Optimizing the Paths for Geodesic Interpolating Splines

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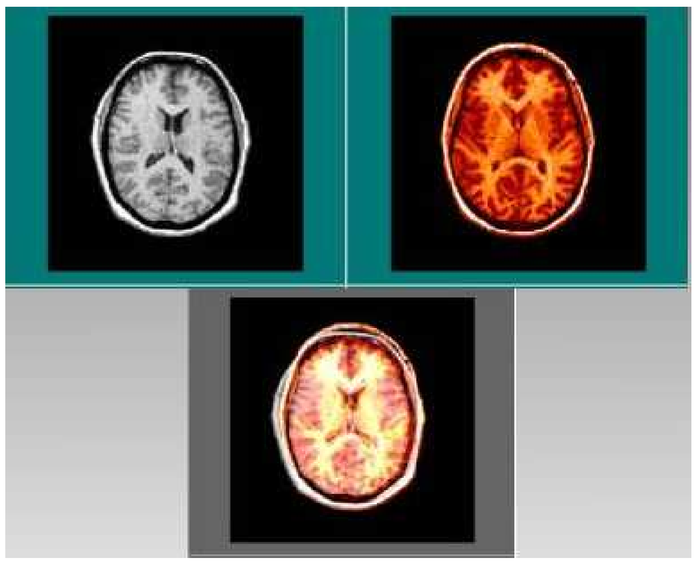
Introduction

- We may wish to compare brain images
  - to monitor disease progression,
  - to detect abnormality, by comparing with an 'atlas' brain.
- Nonlinear deformation of the brain occurs
  - over time,
  - with movement,
  - across populations.

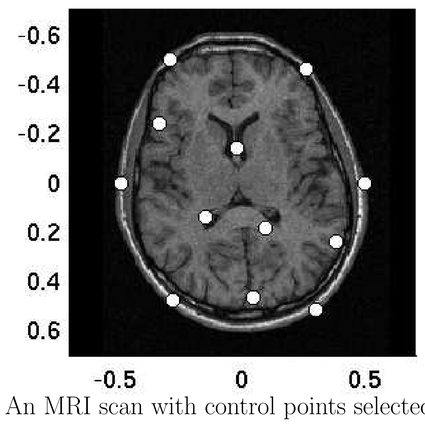


The image shows nonlinear variation between two brain scans, with an overlay shown where the nonlinear variation can clearly be seen.(Ding 2001)

- To eliminate these external variations, we generate suitable nonlinear mappings, or *warps*.
- This process is known as *non-rigid registration*.

The Control Point Method

A medical image has certain important points known as *landmarks* which we use as control points for the warp.



An MRI scan with control points selected.

- The control points can be selected manually or automatically.
- Gives a set of n_c control points,
- start points (P_i) on the first image, the *floating* image,
- end points (Q_i) on the second image, the *reference* image.

We seek a warp $\Phi(x)$ such that $\Phi(P_i) = Q_i$.

Optimizing the warp

Define $\Phi(y) = \mathbf{x}(1)$. We select a suitable mapping, Φ by solving the differential equation
$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}(t, \mathbf{x})$, with the initial condition $\mathbf{x}(0) = \mathbf{y}$,
where $\mathbf{v}$ is a deformation field to be determined by minimizing:

$$l(\mathbf{x}_i(t), \mathbf{v}(t, \mathbf{x})) = \int_0^1 \int_B \|\nabla^2 \mathbf{v}(t, \mathbf{x})\|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}t,$$

over deformation fields, $\mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^d$ and paths, $\mathbf{x}_i(t) \in B$ for $i = 1, \ldots, n_c$

$$\text{such that } \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{v}(t, \mathbf{x}_i(t)), \qquad 0 \le t \le 1,$$
$$\text{and } \mathbf{x}_i(0) = P_i, \text{ and } \mathbf{x}_i(1) = Q_i, i = 1, \ldots, n_c,$$

where the operator ∇^2 approximates the *Willmore energy* which quantifies the "bending energy" of the warp.

We expand this system of equations and discretize in time to give the optimization problem:

Minimize

$$l\{\boldsymbol{\alpha}_i(\tau), \mathbf{q}_i(\tau)\} =$$
$$\sum_{i,j=1}^{n_c} \sum_{\tau=1}^{T} (\boldsymbol{\alpha}_i^\top(\tau) G(\mathbf{q}_i(\tau), \mathbf{q}_j(\tau))) \boldsymbol{\alpha}_j(\tau)$$

over $\boldsymbol{\alpha}_i(\tau), \mathbf{q}_i(\tau)$ $i = 1, \ldots, n_c$, such that

$$T(\mathbf{q}_i(\tau + 1) - \mathbf{q}_i(\tau)) = \sum_{j=1}^{n_c} \boldsymbol{\alpha}_j(\tau) G(\mathbf{q}_i(t), \mathbf{q}_j(t)),$$

with conditions

$$\mathbf{q}_i(1) = P_i, \quad \text{and} \quad \mathbf{q}_i(T + 1) = Q_i \qquad i = 1, \ldots, n_c,$$

where $G(\mathbf{x}, \mathbf{y})$ is the clamped plate biharmonic Green's function,

$$G(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2 \left(\frac{1}{2}(A^2 - 1) - \ln A \right),$$

where

$$A(\mathbf{x}, \mathbf{y}) = \frac{\sqrt{|\mathbf{x}|^2 |\mathbf{y}|^2 - 2\mathbf{x}.\mathbf{y} + 1}}{|\mathbf{x} - \mathbf{y}|},$$

and where $\mathbf{x}$ and $\mathbf{y}$ are two-dimensional vectors.

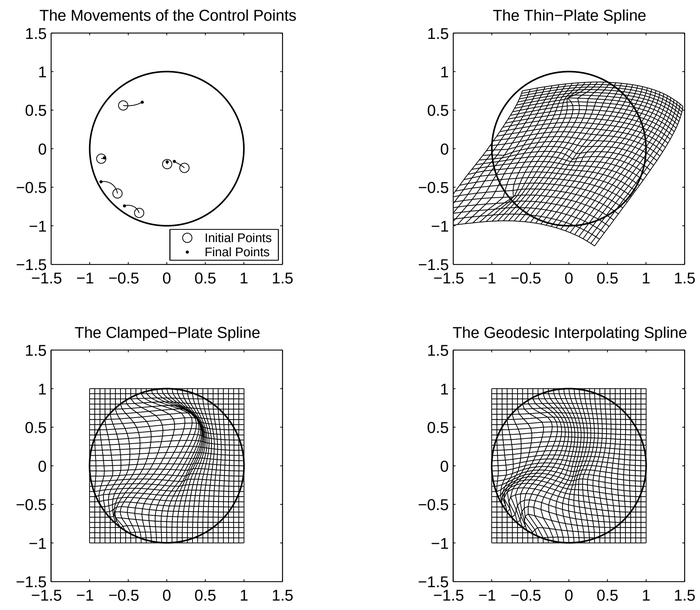
- solved as a *constrained* problem,
- more efficient than an *unconstrained* approach (Mills 2003).

We can exploit *partial separability* of the problem, using the package, GALAHAD (Gould and et al 2002): Examining the objective function, for instance, we see that there is a natural division into T groups, each group being given by, for the τ^{th} group

$$l\{\boldsymbol{\alpha}_i(\tau), \mathbf{q}_i(\tau)\} = \sum_{i,j=1}^{n_c} (\boldsymbol{\alpha}_i^\top(\tau) G(\mathbf{q}_i(\tau), \mathbf{q}_j(\tau))) \boldsymbol{\alpha}_j(\tau).$$

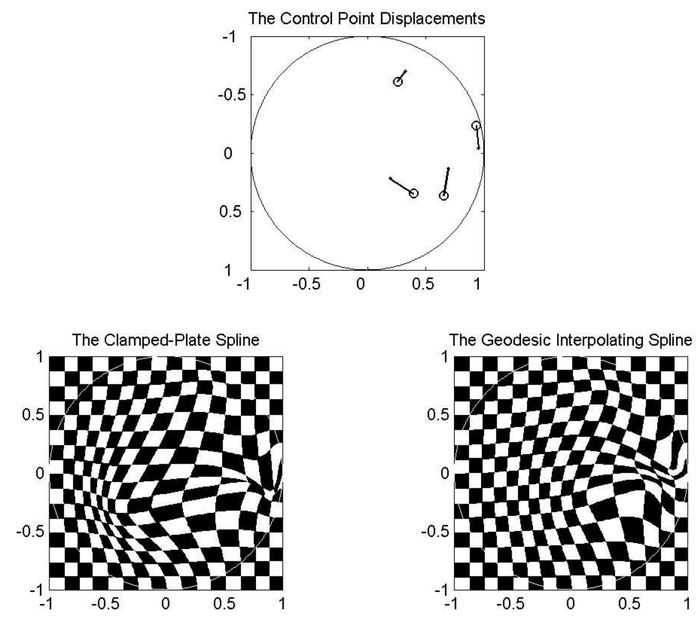
Comparison with other methods

The Geodesic Interpolating Spline performs well compared with other popularly used methods.



A system of 6 control points subjected to random small displacements, and the corresponding thin-plate, clamped-plate and geodesic interpolating splines

- folding in the upper right-hand quadrant of the mapping calculated by the clamped-plate technique,
- thin-plate spline does not impose boundary conditions.



Random small control point movements for 4 control points, and the corresponding clamped-plate and 20 time point geodesic interpolating splines applied to a grid

Further Investigations

- **Is the problem well posed?**
- **Do we guarantee a diffeomorphism?**
- **What limit do we approach as we increase the number of control points?**
- **Can we incorporate a similarity measure?**
- **Can we apply the technique to *groups* of images?**

REFERENCES

Ding, L. 2001. http://www.cs.wright.edu/people/faculty/agoshtas/lijun_thesis.html.

Gould, N., & et al 2002. http://galahad.rl.ac.uk/galahad-www/.

Mills, A. M. 2003. Master's thesis, University of Manchester, http://www.ma.man.ac.uk/~amills/thesis.pdf.

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Table 1:

Fig. 1.—

Fig. 2.—

Fig. 3.—