbf{T}^{(Fw_2^{-1})}$ as in [4], Table 1.1. Each semisimple element of L_P is either central in L_P or regular semisimple and a set of representatives for the conjugacy classes of semisimple elements of L_P is given in Table 5.1.

Table 5.1 *Parametrization of the semisimple conjugacy classes of L_P:*

Representative	Parameters	Number of Classes
$h_1(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0, \dots, q-2$	$q-1$
$h_2(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \dots, q^3-2$ $j = 0, \dots, q-2$ $i \neq (q^2+q+1)l$ or $j \neq 2l$, $l = 0, \dots, q-2$	$\frac{(q^3-2)(q-1)}{2}$
$h_3(i) := h(\tilde{\eta}_3^i, \tilde{\eta}_3^{(q^3+1)i}, \tilde{\eta}_3^{q^4 i}, \tilde{\eta}_3^{q^2 i})$	$i = 0, \dots, q^4-q^3+q-2$ $i \neq (q^3+1)l,\ l = 0, \dots, q-2$	$\frac{q^3(q-1)}{2}$

The maximal torus $T = \mathbf{T}^F$ acts transitively on $X_\alpha - \{1\}$ by conjugation. Thus, L_P has exactly one nontrivial unipotent conjugacy class and $x_\alpha(1)$ is a representative. Using the Bruhat decomposition (see [2], Theorem 2.5.14) and the relations in [9], Tables 2.2–2.4, we get $|C_{L_P}(x_\alpha(1))| = q^3(q-1)$.

Table 5.2 *The conjugacy classes of L_P:*

Notation	Representantive	$\lvert C_{L_P}\rvert$
$c_{1,0}(i)$ $c_{1,1}(i)$	$h_1(i)$ $h_1(i)x_\alpha(1)$	$q^3(q^6-1)(q-1)$ $q^3(q-1)$
$c_{2,0}(i,j)$	$h_2(i,j)$	$(q^3-1)(q-1)$
$c_{3,0}(i)$	$h_3(i)$	$(q^3+1)(q-1)$

By [5], 0.12 and 3.16 and the proof of [2], Proposition 3.3.3 (applied to the connected F–stable algebraic group $\mathbf{P}$) each semisimple element of P is conjugate in P to a semisimple element of L_P. Thus, a set of representatives for the semisimple conjugacy classes of L_P is also one for the semisimple conjugacy classes of P. We parametrize the semisimple conjugacy classes of P according to Table A.7.

The unipotent conjugacy classes of P can be computed using the relations in [9], Tables 2.2-2.4, and using the fusions of the conjugacy classes of B in ${}^3D_4(q)$, which have been determined by M. Geck in [8], Tafel 3.8.1. Representatives for the mixed classes can be obtained by computing the nontrivial unipotent conjugacy classes of the centralizers $C_P(s)$ where s runs through a set of representatives for the nontrivial semisimple conjugacy classes of P. The parabolic subgroup P has exactly $q^4+q^3+q^2+3q+7$ conjugacy classes. Representatives are given in Table A.8.

Now, we start to construct the irreducible characters of P. Our main tool is Clifford theory applied to the Levi decomposition $P = L_P \ltimes U_P$. By the Bruhat decomposition, each element $x \in L_P$ can be written uniquely as $x = x_\alpha(s)h(t_1, t_2, t_1^q, t_1^{q^2})$ or $x = x_\alpha(r)h(t_1, t_2, t_1^q, t_1^{q^2})n_\alpha x_\alpha(s)$ with $r, s \in \mathbb{F}_{q^3}$, $t_1, t_2 \in \mathbb{F}$ and $t_1^{q^3-1} = t_2^{q-1} = 1$. In order to use Clifford theory, we collect some information about U_P. Each element of U_P can be written uniquely as $x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)$ with $d_2, d_3 \in \mathbb{F}_{q^3} \subseteq \mathbb{F}$ and $d_1, d_4, d_5 \in \mathbb{F}_q \subseteq \mathbb{F}$. Hence, $|U_P| = q^9$ and, by [9],

Table 2.2, the center, the derived subgroup and the Frattini subgroup of U_P are all equal to $X_{3\alpha+2\beta}$. The irreducible characters of U_P can be computed similarly to those of extraspecial p–groups (cf. [12], p. 562f). U_P has exactly q^8+q-1 irreducible chararacters: q^8 linear characters $\chi_{\phi_1,\phi_2,\phi_3,\phi_4}$ defined by

$$x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)\mapsto \phi_1(d_1)\phi_2(d_2)\phi_3(d_3)\phi_4(d_4)$$

with $(\phi_1,\phi_2,\phi_3,\phi_4)\in \mathrm{Irr}(\mathbb{F}_q)\times\mathrm{Irr}(\mathbb{F}_{q^3})\times\mathrm{Irr}(\mathbb{F}_{q^3})\times\mathrm{Irr}(\mathbb{F}_q)$ and $q-1$ irreducible characters χ_{ϕ_1} ($\phi_1\in\mathrm{Irr}(\mathbb{F}_q)$ nontrivial) of degree q^4 induced from linear characters of the maximal abelian normal subgroup $X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$ of U_P. The values of χ_{ϕ_1} on the central elements are $\chi_{\phi_1}(x_{3\alpha+2\beta}(d))=q^4\phi_1(d)$ and χ_{ϕ_1} vanishes on all other elements.

The group P acts on $\mathrm{Irr}(U_P)$ by conjugation. For $\psi\in\mathrm{Irr}(U_P)$ let $I_P(\psi):=\{x\in P\,|\,{}^x\psi=\psi\}$ be its inertia subgroup. We usually identify the inertia factor group $\bar{I}_P(\psi):=I_P(\psi)/U_P$ with the corresponding subgroup of L_P. Choose $\psi_0,\psi_1,\psi_2,\psi_3,\psi_5\in\mathrm{Irr}(U_P)$ as follows: $\psi_0:=\chi_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$, $\psi_1:=\chi_{\phi,\mathbf{1},\mathbf{1},\mathbf{1}}$, $\psi_2:=\chi_{\mathbf{1},\phi',\mathbf{1},\mathbf{1}}$, $\psi_3:=\chi_{\phi,\mathbf{1},\mathbf{1},\phi}$ and $\psi_5:=\chi_\phi$.

Proposition 5.3. *There is a linear character $\psi_4\in Irr(U_P)$, such that (a),(b),(c) are true:*

(a) *$\{\psi_0,\dots,\psi_4,\psi_5\}$ ist a set of representatives for the orbits P on $Irr(U_P)$,*

(b) *the inertia factor groups $\bar{I}_j:=\bar{I}_P(\psi_j)$, $j=0,1,\dots,5$, have the orders:*
$|\bar{I}_0|=q^3(q^6-1)(q-1)$, $|\bar{I}_1|=q^3(q^3-1)$, $|\bar{I}_2|=q^2(q-1)$,
$|\bar{I}_3|=2(q^2+q+1)$, $|\bar{I}_4|=2(q^2-q+1)$, $|\bar{I}_5|=q^3(q^6-1)$.

(c) *the inertia factor groups are given by:*

0. $\bar{I}_0=L_P$,
1. $\bar{I}_1=\{h(t^2,t^{q^2+q+1},t^{2q},t^{2q^2})x_\alpha(s)\,|\,t\in\mathbb{F},t^{q^3-1}=1,s\in\mathbb{F}_{q^3}\}$,
2. $\bar{I}_2=\{h(t,t,t^q,t^{q^2})x_\alpha(s)\,|\,t\in\mathbb{F},t^{q-1}=1,s\in\mathbb{F}_{q^3},Tr(s)=0\}$,
3. $\bar{I}_3=\langle h(1,-1,1,1)n_\alpha\,,\,h(t,1,t^q,t^{q^2})\,|\,t\in\mathbb{F},t^{q^2+q+1}=1\rangle$,
4. $\bar{I}_4=\langle n_4\rangle\ltimes\{h(t,1,t^{-q},t^{q-1})\,|\,t\in\mathbb{F},t^{q^2-q+1}=1\}$, *where n_4 is an involution with $C_{\bar{I}_4}(n_4)=\langle n_4\rangle$ and $\{h(t,1,t^{-q},t^{q-1})|t\in\mathbb{F},t^{q^2-q+1}=1\}$ is a cyclic normal subgroup of order q^2-q+1 of $\bar{I}_4$,*
5. $\bar{I}_5=\langle X_\alpha,n_\alpha,h(t,1,t^q,t^{q^2})\,|\,t\in\mathbb{F},t^{q^3-1}=1\rangle$.

Proof: The $q-1$ characters $\chi_{\phi_1}\in\mathrm{Irr}(U_P)$ ($\phi_1\in\mathrm{Irr}(\mathbb{F}_q)$) differ only on the elements $x_{3\alpha+2\beta}(d)$, $d\in\mathbb{F}_q^\times$. By [9], Table 2.3, these elements are permuted transitively by L_P. Thus, the $q-1$ characters χ_{ϕ_1} ($\phi_1\in\mathrm{Irr}(\mathbb{F}_q)$) form one orbit under the action of L_P. Hence, $|\bar{I}_5|=|L_P|/(q-1)=q^3(q^6-1)$. By [9], Tables 2.2–2.4, we get $\langle X_\alpha,n_\alpha,h(t,1,t^q,t^{q^2})\,|\,t\in\mathbb{F},t^{q^3-1}=1\rangle\subseteq\bar{I}_5$. By comparing orders (using Bruhat decomposition), equality holds.

It is not difficult to prove the assertions about $\bar{I}_0$, $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$. We demonstrate the computations for $\bar{I}_3$; the calculations for $\bar{I}_0$, $\bar{I}_1$ and $\bar{I}_2$ are similar. Each $u\in U_P$ can be written uniquely as $u=x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)$ with $d_1,d_4,d_5\in\mathbb{F}_q$ and $d_2,d_3\in\mathbb{F}_{q^3}$. By the Bruhat decomposition, each element $x\in L_P$ can be written uniquely as $x=h(t_1,t_2)x_\alpha(s)$ or $x=x_\alpha(r)h(t_1,t_2)n_\alpha x_\alpha(s)$ with $r,s\in\mathbb{F}_{q^3}$, $t_1\in\mathbb{F}_{q^3}^\times$, $t_2\in\mathbb{F}^\times$. Let us first assume $x=h(t_1,t_2)x_\alpha(s)\in\bar{I}_3$. Then, in particular, $\psi_3(x^{-1}ux)=\psi_3(u)$ for all $u\in U_P$ which is equivalent with

$$\begin{aligned}\phi(d_1t_1^{q^2+q+1}t_2^{-2}+d_4t_1^{-q^2-q-1}t_2-\mathrm{Tr}(s^{q^2}d_3t_1^{-q^2-q-1}t_2)+\\ \mathrm{Tr}(s^{q^2+q}d_2t_1^{-q^2-q-1}t_2)+s^{q^2+q+1}d_1t_1^{-q^2-q-1}t_2)=\phi(d_1+d_4).\end{aligned}$$

for all $d_1, d_4, d_5 \in \mathbb{F}_q$ and $d_2, d_3 \in \mathbb{F}_{q^3}$. Hence,

$$\begin{aligned}(5) \qquad & d_1(t_1^{q^2+q+1}t_2^{-2} + s^{q^2+q+1}t_1^{-q^2-q-1}t_2 - 1) + \mathrm{Tr}(s^{q^2+q}d_2t_1^{-q^2-q-1}t_2) \\ & -\mathrm{Tr}(s^{q^2}d_3t_1^{-q^2-q-1}t_2) + d_4(t_1^{-q^2-q-1}t_2 - 1) \in \ker(\phi)\end{aligned}$$

for all $(d_1, d_2, d_3, d_4) \in \mathbb{F}_q \times \mathbb{F}_{q^3} \times \mathbb{F}_{q^3} \times \mathbb{F}_q$. Plugging in $d_1 = d_3 = d_4 = 0$ and the surjectivity of Tr imply $s = 0$. Then, (5) becomes

$$d_1(t_1^{q^2+q+1}t_2^{-2} - 1) + d_4(t_1^{-q^2-q-1}t_2 - 1) \in \ker(\phi)$$

for all $d_1, d_4 \in \mathbb{F}_q$. Plugging in $d_1 = 0$, $d_4 = 0$ resp. implies $t_1^{q^2+q+1} = t_2 = 1$. On the other hand, by the relations in [9], Tables 2.2 and 2.3 it is straightforward to verify that $x = h(t, 1) \in \bar{I}_3$ for all $t \in \mathbb{F}$ with $t^{q^2+q+1} = 1$. The elements $x = x_\alpha(r)h(t_1, t_2)n_\alpha x_\alpha(s) \in \bar{I}_3$ can be treated analogously. This proves the assertions about $\bar{I}_3$ in (b) and (c).

Finally, we prove the assertions about $\bar{I}_4$. By Table A.8, exactly 6 conjugacy classes of P are contained in U_P, i.e. the action of L_P (by conjugation) on the conjugacy classes of U_P has exactly 6 orbits. By Brauer‘s Permutation Lemma (Satz V.13.5 in [12]), the action of L_P on $\mathrm{Irr}(U_P)$ also has exactly 6 orbits. Thus, the action of L_P on the set of linear characters of U_P has exactly 5 orbits. As $\bar{I}_0, \dots, \bar{I}_3$ have different orders, $\psi_0, \dots, \psi_3$ are in different orbits under the action of L_P. Summation over the size of these orbits shows, that the size of the missing orbit is $q^3(q^3 - 1)(q^2 - 1)/2$ and hence, that the inertia factor group of a representative of this orbit has order $2(q^2 - q + 1)$. By Table 5.1, L_P has a cyclic subgroup $\{h(t, 1, t^{-q}, t^{q-1}) \,|\, t \in \mathbb{F}, t^{q^2-q+1} = 1\}$ of order $q^2 - q + 1$. Let h be a generator of this subgroup. By the orders of the centralizers in Table A.8, there is a non–central conjugacy class of U_P which is invariant under $\langle h\rangle$. By Brauer's Permutation Lemma, there is a nontrivial linear character ψ_4 of U_P which is fixed by h. Hence, $|\bar{I}_4| = 2(q^2 - q + 1)$ and $\{h(t, 1, t^{-q}, t^{q-1}) \,|\, t \in \mathbb{F}, t^{q^2-q+1} = 1\}$ is a cyclic normal subgroup of $\bar{I}_4$ of index 2. Let $n_4 \in \bar{I}_4$ be an involution. Then:

$$\bar{I}_4 = \langle n_4\rangle \ltimes \{h(t, 1, t^{-q}, t^{q-1}) \,|\, t \in \mathbb{F}, t^{q^2-q+1} = 1\}.$$

Once we have proved $C_{\bar{I}_4}(n_4) = \langle n_4\rangle$, Proposition 5.3 will be established in full. Assume the contrary. Then, there would be an element of $C_{L_P}(n_4) - \{1\}$, whose order divides $q^2 - q + 1$. By the orders of the representatives and the centralizers in Tables 5.1 and 5.2 (and because $q^2 - q + 1$ and $(q^3 - 1)(q - 1)$ are relatively prime) it follows that n_4 would be central in L_P. Thus, $\bar{I}_4$ would be cyclic and there would be an element s of order $2(q^2 - q + 1)$, which fixes a nontrivial linear character of U_P. Again by the Permutation Lemma, there would be a non–central conjugacy class C of U_P, which is fixed by the cyclic group $S := \langle s\rangle$ of order $2(q^2 - q + 1)$. By (13.8) in [13], the element s would centralize an element of C, in particular a non–central element of U_P. Because s has order $2(q^2 - q + 1)$, the semisimple element s would be contained in one of the conjugacy classes $c_{11,0}$ of P. But the rows corresponding to $c_{11,0}$, $c_{11,1}$ in Table A.8 show, that all elements of U_P, which are centralized by such semisimple elements, are central in U_P, a contradiction. ■

Using Proposition 5.3, we construct the irreducible characters of P. We proceed in two steps: In the first step, we construct all irreducible characters χ of P with $Z(U_P) = X_{3\alpha+2\beta} \subseteq \ker(\chi)$, i.e. in fact we compute the irreducible characters of $\bar{P} := P/Z(U_P)$. Our main tool in this step is the following Lemma:

Lemma 5.4. *Let H_1 be an abelian normal subgroup of a finite group H and let K be a complement of H_1 in H. Let $\psi_1, \dots, \psi_m$ be representatives for the orbits of H on $Irr(H_1)$. For each $i \in \{1, \dots, m\}$, let $\bar{I}_i$ be the intersection of K with the inertia subgroup of ψ_i in H. For each $\psi \in Irr(\bar{I}_i)$, the map $\psi \ltimes \psi_i : \bar{I}_i H_1 \to \mathbb{C}$, $xu \mapsto \psi(x)\psi_i(u)$ is a character of $\bar{I}_i H_1$ extending ψ_i. Furthermore, for each $i \in \{1, \dots, m\}$, the induced character $(\psi \ltimes \psi_i)^G$ is irreducible and for each $\chi \in Irr(G)$ there is a unique $i \in \{1, \dots, m\}$ and a unique $\psi \in Irr(\bar{I}_i)$ such that $\chi = (\psi \ltimes \psi_i)^G$.*

Proof: This is probably well known. That $\psi \ltimes \psi_i$ is a character follows from the fact that H_1 is abelian. The other assertions follow from [13], (6.11) and (6.17). ■

We apply the Lemma to the abelian normal subgroup $U_P/Z(U_P) = U_P/X_{3\alpha+2\beta}$ of $P/Z(U_P)$ and its complement $L_P Z(U_P)/Z(U_P)$, which we identify with L_P. In the second step, we compute those irreducible characters of P covering ψ_5 (ψ_5 as in Proposition 5.3). For this, we use the restriction of the unipotent characters of ${}^3D_4(q)$, which have been computed by Spaltenstein ([17], Table 2).

The first step: The character table of L_P is given in Tables 5.5 and 5.6. The characters ${}_{L_P}\chi_3(k,l)$, ${}_{L_P}\chi_4(k)$ are (up to sign) generalized Deligne–Lusztig characters and their values can be computed by the character formula in [2], Theorem 7.2.8. The characters ${}_{L_P}\chi_1(k)$ are the linear characters of L_P and their values can be computed directly. The characters ${}_{L_P}\chi_2(k)$ are the products of the Steinberg character of L_P with the linear characters (for detailed computations, see [11], Abschnitt 4.3).

Table 5.5 *Parametrization of the irreducible characters of L_P:*

Character	Parameters	Number of Characters
${}_{L_P}\chi_1(k)$	$k = 0, \dots, q-2$	$q-1$
${}_{L_P}\chi_2(k)$	$k = 0, \dots, q-2$	$q-1$
${}_{L_P}\chi_3(k,l)$	$k = 0, \dots, q^3-2$; $l = 0, \dots, q-2$ $k \neq 0$	$(q^3-2)(q-1)/2$
${}_{L_P}\chi_4(k)$	$k = 0, \dots, q^4-q^3+q-2$ $k \neq (q^3+1)m$, $m = 0, \dots, q-2$	$q^3(q-1)/2$

Table 5.6 *The character table of L_P:*

	$c_{1,0}(i)$	$c_{1,1}(i)$	$c_{2,0}(i,j)$	$c_{3,0}(i)$
${}_{L_P}\chi_1(k)$	ζ_1^{2ik}	ζ_1^{2ik}	ζ_1^{jk}	ζ_1^{ik}
${}_{L_P}\chi_2(k)$	$q^3\zeta_1^{2ik}$	0	ζ_1^{jk}	$-\zeta_1^{ik}$
${}_{L_P}\chi_3(k,l)$	$(q^3+1)\zeta_1^{ik+2il}$	ζ_1^{ik+2il}	$\zeta_3^{ik}\zeta_1^{jl} + \zeta_3^{-ik}\zeta_1^{(k+l)j}$	0
${}_{L_P}\chi_4(k)$	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	0	$-\eta_3^{ik} - \eta_3^{q^3 ik}$

Since $P = L_P \ltimes U_P$, we can extend ${}_{L_P}\chi_1(k)$, ${}_{L_P}\chi_2(k)$, ${}_{L_P}\chi_3(k,l)$, ${}_{L_P}\chi_4(k)$ to P (k, l as in Table 5.5) and obtain the irreducible characters ${}_P\chi_1(k)$, ${}_P\chi_2(k)$, ${}_P\chi_3(k,l)$, ${}_P\chi_4(k)$ of P. These are the irreducible characters of P covering $\psi_0 = \mathbf{1}_{U_P}$.

By Proposition 5.3, $\bar{I}_1$ is the semidirect product of the abelian normal subgroup $H_1 := X_\alpha$ and $K := \{h(t^2, t^{q^2+q+1}, t^{2q}, t^{2q^2}) \,|\, t \in \mathbb{F}, t^{q^3-1} = 1\}$. By Lemma 5.4, $\bar{I}_1$ has exactly q^3 irreducible characters: q^3-1 linear characters and one irreducible character of degree q^3-1. Again, by Lemma 5.4, P has exactly q^3 irreducible characters covering $\psi_1 \in \mathrm{Irr}(U_P)$, namely the q^3-1 characters $(\psi \ltimes \psi_1)^P$, where ψ

runs through the linear characters of $\bar{I}_1$, and $(\psi \ltimes \psi_1)^P$, where ψ is the non–linear irreducible character of $\bar{I}_1$. Hence, the irreducible characters of P covering ψ_1 are ${}_P\chi_5(k) := {}_B\chi_3((q^2+q-1)k)^P$ for $k = 0, \dots, q^3-2$ and ${}_P\chi_6 := {}_B\chi_4{}^P$. (We use the factor q^2+q-1 in the definition of ${}_P\chi_5$ only to get a simpler parametrization of these characters. Note, that q^2+q-1 and q^3-1 are relatively prime.)

Analogously, P has exactly $2q$ irreducible characters covering ψ_2: the characters ${}_P\chi_7(k) := {}_B\chi_5(k)^P$, $k = 0, \dots, q-2$, and ${}_P\chi_8(k) := {}_B\chi_6^P$, $k = 1, \dots, q+1$ (to see that the $q-1$ distinct characters ${}_P\chi_7(k)$ are irreducible and cover ψ_2 one computes ψ_2^P and verifies $({}_P\chi_7(k), {}_P\chi_7(k))_P = ({}_P\chi_7(k), \psi_2^P)_P = 1$ for $k = 0, \dots, q-2$).

By Proposition 5.3, $\bar{I}_3$ is the semidirect product of $K := \langle h(1,-1,1,1)n_\alpha \rangle$ and $H_1 := \{h(t,1,t^q,t^{q^2}) \,|\, t \in \mathbb{F}_{q^3}, t^{q^2+q+1} = 1\}$. By Lemma 5.4, $\bar{I}_3$ has $(q^2+q)/2+2$ irreducible characters: 2 linear characters and $(q^2+q)/2$ irreducible characters of degree 2. Let $\varepsilon_{\bar{I}_3}$ be the nontrivial linear character of $\bar{I}_3$. Again, by Lemma 5.4, P has exactly $(q^2+q)/2+2$ irreducible characters covering ψ_3: namely the characters ${}_P\chi_9 := (\mathbf{1}_{\bar{I}_3} \ltimes \psi_3)^P$ and ${}_P\chi_{10} := (\varepsilon_{\bar{I}_3} \ltimes \psi_3)^P$ and $(\psi \ltimes \psi_3)^P$, where ψ runs through a set $\mathcal{S}$ of representatives of the orbits of K on $\mathrm{Irr}(H_1)$ not containing $\mathbf{1}_{H_1}$.

By the definition of ${}_B\chi_{13}(0)$, we have ${}_B\chi_{13}(0)^P = {}_P\chi_9 + {}_P\chi_{10}$. By construction, ${}_P\chi_9$, ${}_P\chi_{10}$ coincide on all conjugacy classes of P, except for the classes $c_{3,0}, \dots, c_{3,4}$, where the values of ${}_P\chi_9$, ${}_P\chi_{10}$ only differ in the sign. Hence, ${}_P\chi_9(x) = {}_P\chi_{10}(x) = \frac{1}{2}\left({}_B\chi_{13}(0)^P(x)\right)$ for all $x \in P$ with $x \notin c_{3,j}$. Let $x_j \in c_{3,j}$ for $j = 0, \dots, 4$. By the definition of induced characters, ${}_P\chi_9(x_0) = \frac{|C_P(x_0)|}{|C_{I_3}(x_0)|} = \frac{1}{2}(q^3-1)(q-1)$. We get the character values ${}_P\chi_9(x_1), \dots, {}_P\chi_9(x_4)$ from the conditions $({}_P\chi_9, {}_P\chi_1(0))_P = ({}_P\chi_9, {}_P\chi_5(0))_P = ({}_P\chi_9, {}_P\chi_7(0))_P = 0$ and $({}_P\chi_9, [\rho_2]_P)_P \in \mathbb{Z}$, where $[\rho_2]_P$ is the restriction of the unipotent character $[\rho_2] \in \mathrm{Irr}({}^3D_4(q))$ to P (see [17], Table 2). For $k = 1, \dots, q^2+q$, we define ${}_P\chi_{11}(k) := {}_B\chi_{13}(k)^P$. Then,

$$\{{}_P\chi_{11}(k) \,|\, k = 1, \dots, q^2+q\} = \{(\psi \ltimes \psi_3)^P \,|\, \psi \in \mathcal{S}\}.$$

${}_P\chi_9$, ${}_P\chi_{10}$, ${}_P\chi_{11}(k)$ are the irreducible characters of P covering $\psi_3 \in \mathrm{Irr}(U_P)$.

$\bar{I}_4$ is the semidirect product of $H_1 := \{h(t,1,t^{-q},t^{q-1}) \,|\, t \in \mathbb{F}, t^{q^2-q+1} = 1\}$ and $K := \langle n_4 \rangle$. By Lemma 5.4, we get: P has exactly $(q^2-q)/2+2$ irreducible characters covering ψ_4: namely ${}_P\chi_{12} := (\mathbf{1}_{\bar{I}_4} \ltimes \psi_4)^P$ and ${}_P\chi_{13} := (\varepsilon_{\bar{I}_4} \ltimes \psi_4)^P$ and the $(q^2-q)/2$ irreducible characters $(\psi \ltimes \psi_4)^P$, where ψ runs through a set $\mathcal{S}$ of representatives of the orbits of K on $\mathrm{Irr}(H_1)$ not containing $\mathbf{1}_{H_1}$. Let $H := H_1U_P$. For $k = 0, \dots, q^2-q$, the map $\vartheta_k : H \to \mathbb{C}^\times$, $h(\tilde{\varphi}_6^i, 1, \tilde{\varphi}_6^{-qi}, \tilde{\varphi}_6^{(q-1)i})u \mapsto \varphi_6^{ik}\psi_4(u)$ is an irreducible character of H, and we define ${}_P\chi_{14}(k) := \vartheta_k^P$. By construction, we have $\{{}_P\chi_{14}(k) \,|\, k = 1, \dots, q^2-q\} = \{(\psi \ltimes \psi_4)^P \,|\, \psi \in \mathcal{S}\}$. The values of ${}_P\chi_{14}(k)$ can be determined as follows: ${}_P\chi_{14}(k)$ has degree $q^3(q^3-1)(q^2-1)$. Because of $X_{3\alpha+2\beta} \subseteq \ker({}_P\chi_{14}(k))$, the character ${}_P\chi_{14}(k)$ has the value $q^3(q^3-1)(q^2-1)$ on the conjugacy classes $c_{1,0}$ and $c_{1,1}$ of P. We denote the values of ${}_P\chi_{14}(k)$ on the classes $c_{1,2}$, $c_{1,3}$, $c_{1,5}$, $c_{1,7}$ by x_2, x_3, x_5, x_7 resp. By the construction of ${}_P\chi_{14}(k)$, the x_i do not depend on k. By the definition of induced characters, ${}_P\chi_{14}(k)$ has the values $(q^2-1)(\varphi_6^{ik} + \varphi_6^{-ik})$ on the classes $c_{10,0}(i)$ and $c_{10,1}(i)$. Again by the definition of induced characters, ${}_P\chi_{14}(k)$ has the values $x_{10}(\varphi_6^{ik} + \varphi_6^{-ik})$ on the classes $c_{10,2}(i)$ with a suitable $x_{10} \in \mathbb{C}$ independent from i and k. Furthermore, ${}_P\chi_{14}(k)$ vanishes on all other conjugacy classes. We get the x_i's and hence the remaining values of ${}_P\chi_{14}(k)$ from the conditions $({}_P\chi_{14}(k), {}_P\chi_1(0)) = ({}_P\chi_{14}(k), {}_P\chi_2(0)) = ({}_P\chi_{14}(k), {}_P\chi_5(0)) = ({}_P\chi_{14}(k), {}_P\chi_7(0)) = ({}_P\chi_{14}(k), {}_P\chi_9) = 0$.

Then, we get the values of ${}_P\chi_{12}$ and ${}_P\chi_{13}$ from the condition ${}_P\chi_{12}+{}_P\chi_{13} = {}_P\chi_{14}(0)$ analogously to the values of ${}_P\chi_9$ and ${}_P\chi_{10}$. The characters ${}_P\chi_{12}$, ${}_P\chi_{13}$, ${}_P\chi_{14}(k)$ are the irreducible characters of P covering ψ_4.

To construct the characters of P covering $\psi_5 \in \mathrm{Irr}(U_P)$, we use the unipotent characters $\mathbf{1}$, $[\varepsilon_1]$, $[\rho_1]$, $[\rho_2]$, ${}^3D_4[-1]$, ${}^3D_4[1]$, $[\varepsilon_2]$, $\mathbf{St}$ of ${}^3D_4(q)$ given in [17], Table 2. For any character χ of ${}^3D_4(q)$, we denote its restriction to P by χ_P.

Let ${}_P\chi_{15} := [\varepsilon_1]_P - {}_P\chi_1(0) - {}_P\chi_5(0)$ and ${}_P\chi_{21}(k) := {}_B\chi_{17}(k)^P$, $k = 0, \dots, q^3-2$. Then, let ${}_P\chi_{16} := {}_P\chi_{21}(0) - {}_P\chi_{15}$ and ${}_P\chi_{22}(k) := {}_P\chi_4(k) \cdot {}_P\chi_{15}$, $k = 0, \dots, q^3$. Next, let ${}_P\chi_{17} := {}^3D_4[1]_P - {}_P\chi_{10} - {}_P\chi_{15} - \sum' {}_P\chi_{21}(k)$, where $\sum'$ means the sum over all *different* characters ${}_P\chi_{21}(k)$ with $q^2+q+1|k$, $k \neq 0, (q^3-1)/2$. Then, we set ${}_P\chi_{18} := {}_P\chi_{21}((q^3-1)/2) - {}_P\chi_{17}$. Finally, we set

$$ {}_P\chi_{19} := {}^3D_4[-1]_P - {}_P\chi_{13} - \sum\nolimits'' {}_P\chi_{22}(k). $$

and

$$ {}_P\chi_{20} := [\rho_2]_P - \mathbf{1}_P - {}_P\chi_2(0) - {}_P\chi_5(0) - {}_P\chi_7(0) - {}_P\chi_{12} - {}_P\chi_{16} - \sum\nolimits'' {}_P\chi_{22}(k) $$

where $\sum''$ is the sum over all *different* ${}_P\chi_{22}(k)$ with $q^2-q+1|k$, $k \neq 0, (q^3+1)/2$.

Theorem 5.7. *The character table of the parabolic subgroup P is given by Tables A.9 and A.10 in the Appendix.*

Proof: Computing scalar products with CHEVIE, we see that ${}_P\chi_1(k)$, …, ${}_P\chi_{22}(k)$ are $q^4+q^3+q^2+3q+7$ irreducible and pairwise different characters. ∎

Corollary 5.8. *For $j = 0, \dots, 5$ there exists $\theta_j \in Irr(I_j)$ with $(\theta_j)_{U_P} = \psi_j$.*

Proof: For $j = 0, \dots, 4$ this follows from Lemma 5.4. For ψ_5 it follows from the degree of of ${}_P\chi_{15}$. ∎

6. The Character Table of a Maximal Parabolic Subgroup Q

Let $\mathbf{Q} = \langle \mathbf{B}, n_{r_2} \rangle$ be the F–stable maximal parabolic subgroup of $\mathbf{G}$ corresponding to the subset $\{r_2\} \subseteq \Delta$ and $Q := \mathbf{Q}^F$ be the corresponding maximal parabolic subgroup of $\mathbf{G}^F = {}^3D_4(q)$. Then, Q is generated by B and n_β and $|Q| = q^{12}(q^3-1)(q^2-1)$. In this section, we determine the conjugacy classes and the character table of Q.

Q is the semidirect product of the Levi complement $L_Q = \langle \mathbf{T}^F, X_\beta, X_{-\beta} \rangle$ and the unipotent radical $U_Q := X_\alpha X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. The conjugacy classes of Q can be computed by the same methods as those that we have used for the conjugacy classes of P. The Levi complement $\mathbf{L}_Q = \langle \mathbf{T}, X_{\pm r_2} \rangle$ of $\mathbf{Q}$ is F–stable and $L_Q = (\mathbf{L}_Q)^F$ is a finite group of Lie type of order $|L_Q| = q(q^3-1)(q^2-1)$. The Weyl group $\mathbf{W}_{\mathbf{L}_Q} = \langle w_{r_2} \rangle$ of $\mathbf{L}_Q$ has order 2 and $w_0 = 1$ and $w_1 = w_{r_2}$ are representatives for the F–conjugacy classes of $\mathbf{W}_{\mathbf{L}_Q}$. Thus, $\mathbf{L}_Q$ has exactly two L_Q–conjugacy classes of F–stable maximal tori, and we can choose representatives $\mathbf{T}_0 = \mathbf{T}$ and $\mathbf{T}_1$ such that the corresponding maximal tori $\mathbf{T}^F$ and $\mathbf{T}_1^F$ of L_Q are given by $T_0 = \mathbf{T}^F$ and $T_1 = \mathbf{T}^{(Fw_1^{-1})}$ as in Section 3. The conjugacy classes of Q can be determined analogously to those of P by first determining the conjugacy classes of L_Q and then investigating the splitting of these classes in Q. The parabolic subgroup Q has exactly $q^4+2q^3+q^2+3q+7$ conjugacy classes. Representatives are given in Tables A.11 and A.12.

We use Clifford theory applied to the Levi decomposition $Q = L_Q \ltimes U_Q$ to construct the irreducible characters of Q. The structure of U_Q is more complicated than the structure of U_P because U_Q has nilpotency class 3. It appears to be difficult to determine the character table and even the conjugacy classes of U_Q generically. But we can compute those $\chi \in \mathrm{Irr}(U_Q)$ with $Z(U_Q) = X_{3\alpha+\beta}X_{3\alpha+2\beta} \subseteq \ker(\chi)$ and determine the number and degrees of the remaining irreducible characters of U_Q.

We start with some group theoretical properties of U_Q: Each $u \in U_Q$ can be written uniquely as $u = x_\alpha(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5)$ with $d_1, d_2, d_3 \in \mathbb{F}_{q^3} \subseteq \mathbb{F}$ and $d_4, d_5 \in \mathbb{F}_q \subseteq \mathbb{F}$. Hence, $|U_Q| = q^{11}$. The relations in [9], Table 2.2 show, that the center, derived subgroup and the Frattini subgroup of $\overline{U}_Q := U_Q/Z(U_Q)$ are equal to $\overline{X}_{2\alpha+\beta} := X_{2\alpha+\beta}Z(U_Q)/Z(U_Q)$. The irreducible characters of $\overline{U}_Q$ can be computed analogously to those of U_P. We get: U_Q has exactly q^6+q^3-1 irreducible characters χ with $Z(U_Q) \subseteq \ker(\chi)$, namely: q^6 linear characters χ_{ϕ_1,ϕ_2} defined by

$$x_\alpha(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5) \mapsto \phi_1(d_1)\phi_2(d_2)$$

with $(\phi_1, \phi_2) \in \mathrm{Irr}(\mathbb{F}_{q^3})\times\mathrm{Irr}(\mathbb{F}_{q^3})$ and q^3-1 irreducible characters χ_{ϕ_1} ($\phi_1 \in \mathrm{Irr}(\mathbb{F}_{q^3})$ nontrivial) of degree q^3 mapping $x_{2\alpha+\beta}(d_1)x_{3\alpha+\beta}(d_2)x_{3\alpha+2\beta}(d_3)$ to $q^3\phi_1(d_1)$ and vanishing on all other elements of U_Q. We get the number and degrees of the remaining irreducible characters of U_Q as follows: Using the commutator relations in [9], Table 2.2, it is straightforward to verify that the derived subgroup of U_Q is equal to $X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$ and that $U_Q/X_{3\alpha+2\beta}$ has exactly q^6+q^4-1 conjugacy classes. Since $X_{\alpha+\beta}X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}/X_{3\alpha+2\beta}$ is a normal abelian subgroup of index q^3 of $U_Q/X_{3\alpha+2\beta}$, Ito's Theorem [13], (6.15) and [13], (2.7) imply that $U_Q/X_{3\alpha+2\beta}$ has q^6 linear characters and q^4-1 irreducible characters of degree q^3. By [9], Tables 2.3 and 2.4, L_Q acts transitively on $\mathrm{Irr}(Z(U_Q)) - \{\mathbf{1}_{Z(U_Q)}\}$. Hence, each irreducible character of U_Q is conjugate (in Q) to some $\chi \in \mathrm{Irr}(U_Q)$ with $X_{3\alpha+2\beta} \subseteq \ker(\chi)$. Thus, U_Q has exactly q^6+q^5-1 irreducible characters: q^6 linear characters and q^5-1 irreducible characters of degree q^3.

We choose $\phi' \in \mathrm{Irr}(\mathbb{F}_{q^3})$, $\phi \in \mathrm{Irr}(\mathbb{F}_q)$, $\pi \in \mathbb{F}_q^\times$ as in Section 4 and we define $\psi_0, \psi_1, \psi_3 \in \mathrm{Irr}(U_Q)$ by: $\psi_0 := \chi_{\mathbf{1},\mathbf{1}}$, $\psi_1 := \chi_{\phi',\mathbf{1}}$ and $\psi_3 := \chi_{\phi'}$. Furthermore, we define linear characters φ_4, φ_5, φ_6 of $X_{\alpha+\beta}X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$ by mapping $x_{\alpha+\beta}(d_1)x_{2\alpha+\beta}(d_2)x_{3\alpha+\beta}(d_3)x_{3\alpha+2\beta}(d_4)$ to $\phi(d_3)$, $\phi'(d_1+d_3)$, $\phi'(d_1+\pi d_3)$ respectively. Let $\psi_j := \varphi_j^{U_Q}$ for $j = 4, 5, 6$. For $\psi \in \mathrm{Irr}(U_Q)$, let $I_Q(\psi) := \{x \in Q \,|\, {}^x\psi = \psi\}$ be the inertia subgroup of ψ in Q and we usually identify the inertia factor group $\bar{I}_Q(\psi) := I_Q(\psi_j)/U_Q$ with the corresponding subgroup of L_Q.

Proposition 6.1. *There is a linear character $\psi_2 \in Irr(U_Q)$, such that (a),(b),(c) are true:*

(a) $\{\psi_0, \dots, \psi_6\}$ *ist a set of representatives for the orbits of Q on $Irr(U_Q)$,*

(b) *the inertia factor groups $\bar{I}_j := \bar{I}_Q(\psi_j)$, $j = 0, 1, \dots, 6$, have the orders:*
$|\bar{I}_0| = q(q^3-1)(q^2-1)$, $|\bar{I}_1| = q(q-1)$, $|\bar{I}_2| = 1$, $|\bar{I}_3| = q(q^2-1)$,
$|\bar{I}_4| = q(q^3-1)$, $|\bar{I}_5| = 2q$, $|\bar{I}_6| = 2q$.

(c) *the inertia factor groups are given by:*
0. $\bar{I}_0 = L_Q$,
1. $\bar{I}_1 = \{h(t,t^2)x_\beta(s) \,|\, t \in \mathbb{F}, t^{q-1} = 1, s \in \mathbb{F}_q\}$,
2. $\bar{I}_2 = \{1\}$,
3. $\bar{I}_3 = \langle X_\beta, n_\beta, h(1,t) \,|\, t \in \mathbb{F}, t^{q-1} = 1\rangle$,
4. $\bar{I}_4 = \{h(t, t^{q^2+q+1})x_\beta(s) \,|\, t \in \mathbb{F}, t^{q^3-1} = 1, s \in \mathbb{F}_q\}$,

5. $\bar{I}_5 = \langle h(-1,-1)\rangle X_\beta$,
6. $\bar{I}_6 = \bar{I}_5$.

Proof: By construction, $\psi_0, \ldots, \psi_3 \in \mathrm{Irr}(U_Q)$ and $Z(U_Q) \subseteq \ker(\psi_0), \ldots, \ker(\psi_3)$. Hence, we can interpret $\psi_0, \ldots, \psi_3$ as irreducible characters of $\overline{U}_Q$. Let $\overline{Q} := Q/Z(U_Q)$ and $\overline{L}_Q := L_Q Z(U_Q)/Z(U_Q)$ (which we usually identify with L_Q). Using the decomposition $\overline{Q} = \bar{L}_Q \ltimes \overline{U}_Q$, the assertions about $\psi_0, \ldots, \psi_3$ can be proved analogously to Proposition 5.3.

The assertions about ψ_4, ψ_5, ψ_6 can be proved as follows: As $\psi_4(x_{3\alpha+\beta}(d_1)) = \psi_5(x_{3\alpha+\beta}(d_1)) = q^3\phi(d_1)$ and $\psi_6(x_{3\alpha+\beta}(d_1)) = q^3\phi(\pi \cdot d_1)$ for all $d_1 \in \mathbb{F}_q$, it follows $Z(U_Q) \not\subseteq \ker(\psi_4), \ker(\psi_5), \ker(\psi_6)$. Since $\psi_4(1) = \psi_5(1) = \psi_6(1) = q^3$, our classification of the irreducible characters of U_Q implies $\psi_4, \psi_5, \psi_6 \in \mathrm{Irr}(U_Q)$ and ψ_4, ψ_5, ψ_6 are not conjugate to $\psi_0, \ldots, \psi_3$. By Table A.12, exactly 7 unipotent conjugacy classes of Q are contained in U_Q. Hence, Brauer's Permutation Lemma implies, that the action of L_Q on $\mathrm{Irr}(U_Q)$ has exactly 7 orbits. Assume, that ψ_5 and ψ_6 are conjugate to each other under the action of L_Q. Then $\psi_5^Q = \psi_6^Q$. Using (3) and the class fusions in of B in Q (see Table A.4), it is easy to compute the values of $\psi_5^Q + \psi_6^Q$. Let $[\rho_1]_Q$ be the restriction of the unipotent character $[\rho_1]$ of ${}^3D_4(q)$ to Q (see [17], Table 2). By CHEVIE, we get $(\psi_5^Q + \psi_6^Q, [\rho_1]_Q)_Q = q$ which is odd. Hence, ψ_5 and ψ_6 are not L_Q–conjugate. Similarly, ψ_4 is not L_Q–conjugate to ψ_5 and not to ψ_6. This proves (a).

Using the relations in [9], Tables 2.2 and 2.3, it is straightforward to verify $\{h(t, t^{q^2+q+1})x_\beta(s) \,|\, t \in \mathbb{F}, t^{q^3-1} = 1, s \in \mathbb{F}_q\} \subseteq \bar{I}_4$ and $\langle h(-1,-1)\rangle X_\beta \subseteq \bar{I}_5, \bar{I}_6$. Hence, $|\bar{I}_4| \geq q(q^3-1)$ and $|\bar{I}_5|, |\bar{I}_6| \geq 2q$. By summation over the size of the orbits of L_Q on $\mathrm{Irr}(U_Q)$ we get equality. This completes the proof of (b) and (c). ■

The character table of L_Q can be computed by the same methods as that of L_P. Since $Q = L_Q \ltimes U_Q$ we can extend the characters of L_Q to Q. The characters ${}_Q\chi_1(k)$ in Tables A.13 and A.14 are the extensions of the linear characters of L_Q to Q. The characters ${}_Q\chi_2(k)$ are the extensions of the products of the linear characters with the Steinberg character of L_Q. The characters ${}_Q\chi_3(k,l)$, ${}_Q\chi_4(k)$ are (up to sign) extensions of irreducible generalized Deligne–Lusztig characters of L_Q. We have obtained all irreducible characters of Q covering $\psi_0 = \mathbf{1}_{U_Q}$.

By Proposition 6.1, $\bar{I}_1$ is the semidirect product of the abelian normal subgroup $H_1 := X_\beta$ and $K := \{h(t,t^2) \,|\, t \in \mathbb{F}, t^{q-1} = 1\}$. By Lemma 5.4, $\bar{I}_1$ has exactly q irreducible characters: $q-1$ linear characters and one irreducible character of degree $q-1$. Again, by Lemma 5.4, Q has exactly q irreducible characters covering $\psi_1 \in \mathrm{Irr}(U_Q)$, namely ${}_Q\chi_5(k) := {}_B\chi_2(k)^P$ for $k = 0, \ldots, q-2$ and ${}_Q\chi_6 := {}_B\chi_4{}^Q$.

Because of $\bar{I}_2 = \{1\}$, there is exactly one irreducible character ${}_Q\chi_7$ of Q, covering ψ_2. Since ${}_Q\chi_7$ is induced by a linear character of U_Q, it vanishes on all non–unipotent conjugacy classes of Q and on the unipotent classes $c_{1,2}$, $c_{1,5}$, $c_{1,6}$, $c_{1,7}$, $c_{1,9}$, $c_{1,10}$, $c_{1,12}$, $c_{1,14}$ of Q, which are not contained in U_Q. Because of $X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta} \subseteq \ker({}_Q\chi_7)$, the value of ${}_Q\chi_7$ on the classes $c_{1,0}$, $c_{1,1}$, $c_{1,3}$ is $q(q^2-1)(q^3-1)$. Hence, the only values of ${}_Q\chi_7$, which we do not know so far, are the values on the classes $c_{1,4}$, $c_{1,8}$, $c_{1,11}$, $c_{1,13}$. Due to its kernel, ${}_Q\chi_7$ has the same value, let us call it x, on the classes $c_{1,4}$, $c_{1,8}$ and $c_{1,11}$. Let y be the value of ${}_Q\chi_7$ on the conjugacy class $c_{1,13}$. Then, we get x and y by the conditions $({}_Q\chi_1(0), {}_Q\chi_7)_Q = ({}_Q\chi_5(0), {}_Q\chi_7)_Q = 0$. Hence, we have computed all values of ${}_Q\chi_7$ and have determined all irreducible characters of Q covering ψ_2.

For $k = 0, \dots, q-2$, we define ${}_Q\chi_{14}(k) := {}_B\chi_7(k)^Q$. Then, $Z(U_Q) \subseteq \ker({}_Q\chi_{14}(k))$ and by computing scalar products, we see that ${}_Q\chi_{14}(k)$ is irreducible for all $k \neq 0, (q-1)/2$. Hence, this gives us $(q-3)/2$ different irreducible characters of Q covering ψ_3. Next, we prove that ψ_3 extends to I_3. We define the linear character $\varphi : x_\beta(d_1)x_{\alpha+\beta}(d_2)x_{2\alpha+\beta}(d_3)x_{3\alpha+\beta}(d_4)x_{3\alpha+2\beta}(d_5) \mapsto \phi'(d_3)$ of the subgroup $N := X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. After computing the character values, we see that φ^U is an extension of ψ_3 to U (notice that φ^U vanishes on all elements of $U - N$). Corollary (8.16) and (11.31) in [13] imply that ψ_3 extends to I_3. Hence, by (6.11) and (6.17) in [13], there is a natural bijection between $\mathrm{Irr}(\bar{I}_3)$ and the set of irreducible characters of Q covering ψ_3. The character table of $\bar{I}_3$ is the same as the character table of $SL_2(q)$ and can be computed as in [16], p. 127f. This gives us a classification of the irreducible characters of Q covering ψ_3: one irreducible character ${}_Q\chi_8$ of degree $q^3(q^3-1)$, one irreducible character ${}_Q\chi_9$ of degree $q^4(q^3-1)$, two irreducible characters ${}_Q\chi_{10}$ and ${}_Q\chi_{11}$ of degree $\frac{1}{2}q^3(q^3-1)(q+1)$, two irreducible characters ${}_Q\chi_{12}$ and ${}_Q\chi_{13}$ of degree $\frac{1}{2}q^3(q^3-1)(q-1)$ and $\frac{q-3}{2}$ irreducible characters ${}_Q\chi_{14}(k)$ of degree $q^3(q^3-1)(q+1)$ and $\frac{q-1}{2}$ irreducible characters ${}_Q\chi_{15}(k)$ of degree $q^3(q^3-1)(q-1)$. We have already computed the characters ${}_Q\chi_{14}(k)$. We have $({}_Q\chi_{14}(0), {}_Q\chi_{14}(0))_Q = ({}_Q\chi_{14}(\frac{q-1}{2}), {}_Q\chi_{14}(\frac{q-1}{2}))_Q = 2$. Using (6.11) and (6.17) in [13], the character table of $\bar{I}_3$, the orthogonality relations for the character table of Q and the conditions ${}_Q\chi_{14}(0) = {}_Q\chi_{10} + {}_Q\chi_{11}$ and ${}_Q\chi_{14}(\frac{q-1}{2}) = {}_Q\chi_8 + {}_Q\chi_9$, one can compute ${}_Q\chi_8$, ${}_Q\chi_9$, ${}_Q\chi_{10}$, ${}_Q\chi_{11}$ explicitly. Then, we get the characters ${}_Q\chi_{15}(k)$ by ${}_Q\chi_{15}(k) = {}_Q\chi_4(k) \cdot {}_Q\chi_8$ for $k = 0, \dots, q$, which are irreducible for $k \neq 0, (q+1)/2$. Thus, we have constructed all irreducible characters χ of Q with $X_{3\alpha+\beta}X_{3\alpha+2\beta} \subseteq \ker(\chi)$ except for the two characters ${}_Q\chi_{12}$ and ${}_Q\chi_{13}$. We interpret the irreducible characters χ of Q with $X_{3\alpha+\beta}X_{3\alpha+2\beta} \subseteq \ker(\chi)$ as the irreducible characters of $\bar{Q} := Q/X_{3\alpha+\beta}X_{3\alpha+2\beta}$. The values of ${}_Q\chi_{12}$ and ${}_Q\chi_{13}$ follow from (6.11) and (6.17) in [13], the character table of $\bar{I}_3$, the row and column orthogonality relations for the character table of $\bar{Q}$ and the condition ${}_Q\chi_{15}(\frac{q+1}{2}) = {}_Q\chi_{12} + {}_Q\chi_{13}$. Hence, we have computed all irreducible characters of Q covering ψ_3.

Let ${}_Q\chi_{16}(k) := {}_B\chi_{12}(k)^Q$, $k = 0, \dots, q^3-2$, and let ${}_Q\chi_{17}(k) := {}_B\chi_{13}(k)^Q$, $k = 0, \dots, q^2+q$. Computing scalar products, we see that ${}_Q\chi_{16}(k)$ and ${}_Q\chi_{17}(k)$ are irreducible and pairwise different. These are the irreducible characters of Q covering ψ_4.

Let ${}_Q\chi_{18}(k) := {}_B\chi_{14}(k)^Q$ and ${}_Q\chi_{19}(k) := {}_B\chi_{15}(k)^Q$ for $k = 0, 1$, and let ${}_Q\chi_{20}(k) := {}_B\chi_{16}(k)^Q$ for $k = 1, \dots, q-1$. Using CHEVIE, we compute scalar products $(\sum_{k=0}^{1} {}_Q\chi_{18}(k), \sum_{k=0}^{1} {}_Q\chi_{18}(k))_Q = 2$, $(\sum_{k=0}^{1} {}_Q\chi_{19}(k), \sum_{k=0}^{1} {}_Q\chi_{19}(k))_Q = 2$, $(\sum_{k=0}^{1} {}_Q\chi_{18}(k), \sum_{k=0}^{1} {}_Q\chi_{19}(k))_Q = 0$ and $(\sum_{k=1}^{q-1} {}_Q\chi_{20}(k), \sum_{k=1}^{q-1} {}_Q\chi_{20}(k))_Q = q-1$. Hence, ${}_Q\chi_{18}(k)$, ${}_Q\chi_{19}(k)$, ${}_Q\chi_{20}(k)$ are irreducible and pairwise different.

Theorem 6.2. *The character table of the parabolic subgroup Q is given by Tables A.13 and A.14 in the Appendix.*

Proof: Computing scalar products with CHEVIE, we see that ${}_Q\chi_1(k), \dots, {}_Q\chi_{20}(k)$ are $q^4 + 2q^3 + q^2 + 3q + 7$ irreducible and pairwise different characters. ∎

Corollary 6.3. *For $j = 0, \dots, 6$ there exists $\theta_j \in Irr(I_j)$ with $(\theta_j)_{U_Q} = \psi_j$.*

Proof: For $j = 0, \dots, 2$ this follows from Lemma 5.4. For $j = 3, \dots, 6$ it follows from the degrees of ${}_Q\chi_8$, ${}_Q\chi_{16}(k)$, ${}_Q\chi_{18}(k)$ and ${}_Q\chi_{19}(k)$. ∎

Appendix

Table A.1 (cf. Deriziotis and Michler [4], Table 2.1) *Parametrization of the semisimple conjugacy classes of $^3D_4(q)$, q odd. (c is the multiplicative inverse of q^2+q-1 modulo $(q^3-1)(q+1)$.)*

Representative	Parameters	Number of Classes
$h_1 := h(1,1,1,1)$		1
$h_2 := h(-1,1,-1,-1)$		1
$h_3(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0,\dots,q-2$ $i \neq 0, \frac{q-1}{2}$	$\frac{q-3}{2}$
$h_4(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0,\dots,q^2+q$ $i \neq 0$	$\frac{q^2+q}{2}$
$h_5(i) := h(\tilde{\zeta}_3^i, 1, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2$ $i \neq (q-1)l,\ l = 0,\dots,q^2+q+1$ $i \neq \frac{q^3-1}{2}$	$\frac{q^3-q^2-q-3}{2}$
$h_6(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2;\ j = 0,\dots,q-2$ $i,j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l,\quad l = 0,\dots,q-2$ $i \neq (q^2+q+1)l$ or $j \neq 2l,\quad l = 0,\dots,q-2$ $i \neq j+(q-1)l,\quad l = 0,\dots,q^2+q$ $i \neq 2j+(q-1)l,\quad l = 0,\dots,q^2+q$	$\frac{1}{12}(q^4-4q^3$ $+2q^2-2q+15)$
$h_7(i) := h(1, \tilde{\xi}_1^{ci}, 1, 1)$	$i = 0,\dots,q$ $i \neq 0, \frac{q+1}{2}$	$\frac{q-1}{2}$
$h_8(i) := h(\tilde{\mu}_3^{(q^2+q)ci}, \tilde{\mu}_3^{(q^2+q+1)ci},$ $\tilde{\mu}_3^{(q^3+q^2)ci}, \tilde{\mu}_3^{(q+1)ci})$	$i = 0,\dots,q^4+q^3-q-2$ $i \neq (q+1)l,\ l = 0,\dots,q^3-2$ $i \neq (q^3-1)l,\ l = 0,\dots,q$	$\frac{q^4-2q+1}{4}$
$h_9(i) := h(\tilde{\varphi}_6^i, 1, \tilde{\varphi}_6^{-qi}, \tilde{\varphi}_6^{(q-1)i})$	$i = 0,\dots,q^2-q$ $i \neq 0$	$\frac{q^2-q}{2}$
$h_{10}(i) := h(\tilde{\xi}_3^i, 1, \tilde{\xi}_3^{-qi}, \tilde{\xi}_3^{q^2 i})$	$i = 0,\dots,q^3$ $i \neq 0, \frac{q^3+1}{2}$ $i \neq (q+1)l,\ l = 0,\dots,q^2-q$	$\frac{q^3-q^2+q-1}{2}$
$h_{11}(i) := h(\tilde{\eta}_3^i, \tilde{\eta}_3^{(q^3+1)i}, \tilde{\eta}_3^{q^4 i}, \tilde{\eta}_3^{q^2 i})$	$i = 0,\dots,q^4-q^3+q-2$ $i \neq (q-1)l,\ l = 0,\dots,q^3$ $i \neq (q^3+1)l,\ l = 0,\dots,q-2$	$\frac{q^4-2q^3+1}{4}$
$h_{12}(i,j) :=$ $h(\tilde{\varphi}_3^i, \tilde{\varphi}_3^j, \tilde{\varphi}_3^{qi+j}, \tilde{\varphi}_3^{(q+1)(j-i)})$	$i = 0,\dots,q^2+q$ $j = 0,\dots,q^2+q$ $j \neq 0, -2qi$ $2i \neq j, (1-q^2)j$	$\frac{q^4+2q^3-q^2-2q}{24}$
$h_{13}(i,j) :=$ $h(\tilde{\varphi}_6^i, \tilde{\varphi}_6^j, \tilde{\varphi}_6^{-qi+j}, \tilde{\varphi}_6^{(q-1)(i-j)})$	$i = 0,\dots,q^2-q$ $j = 0,\dots,q^2-q$ $j \neq 0, 2qi$ $2i \neq j, (1-q^2)j$	$\frac{q^4-2q^3-q^2+2q}{24}$
$h_{14}(i) := h(\tilde{\varphi}_{12}^i, \tilde{\varphi}_{12}^{(q^3+1)i}, \tilde{\varphi}_{12}^{qi}, \tilde{\varphi}_{12}^{q^2 i})$	$i = 0,\dots,q^4-q^2$ $i \neq 0$	$\frac{q^4-q^2}{4}$
$h_{15}(i,j) := h(\tilde{\xi}_3^i, \tilde{\xi}_1^j, \tilde{\xi}_3^{-qi}, \tilde{\xi}_3^{q^2 i})$	$i = 0,\dots,q^3$ $j = 0,\dots,q$ $i,j \neq 0$ $i \neq (q^2-q+1)l$ or $j \neq l,\quad l = 0,\dots,q$ $i \neq (q^2-q+1)l$ or $j \neq 2l,\quad l = 0,\dots,q$ $i \neq j+(q+1)l,\quad l = 0,\dots,q^2-q$ $i \neq 2j+(q+1)l,\quad l = 0,\dots,q^2-q$	$\frac{1}{12}(q^4-2q^3$ $+2q^2-4q+3)$

Table A.2 (cf. Deriziotis and Michler [4], Table 2.4, Geck [9], Table 3.1) *The conjugacy classes of* ${}^3D_4(q)$, q *odd. (In this table* $\zeta \in \mathbb{F}_q$ *is a non–square,* $a \in \mathbb{F}$ *with* $a^{q^3} = a$, $a^q \neq a$, *furthermore* $s \in \mathbb{F}$ *is a primitive* $2(q^2-1)$*–th root of unity and* $r \in \mathbb{F}$ *a root of the polynomial* $X^q - X + s^{q+1}$*.)*

Notation	Representative	$\lvert C_{{}^3D_4(q)}\rvert$
$c_{1,0}$	1	$q^{12}(q^6-1)^2(q^4-q^2+1)$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^6-1)$
$c_{1,2}$	$x_\alpha(1)$	$q^{10}(q^2-1)$
$c_{1,3}$	$x_\beta(1)x_{2\alpha+\beta}(-1)$	$2q^8(q^2+q+1)$
$c_{1,4}$	$x_\beta(1)x_{2\alpha+\beta}(-\zeta)$	$2q^8(q^2-q+1)$
$c_{1,5}$	$x_\alpha(1)x_{\alpha+\beta}(a)$	q^6
$c_{1,6}$	$x_\alpha(1)x_\beta(1)$	q^4
$c_{2,0}$	h_2	$q^4(q^6-1)(q^2-1)$
$c_{2,1}$	$h_2x_{3\alpha+2\beta}(1)$	$q^4(q^6-1)$
$c_{2,2}$	$h_2x_\alpha(1)$	$q^4(q^2-1)$
$c_{2,3}$	$h_2x_\beta(1)x_{2\alpha+\beta}(-1)$	$2q^4$
$c_{2,4}$	$h_2x_\beta(1)x_{2\alpha+\beta}(-\zeta)$	$2q^4$
$c_{3,0}(i)$	$h_3(i)$	$q^3(q^6-1)(q-1)$
$c_{3,1}(i)$	$h_3(i)x_\alpha(1)$	$q^3(q-1)$
$c_{4,0}(i)$	$h_4(i)$	$q^3(q^3-1)^2(q+1)$
$c_{4,1}(i)$	$h_4(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$
$c_{4,2}(i)$	$h_4(i)x_\beta(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$
$c_{5,0}(i)$	$h_5(i)$	$q(q^3-1)(q^2-1)$
$c_{5,1}(i)$	$h_5(i)x_{3\alpha+2\beta}(1)$	$q(q^3-1)$
$c_{6,0}(i,j)$	$h_6(i,j)$	$(q^3-1)(q-1)$
$c_{7,0}(i)$	$h_7(i)$	$q^3(q^6-1)(q+1)$
$c_{7,1}(i)$	$h_7(i)x_{2\alpha+\beta}(1)$	$q^3(q+1)$
$c_{8,0}(i)$	$h_8(i)$	$(q^3-1)(q+1)$
$c_{9,0}(i)$	$h_9(i)$	$q^3(q^3+1)^2(q-1)$
$c_{9,1}(i)$	$h_9(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3+1)$
$c_{9,2}(i)$	$h_9(i)x_\beta(s)x_{3\alpha+\beta}(-s^q)x_{3\alpha+2\beta}(r)$	$q^2(q^2-q+1)$
$c_{10,0}(i)$	$h_{10}(i)$	$q(q^3+1)(q^2-1)$
$c_{10,1}(i)$	$h_{10}(i)x_{3\alpha+2\beta}(1)$	$q(q^3+1)$
$c_{11,0}(i)$	$h_{11}(i)$	$(q^3+1)(q-1)$
$c_{12,0}(i,j)$	$h_{12}(i,j)$	$(q^2+q+1)^2$
$c_{13,0}(i,j)$	$h_{13}(i,j)$	$(q^2-q+1)^2$
$c_{14,0}(i)$	$h_{14}(i)$	q^4-q^2+1
$c_{15,0}(i,j)$	$h_{15}(i,j)$	$(q^3+1)(q+1)$

Table A.3 (cf. Geck [8], Tafel 3.8.2) *Parametrization of the semisimple conjugacy classes of B.*

Representative	Parameters	Number of Classes
$h_1 := h(1,1,1,1)$		1
$h_2 := h(-1,1,-1,-1)$		1
$h_3 := h(-1,-1,-1,-1)$		1
$h_4 := h(1,-1,1,1)$		1
$h_5(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0, \ldots, q^2+q$ $i \neq 0$	q^2+q
$h_6(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0, \ldots, q-2$ $i \neq 0, \frac{q-1}{2}$	$q-3$
$h_7(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0, \ldots, q-2$ $i \neq 0, \frac{q-1}{2}$	$q-3$
$h_8(i) := h(1, \tilde{\zeta}_1^i, 1, 1)$	$i = 0, \ldots, q-2$ $i \neq 0, \frac{q-1}{2}$	$q-3$
$h_9(i) := h(\tilde{\zeta}_3^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{2qi}, \tilde{\zeta}_3^{2q^2 i})$	$i = 0, \ldots, q^3-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^2+q+1$ $i \neq \frac{q^3-1}{2}$	q^3-q^2-q-3
$h_{10}(i) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^2+q+1$ $i \neq \frac{q^3-1}{2}$	q^3-q^2-q-3
$h_{11}(i) := h(\tilde{\zeta}_3^i, 1, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^2+q+1$ $i \neq \frac{q^3-1}{2}$	q^3-q^2-q-3
$h_{12}(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2$ $j = 0, \ldots, q-2$ $i \neq 0$ $j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l, \quad l = 0, \ldots, q-2$ $i \neq (q^2+q+1)l$ or $j \neq 2l, \quad l = 0, \ldots, q-2$ $i \neq j+(q-1)l, \quad l = 0, \ldots, q^2+q$ $i \neq 2j+(q-1)l, \quad l = 0, \ldots, q^2+q$	$q^4-4q^3+2q^2$ $-2q+15$

Table A.4 (cf. Geck [8], Tafeln 3.8.1,3.8.2) *The conjugacy classes of B. ($\zeta \in \mathbb{F}_q$ is a non–square. The parameters (i,b) in the representatives for the conjugacy classes of type $c_{1,12}$, $c_{1,16}$ run through the sets I_1, I_2 respectively with $|I_1| = \frac{q-3}{2}$ and $|I_2| = \frac{q-1}{2}$ respectively. The parameter a' in the representatives for the conjugacy classes of type $c_{1,17}$ runs through the set I_3 with $|I_3| = q+1$. The sets I_1, I_2, I_3 are defined in Section 4).*

Notation	Representative	$\lvert C_B\rvert$	Fusion in P	Fusion in Q
$c_{1,0}$	1	$q^{12}(q^3-1)(q-1)$	$c_{1,0}$	$c_{1,0}$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^3-1)$	$c_{1,1}$	$c_{1,1}$
$c_{1,2}$	$x_{3\alpha+\beta}(1)$	$q^{11}(q^3-1)$	$c_{1,2}$	$c_{1,1}$
$c_{1,3}$	$x_{\beta}(1)$	$q^8(q^3-1)$	$c_{1,2}$	$c_{1,2}$
$c_{1,4}$	$x_{2\alpha+\beta}(1)$	$q^{10}(q-1)$	$c_{1,3}$	$c_{1,3}$
$c_{1,5}$	$x_{\alpha+\beta}(1)$	$q^8(q-1)$	$c_{1,3}$	$c_{1,4}$
$c_{1,6}$	$x_{\beta}(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(2)$	q^8	$c_{1,3}$	$c_{1,5}$
$c_{1,7}$	$x_{\alpha}(1)$	$q^7(q-1)$	$c_{1,4}$	$c_{1,4}$
$c_{1,8}$	$x_{\beta}(1)x_{3\alpha+\beta}(1)$	$q^8(q^2+q+1)$	$c_{1,5}$	$c_{1,6}$
$c_{1,9}$	$x_{\beta}(1)x_{2\alpha+\beta}(-1)$	$2q^8$	$c_{1,5}$	$c_{1,7}$
$c_{1,10}$	$x_{\alpha+\beta}(1)x_{3\alpha+\beta}(1)$	$2q^8$	$c_{1,5}$	$c_{1,8}$
$c_{1,11}$	$x_{\alpha}(1)x_{3\alpha+2\beta}(-1)$	$2q^7$	$c_{1,6}$	$c_{1,8}$
$c_{1,12}(i,b)$	$x_{\beta}(1)x_{2\alpha+\beta}(\zeta^i)x_{3\alpha+\beta}(b)$	q^8	$c_{1,5}$	$c_{1,9}(i,b)$
$c_{1,13}$	$x_{\beta}(1)x_{2\alpha+\beta}(-\zeta)$	$2q^8$	$c_{1,7}$	$c_{1,10}$
$c_{1,14}$	$x_{\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^8$	$c_{1,7}$	$c_{1,11}$
$c_{1,15}$	$x_{\alpha}(1)x_{3\alpha+2\beta}(-\zeta)$	$2q^7$	$c_{1,8}$	$c_{1,11}$
$c_{1,16}(i,b)$	$x_{\beta}(1)x_{2\alpha+\beta}(\zeta^i)x_{3\alpha+\beta}(b)$	q^8	$c_{1,7}$	$c_{1,12}(i,b)$
$c_{1,17}(a')$	$x_{\alpha}(1)x_{\alpha+\beta}(a')$	q^6	$c_{1,9}(a')$	$c_{1,13}$
$c_{1,18}$	$x_{\alpha}(1)x_{\beta}(1)$	q^4	$c_{1,10}$	$c_{1,14}$
$c_{2,0}$	h_2	$q^4(q^3-1)(q-1)$	$c_{2,0}$	$c_{2,0}$
$c_{2,1}$	$h_2x_{3\alpha+2\beta}(1)$	$q^4(q^3-1)$	$c_{2,1}$	$c_{2,1}$
$c_{2,2}$	$h_2x_{\alpha}(1)$	$q^4(q-1)$	$c_{2,2}$	$c_{2,2}$
$c_{2,3}$	$h_2x_{\alpha}(1)x_{3\alpha+2\beta}(-1)$	$2q^4$	$c_{2,3}$	$c_{2,3}$
$c_{2,4}$	$h_2x_{\alpha}(1)x_{3\alpha+2\beta}(-\zeta)$	$2q^4$	$c_{2,4}$	$c_{2,4}$
$c_{3,0}$	h_3	$q^4(q^3-1)(q-1)$	$c_{3,0}$	$c_{2,0}$
$c_{3,1}$	$h_3x_{\alpha+\beta}(1)$	$q^4(q-1)$	$c_{3,1}$	$c_{2,2}$
$c_{3,2}$	$h_3x_{3\alpha+\beta}(1)$	$q^4(q^3-1)$	$c_{3,2}$	$c_{2,1}$
$c_{3,3}$	$h_3x_{\alpha+\beta}(1)x_{3\alpha+\beta}(1)$	$2q^4$	$c_{3,3}$	$c_{2,3}$
$c_{3,4}$	$h_3x_{\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^4$	$c_{3,4}$	$c_{2,4}$
$c_{4,0}$	h_4	$q^4(q^3-1)(q-1)$	$c_{3,0}$	$c_{3,0}$
$c_{4,1}$	$h_4x_{\beta}(1)$	$q^4(q^3-1)$	$c_{3,2}$	$c_{3,1}$
$c_{4,2}$	$h_4x_{2\alpha+\beta}(1)$	$q^4(q-1)$	$c_{3,1}$	$c_{3,2}$
$c_{4,3}$	$h_4x_{\beta}(1)x_{2\alpha+\beta}(-1)$	$2q^4$	$c_{3,3}$	$c_{3,3}$
$c_{4,4}$	$h_4x_{\beta}(1)x_{2\alpha+\beta}(-\zeta)$	$2q^4$	$c_{3,4}$	$c_{3,4}$
$c_{5,0}(i)$	$h_5(i)$	$q^3(q^3-1)(q-1)$	$c_{6,0}(i)$	$c_{6,0}(i)$
$c_{5,1}(i)$	$h_5(i)x_{\beta}(1)$	$q^2(q^3-1)$	$c_{6,1}(i)$	$c_{6,1}(i)$
$c_{5,2}(i)$	$h_5(i)x_{3\alpha+\beta}(1)$	$q^2(q^3-1)$	$c_{6,2}(i)$	$c_{6,2}(i)$
$c_{5,3}(i)$	$h_5(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$	$c_{6,3}(i)$	$c_{6,2}(i)$
$c_{5,4}(i)$	$h_5(i)x_{\beta}(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$	$c_{6,4}(i)$	$c_{6,3}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^3(q^3-1)(q-1)$	$c_{4,0}(i)$	$c_{4,0}(i)$
$c_{6,1}(i)$	$h_6(i)x_{\alpha}(1)$	$q^3(q-1)$	$c_{4,1}(i)$	$c_{4,1}(i)$

Table A.4 (continued) *Conjugacy classes of B.*

Notation	Representative	$\lvert C_B\rvert$	Fusion in P	Fusion in Q
$c_{7,0}(i)$	$h_7(i)$	$q^3(q^3-1)(q-1)$	$c_{5,0}(i)$	$c_{4,0}(i)$
$c_{7,1}(i)$	$h_7(i)x_{\alpha+\beta}(1)$	$q^3(q-1)$	$c_{5,1}(i)$	$c_{4,1}(i)$
$c_{8,0}(i)$	$h_8(i)$	$q^3(q^3-1)(q-1)$	$c_{5,0}(i)$	$c_{5,0}(i)$
$c_{8,1}(i)$	$h_8(i)x_{2\alpha+\beta}(1)$	$q^3(q-1)$	$c_{5,1}(i)$	$c_{5,1}(i)$
$c_{9,0}(i)$	$h_9(i)$	$q(q^3-1)(q-1)$	$c_{7,0}\left((q^2+q-1)i\right)$	$c_{7,0}(i)$
$c_{9,1}(i)$	$h_9(i)x_{\beta}(1)$	$q(q^3-1)$	$c_{7,1}\left((q^2+q-1)i\right)$	$c_{7,1}(i)$
$c_{10,0}(i)$	$h_{10}(i)$	$q(q^3-1)(q-1)$	$c_{7,0}(i)$	$c_{8,0}(i)$
$c_{10,1}(i)$	$h_{10}(i)x_{3\alpha+\beta}(1)$	$q(q^3-1)$	$c_{7,1}(i)$	$c_{8,1}(i)$
$c_{11,0}(i)$	$h_{11}(i)$	$q(q^3-1)(q-1)$	$c_{8,0}(i)$	$c_{8,0}(i)$
$c_{11,1}(i)$	$h_{11}(i)x_{3\alpha+2\beta}(1)$	$q(q^3-1)$	$c_{8,1}(i)$	$c_{8,1}(i)$
$c_{12,0}(i,j)$	$h_{12}(i,j)$	$(q^3-1)(q-1)$	$c_{9,0}(i,j)$	$c_{9,0}(i,j)$

Table A.5 *Parametrization of the irreducible characters of B.*

Character	Parameters	Number of Characters
${}_B\chi_1(k,l)$	$k=0,\dots,q^3-2$; $l=0,\dots,q-2$	$(q^3-1)(q-1)$
${}_B\chi_2(k)$	$k=0,\dots,q-2$	$q-1$
${}_B\chi_3(k)$	$k=0,\dots,q^3-2$	q^3-1
${}_B\chi_4$		1
${}_B\chi_5(k)$	$k=0,\dots,q-2$	$q-1$
${}_B\chi_6(k)$	$k=1,\dots,q+1$	$q+1$
${}_B\chi_7(k)$	$k=0,\dots,q-2$	$q-1$
${}_B\chi_8$		1
${}_B\chi_9$		1
${}_B\chi_{10}$		1
${}_B\chi_{11}$		1
${}_B\chi_{12}(k)$	$k=0,\dots,q^3-2$	q^3-1
${}_B\chi_{13}(k)$	$k=0,\dots,q^2+q$	q^2+q+1
${}_B\chi_{14}(k)$	$k=0,1$	2
${}_B\chi_{15}(k)$	$k=0,1$	2
${}_B\chi_{16}(k)$	$k=1,\dots,q-1$	$q-1$
${}_B\chi_{17}(k)$	$k=0,\dots,q^3-2$	q^3-1
${}_B\chi_{18}$		1
${}_B\chi_{19}$		1
${}_B\chi_{20}$		1
${}_B\chi_{21}$		1

Table A.6 *The character table of B. (In this table, zeros are replaced by dots. The parameter $\delta \in \{1, -1\}$ depends on the residue class of q modulo 4 and is defined in Section 2. See Table 2.2 for notation for the irrational character values.)*

	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$	$c_{1,6}$	$c_{1,7}$
${}_B\chi_1(k,l)$	1	1	1	1	1	1	1	1
${}_B\chi_2(k)$	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1	-1
${}_B\chi_3(k)$	$q-1$	$q-1$	$q-1$	-1	$q-1$	$q-1$	-1	$q-1$
${}_B\chi_4$	$(q^3-1)(q-1)$	$(q^3-1)(q-1)$	$(q^3-1)(q-1)$	$-(q^3-1)$	$(q^3-1)(q-1)$	$(q^3-1)(q-1)$	$-(q^3-1)$	$-(q-1)$
${}_B\chi_5(k)$	$q(q^3-1)$	$q(q^3-1)$	$q(q^3-1)$	.	$q(q^3-1)$	$-q$	.	$q(q^2-1)$
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	$q(q+1)(q^3-1)(q-1)$	$q(q+1)(q^3-1)(q-1)$	$q(q+1)(q^3-1)(q-1)$	.	$q(q+1)(q^3-1)(q-1)$	$-(q^3-q)$	.	$-(q^3-q)$
${}_B\chi_7(k)$	$q^3(q^3-1)$	$q^3(q^3-1)$	$q^3(q^3-1)$	.	$-q^3$	.	q^3	.
${}_B\chi_8$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	.	$-\frac{1}{2}q^2(q+1)$	.
${}_B\chi_9$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	.	$-\frac{1}{2}q^2(q+1)$	.
${}_B\chi_{10}$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$-\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	.	$-\frac{1}{2}q^2(q-1)$	.
${}_B\chi_{11}$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$\frac{1}{2}q^3(q^3-1)(q-1)$	$-\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	.	$-\frac{1}{2}q^2(q-1)$	.
${}_B\chi_{12}(k)$	$q^3(q-1)$	$q^3(q-1)$	$-q^3$	$-(q^3-q)$	.	.	q	.
${}_B\chi_{13}(k)$	$q^3(q-1)^2$	$q^3(q-1)^2$	$-q^3(q-1)$	$q(q-1)(q+1)$	.	.	$-q$	.
$\sum_{k=0}^{1} {}_B\chi_{14}(k)$	$q^3(q-1)(q^3-1)$	$q^3(q-1)(q^3-1)$	$-q^3(q^3-1)$	$q(q^3-1)$	.	.	$-q(q^2+1)$	.
$\sum_{k=0}^{1} {}_B\chi_{15}(k)$	$q^3(q-1)(q^3-1)$	$q^3(q-1)(q^3-1)$	$-q^3(q^3-1)$	$q(q^3-1)$	.	.	$-q(q^2+1)$	.
$\sum_{k=0}^{q-1} {}_B\chi_{16}(k)$	$q^3(q-1)^2(q^3-1)$	$q^3(q-1)^2(q^3-1)$	$-q^3(q^3-1)(q-1)$	$-q(q^3-1)$	.	.	$q(q^2+1)$	.
${}_B\chi_{17}(k)$	$q^4(q-1)$	$-q^4$	.	.	.	.	.	.
${}_B\chi_{18}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$-\frac{1}{2}q^4(q^3-1)$	.	.	.	.	.	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_{19}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$-\frac{1}{2}q^4(q^3-1)$	.	.	.	.	.	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_{20}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$-\frac{1}{2}q^4(q^3-1)$	.	.	.	.	.	$\frac{1}{2}q^3(q-1)$
${}_B\chi_{21}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$-\frac{1}{2}q^4(q^3-1)$	.	.	.	.	.	$\frac{1}{2}q^3(q-1)$

Table A.6 (continued) *The character table of B.*

	$c_{1,8}$	$c_{1,9}$	$c_{1,10}$	$c_{1,11}$	$c_{1,12}(i,b)$	$c_{1,13}$	$c_{1,14}$	$c_{1,15}$	$c_{1,16}(i,b)$
${}_B\chi_1(k,l)$	1	1	1	1	1	1	1	1	1
${}_B\chi_2(k)$	q^3-1	q^3-1	q^3-1	-1	q^3-1	q^3-1	q^3-1	-1	q^3-1
${}_B\chi_3(k)$	-1	-1	$q-1$	$q-1$	-1	-1	$q-1$	$q-1$	-1
${}_B\chi_4$	$-(q^3-1)$	$-(q^3-1)$	$(q^3-1)(q-1)$	$-(q-1)$	$-(q^3-1)$	$-(q^3-1)$	$(q^3-1)(q-1)$	$-(q-1)$	$-(q^3-1)$
${}_B\chi_5(k)$	.	.	$-q$	$q(q^2-1)$	.	.	$-q$	$q(q^2-1)$	.
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	.	.	$-(q^3-q)$	$-(q^3-q)$	.	.	$-(q^3-q)$	$-(q^3-q)$	.
${}_B\chi_7(k)$	.	$q^3\delta$	.	.	$(-1)^i q^3$	$-q^3\delta$	.	.	$(-1)^i q^3$
${}_B\chi_8$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}(q\delta+1)q^2$	.	.	$-\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$	$\frac{1}{2}(q\delta-1)q^2$	.	.	$-\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$
${}_B\chi_9$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}(q\delta+1)q^2$	.	.	$-\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$	$\frac{1}{2}(q\delta-1)q^2$	.	.	$-\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$
${}_B\chi_{10}$	$-\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}(q\delta-1)q^2$	.	.	$\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$	$\frac{1}{2}(q\delta+1)q^2$	.	.	$\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$
${}_B\chi_{11}$	$-\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}(q\delta-1)q^2$	.	.	$\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$	$\frac{1}{2}(q\delta+1)q^2$	.	.	$\frac{1}{2}q^2-\frac{1}{2}(-1)^i q^3$
${}_B\chi_{12}(k)$	$q(q+1)$	$q(q+1)$	q^2	.	$q(q+1)$	$-q(q-1)$	$-q^2$	.	$-q(q-1)$
${}_B\chi_{13}(k)$	$-q(q+1)$	$-q(q+1)$	$q^2(q-1)$	.	$-q(q+1)$	$q(q-1)$	$-q^2(q-1)$	.	$q(q-1)$
$\sum_{k=0}^{1} {}_B\chi_{14}(k)$	$-q(q^2+q+1)$	$q(q^3-q^2-q-1)$	$-q^2$	.	$-q(q^2+q+1)$	$q(q-1)(q^2+1)$	q^2	.	$-q(q^2-q+1)$
$\sum_{k=0}^{1} {}_B\chi_{15}(k)$	$-q(q^2+q+1)$	$q(q^3-q^2-q-1)$	$-q^2$	.	$-q(q^2+q+1)$	$q(q-1)(q^2+1)$	q^2	.	$-q(q^2-q+1)$
$\sum_{k=0}^{q-1} {}_B\chi_{16}(k)$	$q(q^2+q+1)$	$-q(q^3-q^2-q-1)$	$-q^2(q-1)$	.	$q(q^2+q+1)$	$-q(q-1)(q^2+1)$	$q^2(q-1)$	.	$q(q^2-q+1)$
${}_B\chi_{17}(k)$	.	.	.	q^2	.	.	.	$-q^2$	.
${}_B\chi_{18}$	.	.	.	$\frac{1}{2}q^2(q-1)$	.	.	.	$\frac{1}{2}q^2(q+1)$	.
${}_B\chi_{19}$	.	.	.	$\frac{1}{2}q^2(q-1)$	.	.	.	$\frac{1}{2}q^2(q+1)$	.
${}_B\chi_{20}$	.	.	.	$-\frac{1}{2}q^2(q+1)$	.	.	.	$-\frac{1}{2}q^2(q-1)$	.
${}_B\chi_{21}$	.	.	.	$-\frac{1}{2}q^2(q+1)$	.	.	.	$-\frac{1}{2}q^2(q-1)$	.

Table A.6 (continued) *The character table of B.*

	$c_{1,17}(a')$	$c_{1,18}$	$c_{2,0}$	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$c_{2,4}$	$c_{3,0}$	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$
${}_B\chi_1(k,l)$	1	1	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^{k+l}$	$(-1)^{k+l}$	$(-1)^{k+l}$	$(-1)^{k+l}$
${}_B\chi_2(k)$	-1	-1	$(q^3-1)(-1)^k$	$(q^3-1)(-1)^k$	$-(-1)^k$	$-(-1)^k$	$-(-1)^k$	.	.	.	.
${}_B\chi_3(k)$	$q-1$	-1	.	.	.	.	.	.	.	.	.
${}_B\chi_4$	$-(q-1)$	1	.	.	.	.	.	.	.	.	.
${}_B\chi_5(k)$	$-q$	.	.	.	.	.	.	$(-1)^k(q^3-1)$	$-(-1)^k$	$(q^3-1)(-1)^k$	$-(-1)^k$
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	q	.	.	.	.	.	.	.	.	.	.
${}_B\chi_7(k)$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_8$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_9$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{10}$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{11}$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{12}(k)$	.	.	.	.	.	.	.	$(-1)^k(q-1)$	$(-1)^k(q-1)$	$-(-1)^k$	$-(-1)^k$
${}_B\chi_{13}(k)$	.	.	.	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1} {}_B\chi_{14}(k)$	.	.	.	.	.	.	.	$(q-1)(q^3-1)$	$-(q-1)$	$-(q^3-1)$	1
$\sum_{k=0}^{1} {}_B\chi_{15}(k)$	.	.	.	.	.	.	.	$-(q-1)(q^3-1)$	$q-1$	q^3-1	-1
$\sum_{k=0}^{q-1} {}_B\chi_{16}(k)$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{17}(k)$	.	.	$(-1)^k(q-1)$	$-(-1)^k$	$(-1)^k(q-1)$	$-(-1)^k$	$-(-1)^k$	.	.	.	.
${}_B\chi_{18}$	.	.	$\frac{1}{2}(q-1)(q^3-1)$	$-\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q-1)$	$\frac{1}{2}-\frac{1}{2}q^2\delta$	$\frac{1}{2}+\frac{1}{2}q^2\delta$	.	.	.	.
${}_B\chi_{19}$	.	.	$-\frac{1}{2}(q-1)(q^3-1)$	$\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}+\frac{1}{2}q^2\delta$	$-\frac{1}{2}-\frac{1}{2}q^2\delta$	.	.	.	.
${}_B\chi_{20}$	.	.	$\frac{1}{2}(q-1)(q^3-1)$	$-\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q-1)$	$\frac{1}{2}+\frac{1}{2}q^2\delta$	$\frac{1}{2}-\frac{1}{2}q^2\delta$	.	.	.	.
${}_B\chi_{21}$	.	.	$-\frac{1}{2}(q-1)(q^3-1)$	$\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}-\frac{1}{2}q^2\delta$	$-\frac{1}{2}+\frac{1}{2}q^2\delta$	.	.	.	.

Table A.6 (continued) *The character table of B.*

	$c_{3,4}$	$c_{4,0}$	$c_{4,1}$	$c_{4,2}$	$c_{4,3}$	$c_{4,4}$	$c_{5,0}(i)$	$c_{5,1}(i)$	$c_{5,2}(i)$	$c_{5,3}(i)$
${}_B\chi_1(k,l)$	$(-1)^{k+l}$	$(-1)^l$	$(-1)^l$	$(-1)^l$	$(-1)^l$	$(-1)^l$	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}
${}_B\chi_2(k)$	.	.	.	.	.	.	.	.	.	.
${}_B\chi_3(k)$	.	$(-1)^k(q-1)$	$-(-1)^k$	$(-1)^k(q-1)$	$-(-1)^k$	$-(-1)^k$	${\varphi_3}^{-\frac{q^2+q}{2}ik}(q-1)$	$-{\varphi_3}^{-\frac{q^2+q}{2}ik}$	${\varphi_3}^{-\frac{q^2+q}{2}ik}(q-1)$	${\varphi_3}^{-\frac{q^2+q}{2}ik}(q-1)$
${}_B\chi_4$	.	.	.	.	.	.	.	.	.	.
${}_B\chi_5(k)$	$-(-1)^k$	.	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	.	.	.	.	.	.	.	.	.	.
${}_B\chi_7(k)$	.	$(q^3-1)(-1)^k$	$(q^3-1)(-1)^k$	$-(-1)^k$	$-(-1)^k$	$-(-1)^k$	.	.	.	.
${}_B\chi_8$	.	$-\frac{1}{2}(q^3-1)(q-1)$	$\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q^2+1)$	$\frac{1}{2}(q^2-1)$	.	.	.	.
${}_B\chi_9$	.	$\frac{1}{2}(q^3-1)(q-1)$	$-\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q-1)$	$\frac{1}{2}(q^2+1)$	$-\frac{1}{2}(q^2-1)$	.	.	.	.
${}_B\chi_{10}$	.	$-\frac{1}{2}(q^3-1)(q-1)$	$\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q^2-1)$	$-\frac{1}{2}(q^2+1)$	.	.	.	.
${}_B\chi_{11}$	.	$\frac{1}{2}(q^3-1)(q-1)$	$-\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q-1)$	$-\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2+1)$	.	.	.	.
${}_B\chi_{12}(k)$	$-(-1)^k$	.	.	.	.	.	$(q-1)\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$	$-\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$
${}_B\chi_{13}(k)$	.	.	.	.	.	.	$(q-1)^2\varphi_3^{ik}$	$-(q-1)\varphi_3^{ik}$	$-(q-1)\varphi_3^{ik}$	$(q-1)^2\varphi_3^{ik}$
$\sum_{k=0}^{1} {}_B\chi_{14}(k)$	1	.	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1} {}_B\chi_{15}(k)$	-1	.	.	.	.	.	.	.	.	.
$\sum_{k=0}^{q-1} {}_B\chi_{16}(k)$	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{17}(k)$	.	.	.	.	.	.	$q(q-1)\varphi_3^{ik}$	.	.	$-q\varphi_3^{ik}$
${}_B\chi_{18}$	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{19}$	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{20}$	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{21}$	.	.	.	.	.	.	.	.	.	.

Table A.6 (continued) *The character table of B.*

	$c_{5,4}(i)$	$c_{6,0}(i)$	$c_{6,1}(i)$	$c_{7,0}(i)$	$c_{7,1}(i)$	$c_{8,0}(i)$	$c_{8,1}(i)$	$c_{9,0}(i)$	$c_{9,1}(i)$	$c_{10,0}(i)$	$c_{10,1}(i)$	$c_{11,0}(i)$	$c_{11,1}(i)$	$c_{12,0}(i,j)$
${}_B\chi_1(k,l)$	φ_3^{ik}	ζ_1^{ik+2il}	ζ_1^{ik+2il}	ζ_1^{ik+il}	ζ_1^{ik+il}	ζ_1^{il}	ζ_1^{il}	$\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{il}$	ζ_3^{ik}	ζ_3^{ik}	$\zeta_3^{ik}\zeta_1^{jl}$
${}_B\chi_2(k)$	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_3(k)$	$-\varphi_3^{-\frac{q^2+q}{2}ik}$	.	.	.	.	.	.	$\zeta_3^{ik}(q-1)$	$-\zeta_3^{ik}$	.	.	.	.	.
${}_B\chi_4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_5(k)$	.	.	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_7(k)$	.	.	.	.	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.	.	.	.	.	.
${}_B\chi_8$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_9$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{10}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{11}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{12}(k)$	$-\varphi_3^{ik}$	.	.	.	.	.	.	.	.	$\zeta_3^{ik}(q-1)$	$-\zeta_3^{ik}$	.	.	.
${}_B\chi_{13}(k)$	φ_3^{ik}	.	.	.	.	.	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1} {}_B\chi_{14}(k)$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1} {}_B\chi_{15}(k)$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\sum_{k=0}^{q-1} {}_B\chi_{16}(k)$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{17}(k)$	.	.	.	.	.	.	.	.	.	.	.	$\zeta_3^{ik}(q-1)$	$-\zeta_3^{ik}$	.
${}_B\chi_{18}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{19}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{20}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{21}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table A.7 *Parametrization of the semisimple conjugacy classes of P.*

Representative	Parameters	Number of Classes
$h_1 := h(1,1,1,1)$		1
$h_2 := h(-1,1,-1,-1)$		1
$h_3 := h(-1,-1,-1,-1)$		1
$h_4(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0,\dots,q-2;\ i \neq 0, (q-1)/2$	$q-3$
$h_5(i) := h(1, \tilde{\zeta}_1^i, 1, 1)$	$i = 0,\dots,q-2;\ i \neq 0, (q-1)/2$	$q-3$
$h_6(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0,\dots,q^2+q;\ i \neq 0$	$\frac{q^2+q}{2}$
$h_7(i) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2;\ i \neq (q^3-1)/2$ $i \neq (q-1)l,\ l = 0,\dots,q^2+q+1$	q^3-q^2-q-3
$h_8(i) := h(\tilde{\zeta}_3^i, 1, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2;\ i \neq (q^3-1)/2$ $i \neq (q-1)l,\ l = 0,\dots,q^2+q+1$	$(q^3-q^2-q-3)/2$
$h_9(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2$ $j = 0,\dots,q-2$ $i, j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l$, $i \neq (q^2+q+1)l$ or $j \neq 2l$, $l = 0,\dots,q-2$ $i \neq j+(q-1)l, \quad l = 0,\dots,q^2+q$ $i \neq 2j+(q-1)l, \quad l = 0,\dots,q^2+q$	$\frac{q^4-4q^3+2q^2-2q+15}{2}$
$h_{10}(i) := h(\tilde{\varphi}_6^i, 1, \tilde{\varphi}_6^{-qi}, \tilde{\varphi}_6^{(q-1)i})$	$i = 0,\dots,q^2-q;\ i \neq 0$	$\frac{q^2-q}{2}$
$h_{11}(i) := h(\tilde{\xi}_3^i, 1, \tilde{\xi}_3^{-qi}, \tilde{\xi}_3^{q^2 i})$	$i = 0,\dots,q^3;\ i \neq 0, \frac{q^3+1}{2}$ $i \neq (q+1)l,\ l = 0,\dots,q^2-q$	$\frac{q^3-q^2+q-1}{2}$
$h_{12}(i) := h(\tilde{\eta}_3^i, \tilde{\eta}_3^{(q^3+1)i}, \tilde{\eta}_3^{q^4 i}, \tilde{\eta}_3^{q^2 i})$	$i = 0,\dots,q^4-q^3+q-2$ $i \neq (q-1)l,\ l = 0,\dots,q^3$ $i \neq (q^3+1)l,\ l = 0,\dots,q-2$	$\frac{q^4-2q^3+1}{2}$

Table A.8 *The conjugacy classes of P. ($\zeta \in \mathbb{F}_q$ is a non–square. The parameter a' in the representatives for the conjugacy classes of type $c_{1,9}$ runs through the set I_3 with $|I_3| = q+1$. The parameter set I_3 is defined in Section 4).*

Notation	Representative	$\lvert C_P \rvert$	Fusion in ${}^3D_4(q)$
$c_{1,0}$	1	$q^{12}(q^6-1)(q-1)$	$c_{1,0}$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^6-1)$	$c_{1,1}$
$c_{1,2}$	$x_{3\alpha+\beta}(1)$	$q^{11}(q^3-1)$	$c_{1,1}$
$c_{1,3}$	$x_{2\alpha+\beta}(1)$	$q^{10}(q-1)$	$c_{1,2}$
$c_{1,4}$	$x_\alpha(1)$	$q^7(q-1)$	$c_{1,2}$
$c_{1,5}$	$x_\beta(1)x_{3\alpha+\beta}(1)$	$2q^8(q^2+q+1)$	$c_{1,3}$
$c_{1,6}$	$x_\alpha(1)x_{3\alpha+2\beta}(-1)$	$2q^7$	$c_{1,3}$
$c_{1,7}$	$x_{\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^8(q^2-q+1)$	$c_{1,4}$
$c_{1,8}$	$x_\alpha(1)x_{3\alpha+2\beta}(-\zeta)$	$2q^7$	$c_{1,4}$
$c_{1,9}(a')$	$x_\alpha(1)x_{\alpha+\beta}(a')$	q^6	$c_{1,5}$
$c_{1,10}$	$x_\alpha(1)x_\beta(1)$	q^4	$c_{1,6}$
$c_{2,0}$	h_2	$q^4(q^6-1)(q-1)$	$c_{2,0}$
$c_{2,1}$	$h_2 x_{3\alpha+2\beta}(1)$	$q^4(q^6-1)$	$c_{2,1}$
$c_{2,2}$	$h_2 x_\alpha(1)$	$q^4(q-1)$	$c_{2,2}$
$c_{2,3}$	$h_2 x_\alpha(1)x_{3\alpha+2\beta}(-1)$	$2q^4$	$c_{2,3}$
$c_{2,4}$	$h_2 x_\alpha(1)x_{3\alpha+2\beta}(-\zeta)$	$2q^4$	$c_{2,4}$
$c_{3,0}$	h_3	$q^4(q^3-1)(q-1)$	$c_{2,0}$
$c_{3,1}$	$h_3 x_{\alpha+\beta}(1)$	$q^4(q-1)$	$c_{2,2}$
$c_{3,2}$	$h_3 x_{3\alpha+\beta}(1)$	$q^4(q^3-1)$	$c_{2,1}$
$c_{3,3}$	$h_3 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(1)$	$2q^4$	$c_{2,3}$
$c_{3,4}$	$h_3 x_{\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^4$	$c_{2,4}$

Table A.8 (continued)

Notation	Representative	$\lvert C_P\rvert$	Fusion in ${}^3D_4(q)$
$c_{4,0}(i)$	$h_4(i)$	$q^3(q^6-1)(q-1)$	$c_{3,0}(i)$
$c_{4,1}(i)$	$h_4(i)x_\alpha(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{5,0}(i)$	$h_5(i)$	$q^3(q^3-1)(q-1)$	$c_{3,0}(i)$
$c_{5,1}(i)$	$h_5(i)x_{2\alpha+\beta}(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^3(q^3-1)(q-1)$	$c_{4,0}(i)$
$c_{6,1}(i)$	$h_6(i)x_\beta(1)$	$q^2(q^3-1)$	$c_{4,1}(i)$
$c_{6,2}(i)$	$h_6(i)x_{3\alpha+\beta}(1)$	$q^2(q^3-1)$	$c_{4,1}(i)$
$c_{6,3}(i)$	$h_6(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$	$c_{4,1}(i)$
$c_{6,4}(i)$	$h_6(i)x_\beta(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$	$c_{4,2}(i)$
$c_{7,0}(i)$	$h_7(i)$	$q(q^3-1)(q-1)$	$c_{5,0}(i)$
$c_{7,1}(i)$	$h_7(i)x_{3\alpha+\beta}(1)$	$q(q^3-1)$	$c_{5,1}(i)$
$c_{8,0}(i)$	$h_8(i)$	$q(q^3-1)(q-1)$	$c_{5,0}(i)$
$c_{8,1}(i)$	$h_8(i)x_{3\alpha+2\beta}(1)$	$q(q^3-1)$	$c_{5,1}(i)$
$c_{9,0}(i,j)$	$h_9(i,j)$	$(q^3-1)(q-1)$	$c_{6,0}(i,j)$
$c_{10,0}(i)$	$h_{10}(i)$	$q^3(q^3+1)(q-1)$	$c_{9,0}(i)$
$c_{10,1}(i)$	$h_{10}(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3+1)$	$c_{9,1}(i)$
$c_{10,2}(i)$	$h_{10}(i)x_\beta(s)x_{3\alpha+\beta}(-s^q)x_{3\alpha+2\beta}(r)$	$q^2(q^2-q+1)$	$c_{9,2}(i)$
$c_{11,0}(i)$	$h_{11}(i)$	$q(q^3+1)(q-1)$	$c_{10,0}(i)$
$c_{11,1}(i)$	$h_{11}(i)x_{3\alpha+2\beta}(1)$	$q(q^3+1)$	$c_{10,1}(i)$
$c_{12,0}(i)$	$h_{12}(i)$	$(q^3+1)(q-1)$	$c_{11,0}(i)$

Table A.9 *Parametrization of the irreducible characters of P.*

Character	Parameters	Number of Characters
${}_P\chi_1(k)$	$k=0,\dots,q-2$	$q-1$
${}_P\chi_2(k)$	$k=0,\dots,q-2$	$q-1$
${}_P\chi_3(k,l)$	$k=0,\dots,q^3-2;\ l=0,\dots,q-2;\ k\neq 0$	$(q^3-2)(q-1)/2$
${}_P\chi_4(k)$	$k=0,\dots,q^4-q^3+q-2$ $k\neq(q^3+1)m,\ m=0,\dots,q-2$	$\frac{1}{2}q^3(q-1)$
${}_P\chi_5(k)$	$k=0,\dots,q^3-2$	q^3-1
${}_P\chi_6$		1
${}_P\chi_7(k)$	$k=0,\dots,q-2$	$q-1$
${}_P\chi_8(k)$	$k=1,\dots,q+1$	$q+1$
${}_P\chi_9$		1
${}_P\chi_{10}$		1
${}_P\chi_{11}(k)$	$k=0,\dots,q^2+q;\ k\neq 0$	$(q^2+q)/2$
${}_P\chi_{12}$		1
${}_P\chi_{13}$		1
${}_P\chi_{14}(k)$	$k=0,\dots,q^2-q;\ k\neq 0$	$\frac{1}{2}(q^2-q)$
${}_P\chi_{15}$		1
${}_P\chi_{16}$		1
${}_P\chi_{17}$		1
${}_P\chi_{18}$		1
${}_P\chi_{19}$		1
${}_P\chi_{20}$		1
${}_P\chi_{21}(k)$	$k=0,\dots,q^3-2;\ k\neq 0,(q^3-1)/2$	$(q^3-3)/2$
${}_P\chi_{22}(k)$	$k=0,\dots,q^3;\ k\neq 0,(q^3+1)/2$	$(q^3-1)/2$

Table A.10 *The character table of P. (In this table, zeros are replaced by dots. The parameter $\delta \in \{1, -1\}$ depends on the residue class of q modulo 4 and is defined in Section 2. See Table 2.2 for notation for the irrational character values.)*

	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$	$c_{1,6}$
${}_P\chi_1(k)$	1	1	1	1	1	1	1
${}_P\chi_2(k)$	q^3	q^3	q^3	q^3	.	q^3	.
${}_P\chi_3(k,l)$	q^3+1	q^3+1	q^3+1	q^3+1	1	q^3+1	1
${}_P\chi_4(k)$	q^3-1	q^3-1	q^3-1	q^3-1	-1	q^3-1	-1
${}_P\chi_5(k)$	$(q-1)(q^3+1)$	$(q-1)(q^3+1)$	$-(q^3-q+1)$	$q-1$	$q-1$	q^2+q-1	$q-1$
${}_P\chi_6$	$(q^6-1)(q-1)$	$(q^6-1)(q-1)$	$-(q^3-1)(q^3-q+1)$	$(q-1)^2(q^2+q+1)$	$-(q-1)$	$(q^3-1)(q^2+q-1)$	$-(q-1)$
${}_P\chi_7(k)$	$q(q^3-1)(q^3+1)$	$q(q^3-1)(q^3+1)$	$q(q^3-1)$	$q(q^3-q^2-1)$	$q(q-1)(q+1)$	$-q(q^2+q+1)$	$q(q-1)(q+1)$
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	$q(q-1)(q+1)(q^6-1)$	$q(q-1)(q+1)(q^6-1)$	$q(q-1)(q+1)(q^3-1)$	$q(q+1)(q-1)(q^3-q^2-1)$	$-q(q-1)(q+1)$	$-q(q+1)(q^3-1)$	$-q(q-1)(q+1)$
${}_P\chi_9$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q^3-2q+1)$	$-\frac{1}{2}q^3(q-1)$	.	q^3	.
${}_P\chi_{10}$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q^3-2q+1)$	$-\frac{1}{2}q^3(q-1)$	.	q^3	.
${}_P\chi_{11}(k)$	$q^3(q-1)^2(q^3+1)$	$q^3(q-1)^2(q^3+1)$	$q^3(q^3-2q+1)$	$-q^3(q-1)$	.	$2q^3$	.
${}_P\chi_{12}$	$\frac{1}{2}q^3(q^2-1)(q^3-1)$	$\frac{1}{2}q^3(q^3-1)(q^2-1)$	$-\frac{1}{2}(q^6-q^3)$	$-\frac{1}{2}(q^4-q^3)$	.	.	.
${}_P\chi_{13}$	$\frac{1}{2}q^3(q^2-1)(q^3-1)$	$\frac{1}{2}q^3(q^2-1)(q^3-1)$	$-\frac{1}{2}q^3(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	.	.	.
${}_P\chi_{14}(k)$	$q^3(q^2-1)(q^3-1)$	$q^3(q^2-1)(q^3-1)$	$-q^3(q^3-1)$	$-q^3(q-1)$	.	.	.
${}_P\chi_{15}$	$q^4(q-1)$	$-q^4$	.	.	.	.	q^2
${}_P\chi_{16}$	$q^7(q-1)$	$-q^7$	.	.	.	.	.
${}_P\chi_{17}$	$\frac{1}{2}q^4(q-1)(q^3+1)$	$-\frac{1}{2}(q^7+q^4)$	.	.	$-\frac{1}{2}(q^4-q^3)$	.	$\frac{1}{2}(q^3+q^2)$
${}_P\chi_{18}$	$\frac{1}{2}q^4(q-1)(q^3+1)$	$-\frac{1}{2}(q^7+q^4)$	.	.	$\frac{1}{2}(q^4-q^3)$	.	$-\frac{1}{2}(q^3-q^2)$
${}_P\chi_{19}$	$\frac{1}{2}q^4(q-1)(q^3-1)$	$-\frac{1}{2}(q^7-q^4)$	.	.	$-\frac{1}{2}(q^4-q^3)$	.	$\frac{1}{2}(q^3-q^2)$
${}_P\chi_{20}$	$\frac{1}{2}q^4(q-1)(q^3-1)$	$-\frac{1}{2}(q^7-q^4)$	.	.	$\frac{1}{2}(q^4-q^3)$	.	$-\frac{1}{2}(q^3+q^2)$
${}_P\chi_{21}(k)$	$q^4(q-1)(q^3+1)$	$-q^4(q^3+1)$	.	.	.	.	q^2
${}_P\chi_{22}(k)$	$q^4(q-1)(q^3-1)$	$-q^4(q^3-1)$	.	.	.	.	$-q^2$

Table A.10 (continued) *The character table of P.*

	$c_{1,7}$	$c_{1,8}$	$c_{1,9}(a')$	$c_{1,10}$	$c_{2,0}$	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$c_{2,4}$
${}_P\chi_1(k)$	1	1	1	1	1	1	1	1	1
${}_P\chi_2(k)$	q^3	.	.	.	q^3	q^3	.	.	.
${}_P\chi_3(k,l)$	q^3+1	1	1	1	$(q^3+1)(-1)^k$	$(q^3+1)(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$
${}_P\chi_4(k)$	q^3-1	-1	-1	-1	$(q^3-1)(-1)^k$	$(q^3-1)(-1)^k$	$-(-1)^k$	$-(-1)^k$	$-(-1)^k$
${}_P\chi_5(k)$	$-(q^2-q+1)$	$q-1$	$q-1$	-1	.	.	.	.	.
${}_P\chi_6$	$-(q^3-1)(q^2-q+1)$	$-(q-1)$	$-(q-1)$	1	.	.	.	.	.
${}_P\chi_7(k)$	$-q(q^2-q+1)$	$q(q-1)(q+1)$	$-q$	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	$-q(q-1)(q^3+1)$	$-q(q-1)(q+1)$	q	.	.	.	.	.	.
${}_P\chi_9$	.	.	.	.	.	.	.	.	.
${}_P\chi_{10}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{11}(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_{12}$	q^3	.	.	.	.	.	.	.	.
${}_P\chi_{13}$	q^3	.	.	.	.	.	.	.	.
${}_P\chi_{14}(k)$	$2q^3$	.	.	.	.	.	.	.	.
${}_P\chi_{15}$	.	$-q^2$	.	.	$q-1$	-1	$q-1$	-1	-1
${}_P\chi_{16}$	.	.	.	.	$q^3(q-1)$	$-q^3$	.	.	.
${}_P\chi_{17}$	.	$\frac{1}{2}(q^3-q^2)$	.	.	$\frac{1}{2}\delta(q-1)(q^3+1)$	$-\frac{1}{2}q^3\delta-\frac{1}{2}\delta$	$\frac{1}{2}q\delta-\frac{1}{2}\delta$	$-\frac{1}{2}\delta-\frac{1}{2}q^2$	$-\frac{1}{2}\delta+\frac{1}{2}q^2$
${}_P\chi_{18}$	.	$-\frac{1}{2}(q^3+q^2)$	.	.	$\frac{1}{2}\delta(q-1)(q^3+1)$	$-\frac{1}{2}(q^3+1)\delta$	$\frac{1}{2}(q-1)\delta$	$-\frac{1}{2}\delta+\frac{1}{2}q^2$	$-\frac{1}{2}\delta-\frac{1}{2}q^2$
${}_P\chi_{19}$	.	$\frac{1}{2}(q^3+q^2)$	.	.	$-\frac{1}{2}(q^3-1)(q-1)\delta$	$\frac{1}{2}(q^3-1)\delta$	$\frac{1}{2}(q-1)\delta$	$-\frac{1}{2}\delta+\frac{1}{2}q^2$	$-\frac{1}{2}\delta-\frac{1}{2}q^2$
${}_P\chi_{20}$	.	$-\frac{1}{2}(q^3-q^2)$	.	.	$-\frac{1}{2}(q^3-1)(q-1)\delta$	$\frac{1}{2}(q^3-1)\delta$	$\frac{1}{2}(q-1)\delta$	$-\frac{1}{2}\delta-\frac{1}{2}q^2$	$-\frac{1}{2}\delta+\frac{1}{2}q^2$
${}_P\chi_{21}(k)$	.	$-q^2$	.	.	$(q-1)(-1)^k(q^3+1)$	$-(q^3+1)(-1)^k$	$(-1)^k(q-1)$	$-(-1)^k$	$-(-1)^k$
${}_P\chi_{22}(k)$	.	q^2	.	.	$(q^3-1)(q-1)(-1)^k$	$-(q^3-1)(-1)^k$	$-(-1)^k(q-1)$	$(-1)^k$	$(-1)^k$

Table A.10 (continued) *The character table of P.*

	$c_{3,0}$	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$c_{3,4}$	$c_{4,0}(i)$	$c_{4,1}(i)$	$c_{5,0}(i)$	$c_{1,1}(i)$
${}_P\chi_1(k)$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	ζ_1^{2ik}	ζ_1^{2ik}	ζ_1^{ik}	ζ_1^{ik}
${}_P\chi_2(k)$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$q^3\zeta_1^{2ik}$	.	ζ_1^{ik}	ζ_1^{ik}
${}_P\chi_3(k,l)$	$(-1)^{k+l}+(-1)^l$	$(-1)^{k+l}+(-1)^l$	$(-1)^{k+l}+(-1)^l$	$(-1)^{k+l}+(-1)^l$	$(-1)^{k+l}+(-1)^l$	$(q^3+1)\zeta_1^{ik+2il}$	ζ_1^{ik+2il}	$\zeta_1^{ik+il}+\zeta_1^{il}$	$\zeta_1^{ik+il}+\zeta_1^{il}$
${}_P\chi_4(k)$	.	.	.	.	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.
${}_P\chi_5(k)$	$(-1)^k(q-1)$	$(-1)^k(q-1)$	$-(-1)^k$	$-(-1)^k$	$-(-1)^k$	.	.	.	.
${}_P\chi_6$	.	.	.	.	.	.	.	.	.
${}_P\chi_7(k)$	$(q^3-1)(-1)^k$	$-(-1)^k$	$(q^3-1)(-1)^k$	$-(-1)^k$	$-(-1)^k$	.	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_9$	$\frac{1}{2}(q-1)(q^3-1)$	$-\frac{1}{2}(q-1)$	$-\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q^2+1)$	$-\frac{1}{2}(q^2-1)$	.	.	.	.
${}_P\chi_{10}$	$-\frac{1}{2}(q-1)(q^3-1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q^2+1)$	$\frac{1}{2}(q^2-1)$	.	.	.	.
${}_P\chi_{11}(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_{12}$	$\frac{1}{2}(q-1)(q^3-1)$	$-\frac{1}{2}(q-1)$	$-\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2+1)$	.	.	.	.
${}_P\chi_{13}$	$-\frac{1}{2}(q-1)(q^3-1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q^2-1)$	$-\frac{1}{2}(q^2+1)$	.	.	.	.
${}_P\chi_{14}(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_{15}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{16}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{17}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{18}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{19}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{20}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{21}(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_{22}(k)$	.	.	.	.	.	.	.	.	.

Table A.10 (continued) *The character table of P.*

	$c_{6,0}(i)$	$c_{6,1}(i)$	$c_{6,2}(i)$	$c_{6,3}(i)$	$c_{6,4}(i)$	$c_{7,0}(i)$	$c_{7,1}(i)$
${}_P\chi_1(k)$	1	1	1	1	1	ζ_1^{ik}	ζ_1^{ik}
${}_P\chi_2(k)$	1	1	1	1	1	ζ_1^{ik}	ζ_1^{ik}
${}_P\chi_3(k,l)$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\zeta_3^{ik}\zeta_1^{il}+\zeta_3^{(q+1)qik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{il}+\zeta_3^{(q+1)qik}\zeta_1^{il}$
${}_P\chi_4(k)$	.	.	.	.	.	.	.
${}_P\chi_5(k)$	$(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	$q\varphi_3^{ik}-\varphi_3^{ik}-\varphi_3^{-ik}$	$q\varphi_3^{-ik}-\varphi_3^{-ik}-\varphi_3^{ik}$	$(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	$-\varphi_3^{ik}-\varphi_3^{-ik}$	$(q-1)\zeta_3^{ik}$	$-\zeta_3^{ik}$
${}_P\chi_6$	.	.	.	.	.	.	.
${}_P\chi_7(k)$	.	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	.	.	.	.	.	.	.
${}_P\chi_9$	$(q-1)^2$	$-(q-1)$	$-(q-1)$	$(q-1)^2$	1	.	.
${}_P\chi_{10}$	$(q-1)^2$	$-(q-1)$	$-(q-1)$	$(q-1)^2$	1	.	.
${}_P\chi_{11}(k)$	$(q-1)^2(\varphi_3^{ik}+\varphi_3^{-ik})$	$-(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	$-(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	$(q-1)^2(\varphi_3^{ik}+\varphi_3^{-ik})$	$\varphi_3^{ik}+\varphi_3^{-ik}$	.	.
${}_P\chi_{12}$	.	.	.	.	.	.	.
${}_P\chi_{13}$	.	.	.	.	.	.	.
${}_P\chi_{14}(k)$	.	.	.	.	.	.	.
${}_P\chi_{15}$	q^2-q	.	.	$-q$	.	.	.
${}_P\chi_{16}$	$q(q-1)$	.	.	$-q$	.	.	.
${}_P\chi_{17}$	q^2-q	.	.	$-q$	.	.	.
${}_P\chi_{18}$	q^2-q	.	.	$-q$	.	.	.
${}_P\chi_{19}$	.	.	.	.	.	.	.
${}_P\chi_{20}$	.	.	.	.	.	.	.
${}_P\chi_{21}(k)$	$q(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	.	.	$-q(\varphi_3^{ik}+\varphi_3^{-ik})$	.	.	.
${}_P\chi_{22}(k)$	.	.	.	.	.	.	.

Table A.10 (continued) *The character table of P.*

	$c_{8,0}(i)$	$c_{8,1}(i)$	$c_{9,0}(i,j)$	$c_{10,0}(i)$	$c_{10,1}(i)$	$c_{10,2}(i)$	$c_{11,0}(i)$	$c_{11,1}(i)$	$c_{12,0}(i)$
${}_P\chi_1(k)$	1	1	ζ_1^{jk}	1	1	1	1	1	ζ_1^{ik}
${}_P\chi_2(k)$	1	1	ζ_1^{jk}	-1	-1	-1	-1	-1	$-\zeta_1^{ik}$
${}_P\chi_3(k,l)$	$\zeta_3^{ik}+\zeta_3^{-ik}$	$\zeta_3^{ik}+\zeta_3^{-ik}$	$\zeta_3^{ik}\zeta_1^{jl}+\zeta_3^{-ik}\zeta_1^{(k+l)j}$	.	.	.	.	.	.
${}_P\chi_4(k)$	.	.	.	$-\varphi_6^{ik}-\varphi_6^{-ik}$	$-\varphi_6^{ik}-\varphi_6^{-ik}$	$-\varphi_6^{ik}-\varphi_6^{-ik}$	$-\xi_3^{-ik}-\xi_3^{ik}$	$-\xi_3^{-ik}-\xi_3^{ik}$	$-\eta_3^{q^3ik}-\eta_3^{ik}$
${}_P\chi_5(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_6$	.	.	.	.	.	.	.	.	.
${}_P\chi_7(k)$	.	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_9$	.	.	.	.	.	.	.	.	.
${}_P\chi_{10}$	.	.	.	.	.	.	.	.	.
${}_P\chi_{11}(k)$	.	.	.	.	.	.	.	.	.
${}_P\chi_{12}$	.	.	.	q^2-1	q^2-1	-1	.	.	.
${}_P\chi_{13}$	.	.	.	q^2-1	q^2-1	-1	.	.	.
${}_P\chi_{14}(k)$	.	.	.	$(q^2-1)(\varphi_6^{ik}+\varphi_6^{-ik})$	$(q^2-1)(\varphi_6^{ik}+\varphi_6^{-ik})$	$-\varphi_6^{ik}-\varphi_6^{-ik}$	.	.	.
${}_P\chi_{15}$	$q-1$	-1	.	$-(q^2-q)$	q	.	$q-1$	-1	.
${}_P\chi_{16}$	$q-1$	-1	.	$q(q-1)$	$-q$	.	$-(q-1)$	1	.
${}_P\chi_{17}$	$q(-1)^i-(-1)^i$	$-(-1)^i$	.	.	.	.	.	.	.
${}_P\chi_{18}$	$q(-1)^i-(-1)^i$	$-(-1)^i$	.	.	.	.	.	.	.
${}_P\chi_{19}$	.	.	.	q^2-q	$-q$	.	$-(-1)^iq+(-1)^i$	$(-1)^i$	.
${}_P\chi_{20}$	.	.	.	q^2-q	$-q$	.	$-(-1)^iq+(-1)^i$	$(-1)^i$	.
${}_P\chi_{21}(k)$	$(q-1)(\zeta_3^{ik}+\zeta_3^{-ik})$	$-\zeta_3^{ik}-\zeta_3^{-ik}$	.	.	.	.	.	.	.
${}_P\chi_{22}(k)$	.	.	.	$(q^2-q)(\varphi_6^{ik}+\varphi_6^{-ik})$	$-q(\varphi_6^{ik}+\varphi_6^{-ik})$	.	$-(q-1)(\xi_3^{ik}+\xi_3^{-ik})$	$\xi_3^{ik}+\xi_3^{-ik}$	.

Table A.11 *Parametrization of the semisimple conjugacy classes of Q. (c is the multiplicative inverse of q^2+q-1 modulo $(q^3-1)(q+1)$.)*

Representative	Parameters	Number of Classes
$h_1 := h(1,1,1,1)$		1
$h_2 := h(-1,1,-1,-1)$		1
$h_3 := h(1,-1,1,1)$		1
$h_4(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0, \ldots, q-2\,;\ i \neq 0, \frac{q-1}{2}$	$q-3$
$h_5(i) := h(1, \tilde{\zeta}_1^i, 1, 1)$	$i = 0, \ldots, q-2\,;\ i \neq 0, \frac{q-1}{2}$	$\frac{q-3}{2}$
$h_6(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0, \ldots, q^2+q\,;\ i \neq 0$	q^2+q
$h_7(i) := h(\tilde{\zeta}_3^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{2qi}, \tilde{\zeta}_3^{2q^2 i})$	$i = 0, \ldots, q^3-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^2+q+1$ $i \neq \frac{q^3-1}{2}$	q^3-q^2-q-3
$h_8(i) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^2+q+1$ $i \neq \frac{q^3-1}{2}$	q^3-q^2-q-3
$h_9(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2;\ j = 0, \ldots, q-2$ $i, j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l, \quad l = 0, \ldots, q-2$ $i \neq (q^2+q+1)l$ or $j \neq 2l, \quad l = 0, \ldots, q-2$ $i \neq j + (q-1)l, \quad l = 0, \ldots, q^2+q$ $i \neq 2j + (q-1)l, \quad l = 0, \ldots, q^2+q$	$\frac{q^4-4q^3+2q^2-2q+15}{2}$
$h_{10}(i) := h(1, \tilde{\xi}_1^{ci}, 1, 1)$	$i = 0, \ldots, q\,;\ i \neq 0, \frac{q+1}{2}$	$\frac{q-1}{2}$
$h_{11}(i) := h(\tilde{\mu}_3^{(q^2+q)ci},$ $\tilde{\mu}_3^{(q^2+q+1))ci}, \tilde{\mu}_3^{(q^3+q^2)ci}, \tilde{\mu}_3^{(q+1)ci})$	$i = 0, \ldots, q^4+q^3-q-2$ $i \neq (q+1)l,\ l = 0, \ldots, q^3-2$ $i \neq (q^3-1)l,\ l = 0, \ldots, q$	$\frac{q^4-2q+1}{2}$

Table A.12 *The conjugacy classes of Q. ($\zeta \in \mathbb{F}_q$ is a non–square, $a \in \mathbb{F}$ with $a^{q^3} = a$, $a^q \neq a$. The parameters (i,b) for the classes of type $c_{1,9}$, $c_{1,12}$ run through the sets I_1, I_2 respectively, with $|I_1| = \frac{q-3}{2}$, $|I_2| = \frac{q-1}{2}$, cf. Section 4).*

Notation	Representative	$\|C_Q\|$	Fusion in ${}^3D_4(q)$
$c_{1,0}$	1	$q^{12}(q^3-1)(q^2-1)$	$c_{1,0}$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^3-1)$	$c_{1,1}$
$c_{1,2}$	$x_\beta(1)$	$q^8(q^3-1)$	$c_{1,1}$
$c_{1,3}$	$x_{2\alpha+\beta}(1)$	$q^{10}(q^2-1)$	$c_{1,2}$
$c_{1,4}$	$x_{\alpha+\beta}(1)$	$q^8(q-1)$	$c_{1,2}$
$c_{1,5}$	$x_\beta(1)x_{2\alpha+\beta}(1)x_{3\alpha+2\beta}(2)$	q^8	$c_{1,2}$
$c_{1,6}$	$x_\beta(1)x_{3\alpha+\beta}(1)$	$q^8(q^2+q+1)$	$c_{1,3}$
$c_{1,7}$	$x_\beta(1)x_{2\alpha+\beta}(-1)$	$2q^8$	$c_{1,3}$
$c_{1,8}$	$x_{\alpha+\beta}(1)x_{3\alpha+\beta}(1)$	$2q^8$	$c_{1,3}$
$c_{1,9}(i,b)$	$x_\beta(1)x_{2\alpha+\beta}(\zeta^i)x_{3\alpha+\beta}(b)$	q^8	$c_{1,3}$
$c_{1,10}$	$x_\beta(1)x_{2\alpha+\beta}(-\zeta)$	$2q^8$	$c_{1,4}$
$c_{1,11}$	$x_{\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^8$	$c_{1,4}$
$c_{1,12}(i,b)$	$x_\beta(1)x_{2\alpha+\beta}(\zeta^i)x_{3\alpha+\beta}(b)$	q^8	$c_{1,4}$
$c_{1,13}$	$x_\alpha(1)x_{\alpha+\beta}(a)$	q^6	$c_{1,5}$
$c_{1,14}$	$x_\alpha(1)x_\beta(1)$	q^4	$c_{1,6}$
$c_{2,0}$	h_2	$q^4(q^3-1)(q-1)$	$c_{2,0}$
$c_{2,1}$	$h_2x_{3\alpha+2\beta}(1)$	$q^4(q^3-1)$	$c_{2,1}$
$c_{2,2}$	$h_2x_\alpha(1)$	$q^4(q-1)$	$c_{2,2}$
$c_{2,3}$	$h_2x_\alpha(1)x_{3\alpha+2\beta}(-1)$	$2q^4$	$c_{2,3}$
$c_{2,4}$	$h_2x_\alpha(1)x_{3\alpha+2\beta}(-\zeta)$	$2q^4$	$c_{2,4}$

Table A.12 (continued) *Conjugacy classes of Q.*

Notation	Representative	$\|C_Q\|$	Fusion in ${}^3D_4(q)$
$c_{3,0}$	h_3	$q^4(q^3-1)(q^2-1)$	$c_{2,0}$
$c_{3,1}$	$h_3x_\beta(1)$	$q^4(q^3-1)$	$c_{2,1}$
$c_{3,2}$	$h_3x_{2\alpha+\beta}(1)$	$q^4(q^2-1)$	$c_{2,2}$
$c_{3,3}$	$h_3x_\beta(1)x_{2\alpha+\beta}(-1)$	$2q^4$	$c_{2,3}$
$c_{3,4}$	$h_3x_\beta(1)x_{2\alpha+\beta}(-\zeta)$	$2q^4$	$c_{2,4}$
$c_{4,0}(i)$	$h_4(i)$	$q^3(q^3-1)(q-1)$	$c_{3,0}(i)$
$c_{4,1}(i)$	$h_4(i)x_\alpha(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{5,0}(i)$	$h_5(i)$	$q^3(q^3-1)(q-1)$	$c_{3,0}(i)$
$c_{5,1}(i)$	$h_5(i)x_{2\alpha+\beta}(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^3(q^3-1)(q^2-1)$	$c_{4,0}(i)$
$c_{6,1}(i)$	$h_6(i)x_\beta(1)$	$q^2(q^3-1)$	$c_{4,1}(i)$
$c_{6,2}(i)$	$h_6(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$	$c_{4,1}(i)$
$c_{6,3}(i)$	$h_6(i)x_\beta(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$	$c_{4,2}(i)$
$c_{7,0}(i)$	$h_7(i)$	$q(q^3-1)(q^2-1)$	$c_{5,0}((q^2+q-1)i)$
$c_{7,1}(i)$	$h_7(i)x_\beta(1)$	$q(q^3-1)$	$c_{5,1}((q^2+q-1)i)$
$c_{8,0}(i)$	$h_8(i)$	$q(q^3-1)(q-1)$	$c_{5,0}(i)$
$c_{8,1}(i)$	$h_8(i)x_{3\alpha+\beta}(1)$	$q(q^3-1)$	$c_{5,1}(i)$
$c_{9,0}(i,j)$	$h_9(i,j)$	$(q^3-1)(q-1)$	$c_{6,0}(i,j)$
$c_{10,0}(i)$	$h_{10}(i)$	$q^3(q^3-1)(q+1)$	$c_{7,0}(i)$
$c_{10,1}(i)$	$h_{10}(i)x_{2\alpha+\beta}(1)$	$q^3(q+1)$	$c_{7,1}(i)$
$c_{11,0}(i)$	$h_{11}(i)$	$(q^3-1)(q+1)$	$c_{8,0}(i)$

Table A.13 *Parametrization of the irreducible characters of Q.*

Character	Parameters	Number of Characters
${}_Q\chi_1(k)$	$k=0,\dots,q^3-2$	q^3-1
${}_Q\chi_2(k)$	$k=0,\dots,q^3-2$	q^3-1
${}_Q\chi_3(k,l)$	$k=0,\dots,q^3-2$ $l=0,\dots,q-2;\ l\neq 0$	$\frac{1}{2}(q^3-1)(q-2)$
${}_Q\chi_4(k)$	$k=0,\dots,q^4+q^3-q-2$ $k\neq(q+1)m,\ m=0,\dots,q^3-2$	$\frac{1}{2}q(q^3-1)$
${}_Q\chi_5(k)$	$k=0,\dots,q-2$	$q-1$
${}_Q\chi_6$		1
${}_Q\chi_7$		1
${}_Q\chi_8$		1
${}_Q\chi_9$		1
${}_Q\chi_{10}$		1
${}_Q\chi_{11}$		1
${}_Q\chi_{12}$		1
${}_Q\chi_{13}$		1
${}_Q\chi_{14}(k)$	$k=0,\dots,q-2;\ k\neq 0,(q-1)/2$	$(q-3)/2$
${}_Q\chi_{15}(k)$	$k=0,\dots,q;\ k\neq 0,(q+1)/2$	$(q-1)/2$
${}_Q\chi_{16}(k)$	$k=0,\dots,q^3-2$	q^3-1
${}_Q\chi_{17}(k)$	$k=0,\dots,q^2+q$	q^2+q+1
${}_Q\chi_{18}(k)$	$k=0,1$	2
${}_Q\chi_{19}(k)$	$k=0,1$	2
${}_Q\chi_{20}(k)$	$k=1,\dots,q-1$	$q-1$

Table A.14 *The character table of Q. (In this table, zeros are replaced by dots. The parameter $\delta \in \{1, -1\}$ depends on the residue class of q modulo 4 and is defined in Section 2. See Table 2.2 for notation for the irrational character values.)*

	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$	$c_{1,6}$
${}_Q\chi_1(k)$	1	1	1	1	1	1	1
${}_Q\chi_2(k)$	q	q	.	q	q	.	.
${}_Q\chi_3(k,l)$	$q+1$	$q+1$	1	$q+1$	$q+1$	1	1
${}_Q\chi_4(k)$	$q-1$	$q-1$	-1	$q-1$	$q-1$	-1	-1
${}_Q\chi_5(k)$	$(q+1)(q^3-1)$	$(q-1)(q+1)(q^2+q+1)$	q^3-1	$(q+1)(q^3-1)$	q^3-q-1	q^3-1	q^3-1
${}_Q\chi_6$	$(q^3-1)(q^2-1)$	$(q^3-1)(q^2-1)$	$-(q^3-1)$	$(q^3-1)(q^2-1)$	$(q-1)(q^3-q-1)$	$-(q^3-1)$	$-(q^3-1)$
${}_Q\chi_7$	$q(q^2-1)(q^3-1)$	$q(q^2-1)(q^3-1)$	.	$q(q^2-1)(q^3-1)$	$-q(q^2-1)$	.	.
${}_Q\chi_8$	q^6-q^3	q^6-q^3	.	$-q^3$	.	q^3	.
${}_Q\chi_9$	$q^4(q^3-1)$	$q^4(q^3-1)$	.	$-q^4$	.	.	.
${}_Q\chi_{10}$	$\frac{1}{2}q^3(q+1)(q^3-1)$	$\frac{1}{2}q^3(q+1)(q^3-1)$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q+1)$	.	$\frac{1}{2}(q^3-q^2)$	$\frac{1}{2}q^2(q^3-1)$
${}_Q\chi_{11}$	$\frac{1}{2}q^3(q+1)(q^3-1)$	$\frac{1}{2}q^3(q+1)(q^3-1)$	$-\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q+1)$	.	$\frac{1}{2}(q^3+q^2)$	$-\frac{1}{2}q^2(q^3-1)$
${}_Q\chi_{12}$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	.	$-\frac{1}{2}(q^3+q^2)$	$\frac{1}{2}q^2(q^3-1)$
${}_Q\chi_{13}$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$-\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	.	$-\frac{1}{2}(q^3-q^2)$	$-\frac{1}{2}q^2(q^3-1)$
${}_Q\chi_{14}(k)$	$q^3(q+1)(q^3-1)$	$q^3(q-1)(q+1)(q^2+q+1)$	.	$-q^3(q+1)$	.	q^3	.
${}_Q\chi_{15}(k)$	$q^3(q-1)^2(q^2+q+1)$	$q^3(q-1)^2(q^2+q+1)$	.	$-q^3(q-1)$	.	$-q^3$	.
${}_Q\chi_{16}(k)$	$q^3(q-1)(q+1)$	$-q^3$	$-q(q^2-1)$	.	.	q	$q(q+1)$
${}_Q\chi_{17}(k)$	$q^3(q-1)^2(q+1)$	$-q^3(q-1)$	$q(q-1)(q+1)$	.	.	$-q$	$-q(q+1)$
$\sum_{k=0}^{1} {}_Q\chi_{18}(k)$	$q^3(q-1)(q+1)(q^3-1)$	$-q^3(q^3-1)$	$q(q^3-1)$	.	.	$-q(q^2+1)$	$-q(q^2+q+1)$
$\sum_{k=0}^{1} {}_Q\chi_{19}(k)$	$q^3(q-1)(q+1)(q^3-1)$	$-q^3(q^3-1)$	$q(q^3-1)$	.	.	$-q(q^2+1)$	$-q(q^2+q+1)$
$\sum_{k=1}^{q-1} {}_Q\chi_{20}(k)$	$q^3(q-1)(q^3-1)(q^2-1)$	$-q^3(q-1)(q^3-1)$	$-q(q^3-1)$	.	.	$q(q^2+1)$	$q(q^2+q+1)$

Table A.14 (continued) *The character table of Q.*

	$c_{1,7}$	$c_{1,8}$	$c_{1,9}(i,b)$	$c_{1,10}$	$c_{1,11}$	$c_{1,12}(i,b)$	$c_{1,13}$	$c_{1,14}$
${}_Q\chi_1(k)$	1	1	1	1	1	1	1	1
${}_Q\chi_2(k)$	.	q	.	.	q	.	q	.
${}_Q\chi_3(k,l)$	1	$q+1$	1	1	$q+1$	1	$q+1$	1
${}_Q\chi_4(k)$	-1	$q-1$	-1	-1	$q-1$	-1	$q-1$	-1
${}_Q\chi_5(k)$	q^3-1	q^3-q-1	q^3-1	q^3-1	q^3-q-1	q^3-1	$-(q+1)$	-1
${}_Q\chi_6$	$-(q^3-1)$	$(q-1)(q^3-q-1)$	$-(q^3-1)$	$-(q^3-1)$	$(q-1)(q^3-q-1)$	$-(q^3-1)$	$-(q^2-1)$	1
${}_Q\chi_7$	.	$-q(q^2-1)$	.	.	$-q(q^2-1)$	.	q	.
${}_Q\chi_8$	$q^3\delta$	.	$(-1)^iq^3$	$-q^3\delta$	.	$(-1)^iq^3$	.	.
${}_Q\chi_9$	.	.	.	.	.	.	.	.
${}_Q\chi_{10}$	$-\frac{1}{2}q^2+\frac{1}{2}q^3\delta$	.	$-\frac{1}{2}q^2+\frac{1}{2}(-1)^iq^3$	$-\frac{1}{2}q^2-\frac{1}{2}q^3\delta$	.	$-\frac{1}{2}q^2+\frac{1}{2}(-1)^iq^3$	.	.
${}_Q\chi_{11}$	$\frac{1}{2}q^2+\frac{1}{2}q^3\delta$	.	$\frac{1}{2}q^2+\frac{1}{2}(-1)^iq^3$	$\frac{1}{2}q^2-\frac{1}{2}q^3\delta$	.	$\frac{1}{2}q^2+\frac{1}{2}(-1)^iq^3$	.	.
${}_Q\chi_{12}$	$-\frac{1}{2}q^2-\frac{1}{2}q^3\delta$	.	$-\frac{1}{2}q^2-\frac{1}{2}(-1)^iq^3$	$-\frac{1}{2}q^2+\frac{1}{2}q^3\delta$	.	$-\frac{1}{2}q^2-\frac{1}{2}(-1)^iq^3$	.	.
${}_Q\chi_{13}$	$\frac{1}{2}q^2-\frac{1}{2}q^3\delta$	.	$\frac{1}{2}q^2-\frac{1}{2}(-1)^iq^3$	$\frac{1}{2}q^2+\frac{1}{2}q^3\delta$	.	$\frac{1}{2}q^2-\frac{1}{2}(-1)^iq^3$	.	.
${}_Q\chi_{14}(k)$	$q^3\delta$	.	$(-1)^iq^3$	$-q^3\delta$	.	$(-1)^iq^3$	.	.
${}_Q\chi_{15}(k)$	$-q^3\delta$	.	$-(-1)^iq^3$	$q^3\delta$	.	$-(-1)^iq^3$	.	.
${}_Q\chi_{16}(k)$	$q(q+1)$	q^2	$q(q+1)$	$-q(q-1)$	$-q^2$	$-q(q-1)$	.	.
${}_Q\chi_{17}(k)$	$-q(q+1)$	$q^2(q-1)$	$-q(q+1)$	$q(q-1)$	$-q^2(q-1)$	$q(q-1)$	.	.
$\sum_{k=0}^{1}{}_Q\chi_{18}(k)$	$q(q^3-q^2-q-1)$	$-q^2$	$-q(q^2+q+1)$	$q(q-1)(q^2+1)$	q^2	$-q(q^2-q+1)$	.	.
$\sum_{k=0}^{1}{}_Q\chi_{19}(k)$	$q(q^3-q^2-q-1)$	$-q^2$	$-q(q^2+q+1)$	$q(q-1)(q^2+1)$	q^2	$-q(q^2-q+1)$	.	.
$\sum_{k=1}^{q-1}{}_Q\chi_{20}(k)$	$-q(q^3-q^2-q-1)$	$-q^2(q-1)$	$q(q^2+q+1)$	$-q(q-1)(q^2+1)$	$q^2(q-1)$	$q(q^2-q+1)$	.	.

Table A.14 (continued) *The character table of Q.*

	$c_{2,0}$	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$c_{2,4}$	$c_{3,0}$	$c_{3,1}$	$c_{3,2}$
${}_Q\chi_1(k)$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	1	1	1
${}_Q\chi_2(k)$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	$(-1)^k$	q	.	q
${}_Q\chi_3(k,l)$	$(-1)^{l+k}+(-1)^k$	$(-1)^{l+k}+(-1)^k$	$(-1)^{l+k}+(-1)^k$	$(-1)^{l+k}+(-1)^k$	$(-1)^{l+k}+(-1)^k$	$(q+1)(-1)^l$	$(-1)^l$	$(q+1)(-1)^l$
${}_Q\chi_4(k)$	.	.	.	.	.	$(q-1)(-1)^k$	$-(-1)^k$	$(q-1)(-1)^k$
${}_Q\chi_5(k)$	$(q^3-1)(-1)^k$	$(q^3-1)(-1)^k$	$-(-1)^k$	$-(-1)^k$	$-(-1)^k$	.	.	.
${}_Q\chi_6$	.	.	.	.	.	.	.	.
${}_Q\chi_7$	.	.	.	.	.	.	.	.
${}_Q\chi_8$	.	.	.	.	.	$(q^3-1)\delta$	$(q^3-1)\delta$	$-\delta$
${}_Q\chi_9$	.	.	.	.	.	$(q^3-1)q\delta$	.	$-q\delta$
${}_Q\chi_{10}$	.	.	.	.	.	$\frac{1}{2}(q+1)(q^3-1)$	$\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q+1)$
${}_Q\chi_{11}$	.	.	.	.	.	$\frac{1}{2}(q+1)(q^3-1)$	$\frac{1}{2}(q^3-1)$	$-\frac{1}{2}(q+1)$
${}_Q\chi_{12}$	.	.	.	.	.	$-\frac{1}{2}(q-1)(q^3-1)$	$\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q-1)$
${}_Q\chi_{13}$	.	.	.	.	.	$-\frac{1}{2}(q-1)(q^3-1)$	$\frac{1}{2}(q^3-1)$	$\frac{1}{2}(q-1)$
${}_Q\chi_{14}(k)$	.	.	.	.	.	$(q^3-1)(-1)^k(q+1)$	$(q^3-1)(-1)^k$	$-(-1)^k(q+1)$
${}_Q\chi_{15}(k)$	.	.	.	.	.	$\delta(q^3-1)(q-1)(-1)^k$	$-\delta(q^3-1)(-1)^k$	$-(q-1)\delta(-1)^k$
${}_Q\chi_{16}(k)$	$(q-1)(-1)^k$	$-(-1)^k$	$(q-1)(-1)^k$	$-(-1)^k$	$-(-1)^k$	.	.	.
${}_Q\chi_{17}(k)$	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1} {}_Q\chi_{18}(k)$	$(q^3-1)(q-1)$	$-(q^3-1)$	$-(q-1)$	1	1	.	.	.
$\sum_{k=0}^{1} {}_Q\chi_{19}(k)$	$-(q^3-1)(q-1)$	q^3-1	$q-1$	-1	-1	.	.	.
$\sum_{k=1}^{q-1} {}_Q\chi_{20}(k)$	.	.	.	.	.	.	.	.

Table A.14 (continued) *The character table of Q.*

	$c_{3,3}$	$c_{3,4}$	$c_{4,0}(i)$	$c_{4,1}(i)$	$c_{5,0}(i)$	$c_{5,1}(i)$	$c_{6,0}(i)$	$c_{6,1}(i)$	$c_{6,2}(i)$	$c_{6,3}(i)$
${}_Q\chi_1(k)$	1	1	ζ_1^{ik}	ζ_1^{ik}	1	1	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}
${}_Q\chi_2(k)$	.	.	ζ_1^{ik}	ζ_1^{ik}	1	1	$q\varphi_3^{ik}$	.	$q\varphi_3^{ik}$	.
${}_Q\chi_3(k,l)$	$(-1)^l$	$(-1)^l$	$\zeta_1^{il+ik}+\zeta_1^{2il+ik}$	$\zeta_1^{il+ik}+\zeta_1^{2il+ik}$	$\zeta_1^{il}+\zeta_1^{-il}$	$\zeta_1^{il}+\zeta_1^{-il}$	$(q+1)\varphi_3^{ik}$	φ_3^{ik}	$(q+1)\varphi_3^{ik}$	φ_3^{ik}
${}_Q\chi_4(k)$	$-(-1)^k$	$-(-1)^k$	.	.	.	.	$(q-1)\varphi_3^{-ik}$	$-\varphi_3^{-ik}$	$(q-1)\varphi_3^{-ik}$	$-\varphi_3^{-ik}$
${}_Q\chi_5(k)$	.	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.	.	.	.	.
${}_Q\chi_6$	.	.	.	.	.	.	.	.	.	.
${}_Q\chi_7$	.	.	.	.	.	.	.	.	.	.
${}_Q\chi_8$	$-\delta$	$-\delta$	.	.	$(-1)^iq^3-(-1)^i$	$-(-1)^i$	.	.	.	.
${}_Q\chi_9$	.	.	.	.	$(q^3-1)(-1)^i$	$-(-1)^i$	.	.	.	.
${}_Q\chi_{10}$	$\frac{1}{2}(q^2-1)$	$-\frac{1}{2}(q^2+1)$	.	.	q^3-1	-1	.	.	.	.
${}_Q\chi_{11}$	$-\frac{1}{2}(q^2+1)$	$\frac{1}{2}(q^2-1)$	.	.	q^3-1	-1	.	.	.	.
${}_Q\chi_{12}$	$-\frac{1}{2}(q^2+1)$	$\frac{1}{2}(q^2-1)$	.	.	.	.	.	.	.	.
${}_Q\chi_{13}$	$\frac{1}{2}(q^2-1)$	$-\frac{1}{2}(q^2+1)$	.	.	.	.	.	.	.	.
${}_Q\chi_{14}(k)$	$-(-1)^k$	$-(-1)^k$	.	.	$(q^3-1)(\zeta_1^{ik}+\zeta_1^{-ik})$	$-\zeta_1^{ik}-\zeta_1^{-ik}$	.	.	.	.
${}_Q\chi_{15}(k)$	$\delta(-1)^k$	$\delta(-1)^k$	.	.	.	.	.	.	.	.
${}_Q\chi_{16}(k)$	.	.	.	.	.	.	$(q-1)\varphi_3^{ik}(q+1)$	$(q-1)\varphi_3^{ik}$	$-\varphi_3^{ik}$	$-\varphi_3^{ik}$
${}_Q\chi_{17}(k)$	.	.	.	.	.	.	$(q-1)^2\varphi_3^{ik}(q+1)$	$-(q-1)\varphi_3^{ik}$	$-(q-1)\varphi_3^{ik}$	φ_3^{ik}
$\sum_{k=0}^{1} {}_Q\chi_{18}(k)$	.	.	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1} {}_Q\chi_{19}(k)$	.	.	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q-1} {}_Q\chi_{20}(k)$	.	.	.	.	.	.	.	.	.	.

Table A.14 (continued) *The character table of Q.*

	$c_{7,0}(i)$	$c_{7,1}(i,j)$	$c_{8,0}(i)$	$c_{8,1}(i)$	$c_{9,0}(i,j)$	$c_{10,0}(i)$	$c_{10,1}(i)$	$c_{11,0}(i)$
${}_Q\chi_1(k)$	ζ_3^{2ik}	ζ_3^{2ik}	ζ_3^{ik}	ζ_3^{ik}	ζ_3^{ik}	1	1	$\zeta_3^{q^2ik-qik+ik}$
${}_Q\chi_2(k)$	$q\zeta_3^{2ik}$	.	ζ_3^{ik}	ζ_3^{ik}	ζ_3^{ik}	-1	-1	$-\zeta_3^{q^2ik-qik+ik}$
${}_Q\chi_3(k,l)$	$(q+1)\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{ik}+\zeta_3^{ik}\zeta_1^{il}$	$\zeta_3^{ik}+\zeta_3^{ik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{(i-j)l}+\zeta_3^{ik}\zeta_1^{jl}$	.	.	.
${}_Q\chi_4(k)$	$(q-1)\zeta_3^{(q^2+q-1)ik}$	$-\zeta_3^{(q^2+q-1)ik}$	.	.	.	$-\xi_1^{ik}-\xi_1^{-ik}$	$-\xi_1^{ik}-\xi_1^{-ik}$	$-\mu_3^{(q^2+q-1)ik}-\mu_3^{-q(q^2-q-1)ik}$
${}_Q\chi_5(k)$	.	.	.	.	.	.	.	.
${}_Q\chi_6$	.	.	.	.	.	.	.	.
${}_Q\chi_7$	.	.	.	.	.	.	.	.
${}_Q\chi_8$	.	.	.	.	.	$-(q^3-1)(-1)^i$	$(-1)^i$	.
${}_Q\chi_9$	.	.	.	.	.	$(q^3-1)(-1)^i$	$-(-1)^i$	.
${}_Q\chi_{10}$	.	.	.	.	.	.	.	.
${}_Q\chi_{11}$	.	.	.	.	.	.	.	.
${}_Q\chi_{12}$	.	.	.	.	.	q^3-1	-1	.
${}_Q\chi_{13}$	.	.	.	.	.	q^3-1	-1	.
${}_Q\chi_{14}(k)$	.	.	.	.	.	.	.	.
${}_Q\chi_{15}(k)$	.	.	.	.	.	$(q^3-1)(-1)^i(\xi_1^{ik}+\xi_1^{-ik})$	$-(-1)^i(\xi_1^{ik}+\xi_1^{-ik})$	.
${}_Q\chi_{16}(k)$	.	.	$\zeta_3^{ik}(q-1)$	$-\zeta_3^{ik}$	.	.	.	.
${}_Q\chi_{17}(k)$	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1}{}_Q\chi_{18}(k)$	.	.	.	.	.	.	.	.
$\sum_{k=0}^{1}{}_Q\chi_{19}(k)$	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q-1}{}_Q\chi_{20}(k)$	.	.	.	.	.	.	.	.

References

[1] R.W. Carter, *Simple Groups of Lie Type,* A Wiley-Interscience publication, London, 1972.

[2] R.W. Carter, *Finite Groups of Lie Type - Conjugacy Classes and Complex Characters,* A Wiley-Interscience publication, Chichester, 1985.

[3] B.W. Char, K.O. Geddes, G.H. Gonnet, B.L. Leong, M.B. Monagan, S.M. Watt, *Maple V, Language Reference Manual,* Springer, 1991.

[4] D.I. Deriziotis, G.O. Michler, *Character Table and Blocks of Finite Simple Triality Groups ${}^3D_4(q)$,* Trans. Amer. Math. Soc. **303** (1987), 39–70.

[5] F. Digne, J. Michel, *Representations of Finite Groups of Lie Type,* Cambridge University Press, Cambridge, 1990.

[6] H. Enomoto, H. Yamada, *The characters of $G_2(2^n)$,* Japan. J. Math. **12**:2 (1986), 325–377.

[7] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.3*; 2002, (`http://www.gap-system.org`).

[8] M. Geck, *Verallgemeinerte Gelfand-Graev Charaktere und Zerlegungszahlen endlicher Gruppen von Lie–Typ,* Dissertation, RWTH Aachen, 1990.

[9] M. Geck, *Generalized Gelfand–Graev Characters for Steinberg's Triality Groups and their Applications,* Comm. Alg. **19**:12 (1991), 3249–3269.

[10] M. Geck, G. Hiss, F. Lübeck, G. Malle, G. Pfeiffer, *CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras,* Appl. Algebra Engrg. Comm. Comput. **7** (1996), 175–210.

[11] F. Himstedt, *Charaktertafeln parabolischer Untergruppen der Steinbergschen Trialitätsgruppen und Anwendungen auf deren Darstellungstheorie,* Dissertation, RWTH Aachen, 2003.

[12] B. Huppert, *Endliche Gruppen I,* Springer, New York, 1967.

[13] I.M. Isaacs, *Character Theory of Finite Groups,* Dover, New York, 1976.

[14] F. Lübeck, *Charaktertafeln für die Gruppen $CSp_6(q)$ mit ungeradem q und $Sp_6(q)$ mit geradem q,* Dissertation, Universität Heidelberg, 1993.

[15] T. Okuyama, K. Waki *Decomposition numbers of $Sp(4, q)$,* J. Alg. **199** (1998), 433–555.

[16] I. Schur, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen,* J. reine angew. Math. **132** (1907), 85-137.

[17] N. Spaltenstein, *Caractères unipotents de ${}^3D_4(q)$,* Comment. Math. Helvitici **57** (1982), 676–691.

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