# GRAPHS, GEOMETRY, 3-TRANSPOSITIONS, AND SYMPLECTIC $\mathbb{F}_2$-TRANSVECTION GROUPS

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## 1. *Introduction*

In this paper we begin the classification completed in **[12]** of all partial linear spaces $\Pi$, graphs $\Gamma$, and groups $G$ which satisfy one of the following:

I. $\Pi = (\mathcal{P}, \mathcal{L})$ is a connected partial linear space of order 2 in which every pair of intersecting lines lies in a subspace isomorphic to the dual of an affine plane of order 2;

II. $\Gamma$ is a connected graph such that, for each vertex $x$ of $\Gamma$, the set of vertices of $\Gamma$ adjacent to $x$ forms a subgraph isomorphic to a grid graph $L(3, \Delta_x)$, where $\Delta_x$ is a set depending on $x$;

III. $G$ is a centre-free group generated by a conjugacy class of 3-transpositions $D$ such that no three elements of $D$ generate a subgroup of order 18 or 54.

Rather surprisingly, these three somewhat odd looking classifications are equivalent. As is often the case when geometries and groups meet, the common ground is that of linear algebra. Specifically we are largely concerned with the identification and properties of those isometry groups for symplectic $\mathbb{F}_2$-spaces which are generated by transvections.

A partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$ is a set of points $\mathcal{P}$ and a set $\mathcal{L}$ of subsets of $\mathcal{P}$ called lines, such that no two lines meet in two or more points. The space $\Pi$ has order 2 if $|l| = 3$ for all $l \in \mathcal{L}$, and $\Pi$ is connected if in every non-trivial partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ there are points $p_i \in \mathcal{P}_i$ with $p_1$ and $p_2$ collinear in $\Pi$. The space $\Pi' = (\mathcal{P}', \mathcal{L}')$ is a subspace of $\Pi$ if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, and every $l \in \mathcal{L}$ with $|l \cap \mathcal{P}'| \geq 2$ in fact belongs to $\mathcal{L}'$.

Given a set $\Delta$, the grid graph $L(3, \Delta)$ has as vertex set the set of ordered pairs $\{1, 2, 3\} \times \Delta$. Two distinct vertices $(a, b)$ and $(c, d)$ are adjacent precisely when $a = c$ or $b = d$.

The subset $D$ of the group $G$ is a conjugacy class of 3-transpositions if $D$ is a conjugacy class of elements of order 2 in $G$ such that, whenever $d$, $e \in D$, the element $de$ has order 1, 2, or 3.

Versions of these problems have been around for a while. Hale **[10]** was the first to study partial linear spaces as in I. Hale states the classification for finite spaces, although he never published a complete proof. Graphs all of whose vertex neighbourhoods are grids $L(\Omega, \Delta)$ have been a subject of much study, beginning with, and motivated by, the work of Buekenhout and Hubaut **[3]**. The study of groups generated by classes of 3-transpositions was begun by Fischer **[9]** and forms an interesting and important part of finite group theory.

The complete classification for each of I, II, and III is accomplished in this

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paper and its companion [12]. Some remarks should be made on the logical flow of these two papers. The beginning of the argument is geometric and combinatorial and is given in this paper. The conclusion of the argument is group-theoretic and is given in the second paper [12]. The classifications for the geometric I and the combinatorial II are described here in Theorems 1 and 2. The classification for the group-theoretic III is stated in two forms as Theorems 1 and 2 of [12]. In this paper we prove the equivalence of I, II, and III, and we prove a special case of I. This special case (rephrased in group-theoretic language as Theorem 5) leads to the complete resolution of III in [12], and the equivalence then completes all three classifications. In particular, Theorem 1 is the only result stated in this paper whose proof relies on any of the work in [12]. Therefore although we view this paper and [12] as two parts of a whole, each paper can be read almost independently of the other, and they contain dissimilar mathematics.

THEOREM 1. *Let $\Pi = (\mathscr{P}, \mathscr{L})$ be a connected partial linear space as in* I. *Then $\Pi$ is isomorphic to one of the partial linear spaces $\mathscr{S}p(V, f)$ or $\mathscr{N}(V, q)$, with $V$ a vector space over $\mathbb{F}_2$, or to $\mathscr{T}(\Omega, \Omega')$ for disjoint sets $\Omega$ and $\Omega'$.*

THEOREM 2. *Let $\Gamma$ be a connected graph as in* II. *Then there is a $\Pi = (\mathscr{P}, \mathscr{L})$ as in* I *(and so described in Theorem* 1*) such that $\Gamma$ is isomorphic to the line graph $L(\Pi)$ whose vertices are the members of $\mathscr{L}$ with two adjacent when intersecting in a member of $\mathscr{P}$. Conversely, if $\Pi$ is a space as in* I, *then $\Gamma = L(\Pi)$ is a graph as in* II *with $\Delta = \Delta_x$ a fixed set, for all $x \in \Gamma$.*

The spaces of Theorem 1 are described below. The main result proven in this paper is Theorem 3. It is actually a special case of Theorem 1 but is of some interest in its own right.

THEOREM 3. *Let $\Gamma$ be an irreducible cotriangular graph. Then $\Gamma$ is isomorphic to one of the graphs $\overline{sp}(V, f)$ or $n(V, q)$, with $f$ and $q$ non-degenerate on the $\mathbb{F}_2$-space $V$, or to a graph $\bar{t}(\Omega)$ for some set $\Omega$.*

For an explanation of the notation and terminology of Theorem 3, see §2. This theorem extends to the infinite case the classification by Shult of the finite, irreducible, cotriangular graphs [13, 18]. The proof and discussion of Theorem 3 in §2 is essentially self-contained.

Theorem 3 is equivalent to the following important case of Theorem 1.

THEOREM 4. *Let $\Pi$ be as in Theorem* 1. *Assume additionally that the equality*

$$\{x\} \cup \{z \in l \mid y \in l \in \mathscr{L}\} = \{y\} \cup \{z \in l \mid x \in l \in \mathscr{L}\}$$

*for points $x$ and $y$ of $\mathscr{P}$ is valid only in the trivial cases where $x = y$. Then $\Pi$ is isomorphic to one of $\mathscr{S}p(V, f)$ or $\mathscr{N}(V, q)$, with $f$ and $q$ non-degenerate on the $\mathbb{F}_2$-space $V$ or to $\mathscr{T}(\Omega)$ for some set $\Omega$.*

We begin by describing the partial linear spaces which appear in the conclusions of Theorems 2 and 4.

Let $V$ be a vector space over $\mathbb{F}_2$, possibly of infinite dimension. The bilinear map $f\colon V \times V \to \mathbb{F}_2$ is a *symplectic form* on $V$ if $f(x, x) = 0$ for all $x \in V$. The vectors $x$ and $y$ are *perpendicular* if $f(x, y) = 0$. The *radical* of $f$, $\mathrm{Rad}(V, f)$, is the subspace of all vectors perpendicular to all of $V$, and $f$ is *non-degenerate* on $V$ if $\mathrm{Rad}(V, f) = 0$. The map $q$: $V \to \mathbb{F}_2$ is a *quadratic form* on $V$, if $f$, given by

$$f(x, y) = q(x) + q(y) + q(x + y),$$

is a symplectic form on $V$. Then $f$ is the symplectic form *associated with* $q$. By definition the *radical* of $q$, $\mathrm{Rad}(V, q)$, is the radical of the symplectic form $f$ associated with $q$, and $q$ is *non-degenerate* if $f$ is.

If $f$ is a symplectic form on the $\mathbb{F}_2$-space $V$, then the points of the partial linear space $\mathscr{S}p(V, f)$ are the 1-spaces of $V$ not contained in $\mathrm{Rad}(V, f)$. The lines of $\mathscr{S}p(V, f)$ are the 2-spaces $W$ of $V$ (that is, dependent triples of 1-spaces) for which the restriction of $f$ to $W$ is non-degenerate. (Such 2-spaces are called *hyperbolic*.) It is not hard to check that $\mathscr{S}p(V, f)$ is a partial linear space as in I.

If $q$ is a quadratic form on the $\mathbb{F}_2$-space $V$ with associated symplectic form $f$, then $\mathcal{N}(V, q)$ is the subspace of $\mathscr{S}p(V, f)$ composed of those non-radical 1-spaces $\langle x \rangle$ which additionally have $q(x) = 1$. A vector $x$ with $q(x) = 1$ is *non-singular.*

Let $\Omega$ be any set (of cardinality at least 2), and let $\Omega'$ be any second set disjoint from $\Omega$ ($\Omega'$ is allowed to have any cardinality). The points of the partial linear space $\mathscr{T}(\Omega, \Omega')$ are the finite subsets of $\Omega \cup \Omega'$ which meet $\Omega$ in a set of size 2, a 2-subset. The lines of $\mathscr{T}(\Omega, \Omega')$ are those triples of points with empty symmetric difference. Let $\mathscr{T}(\Omega)$ be $\mathscr{T}(\Omega, \varnothing)$. Its points are the 2-subsets of $\Omega$, and its lines are the 3-subsets of $\Omega$ (that is, the three 2-subsets of a 3-subset).

Notice that the dual of the affine plane of order 2, the partial linear space mentioned specifically in I, is isomorphic to $\mathscr{T}(\Omega)$ for $|\Omega| = 4$. We write $\mathscr{T}_4$ for $\mathscr{T}(\{1, 2, 3, 4\})$. The space $\mathscr{T}(\{1, 2, 3, 4\}, \{5\})$ is isomorphic to the space of lines in a projective $\mathbb{F}_2$-space of dimension 3 which are disjoint from some fixed line. We write $\mathscr{W}(D_4)$ for this space because of its connection with the Weyl group of type $D_4$.

Consider the subspace $E\mathbb{F}_2(\Omega, \Omega')$ of $\mathbb{F}_2\Omega \oplus \mathbb{F}_2\Omega'$ composed of those vectors with an even number of 1's in their $\Omega$-coordinates. (These are the vectors whose projections onto $\mathbb{F}_2\Omega$ have even weight.) The usual dot product on $\mathbb{F}_2\Omega$ is symplectic on $E\mathbb{F}_2\Omega$ ($=E\mathbb{F}_2(\Omega, \varnothing)$), and this extends uniquely to a symplectic form $f$ for $E\mathbb{F}_2(\Omega, \Omega')$ containing $\mathbb{F}_2\Omega'$ in its radical. The space $\mathscr{T}(\Omega, \Omega')$ can be thought of as the subspace of $\mathscr{S}p(E\mathbb{F}_2(\Omega, \Omega'), f)$ of points with weight 2 in their $\Omega$-coordinates.

If $|\Omega|$ is odd or infinite, then $\mathbb{F}_2\Omega'$ equals the radical of $f$ on $E\mathbb{F}_2(\Omega, \Omega')$. If $|\Omega|$ is even then $\mathbf{1}$, the vector of $E\mathbb{F}_2\Omega$ composed entirely of 1's, is in the radical of $f$ on $E\mathbb{F}_2\Omega$. In this case, set

$$\bar{E}\mathbb{F}_2\Omega = E\mathbb{F}_2\Omega/\langle \mathbf{1} \rangle \quad \text{and} \quad \bar{E}\mathbb{F}_2(\Omega, \Omega') = \bar{E}\mathbb{F}_2\Omega \oplus \mathbb{F}_2\Omega'.$$

The form $f$ induces a symplectic form, which we also call $f$, on $\bar{E}\mathbb{F}_2(\Omega, \Omega')$ with radical equal to $\mathbb{F}_2\Omega'$. Again $\mathscr{T}(\Omega, \Omega')$ is a subspace of $\mathscr{S}p(\bar{E}\mathbb{F}_2(\Omega, \Omega'), f)$. (If $|\Omega|$ is odd or infinite, we set $\bar{E}\mathbb{F}_2(\Omega, \Omega') = E\mathbb{F}_2(\Omega, \Omega')$.)

To connect I and III we make use of certain automorphisms and automorphism groups of the partial linear spaces concerned. It is clear in §2 of this paper and throughout **[12]** that these groups are also important in the proof of the classifications as well. Let $x$ be a vector of the $\mathbb{F}_2$-space $V$ admitting the symplectic form $f$. The linear transformation $t(x)$, of $V$, given by

$$t(x)\colon\ v \to v + f(v, x)x,$$

is an invertible isometry in that $t(x)$ is non-singular and satisfies $f(u, v) = f(ut(x), vt(x))$ for all $u, v \in V$. Thus $t(x)$ is a non-identity transformation precisely when $x \notin \mathrm{Rad}(V, f)$. In this case, $t(x)$ is called a *symplectic transvection,* and the 1-space $\langle x \rangle$ is the *centre* of $t(x)$. We set

$$\mathrm{TSp}(V, f) = \langle t(x) |\ x \in V - \mathrm{Rad}(V, f) \rangle,$$

the *transvection symplectic group*. The group $\mathrm{TSp}(V, f)$ is an isometry group for $f$ and also can be thought of as an automorphism group of $\mathcal{S}p(V, f)$. However, $\mathrm{TSp}(V, f)$ need not be the full isometry group or full automorphism group, and the symplectic transvections need not all be transvection isometries. In particular, if $x$ is a non-zero vector of $\mathrm{Rad}(V, f)$, then any transvection of $V$ with centre $\langle x \rangle$ is an isometry and an automorphism. (In general, a transvection $t$ of $V$ with centre $\langle x \rangle$ is any invertible linear transformation $t$ with $V(t-1) = \langle x \rangle$ and $V(t-1)^2 = 0$.)

If $q$ is a quadratic form on the $\mathbb{F}_2$-space $V$ with associated symplectic form $f$, then the symplectic transvections $t(x)$ for $f$ which additionally satisfy $q(v) = q(vt(x))$, for all $v \in V$, are precisely those $t(x)$ with $q(x) = 1$. Let $\mathrm{TO}(V, q)$, the *transvection orthogonal group,* be the subgroup of $\mathrm{TSp}(V, f)$ generated by the symplectic transvections $t(x)$ with $q(x) = 1$. As these are precisely the symplectic transvections with centres belonging to the subspace $\mathcal{N}(V, q)$ of $\mathcal{S}p(V, f)$, $\mathrm{TO}(V, q)$ is an automorphism group of $\mathcal{N}(V, q)$.

For $\mathcal{T}(\Omega, \Omega')$ we let $\mathrm{TSym}(\Omega, \Omega')$, the *transvection symmetric group,* be the automorphism group generated by those symplectic transvections of $\mathrm{TSp}(E\mathbb{F}_2(\Omega, \Omega'), f)$ whose centres are points of the subspace $\mathcal{T}(\Omega, \Omega')$ of $\mathcal{S}p(E\mathbb{F}_2(\Omega, \Omega'), f)$. If $\Omega' = \varnothing$, then the symplectic transvections of $\mathrm{TSym}(\Omega) = \mathrm{TSym}(\Omega, \varnothing)$ are the transpositions of the usual symmetric group $\mathrm{Sym}(\Omega)$. Thus $\mathrm{TSym}(\Omega)$ is the subgroup of $\mathrm{Sym}(\Omega)$ generated by its transpositions, the subgroup of all permutations of $\Omega$ with finite support.

Theorem 3 has the following form when stated in the language of III.

THEOREM 5. *Let $G$ be a group which is generated by a conjugacy class of 3-transpositions $D$ such that no three elements of $D$ generate a subgroup of order 18 or 54. Then there is a normal subgroup $N$ such that $G^* = G/N$ is isomorphic to one of the groups* $\mathrm{TSym}(\Omega)$, $\mathrm{TO}(V, q)$, *or* $\mathrm{TSp}(V, f)$, *where $\Omega$ is a set and $V$ is a vector space over $\mathbb{F}_2$ with $q$ and $f$ non-degenerate forms on $V$. This isomorphism can be chosen to map $D^*$ to the collection of symplectic transvections. For $x, y \in D$, $x^* = y^*$ if and only if $C_D(x) = C_D(y)$.*

Theorem 5 is the starting point for the work in **[12]** which completes the proof of Theorem 1.

Section 2 consists of the discussion and proof of Theorem 3. The equivalent Theorem 4 is an immediate consequence. In § 3 the correspondence between I and III is made precise. In particular, Theorem 5 is seen to be equivalent to Theorem 3. Section 4 describes the relationship between I and II and contains the proof of Theorem 2. In § 5, Theorem 3 is used to extend the results of **[11]**, on the representation of finite cotriangular spaces, to include the infinite case. Section 6 contains a discussion of four applications of the results in §§ 2 and 5. Section 6.A completes the proof of the main theorem of **[5]** on groups generated by transvections. Sections 6.B and 6.C discuss certain subspaces of the partial linear spaces studied here and the relation of these subspaces to the representation theory of the spaces and of certain Grassmannian spaces. The graphs of II are here given a geometric context. In § 6.D a sketchy account is given of how the present results can be used to calculate explicitly all derivations from a symplectic transvection group on the $\mathbb{F}_2$-space $V$ into $V$.

A general reference for much of the geometry and linear algebra used here is [**7**]. See also [**12**, § 3].

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## 2. *The classification of graphs with the cotriangle property*

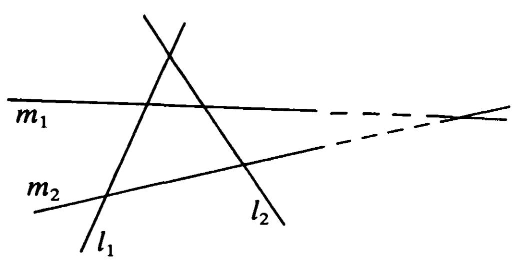
A partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$ is a *cotriangular space* if

(i) each line $l$ of $\mathcal{L}$ has $|l| = 3$,

(ii) for each $a \notin l$, $a$ is collinear with none or all but one of the points of $l$.

It is worth noting that a partial linear space as in I is a cotriangular space, because each subspace $\mathcal{T}_4$ satisfies (ii) always with 'all but one'. Not every cotriangular space $\Pi = (\mathcal{P}, \mathcal{L})$ is as in I. To be so, $\Pi$ must additionally satisfy the Veblen–Young axiom:

(V–Y) If distinct $l_1$, $l_2$, $m_1$, $m_2 \in \mathcal{L}$ with $l_1 \cap l_2 = \varnothing$, $l_i \cap m_j \neq \varnothing$, but $l_1 \cap l_2 \cap m_j = \varnothing$ (for all $i,j$), then $m_1 \cap m_2 \neq \varnothing$.




Indeed it is an easy exercise to see that the spaces of I are exactly the cotriangular spaces which satisfy the Veblen–Young axiom (V–Y). The six intersection points and four lines of the Veblen–Young configuration form a subspace $\mathcal{T}_4$.

The graph $\Gamma$ is *cotriangular* if for every non-edge $(x, y)$ there is a coclique $K(x, y)$ (a subgraph containing no edges) containing $x$ and $y$ and such that

(i) $|K(x, y)| = 3$,

(ii) for each vertex $r$ not in $K(x, y)$, $r$ is adjacent to one or all of the vertices of $K(x, y)$.

If $\Pi = (\mathcal{P}, \mathcal{L})$ is a cotriangular space, then its non-collinearity graph $\bar{P}(\Pi)$ is a cotriangular graph. Here the coclique $K(x, y)$ on the non-adjacent (i.e. collinear) pair $x,y$ is the line of $\mathcal{L}$ containing them.

For $\Gamma$ a graph, we write $a \sim b$ if $a$ and $b$ are adjacent vertices of $\Gamma$. We write $a \not\sim b$ if $a$ and $b$ are distinct and non-adjacent. We let $\Gamma_x = \{a \mid a \sim x\}$, and $\Gamma_A = \bigcap_{x \in A} \Gamma_x$. Then $x^\perp$ is $\{x\} \cup \Gamma_x$ and $A^\perp$ is $\bigcap_{x \in A} x^\perp$. We have $\mathrm{Rad}(A) = A \cap A^\perp$. We write $x \approx y$ if $x^\perp = y^\perp$ and let $x^*$ be the $\approx$-equivalence class of $x$. We use $\Gamma^*$ to denote the factor graph $\Gamma/\approx$. We say that $\Gamma$ is *reduced* if $\Gamma^* = \Gamma$ and *indecomposable* if $\bar{\Gamma}$ is connected (here $\bar{\Gamma}$ is the graph complement of $\Gamma$). Also $\Gamma$ is *irreducible* if it is reduced and indecomposable.

Notice that $\Gamma$ is a cotriangular graph if and only if $\Gamma^*$ is cotriangular (in which case $\Gamma^*$ and the size of the $\approx$-classes uniquely determine $\Gamma$). Similarly $\Gamma$ is a cotriangular graph if and only if, for every connected component $\bar{\Delta}$ of $\bar{\Gamma}$, $\Delta$ is cotriangular (in which case the set of graphs $\Delta$, $\bar{\Delta}$ a connected component of $\bar{\Gamma}$, uniquely determines $\Gamma$). Therefore a complete classification of cotriangular graphs is an immediate consequence of the classification of those which are irreducible. This section is devoted to a proof of the classification.

THEOREM 3. *Let $\Gamma$ be an irreducible cotriangular graph. Then $\Gamma$ is isomorphic to one of the graphs $sp(V, f)$ or $n(V, q)$, with $f$ and $q$ non-degenerate on the $\mathbb{F}_2$-space $V$, or to a graph $\bar{t}(\Omega)$ for some set $\Omega$.*

The vertices of $sp(V, f)$ are the non-radical vectors for the symplectic form $f$ on the $\mathbb{F}_2$-space $V$ with $x$ and $y$ adjacent precisely when $f(x, y) = 0$. The vertices of $n(V, q)$ are the non-radical, non-singular vectors for the quadratic form $q$ on the $\mathbb{F}_2$-space $V$ with $x$ adjacent to $y$ precisely when $q(x+y) = q(x) + q(y)$. The vertices of $\bar{t}(\Omega)$ are the unordered pairs from the set $\Omega$ with $\{a, b\}$ adjacent to $\{c, d\}$ precisely when $\{a, b\} \cap \{c, d\} = \emptyset$.

If the symplectic form $f$ is non-degenerate on the finite-dimensional $\mathbb{F}_2$-space $V$, then $\dim(V)$ is even, $2n$ say, and (up to isometry) the pair $(V, f)$ is uniquely determined by the $\mathbb{F}_2$-dimension $2n$. In this case we may write $sp_{2n}(2)$ for $sp(V, f)$ without confusion. If the quadratic form $q$ is non-degenerate on the finite-dimensional $\mathbb{F}_2$-space $V$, then the pair $(V, q)$ is uniquely determined by the $\mathbb{F}_2$-dimension $2n$ of $V$ and the index $\varepsilon = \pm$ of $q$. We may write $n^{\varepsilon}_{2n}(2)$ for $n(V, q)$. The isomorphism class of $\bar{t}(\Omega)$ is uniquely determined by $|\Omega|$. If $|\Omega| = n$ is finite, we may write $\bar{t}_n$ for $\bar{t}(\Omega)$. In particular, the non-collinearity graph of the cotriangular space $\mathscr{T}_4$ is the graph $\bar{t}_4$, a disjoint union of three edges.

Theorem 3 is an extension of Shult's cotriangle theorem **[13, 19]** to include infinite graphs.

SHULT'S COTRIANGLE THEOREM. *Let $\Gamma$ be a finite, irreducible cotriangular graph. Then, for some $n$, $\Gamma$ is isomorphic to one of the graphs $sp_{2n}(2)$, $n^{\varepsilon}_{2n}(2)$, or $\bar{t}_n$.*

The correspondence between cotriangle spaces and graphs is not perfect. With each cotriangular space we have canonically associated a cotriangular graph, but many cotriangular graphs are associated with no cotriangular space and the same cotriangular graph can be associated with many different (even non-isomorphic) cotriangular spaces. If $\Pi = (\Gamma, \mathscr{A})$ is a cotriangular space then $\Pi$ is, respectively, *reduced* or *irreducible* if the associated cotriangular graph $\Gamma = \bar{P}(\Pi)$ is, respectively, reduced or irreducible. (We continue to call $\Pi$ connected if $\bar{P}(\Pi)$ is indecomposable.) Set $\mathscr{A}^* = \{\{x^*, y^*, z^*\} \mid \{x, y, z\} \in \mathscr{A}\}$, and set $\Pi^* = (\Gamma^*, \mathscr{A}^*)$.

Let $\Gamma$ be a cotriangular graph, and let $x \not\sim y$ in $\Gamma$. Suppose that the cliques $K = \{x, y, z\}$ and $K' = \{x, y, z'\}$ both satisfy (ii) of the definition. Then easily $z \approx z'$. In particular, if $\Gamma$ is reduced then $x$ and $y$ uniquely determine $K(x, y)$; so $\Gamma$ together with its distinguished cocliques forms a cotriangular space. We have the following important observation:

> A reduced cotriangular graph is the non-collinearity graph of a unique cotriangular space.

In particular, if $\Pi = (\Gamma, \mathscr{A})$ is a cotriangular space, then $\Pi^* = (\Gamma^*, \mathscr{A}^*)$ is a reduced cotriangular space.

We have already remarked that to classify all cotriangular graphs it is enough to find those cotriangular graphs which are reduced, indeed irreducible. Now we may, when convenient, restrict our search to the irreducible cotriangular spaces.

If $\Gamma$ is a reduced cotriangular graph then its cocliques $K(x, y)$ distinguished in (ii) are called *hyperbolic lines*. If $\mathscr{A}$ is the set of hyperbolic lines of $\Gamma$, then the *subspaces* of $\Gamma$ are the subgraphs of $\Gamma$ which are subspaces of the cotriangular space $(\Gamma, \mathscr{A})$. The smallest subspace of $\Gamma$ containing $\Omega$ is $\langle \Omega \rangle$, the subspace *generated by* $\Omega$. In particular $K(x, y) = \langle x, y \rangle$. By abuse of terminology we sometimes write $\Gamma$ for the cotriangular space $(\Gamma, \mathscr{A})$.

(2.1) *If $x \sim y$ is an edge of the indecomposable cotriangular graph $\Gamma$, then there is a $z$ with $x \not\sim z \not\sim y$.*

*Proof.* Clearly we need only prove the result for irreducible $\Gamma$.

Let $\Gamma$ be a reduced cotriangular graph, and let $x \sim y$ be an edge of $\Gamma$ such that, for all $z$ of $\Gamma$, $z \sim x$ or $z \sim y$. In particular, $\Gamma - x^\perp \subseteq y^\perp$ and $\Gamma - y^\perp \subseteq x^\perp$. Let $y \in \Delta$ with $\bar{\Delta}$ a connected component of $\bar{\Gamma}_x$.

If $a \in \Gamma$ with $a \not\sim y$, then $a \in \Gamma_x$ and indeed $a \in \Delta$. Suppose $\langle a, b \rangle = \{a, b, c\}$ is any hyperbolic line of $\Gamma$ containing $a$. Then $y \not\sim a$ implies that $y \not\sim b$ or $y \not\sim c$. Thus $b$ or $c$ also belongs to $\Gamma_x$ and $\Delta$. But then in fact the line $\{a, b, c\} \subseteq \Delta$. We conclude that $\Gamma - \Delta \subseteq a^\perp$ as well. In this manner we see that $\bar{\Delta}$ is a connected component of $\bar{\Gamma}$. As $x \notin \Delta$, $\Gamma$ is decomposable.

(2.2) *If $\Gamma$ is a reduced cotriangular graph, then $\Gamma = \langle x^\perp, y \rangle$ for any $y \notin x^\perp$.*

*Proof.* Let $\Delta = \langle x^\perp, y \rangle$, and let $\langle x, y \rangle = \{x, y, z\} \subseteq \Delta$. For any $a \in \Gamma - x^\perp$, $a \not\sim x$; so $a \not\sim w$ for $w = y$ or $z$. If $\langle a, w \rangle = \{a, w, v\}$, then $x^\perp \cap \langle a, w \rangle = \{v\}$. Therefore $a \in \langle v, w \rangle \subseteq \Delta$, giving $\Gamma = \Delta$.

(2.3) *If $\Gamma$ is a cotriangular graph then $a^\perp \subseteq b^\perp$ implies that $a^\perp = b^\perp$ or $b \in \mathrm{Rad}(\Gamma) = \Gamma \cap \Gamma^\perp$.*

*Proof.* Without loss of generality, we may assume that $\Gamma$ is reduced. If $a^\perp$ is proper in $b^\perp$ then for some $y \notin a^\perp$ we have $b^\perp \supseteq \langle a^\perp, y \rangle = \Gamma$ by (2.2).

(2.4) *Let $\Gamma$ be an irreducible cotriangular graph, and let $a \sim b$. Then $ab^{\perp\perp} = \{\{a, b\}^\perp\}^\perp$ is a clique of $\Gamma$ such that, for any $x$ of $\Gamma$, either $ab^{\perp\perp} \subseteq x^\perp$ or $|x^\perp \cap ab^{\perp\perp}|$ is 0 or 1.*

*Proof.* As $a \sim b$, $a,b \in \{a, b\}^{\perp}$, so $ab^{\perp\perp}$ is $\mathrm{Rad}(\{a, b\}^{\perp})$ and in particular is a clique. In fact, by (2.3), $ab^{\perp\perp}$ is $a$ together with the $\approx$-class of $b$ in $\Gamma_a$. Clearly $ab^{\perp\perp} = ba^{\perp\perp}$, and a second application of (2.3) shows that $ab^{\perp\perp} = ac^{\perp\perp}$ for all $c \in ab^{\perp\perp}$, $c \neq a$. We conclude that, for distinct $c,d \in ab^{\perp\perp}$, $ab^{\perp\perp} = cd^{\perp\perp}$. If $x^{\perp} \cap ab^{\perp\perp}$ contains $c$ and $d$, then in $c^{\perp}$ we see that $x^{\perp}$ contains $cd^{\perp\perp} = ab^{\perp\perp}$.

(2.5) *Let $\Gamma$ be an irreducible cotriangular graph, and let $\{x, y, z\}$ be a hyperbolic line of $\Gamma$ contained entirely within $a^{\perp}$. Let $P = ax^{\perp\perp} \cup ay^{\perp\perp} \cup az^{\perp\perp}$. Then $P$ is a subspace of $\Gamma$ and either*

(1) *$P = \{a, x, y, z\}$ or*

(2) *$P - \{a\} \simeq \bar{t}_4$ and $P$, together with its hyperbolic lines and the three lines $ax^{\perp\perp}$, $ay^{\perp\perp}$, and $az^{\perp\perp}$, is a projective plane of order 2.*

*Proof.* We know that $\Gamma_a^*$ is reduced, so $P$ is a subspace of $\Gamma$. For each $b \notin a^{\perp}$, $b^{\perp}$ meets each hyperbolic line of $\Gamma$ contained in $P$ and meets each of $ax^{\perp\perp}$, $ay^{\perp\perp}$, and $az^{\perp\perp}$ at most once. The only possibilities are (1) and (2).

(2.6) *Let $\Gamma$ be a reduced cotriangular graph with a subspace $\Delta \simeq \bar{t}_4$. Then, for any $d \in \Gamma$, $d^{\perp} \cap \Delta$ is one of:*

(1) *an edge,*

(2) *a hyperbolic line,*

(3) *all $\Delta$.*

*Proof.* This is immediate, because $\Gamma$ with its hyperbolic lines is a cotriangular space.

The non-collinearity graph $w(D_4)$ of the cotriangular space $\mathcal{W}(D_4)$ is a disjoint union of three cliques of size 4.

(2.7) *Let $\Gamma$ be a reduced cotriangular graph with a subspace $\Delta \simeq w(D_4)$, and let $C$ be a clique of $\Delta$ of size 4. Then, for any $d \in \Gamma$, $|d^{\perp} \cap C| = 0$, 2, or 4.*

*Proof.* This is an easy consequence of (2.6).

(2.8) *Let $\Gamma$ be a reduced cotriangular graph with an indecomposable subspace $\Delta$.*

(1) *If $\Delta$ is generated by three elements of $\Gamma$, then $\Delta \simeq \bar{t}_3$ or $\bar{t}_4$.*

(2) *If $\Delta$ is generated by four elements of $\Gamma$, then $\Delta \simeq \bar{t}_4$, $\bar{t}_5$, or $w(D_4)$.*

*Proof.* (1) This is equivalent to (V–Y).

(2) Choose three of the elements generating a subspace $\Delta' \simeq \bar{t}_4$. Then $\Delta$ is $\bar{t}_4$, $\bar{t}_5$, or $w(D_4)$ depending on whether the fourth element is in $\Delta'$, adjacent to a hyperbolic line of $\Delta'$, or adjacent to an edge of $\Delta'$ (see (2.6)).

Let $\Pi = (\Gamma, \mathcal{A})$ be a cotriangular space. For each $x \in \Gamma$ define the permutation $t(x)$ of $\Gamma$ by

$$t(x): \begin{cases} y \to y & \text{if } y \in x^{\perp}, \\ y \to z & \text{if } \{x, y, z\} \in \mathcal{A}. \end{cases}$$

Define $D(\Pi) = \{t(x) \mid x \in \Gamma\}$ and $G(\Pi) = \langle D(\Pi) \rangle \leq \mathrm{Sym}(\Gamma)$.

(2.9) *Let* $\Pi = (\Gamma, \mathscr{A})$ *be a cotriangular space.*

(1) *The map* $t(x) \to t(x^*)$, *for* $x \in \Gamma$, *extends to a homomorphism of* $G(\Pi)$ *onto* $G(\Pi^*)$ *taking* $D(\Pi)$ *to* $D(\Pi^*)$.

(2) $G(\Pi)$ *is transitive on* $\Gamma$ *if and only if* $\Gamma$ *is indecomposable.*

(3) *If* $\Delta$ *is a subspace of* $\Pi$ *then the stabilizer of* $\Delta$ *in* $G(\Pi)$ *induces* $G(\Delta)$ *on* $\Delta$.

(4) $G(\Pi) \leq \mathrm{Aut}(\Pi)$ *if and only if* $\Pi$ *satisfies* (V–Y).

*Proof.* (1)–(3) These are clear from the definitions.

(4) For $t(x)$ to belong to $\mathrm{Aut}(\Pi)$ we must check that $l = \{a, b, c\} \in \mathscr{A}$ implies that $lt(x) = \{at(x), bt(x), ct(x)\} \in \mathscr{A}$. If $\{a, b, c\} \subseteq x^{\perp}$, this is clearly true; so we may, without loss of generality, assume that $c = \{a, b, c\} \cap x^{\perp}$. Let $\langle a, x \rangle = \{a, a', x\}$ and $\langle b, x \rangle = \{b, b', x\}$. Then $lt(x) \in \mathscr{A}$ if and only if $\{a', b', c\} \in \mathscr{A}$, which is true if and only if $\{a, b, c, a', b', x\}$ is a subspace $\mathscr{T}_4$ of $\Pi$. Thus $t(x) \in \mathrm{Aut}(\Pi)$ if and only if (V–Y) is true for every pair of lines $l_1$ and $l_2$ from $\mathscr{A}$ which intersect in $x$.

(2.10) *Let* $\Gamma$ *be a cotriangular graph.*

(1) *For all* $x \in \Gamma$ *let* $\bar{x} \in \mathbb{F}_2^{\Gamma}$ *be given by*

$$\bar{x}_y = \begin{cases} 0 & \text{if } y \in x^{\perp}, \\ 1 & \text{if } y \notin x^{\perp}. \end{cases}$$

*If the coclique* $K(x, y) = \{x, y, z\}$ *then* $\bar{x} + \bar{y} + \bar{z} = 0$ *(the zero element) in* $\mathbb{F}_2^{\Gamma}$. *Also* $\bar{x} = \bar{y}$ *if and only if* $x \approx y$. *In particular, if* $\Gamma$ *is reduced then* $\Gamma$ *satisfies* (V–Y). *Set* $\bar{\Gamma} = \{\bar{x} \mid x \in \Gamma\}$, *and let* $V(\Gamma)$ *be the subspace of* $\mathbb{F}_2^{\Gamma}$ *spanned by* $\bar{\Gamma}$.

(2) *Assume that* $\Gamma$ *is reduced. Let* $\pi_x$ *be projection onto the* $x$*-coordinate of* $\mathbb{F}_2^{\Gamma}$ *and* $V(\Gamma)$. *Then the map which, for each* $x \in \Gamma$, *takes* $t(x)$ *to the transvection of* $\mathrm{GL}(V(\Gamma))$ *given by*

$$t(\bar{x}) \colon\ v \to v + \pi_x(v)\bar{x}$$

*extends to an isomorphism of* $G(\Gamma)$ *onto the subgroup* $T(\Gamma) = \langle t(\bar{x}) \mid x \in \Gamma \rangle$ *of* $\mathrm{GL}(V(\Gamma))$. *The subgroup* $T(\Gamma)$ *leaves* $\bar{\Gamma}$ *invariant. This isomorphism and the natural bijection of* $\Gamma$ *and* $\bar{\Gamma}$ *together give a permutation group isomorphism of* $(G(\Gamma), \Gamma)$ *and* $(T(\Gamma), \bar{\Gamma})$.

*Proof.* (1) For any $a \in \Gamma$ the three coordinate entries $x_a$, $y_a$, and $z_a$ contain an even number of 1's as $\Gamma$ is cotriangular.

(2) It is easily checked that with respect to the natural bijection $x \to \bar{x}$ of $\Gamma$ and $\bar{\Gamma}$, the transvection $t(\bar{x})$ induces the same permutation on $\bar{\Gamma}$ as $t(x)$ does on $\Gamma$.

(2.11) *If* $\Gamma$ *is an irreducible cotriangular graph, then* $T(\Gamma)$ *is irreducible on* $V(\Gamma)$.

*Proof.* The subgroup $T(\Gamma)$ is generated by a conjugacy class of transvections whose centres span $V(\Gamma)$. Any proper invariant subspace is therefore contained in $C_{V(\Gamma)}(T(\Gamma))$. But $C_{V(\Gamma)}(t(\bar{x}))$ is $\ker(\pi_x)$, and $\bigcap_{x \in \Gamma} \ker(\pi_x) = 0$.

We now begin in earnest to prove Theorem 3. From now on in § 2 *assume that* $\Gamma$ *is an irreducible cotriangular graph.* The next lemma is an easy consequence of (2.10) but is of fundamental importance.

(2.12) *Let* $\Omega$ *be a finite subset of* $\Gamma$. *Then* $\langle \Omega \rangle$ *is a finite subspace of* $\Gamma$.

*Proof.* The finitely generated subgroup $H = \langle t(\tilde{x}) | \ x \in \Omega \rangle$ is a finite subgroup of $T(\Gamma) \leqslant \mathrm{GL}(V(\Gamma))$. If $\tilde{\Delta} = \{\tilde{x}H | \ x \in \Omega\}$, then $\Delta$ is a finite subspace of $\Gamma$ containing $\Omega$.

(2.13) *The group* $G(\Gamma) \leqslant \mathrm{Aut}(\Gamma)$ *is transitive on* $\Gamma$. *If, for all* $x \in \Gamma$, $\Gamma_x$ *is indecomposable, then* $G(\Gamma)$ *is edge-transitive on* $\Gamma$.

*Proof.* The first sentence is immediate by (2.9) and (2.10). If $\Gamma_x$ is indecomposable, then $G(\Gamma_x)$ is transitive on $\Gamma_x$ by (2.9.2). But by (2.9.3), this group is induced by $G(x^\perp)$ and so by the subgroup of $G(\Gamma)$ which leaves $x$ fixed. This implies edge transitivity.

(2.14) *For all* $x \in \Gamma$, $\Gamma_x$ *is indecomposable.*

*Proof.* By transitivity, it suffices to find a single $x$ with $\Gamma_x$ indecomposable. We first prove this for finite $\Gamma$, and then extend this to general $\Gamma$ using (2.12).

Assume $|\Gamma| < \infty$. Let

$$\mathscr{C} = \{\Delta | \ \Delta \text{ indecomposable, } \Delta \subseteq \Gamma_x, \text{ for some } x\}.$$

Choose (using $|\Gamma| < \infty$) a member $C$ of $\mathscr{C}$ maximal under containment. Let $D = C^\perp \neq \varnothing$ by definition. For any $d \in D$, maximality of $C$ forces the complement $\bar{C}$ to be a connected component of $\bar{\Gamma}_d$. Thus any $e$ with $e^\perp \cap D \neq \varnothing$ has either $e \in C$ or $e \in D$. In particular, $d^\perp \subseteq C \cup D$ for all $d \in D$. If $D$ contains a non-edge $x \not\sim y$ then $\langle x, y \rangle \subseteq D$, and hence $z^\perp \cap D$ is non-empty for all $z$. This forces $D = \Gamma - C$, against the indecomposability of $\Gamma$. We conclude that $D = d^*$ for all $d \in D$; and $D = \{d\}$ as $\Gamma$ is reduced. Thus $C = \Gamma_d$, as required.

Now assume that $\Gamma$ is infinite. Let $x \in \Gamma$. We need to prove that, for all $y$, $z \in \Gamma_x$, $y$ and $z$ are in the same connected component of $\bar{\Gamma}_x$. As $\Gamma$ is indecomposable, (2.1) guarantees a finitely generated indecomposable subspace $\Delta$ of $\Gamma$ which contains $x$, $y$, and $z$ and in which

$$\Delta \cap y^\perp \neq \Delta \cap x^\perp \neq \Delta \cap z^\perp.$$

By (2.12), $\Delta$ is finite; so the previous paragraph guarantees that $y$ and $z$ are in the same connected component of $\bar{\Delta}_x$.

(2.15) *The following are equivalent:*

(1) $\Gamma_x$ *is not reduced;*

(2) *there is an edge* $x \sim y$ *with* $\tilde{x} + \tilde{y} \in \tilde{\Gamma}$;

(3) *the map* $\Gamma \to \tilde{\Gamma}$ *induces an isomorphism of* $\Gamma$ *with* $sp(V(\Gamma), f)$ *for a non-degenerate* $f$.

*Proof.* Clearly (3) implies (1). Suppose as in (1) that $\Gamma_x$ is not reduced, that is, for some $y \in \Gamma_x$, $|xy^{\perp\perp}| > 2$. By (2.5) there is a $z \in P$ with $l = xy^{\perp\perp} = \{x, y, z\}$.

Again by (2.5), for each $d \in \Gamma$, $|d^\perp \cap l|$ is 0, 1, or 3. By (2.6) the possibility $|d^\perp \cap l| = 0$ never occurs. By arguing as for (2.10.1) we conclude that $\bar{x} + \bar{y} + \bar{z} = 0$. Therefore $\bar{x} + \bar{y} = \bar{z} \in \bar{\Gamma}$, and (1) implies (2). Assume that (2) holds. By edge transitivity, for every edge $a \sim b$ of $\Gamma$, $\bar{a} + \bar{b} \in \bar{\Gamma}$. This is already true for non-edges $a \not\sim b$ by (2.10.1), so in fact $V(\Gamma) = 0 \cup \bar{\Gamma}$. By (2.2), for $x \in \Gamma$,

$$\bar{x}^\perp = \langle \bar{y} |\ y \in x^\perp \rangle$$

is always a hyperplane of $V(\Gamma)$. Set $0^\perp = V(\Gamma)$. The map $\bar{x} \to \bar{x}^\perp$, $\bar{x} \in V(\Gamma)$, has the following two properties:

(1) $\bar{x} \in \bar{x}^\perp$, for all $\bar{x} \in V(\Gamma)$;

(2) if $\bar{y} \in \bar{x}^\perp$ then $\bar{x} \in \bar{y}^\perp$.

By a theorem of Brauer [**1**], there is a non-degenerate symplectic form $f$ on $V(\Gamma)$ with, for each $\bar{x} \in V(\Gamma)$, $\bar{x}^\perp = \{\bar{y} \in V(\Gamma |\ f(\bar{x}, \bar{y}) = 0\}$. Therefore $\Gamma \simeq sp(V(\Gamma), f)$. (Brauer's proof is only for finite-dimensional $V(\Gamma)$, but this easily implies the infinite-dimensional case; see [**5**, 3.7] for a fuller discussion of a slightly more general version.)

(2.16) *Assume that, for $x \in \Gamma$, $\Gamma_x$ is reduced and in addition that $\Gamma$ contains a subspace $w(D_4)$. Then the map $\Gamma \to \bar{\Gamma}$ induces an isomorphism of $\Gamma$ with $n(V(\Gamma), q)$ for a non-degenerate $q$.*

*Proof.* By (2.14) for each $x$ of $\Gamma$, $\Gamma_x$ is irreducible. In particular, by arguing as in (2.13), we find that $G(\Gamma)$ is transitive on the triangles of the graph $\Gamma$. By (2.7), every triangle $\{x, y, z\}$ of $\Gamma$ lies in a clique $C = \{x, y, z, w\}$ of four points with $|d^\perp \cap C| \in \{0, 2, 4\}$, for every $d$ of $\Gamma$. Again by arguing as for (2.10.1), we find that $\bar{w} + \bar{x} + \bar{y} + \bar{z} = 0$.

A special consequence of the observation of the previous paragraph is that every edge $y \sim z$ of $\Gamma_x$ is contained in a clique $C = \{w, y, z\}$ of $\Gamma_x$ such that $|C \cap d^\perp|$ is 1 or 3 for all $d$ in $\Gamma_x$. With $\Delta = \Gamma_x$, $\Delta_w$ is not reduced as $y \approx x$ within $\Delta_w$; so (2.15) applies within $\Gamma_x$.

Because $\Gamma_x$ is irreducible, (2.15) guarantees that $\Gamma_x$ is injectively embedded in $\bar{x}^\perp / \langle \bar{x} \rangle$ by $y \to \bar{y} + \langle \bar{x} \rangle$ for $y \in \Gamma_x$. Applying (2.15) within $\Gamma_x$ we see that the image of $\Gamma_x$ in fact covers $(\bar{x}^\perp / \langle \bar{x} \rangle) - \langle \bar{x} \rangle$. We conclude that

(*) every 2-dimensional subspace of $\bar{x}^\perp$ containing $\bar{x}$ meets $\bar{\Gamma}$ exactly twice.

We claim that the map $q$: $V(\Gamma) \to \mathbb{F}_2$, given by

$$q(\bar{x}) = 1, \quad \text{for } x \in \Gamma,$$
$$q(v) = 0 \quad \text{otherwise},$$

is a quadratic form on $V(\Gamma)$. Notice that by (*) and (2.10), for $x$ and $y$ in $\Gamma$, $x \sim y$ if and only if $\bar{x} + \bar{y} \notin \bar{\Gamma}$ and that this is true if and only if

$$q(\bar{x}) + q(\bar{y}) + q(\bar{x} + \bar{y}) = 0.$$

Thus given the claim, $\Gamma$ is isomorphic to $n(V(\Gamma), q)$. Therefore $q$ is non-degenerate as $T(\Gamma)$ respects $q$ and is irreducible on $V(\Gamma)$ (2.11).

It remains to prove that $f$, given by

$$f(u, v) = q(u+v) + q(u) + q(v),$$

is bilinear on $V(\Gamma)$. This can be checked within 3-dimensional subspaces of $V(\Gamma)$. Indeed $f$ is bilinear on $V(\Gamma)$ if and only if $q(W) = \sum_{w \in W} q(w)$ is 0 for all 3-spaces $W$ of $V(\Gamma)$.

If $W \cap \bar{\Gamma}$ is empty, then clearly $q(W) = 0$. So we may assume that $\bar{x} \in W$, for some $x \in \Gamma$. Now $\bar{x}^\perp$ is a hyperplane of $V(\Gamma)$; hence by (*) there is a $y \in \Gamma$ with $\bar{y} \in W$ but $\bar{x} + \bar{y} \notin \Gamma$. Thus $q(\bar{x}) = q(\bar{y}) = 1$ and $q(\bar{x} + \bar{y}) = 0$. If no further vector of $W$ belongs to $\Gamma$, then $q(W) = 0$; so we may assume that there exist a $\bar{z} \in W - \langle \bar{x}, \bar{y} \rangle$, for some $z \in \Gamma$. If $z \in x^\perp$ (respectively, $y^\perp$), then $W \subseteq \bar{x}^\perp$ (respectively, $\bar{y}^\perp$) and, by (*), $q(W) = 0$. If $x \not\sim z \not\sim y$, then by (V–Y) or (2.8.1), $\bar{\Gamma} \cap W$ contains at least six vectors, of necessity $W - \langle \bar{x} + \bar{y} \rangle$, and once again $q(W) = 0$. We conclude that $\Gamma \simeq n(V(\Gamma), q)$.

(2.17) *If $\Gamma$ contains no subspace $w(D_4)$, then $\Gamma \simeq \bar{t}(\Omega)$ for appropriate $\Omega$.*

*Proof.* By (2.1) and (2.8) every non-edge of $\Gamma$ lies in a subspace $\bar{t}_5$ of $\Gamma$. In particular, each vertex of $\Gamma$ is in at least two cocliques of $\Gamma$ of size at least 4 and each non-edge of $\Gamma$ is in at least one coclique of size at least 4. A hyperbolic line is a maximal coclique of $\Gamma$, so any coclique of four or more vertices contains no hyperbolic lines. Set

$$\begin{aligned}\Omega &= \{\text{maximal cocliques of } \Gamma \text{ of size} > 3\}\\ &= \{\text{maximal cocliques of } \Gamma \text{ containing no hyperbolic line}\}.\end{aligned}$$

By Zorn's lemma, such cocliques exist and every coclique not a hyperbolic line is contained in a member of $\Omega$.

For each $x \in \Gamma$, set $x' = \{C \in \Omega |\ x \in C\}$, and define a graph $\Gamma'$ on the vertex set $\{x' |\ x \in \Gamma\}$ by $x' \sim y'$ if and only if $x' \cap y' = \varnothing$.

We wish to prove that

(i) $x \to x'$ is an isomorphism of $\Gamma$ and $\Gamma'$,

(ii) $\Gamma' = \bar{t}(\Omega)$.

Part (i) is clear, as $x \sim y$ in $\Gamma$ if and only if $\{x, y\}$ lies in no coclique of $\Gamma$ and hence no coclique of $\Omega$. For (ii) we have already remarked that $|x'| \geqslant 2$, so it remains to prove that $x$ is not contained in three distinct maximal cocliques from $\Omega$.

If $C$ is a maximal coclique from $\Omega$ containing $x$, then $\hat{C} = \{ct(x) |\ c \in C\}$ is also a maximal coclique on $x$ and belongs to $\Omega$. If $y \neq x$ belongs to $C$, then $\hat{y} = yt(x)$ is in $\hat{C}$ and $\{x, y, \hat{y}\}$ is a hyperbolic line. If $y \in C \cap \hat{C}$, then $\{x, y, \hat{y}\} \subseteq C \cap \hat{C}$ which is not the case; so $\{x\} = C \cap \hat{C}$. For any further $z \in C$, $x \not\sim z \not\sim y$, so $z \sim \hat{y}$. Thus every member $w$ of $C$ which is not $x$ is adjacent to all but one of the members of $\hat{C} - x$, and this member of $\hat{C}$ together with $x$ and $w$ form a hyperbolic line. The same is true with the roles of $C$ $(= \hat{C}t(x))$ and $\hat{C}$ reversed.

Suppose $x$ lies within a third coclique $D$ of $\Omega$ which is not $C$ or $\hat{C}$. The previous paragraph guarantees that $D$ is not contained in $C \cup \hat{C}$, so we may choose a $w \in D - (C \cup \hat{C})$. As $w \notin C \cup \hat{C}$, there are, by maximality, $y \in C$ and $z \in \hat{C}$ with $y \sim w \sim z$. As $w \not\sim x$, $\{x, y, z\}$ is not a hyperbolic line. Again by the previous paragraph, $y \sim z$. But now by (2.8.2), $\langle w, x, y, z \rangle$ is a subspace $w(D_4)$, against assumption. Therefore no third coclique $D$ exists, and the proof is complete.

Theorem 3 now follows from (2.15)–(2.17). The discussion at the beginning of the section makes Theorem 4 an immediate consequence of Theorem 3.

## 3. *3-transposition groups*

In this section we give a precise description of the equivalence of I and III. As a particular consequence of this, Theorem 1 follows immediately from Theorems 1 and 2 of [**12**]. We finish with a discussion of Theorem 5 which is a restatement of Theorems 3 and 4 in the language of III. Theorem 5 is the starting point for the proof in [**12**].

Let $G$ be a group generated by $D$, a conjugacy class of 3-transpositions of $G$ (a conjugacy class of elements of order 2 with all pairwise products from $D$ of order 1, 2, or 3). Let $\mathscr{L}(G, D)$ be the set of triples $\{x, y, z\}$ from $D$ such that $\langle x, y, z\rangle \simeq S_3$, and let $\Pi(G, D)$ be the pair $(D, \mathscr{L}(G, D))$.

Our first result is essentially due to Fischer [**9**] who introduced, and was the first to study, 3-transpositions.

(3.1) *$\Pi(G, D)$ is a connected partial linear space of order* 2 *in which every pair of intersecting lines lies in a subspace isomorphic to $\mathscr{T}_4$ or to the affine plane of order* 3, $AG(2, 3)$. *If $G$ has no subgroups of order* 18 *or* 54 *generated by three elements of $D$, then every pair of intersecting lines lies in a subspace isomorphic to $\mathscr{T}_4$.*

*Proof.* Set $\Pi = \Pi(G, D)$. As $S_3$ is generated by any two of its transpositions, $\Pi$ is a partial linear space of order 2. As $D$ is a conjugacy class of $G$, $\Pi$ is connected. Every pair of intersecting lines of $\Pi$ lies in a subspace $\Pi(H, D \cap H)$ where $H$ is generated by three members of $D$, the point of intersection and an additional point from each line. Fischer [**9**, 1.6] describes all possibilities for $H$ generated by three 3-transpositions $\{a, b, c\}$. If $\{a, b, c\}$ is not a line of $\Pi$ then either $H = \langle a, b, c\rangle \simeq S_4$ and $\Pi(H, D \cap H)$ is isomorphic to $\mathscr{T}_4$ or $H = \langle a, b, c\rangle$ has order 18 or 54 and $\Pi(H, D \cap H)$ is isomorphic to $AG(2, 3)$. (This is also an easy consequence of results on Coxeter groups. See (2.2.1) and (2.2.2) of [**12**].)

Result (3.1) allows us to get from the situation of III to that of I. The machinery to get from I back to III has already been constructed in §2. Given a space $\Pi$ as in I, we constructed a pair $(G(\Pi), D(\Pi))$, where $G$ is a group and $D$ is a conjugacy class of elements of $G$. This pair is in fact as in III. Here we give a stronger result due originally to Buekenhout [**2**]. If $\Pi = (\mathscr{P}, \mathscr{L})$ is any partial linear space of order 2, we (as in §2) define the group $G(\Pi) \leqslant \mathrm{Sym}(\mathscr{P})$ to be $\langle t(x) |\ x \in \mathscr{P}\rangle$ where $t(x)$ fixes $x$ and all points $y$ not collinear with $x$, and $t(x)$ interchanges $y$ and $z$ when $\{x, y, z\} \in \mathscr{L}$. Additionally set $D(\Pi) = \{t(x) |\ x \in P\}$.

(3.2) *Let $\Pi = (\mathscr{P}, \mathscr{L})$ be a connected partial linear space of order* 2 *with $|\mathscr{P}| > 1$. Then $G(\Pi) \leqslant \mathrm{Aut}(\Pi)$ if and only if every pair of intersecting lines of $\Pi$ lies in a subspace $\mathscr{T}_4$ or $AG(2, 3)$. In this case $D(\Pi)$ is a conjugacy class of* 3*-transpositions generating $G(\Pi)$, and the centre of $G(\Pi)$ is trivial.*

*Proof.* If every pair of intersecting lines of $\Pi$ lies in a subspace $\mathscr{T}_4$ or $AG(2, 3)$ then $G(\Pi) \leqslant \mathrm{Aut}(\Pi)$. As in the proof of (2.5.4) this follows easily from its truth in the two special cases $\Pi = \mathscr{T}_4$ and $\Pi = \mathrm{AG}(2, 3)$.

Now assume that $G = G(\Pi) \leq \mathrm{Aut}(\Pi)$. We may assume that $|\mathcal{P}| > 3$. By definition $t(x)^2 = 1$, for all $x \in \mathcal{P}$. Now $x$ is the intersection of all lines of $\mathcal{L}$ fixed globally but not pointwise by $t(x)$ and these lines are in fact all lines of $\mathcal{L}$ containing $x$. Thus the stabilizer of $x$ in $G(\Pi)$ equals the centralizer of $t(x)$ in $G(\Pi)$. In particular, $t(y)t(x)t(y) = t(x)$ for $y$ not collinear with $x$; that is, $|t(x)t(y)| = 2$. If $\{x, y, z\} \in \mathcal{L}$, then

$$t(y)t(x)t(y) = t(z) = t(x)t(y)t(x)$$

has order 2; in particular $|t(x)t(y)| = 3$.

The normal set $D = D(\Pi)$ is a conjugacy class of $G$ as $\Pi$ is connected, so $D$ is a conjugacy class of 3-transpositions which generates $G$. If $z$ is central in $G$, then $z$ centralizes each $t(x)$, for $x \in \mathcal{P}$, and so stabilizes each $x$ of $\mathcal{P}$; but $G \leq \mathrm{Sym}(\mathcal{P})$, whence $z = 1$. The proof of (3.2) is now completed using (3.1).

We now have a map $\Pi$ from 3-transposition groups with trivial centre to partial linear spaces in which pairs of intersecting lines give rise to subspaces $\mathcal{T}_4$ or $AG(2, 3)$. We also have a map $G$ going in the other direction. From (3.1) and (3.2) it is clear that:

(a) $\Pi$ takes groups as in III to partial linear spaces as in I, and $G$ takes partial linear spaces as in I to groups as in III;

(b) if $\Pi$ is a partial linear space as in I, then $\Pi \simeq \Pi(G(\Pi), D(\Pi))$, and if $(G, D)$ is a 3-transposition group as in III, then $(G, D) \simeq (G(\Pi(G, D)), D(\Pi(G, D)))$.

This gives us a precise correspondence between I and III. Theorem 5 presents the results of Theorems 3 and 4 phrased in the language of III. If $\Pi = \Pi(G, D)$, then there is a homomorphism $\varphi$ which maps $d$ of $D$ to the symplectic transvection $t(\tilde{d})$ of $T(\Pi)$. The normal subgroup $N$ of Theorem 5 is the kernel of $\varphi$. We need not assume that $Z(G) = 1$ in Theorem 5 as (3.1) requires no such assumption. Thus $Z(G)$ sits inside $N$.

## 4. *Locally grid graphs; the proof of Theorem* 2

If $\Delta$ is a set, then the grid graph $L(3, \Delta)$ is the graph whose vertex set is the collection of ordered pairs $\{1, 2, 3\} \times \Delta$ with edges $(a, b) \sim (c, d)$ if and only if $a = c$ or $b = d$.

(4.1) *Let* $\Pi = (\mathcal{P}, \mathcal{L})$ *be a connected partial linear space as in* I. *Then* $\Gamma = L(\Pi)$, *the line graph of* $\Pi$, *is a connected graph in which, for all* $x$ *of* $\Gamma$, $\Gamma_x$ *is isomorphic to* $L(3, \Delta(x))$. *Here* $\Delta(x)$ *is the set of all subspaces of* $\Pi$ *which are generated by the line* $x$ *and a second line meeting* $x$ *in a point.*

*Proof.* If $\Pi$ is a point, a line, or isomorphic to $\mathcal{T}_4$, the result is clear; so we may assume that $|\Gamma| > 4$. For $x$ a line of $\Pi$, $\Gamma_x$ is the set of those lines which share a point of $\mathcal{P}$ with $x$. If $p$ is a point of $\Pi$, then $\Gamma(p)$, the collection of lines on $p$, is a maximal clique of $\Gamma$. If $x = \{a, b, c\}$, then $\Gamma_x$ is the disjoint union of the three cliques $\Gamma(a) - x$, $\Gamma(b) - x$, and $\Gamma(c) - x$. For $x$ in $\Gamma$ let $\Delta(x)$ be the set of subspaces $\mathcal{T}_4$ of $\Pi$ containing the line $x$. If $\Gamma(P)$ is the set of four lines in a member $P$ of $\Delta(x)$, then $\Gamma(P)$ is a maximal clique of $\Gamma$ of size 4. The set $\Gamma_x$ is the disjoint union over $P$ from $\Delta(x)$ of $\Gamma(P)-x$. Furthermore, for each point $p$ of $x$ and each $P$ of $\Delta(x)$, $(\Gamma(P)\cap\Gamma(p))-x$ has size 1. Thus $\Gamma_x \simeq L(3, \Delta(x))$ as claimed. Finally $\Gamma$ is connected as $\Pi$ is.

(4.2) *Let $\Gamma$ be a connected graph such that, for each $x \in \Gamma$, there is a set $\Delta(x)$ with $\Gamma_x \simeq L(3, \Delta(x))$.*

(1) *We may take $\Delta(x)$ to be some fixed set $\Delta$, for all $x \in \Gamma$.*

(2) *If $|\Delta| \neq 3$, then there is a connected partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$ as in* I *with $\Gamma \simeq L(\Pi)$.*

*Proof.* (1) Let $x \sim y$ be an edge of $\Gamma$. Then by examining $x^\perp$ we see that this edge is in exactly two maximal cliques of $\Gamma$ and these have cardinality 4 and $|\Delta(x)|+1$. By examining $y^\perp$ we see that the maximal cliques of $\Gamma$ containing the same edge have cardinality 4 and $|\Delta(y)|+1$. We conclude that $|\Delta(x)| = |\Delta(y)|$. As $\Gamma$ is connected, $|\Delta(z)|$ is uniform over all $z \in \Gamma$. We may therefore choose a fixed $\Delta$ for all $\Delta(z)$.

(2) Let $\mathcal{L}$ be the vertex set of $\Gamma$, and let $\mathcal{P}$ be the set of maximal cliques of $\Gamma$ of size $|\Delta|+1$. We view $p$ of $\mathcal{P}$ as being contained in $l$ of $\mathcal{L}$ if the clique $p$ of $\Gamma$ contains the vertex $l$ of $\Gamma$. Each edge of $\Gamma$ lies in a unique clique of $\mathcal{P}$ (since $|\Delta| \neq 3$), so $\Pi = (\mathcal{P}, \mathcal{L})$ is a partial linear space of order 2, connected as $\Gamma$ is. Evidently, $\Gamma \simeq L(\Pi)$. It remains to consider the subspaces of $\Pi$ which are generated by a pair of intersecting lines. Such a pair is precisely an edge of $\Gamma$.

Each edge $x \sim y$ of $\Gamma$ lies in a unique maximal clique $C$ of size 4. In $\Pi$ the clique $C$ is a set of four lines which pairwise intersect non-trivially. The points on the four lines of $C$ are exactly those six cliques from $\mathcal{P}$ meeting $C$ non-trivially (and so in an edge). Any two of those points of $\mathcal{P}$ are collinear in $\Pi$ precisely when as cliques of $\Gamma$ they share a common vertex of $C$. Thus $C$ gives rise to a subspace $P$ of $\Pi$ which is the dual of the partial linear space of vertices and edges of a clique of size 4, that is, $P$ is isomorphic to $\mathcal{T}_4$.

(4.3) *Result* (4.2.2) *remains true when $|\Delta| = 3$.*

*Proof.* In fact if $|\Delta| = 3$, then $\Gamma$ is isomorphic to the line graph of a cotriangular space of type $\mathcal{T}(\{1, \dots, 6\})$ or $\mathcal{W}(D_4)$. This was first proven by Buekenhout and Hubaut [3]. There is also a proof by Hall and Shult in [13]. The proof of (4.2.2) actually works in this case as well if the book-keeping is done a bit more carefully. Let $\Pi' = (\mathcal{P}', \mathcal{L}')$ where $\mathcal{P}'$ is composed of the cliques of size 4 in $\Gamma$ and $\mathcal{L}'$ is composed of those triples from $\mathcal{P}'$ with both total and pairwise intersection of size 1. Then $\Pi$ is any connected component of the partial linear space $\Pi'$. Finally, (4.3) can also be easily proved using ad hoc methods.

The three results of this section furnish a proof of Theorem 2. We could write down a correspondence between I and II, as we have done for I and III in §3, but the complications of the case $|\Delta| = 3$ from (4.3) would make this awkward.

## 5. *Faithful representations of cotriangular spaces*

For any cotriangular space $\Pi = (\mathcal{P}, \mathcal{L})$ we found in §2 an $\mathbb{F}_2$-vector space $V$ and a map $\varphi\colon \mathcal{P} \to V$ with the property that

$$\varphi(x) + \varphi(y) + \varphi(z) = 0 \quad \text{if } \{x, y, z\} \in \mathcal{L}.$$

Such a map $\varphi$ is an $\mathbb{F}_2$-*linear representation* of $\Pi$ in $V$ (a representation, for short). We say that $\varphi$ is *faithful* if it is injective on $\mathcal{P}$, and that $\varphi$ is *non-trivial* if $\langle\varphi(\mathcal{P})\rangle \neq 0$ and *full* if $\langle\varphi(\mathcal{P})\rangle = V$. Given two cotriangular spaces $\Pi_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\Pi_2 = (\mathcal{P}_2, \mathcal{L}_2)$ and two representations $\varphi_1$: $\mathcal{P}_1 \to V_1$ and $\varphi_2$: $\mathcal{P}_2 \to V_2$, $\varphi_1$ and $\varphi_2$ are *equivalent* if there are an isomorphism $\alpha$: $\Pi_2 \to \Pi_1$ and an invertible linear transformation $R$: $V_1 \to V_2$ with $\varphi_2 = \alpha\varphi_1 R$.

In [**11**] we classified all finite cotriangular spaces $\Pi$ which possess faithful representations and classified up to equivalence all such representations. In this section we use Theorem 4 to extend the work of [**11**] to include all infinite cotriangular spaces with faithful representations. As this condition easily forces $\Pi$ to satisfy (V–Y), this section can in part be viewed as providing a proof of Theorem 1 under the additional assumption of faithful representation. We note that our present terminology differs slightly from that of [**11**]. In that paper all representations were required to be faithful. The results of our §2 make it clear that non-faithful representations can still be of great use.

The concept of representation for cotriangular spaces deserves and possesses a more general context. Certainly our present definitions and terminology regarding representation go over unaltered to the case of any partial linear space of order 2. Furthermore, a great deal of work has been done on various more general questions about the embedding of partial linear spaces as points and lines of projective spaces or as 1-spaces and 2-spaces of affine spaces. See [**4, 6, 8, 11, 14, 16, 17, 19, 20**].

The representation $\hat{\varphi}$: $\mathcal{P} \to \hat{V}$ is a *universal representation* of the partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$ if every representation factors through $\hat{\varphi}$. That is, whenever $\varphi$: $\mathcal{P} \to V$ is a representation of $\Pi$, there is a linear transformation $T$ from $\hat{V}$ to $V$ with $\varphi = \hat{\varphi}T$. If a universal representation exists, then it is clearly full and unique up to equivalence.

In a remark in [**11**] we observed that every partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$ of order 2 has a universal representation. Let $R$ be the subspace of $\mathbb{F}_2^{\mathcal{P}}$ generated by $\{x+y+z \mid \{x, y, z\} \in \mathcal{L}\}$. (Here we identify each member of $\mathcal{P}$ with its characteristic function in $\mathbb{F}_2^{\mathcal{P}}$.) Then the map $\hat{\varphi}$: $\mathcal{P} \to \mathbb{F}_2^{\mathcal{P}}/R$ given by $x\varphi = x + R$, for all $x \in \mathcal{P}$, is by design a representation of $\Pi$. Letting $\hat{V}$ be the span of the $x\hat{\varphi}$, we have a full representation $\hat{\varphi}$: $\mathcal{P} \to \hat{V}$. Clearly $\hat{\varphi}$ is universal. (We must note, as was pointed out to the author by A. Wells, that the implication in [**11**] that $\hat{\varphi}$ is faithful is certainly not in general true.)

We now return to the representation of cotriangular spaces. The first four results of this section correspond to the four theorems given in the introduction of [**11**]. These provide a complete classification of all full, faithful representations of indecomposable cotriangular spaces with no finiteness assumption. Theorem 1 of [**11**] was proven for general cotriangular spaces. We restate it here in greater precision, incorporating some of the results from §1 of [**11**].

We first need a construction. Let $\varphi$: $\mathcal{P} \to V$ be a full, faithful representation of the cotriangular space $\Pi = (\mathcal{P}, \mathcal{L})$ in the $\mathbb{F}_2$-space $V$, and let $W$ be an $\mathbb{F}_2$-space disjoint from $V$. Set

$$\mathcal{P} \oplus W = \{(\varphi(p), w) \mid p \in \mathcal{P}, w \in W\}$$

and

$$\begin{aligned}\mathcal{L} \oplus W = \{&\{(\varphi(x), 0), (\varphi(y), 0), (\varphi(z), 0)\}\{(\varphi(x), w), (\varphi(y), w), (\varphi(z), 0)\}\\ &\{(\varphi(x), w), (\varphi(y), 0), (\varphi(z), w)\}\\ &\{(\varphi(x), 0), (\varphi(y), w), (\varphi(z), w)\} \mid \{x, y, z\} \in \mathcal{L}, w \in W\}.\end{aligned}$$

Then $\Pi \oplus W = (\mathcal{P} \oplus W, \mathcal{L} \oplus W)$ is a cotriangular space and the inclusion of $\mathcal{P} \oplus W$ in $V \oplus W$ is a full, faithful representation of $\Pi \oplus W$.

(5.1) *Let $\varphi$: $\mathcal{P} \to V$ be a full, faithful representation of the indecomposable cotriangular space $\Pi = (\mathcal{P}, \mathcal{L})$. For all $x \in \mathcal{P}$, define*

$$W(x) = \langle \varphi(x) + \varphi(y) |\ x \approx y \rangle.$$

(1) *For all $x, y \in \mathcal{P}$, $W(x) = W(y) = W$, say.*

(2) *Define $V^* = V/W$ and $\varphi^*(x^*) = \varphi(x) + W$, for $x \in \mathcal{P}$. Then $\varphi^*$: $\mathcal{P}^* \to V^*$ is a full, faithful representation of the irreducible cotriangular space $\Pi^* = (\mathcal{P}^*, \mathcal{L}^*)$.*

(3) *$\Pi$ is isomorphic to $\Pi^* \oplus W$, and $\varphi$ is equivalent to the inclusion*

$$i\colon\ \mathcal{P} \oplus W \to V^* \oplus W.$$

As a consequence of (5.1), the equivalence class of a representation $\varphi$ of $\Pi$ is classified by the dimension of the space $W$ and the equivalence class of the representation $\varphi^*$. This reduces the classification of representations to the case where $\Pi$ is irreducible. By Theorem 4 of this paper such a $\Pi$ has type $\mathcal{N}$, $\mathcal{S}p$, or $\mathcal{T}$.

(5.2) *Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a cotriangular space isomorphic to $\mathcal{N}(V, q)$ for $q$ non-degenerate on the $\mathbb{F}_2$-space $V$ of dimension at least* 6. *If $\mathcal{N}(V, q)$ does not have type $\mathcal{N}_6^+(2)$, then every full, faithful representation of $\Pi$ is equivalent to the natural representation of $\Pi$ in $V$.*

(5.3) *Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a cotriangular space isomorphic to $\mathcal{S}p(V, f)$ for $f$ non-degenerate on the $\mathbb{F}_2$-space $V$ of dimension at least* 4.

(1) *There is an orthogonal $\mathbb{F}_2$-space $(\tilde{V}, \tilde{q})$ with radical non-singular of dimension* 1 *and $\tilde{V}/\mathrm{Rad}(\tilde{V}, \tilde{q})$ isometric to $(V, f)$ as symplectic space, and hence*

$$\mathcal{S}p(V, f) = \mathcal{N}(\tilde{V}, \tilde{q}).$$

(2) *Every full, faithful representation of $\Pi$ is equivalent either to the natural symplectic representation of $\Pi$ in $V$ or to the natural orthogonal representation of $\Pi$ in $\tilde{V}$, both full and faithful.*

(5.4) *There is a one-to-one correspondence between the equivalence classes of full, faithful representations of $\mathcal{T}(\Omega)$ for $|\Omega| \geq 2$ and the equivalence classes of binary, even, finitary linear codes of length $|\Omega|$ with minimal distance at least* 6. (A code is *finitary* if all its words have finite weight.)

The irreducible spaces of type $\mathcal{S}p$ and $\mathcal{N}$ excluded under (5.2) and (5.3) are isomorphic to spaces of type $\mathcal{T}$ and so are included in (5.4).

The proofs of (5.2) and (5.4) are effectively identical to the proofs of Theorems 2 and 4 of **[11]**. The proofs may be thought of as having two steps:

(i) identification of a universal representation $\hat{\varphi}$: $\mathcal{P} \to \hat{V}$ for $\Pi$;

(ii) classification (up to equivalence) of those subspaces $C$ of $\hat{V}$ for which the representation $\mathcal{P} \to \hat{V} \to \hat{V}/C$ is faithful.

For the orthogonal case (5.2), arguing as in §2 of **[11]**, we find that $\hat{V}$ is equivalent to $V$ and $C = 0$ is the only possibility. (Many of the arguments of **[11**,

§2] contain arithmetic remarks, however these mainly play a book-keeping role.) In the symmetric case (5.4), $\hat{V}$ is equivalent to $E\mathbb{F}_2\Omega$, and the codes of (5.4) give the various $C$. (See §4 of [**11**].)

The argument of §3 of [**11**] shows essentially that for symplectic $\Pi$ either $\hat{V}$ is equivalent to $V$ or $\hat{V}$ is equivalent to a $\tilde{V}$ as described in (5.3.1). The existence of $\tilde{V}$ is well-known for finite $V$. For general $V$, $\tilde{V}$ is constructed in (3.7.1) of [**12**]. We sketch an alternative construction (effectively reversing some of the arguments of [**12**, §3]). Choose a basis $B$ of $V$. The map $q(b)=1$, for all $b \in B$, extends uniquely to a quadratic form $q$ on $V$ associated with $f$ via

$$q(x+y)=q(x)+q(y)+f(x,y), \quad \text{for all } x,y \in V.$$

The set $\{x \in V \mid q(x)=1\}$ is then a naturally embedded subspace $\mathcal{N}(V,q)$ of $\mathcal{S}p(V,f)$. For $\langle v\rangle$ a 1-dimensional $\mathbb{F}_2$-space disjoint from $V$, we set $\tilde{V}=\langle v\rangle \oplus V$. On $\tilde{V}$ define $\tilde{q}$ by

$$\tilde{q}(\alpha v \oplus x)=\alpha+q(x) \quad (\alpha \in \mathbb{F}_2, x \in V).$$

Then $(\tilde{V},\tilde{q})$ is an orthogonal space as in (5.3.1).

Results (5.1), (5.2), and (5.4) give the following:

(5.5) *Let $\Pi=(\mathcal{P},\mathcal{L})$ be a cotriangular space such that $\Pi^*$ is not isomorphic to $\mathcal{T}(\Omega)$ for any $\Omega$. If $\varphi$: $\mathcal{P}\to V$ is a full, faithful representation of $\Pi$ then there is a form $f$ or $q$ on $V$ such that $\Pi$ is isomorphic either to $\mathcal{N}(V,q)$ or to $\mathcal{S}p(V,f)$.*

(5.6) *Let $\Pi=(\mathcal{P},\mathcal{L})$ be a cotriangular space with $\Pi^*\simeq\mathcal{T}(\Omega)$ and with full, faithful representation $\varphi$: $\mathcal{P}\to V$. Let $\varphi^*$: $\mathcal{P}^*\to V^*$ be the corresponding representation of $\Pi^*$ (see (5.1.2)), and assume that* $\mathrm{GL}(V^*)$ *induces on $\Pi^*$ a doubly transitive subgroup of* $\mathrm{Aut}(\Pi^*)\simeq\mathrm{Sym}(\Omega)$. *Then, for some $\Omega'$, $\varphi$ is equivalent to the standard representation of $\Pi\simeq\mathcal{T}(\Omega,\Omega')$ in $E\mathbb{F}_2(\Omega,\Omega')$ or $\bar{E}\mathbb{F}_2(\Omega,\Omega')$.*

*Proof.* Under the given hypothesis, $E\mathbb{F}_2\Omega$ and $\bar{E}\mathbb{F}_2\Omega$ are the only factors of $\hat{V}$ left invariant by $\mathrm{Aut}(\Pi^*)\cap\mathrm{GL}(V^*)$.

We also have from (5.2)–(5.4):

(5.7) *Let $\Pi=(\mathcal{P},\mathcal{L})$ be an irreducible cotriangular space and let $\varphi$: $\mathcal{P}\to V$ be a non-trivial, full representation of $\Pi$. Then $\varphi$ is faithful.*

## 6. *Applications*

In §§2 and 5 we have classified the partial linear spaces $\Pi$ of I subject to either of the additional hypotheses:

(i) $\Pi$ is reduced;

(ii) $\Pi$ has a faithful representation.

In this section we discuss some applications of these results.

### A. *Groups generated by transvections*

The following theorem completes the argument of Case 4 in the proof of [**5**, Theorem 2].

THEOREM 6. *Let $G$ be a group of linear transformations of the $\mathbb{F}_2$-vector space $V$ which is generated by a conjugacy class $D$ of transvections with the property that every 1-space of $V$ is the centre of at most one transvection of $D$. Assume additionally that $V$ is spanned by the centres of the members of $V$. Then either*

(1) *there is a form $f$ or $q$ on $V$ and $G$ is* $\mathrm{TSp}(V, f)$ *or* $\mathrm{TO}(V, q)$ *with $D$ the class of symplectic transvections, or*

(2) *$G$ is isomorphic to* $\mathrm{TSym}(\Omega, \Omega')$ *with $V$ isomorphic as a $G$-space to* $E\mathbb{F}_2(\Omega, \Omega')$ *or* $\bar{E}\mathbb{F}_2(\Omega, \Omega')$.

*Proof.* If an element of $D$ has centre $\langle s \rangle$, then by assumption we may identify the element unambiguously as $d(s)$. The restriction on centres implies that if $d(s)$ and $d(t)$ of $D$ do not commute, then $d(s)d(t)d(s) = d(t+s) \in D$. Therefore $D$ is a conjugacy class of 3-transpositions of $G$. As in §3 we may consider the connected partial linear space $\Pi = \Pi(G, D)$ of order 2 whose point set is $D$ and whose line set is composed of all triples $\{d(s), d(t), d(s+t)\}$ from $D$ where $|d(s)d(t)| = 3$.

The map $\varphi$: $D \to V$ given by $d(s) \to s$ is clearly a representation of $\Pi$. In addition, $\varphi$ is faithful and full by assumption. Three centres from $D$ lie in a 3-space of $V$, so $\Pi$ is in fact a cotriangular space with (V–Y). By construction the members of $D$ act as symplectic transvections of $\Pi^*$, so that if $\Pi^* = \mathcal{T}(\Omega)$, then $G$ induces $\mathrm{TSym}(\Omega)$ on $\Pi^*$. Thus by (5.5) and (5.6) either $\Pi$ is $\mathcal{S}p(V, f)$ or $\mathcal{N}(V, q)$ and $\varphi$ is equivalent to the natural representation or $\Pi$ is $\mathcal{T}(\Omega, \Omega')$ and $\varphi$ is equivalent to one of the natural representations in $E\mathbb{F}_2(\Omega, \Omega')$ or $\bar{E}\mathbb{F}_2(\Omega, \Omega')$. The equivalence identifies the class $D = D(\Pi)$ with the class of symplectic transvections of $G = G(\Pi) \leq \mathrm{Aut}(\Pi)$. Thus $G$ is respectively $\mathrm{TSp}(V, f)$, $\mathrm{TO}(V, q)$, or $\mathrm{TSym}(\Omega, \Omega')$.

B. *Subspaces of irreducible cotriangular spaces*

Each of the irreducible cotriangular spaces $\Pi = (\mathcal{P}, \mathcal{L})$ is isomorphic to a subspace of $\mathcal{S}p(V, f)$ for some $\mathbb{F}_2$-space $V$ and non-degenerate $f$ in such a way that the image of $\mathcal{P}$ in $V$ spans $V$.

(6.B.1) *Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a subspace of $\mathcal{S}p(V, f)$, with $f$ non-degenerate on the $\mathbb{F}_2$-space $V$, such that $V$ is the $\mathbb{F}_2$-span of $\mathcal{P}$. Then we have one of:*

(1) $\Pi = \mathcal{S}p(V, f)$;

(2) *$\Pi$ is reduced but not connected;*

(3) *$\Pi$ is $\mathcal{N}(V, q)$ naturally embedded in $\mathcal{S}p(V, f)$ for some $q$ associated with $f$;*

(4) *$\Pi$ is isomorphic to $\mathcal{T}(\Omega)$ so that the representation induced by the injection $\mathcal{P} \to V$ is equivalent to the standard representation of $\mathcal{T}(\Omega)$ in $\bar{E}\mathbb{F}_2\Omega$.*

*Proof.* For $\Pi$ the $W$ of (5.1.1) must sit inside $\mathrm{Rad}(V, f) = 0$ as $\mathcal{P}$ spans $V$. Thus $\Pi$ is reduced. If $\Pi$ is connected, then the injection $i$: $\mathcal{P} \to V$ induces a full, faithful representation of $\Pi$. When $\Pi \not\simeq \mathcal{T}(\Omega)$, we have (1) or (3) by (5.5). If $\Pi \simeq \mathcal{T}(\Omega)$, then the argument of [**11**, 4.6] is valid for all $\Omega$ and gives (4). (As $f$ is non-degenerate, $\dim(V)$ is either even or infinite.)

If $\Pi = (\mathcal{P}, \mathcal{L})$ is a partial linear space, the proper subspace $\Delta = (\mathcal{P}', \mathcal{L}')$ is a *hyperplane* of $\Pi$ if every line of $\mathscr{L}$ meets $\mathscr{P}'$ non-trivially. This terminology comes from the observation that if $\varphi$: $\mathscr{P} \to V$ is a full representation of $\Pi$ in the space $V$ then $\varphi^{-1}(H)$ supports a hyperplane of $\Pi$ for each hyperplane $H$ of $V$. For partial linear spaces $\Pi$ of order 2 the correspondence is perfect; every hyperplane $\Delta$ of $\Pi$ arises as a hyperplane section of a universal representation of $\Pi$. To see this we only need observe that the map sending the points of $\Delta$ to 0 and the rest of the points of $\Pi$ to 1 is a full representation of $\Pi$ in $\mathbb{F}_2$ and so is associated with a non-zero linear functional on $\hat{V}(\Pi)$, the universal representing space. Hyperplanes played an important part in the work of Teirlinck on representations of Steiner triple systems **[19]** and have appeared in various guises in much of the work on representation of partial linear spaces (see, for instance, $\Delta$ of Lemma 3.4 in **[11]**). Nevertheless it seems to have been left to Mark Ronan (see **[16]**) to observe and emphasize the central role played by hyperplanes in the representation theory of all partial linear spaces.

In some sense knowledge of a universal representation of the partial linear space $\Pi$ of order 2 is equivalent to knowledge of all hyperplanes of $\Pi$. Thus it is not surprising to find the next result as a consequence of (6.B.1). Of course it could also be proven by examining the hyperplane sections of a universal representing space.

(6.B.2) *Let $\Delta$ be a hyperplane of $\mathscr{S}p(V,f)$, with $f$ non-degenerate on the $\mathbb{F}_2$-space $V$. Then either $\Delta$ is a hyperplane section of $V$ or $\Delta$ is a subspace $\mathscr{N}(V, q)$ for some $q$ associated with $f$.*

*Proof.* Let $\Pi = \mathscr{S}p(V,f)$. If the points of $\Delta$ do not span $V$, they lie within some hyperplane $H$ of $V$. However every hyperplane of a partial linear space is a maximal subspace, so this forces $\Delta$ to be the section of $\Pi$ by $H$. Thus we may assume that $\Delta$ spans $V$ and so is one of (2), (3), (4) of (6.B.1). We may assume that $\Delta$ is not $\mathscr{T}(\Omega)$ as in (4) because there is always a $q$ associated with the usual dot-product $f$ such that $\mathscr{T}(\Omega) \subset \mathscr{N}(\bar{E}\mathbb{F}_2\Omega, q) \subseteq \mathscr{S}p(\bar{E}\mathbb{F}_2\Omega, f)$. (See the discussion of (5.3.1).) If $\Delta$ is disconnected as in (2) with $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1 = (\mathscr{P}_1, \mathscr{L}_1)$ is not connected to $\Delta_2 = (\mathscr{P}_2, \mathscr{L}_2)$, then $V$ is the perpendicular direct sum of the subspaces $V_1 = \langle \mathscr{P}_1 \rangle$ and $V_2 = \langle \mathscr{P}_2 \rangle$. But then we must have $\Delta = \mathscr{S}p(\Delta_1, f_1) \cup \mathscr{S}p(V_2, f_2)$ with $f_i$ being $f$ restricted to $V_i$. However, such a $\Delta$ while maximal is not a hyperplane except when $\dim(V) = 4$ and $\Delta = \mathscr{S}p_2(2) \cup \mathscr{S}p_2(2)$ which is in fact $\mathscr{N}_4^+(2)$.

Hyperplanes of $\mathscr{N}(V, q)$ and $\mathscr{T}(\Omega)$ either must arise from hyperplane sections of $\mathscr{S}p(V,f)$ and $\mathscr{S}p(\bar{E}\mathbb{F}_2\Omega, f)$ or must be spanning subspaces. Thus from (6.B.1) we can also find

(6.B.3) *Let $\Delta$ be a hyperplane of $\Pi = \mathscr{N}(V, q)$, with $q$ non-degenerate on $V$ of dimension at least 6 over $\mathbb{F}_2$ but $\mathscr{N}(V, q)$ not of type $\mathscr{N}_6^+(2)$. Then $\Delta$ is a hyperplane section of $V$.*

(6.B.4) *Let $\Delta$ be a hyperplane of $\mathscr{T}(\Omega)$. Then there is a partition $\Omega = \Omega_1 \cup \Omega_2$ with $\Delta = \mathscr{T}(\Omega_1) \cup \mathscr{T}(\Omega_2)$.*

C. *Locally grid graphs and hyperplanes of the line Grassmannian*

For $V$ a vector space, let $G_2(V)$ be the *line Grassmannian* of $V$, the partial linear space whose points are the 2-spaces of $V$ and whose lines are the set of 2-spaces containing a given 1-space and contained within a given 3-space. The collinearity graph of $G_2(V)$ is thus precisely the line graph of the projective space underlying $V$.

If $V$ is a vector space over $\mathbb{F}_2$, then in the collinearity graph $\Gamma = P(G_2(V))$ of $G_2(V)$, each $\Gamma_x$ has all $\approx$-classes of size 2 and $\Gamma_x^* \simeq L(3, \Omega)$, where $\Omega$ is the set of 3-spaces of $V$ containing the $L$-space $x$.

A consequence of Theorems 1 and 2 is that each graph which is locally a grid as in II is in fact an induced subgraph of the collinearity graph $P(G_2(V))$ of $G_2(V)$ for an appropriate $\mathbb{F}_2$-space $V$.

Let $V$ be an $\mathbb{F}_2$-space, and let $\Delta$ be a collection of points of $G_2(V)$ (that is, 2-spaces of $V$) such that $P(G_2(V))$ induces on $\Delta$ a graph with $\Delta_x$ isomorphic to $L(3, \Omega(x))$ for all $x \in \Delta$. Then every 3-space of $V$ containing a point of $\Delta$ either contains only that one point of $\Delta$ or contains exactly three additional points of $\Delta$. Stated differently we have

(6.C.1) *Let $\Sigma$ be a subset of the points of the Grassmannian space $G_2(V)$, where $V$ is an $\mathbb{F}_2$-space. A necessary and sufficient condition that the collinearity graph of $G_2(V)$ induce on the points of $G_2(V)$ not in $\Sigma$ a graph as in* III *is that every* 3*-space $W$ of $V$ meet $\Sigma$ in one of:*

(1) *all seven points of $G_2(V)$ which are contained in $W$;*

(2) *the three points of a line of $G_2(V)$;*

(3) *six of the points of $G_2(V)$ which are contained in $W$.*

Of particular interest is the case when the third possibility of (6.C.1) does not occur. Then $\Sigma$ is in fact (the point set of) a subspace of $G_2(V)$ which is readily seen to be a hyperplane. We conclude that the complement $\Delta$ of a hyperplane $H$ of $G_2(V)$ supports a graph such that for each $x \in \Delta$, $\Delta_x$ is isomorphic to $L(3, \Delta(x))$ where $\Delta(x)$ is the set of 3-spaces of $V$ containing the 2-space $x$. Thus we could use Theorems 1 and 2 to find $\Delta$ and then recover all possible $H$; however the presence of $V$ in the background makes this unnecessary. If we view the points of $\Delta$ again as 2-spaces of $V$, then they are the 2-spaces corresponding to lines of a cotriangular space represented in $V$; hence the results of §5, and in particular (5.5), are available. As $H$ is a subspace of $G_2(V)$, it is easy to check that $\Delta$ must be of symplectic type $\mathcal{S}p(V, f)$ for some non-zero symplectic form of $V$. Conversely if $\Delta = \mathcal{S}p(V, f)$ then the complement $H$ of $\Delta$ in $G_2(V)$ is indeed a subspace and is a hyperplane provided $f$ is not identically 0.

(6.C.2) *The standard representation of $G_2(V)$, with $V$ an $\mathbb{F}_2$-space, in the exterior square $V \wedge V$ of $V$ is a universal representation of $G_2(V)$.*

*Proof.* The representation maps the 2-space $\langle x, y\rangle$ to $\langle x\rangle \wedge \langle y\rangle$. If $\lambda$ is a linear functional on $V \wedge V$ then $f$, given by $f(x, y) = \lambda(x \wedge y)$, is a symplectic form on $V$. Conversely if $f$ is any symplectic form on $V$ then $\lambda(x \wedge y) = f(x, y)$ defines a linear functional on $V \wedge V$. In particular, all subspaces $\mathcal{S}p(V, f)$ of

$G_2(V)$ are associated with hyperplane sections of $G_2(V)$ in $V \wedge V$. Therefore $V \wedge V$ is a universal representing space for $G_2(V)$.

Result (6.C.2) was proven for finite-dimensional $V$ by Wells [**20**] and independently by Ronan and Smith [**17**] as a special case of a more general result allowing fields other than $\mathbb{F}_2$ and Grassmannians other than the line Grassmannian. The corresponding result for the line Grassmannian in any dimension and over fields bigger than $\mathbb{F}_2$ can be given a proof similar to that of (6.C.2) using the results of Farmer and Hale [**8**] and Lefevre-Percsy [**14**].

D. *Derivations on symplectic $\mathbb{F}_2$-transvection groups*

Our last application is more algebraic and abstract than the others, and we only describe it briefly. We hope at some later date to discuss it in greater detail and generality.

Let $G$ be a group of linear transformations of the $F$-space $V$. A *derivation* $\delta$ from $G$ to $V$ is a map $\delta$: $G \to V$ such that

$$(*) \qquad \delta(gh) = \delta(g)h + \delta(h), \quad \text{for all } g,h \in G.$$

A special example of a derivation is an *inner derivation* $\delta_v$ $(v \in V)$, given by

$$\delta_v(g) = [v, g] = v(g-1), \quad \text{for all } g \in G.$$

The set $\mathrm{Der}(G, V)$ of all derivations from $G$ to $V$ has a natural $F$-space structure and the set $\mathrm{IDer}(G, V)$ of all inner derivations is clearly a subspace of $\mathrm{Der}(G, V)$. The factor space $\mathrm{Der}(G, V)/\mathrm{IDer}(G, V)$ is in fact the first cohomology group $H^1(G, V)$.

Now let $G$ be one of the groups $\mathrm{TSp}(V, f)$, $\mathrm{TO}(V, q)$, or $\mathrm{TSym}(\Omega)$ with $f$ and $q$ non-degenerate on the $\mathbb{F}_2$-space $V$. If $G$ is $\mathrm{TSym}(\Omega)$, let $V = \bar{E}\mathbb{F}_2\Omega$. Let $\delta \in \mathrm{Der}(G, V)$. By (*), $\delta$ is completely determined by its values at all the symplectic transvections of $G$. It is not difficult using (*) to prove that for each symplectic transvection $t(x)$ of $G$, $\delta(t(x)) \leq \langle x \rangle$. Thus associated with $\delta$ we can define a map $\lambda_\delta$: $V \to \mathbb{F}_2$ given by $\delta(t(v)) = \lambda_\delta(v)v$. Then it can be proven that $\ker(\lambda_\delta) = \{v \mid \lambda_\delta(v) = 0\}$ is a hyperplane of the cotriangular space $\Pi$ associated with the transvection class of $G$, as represented in $V$ by centres. Conversely, it can be checked that every 'cotriangular linear functional' $\lambda$ (0 on a hyperplane of $\Pi$ and 1 off the hyperplane) is equal to $\lambda_\delta$ for some derivation $\delta \in \mathrm{Der}(G, V)$. (See [**15**, 1.6].) In view of the results of §6.B this gives a concise and concrete description of $\mathrm{Der}(G, V)$ and so of $H^1(G, V)$. If $\delta = \delta_v$ then $\lambda_\delta$ is the linear functional on $V$ with kernel $v^\perp$. So, for instance, we recover the well-known result (due originally to Pollatsek [**15**]) for the finite symplectic groups over $\mathbb{F}_2$ that $H^1(G, V)$ is $\mathbb{F}_2$ for $\dim(V) \geq 4$; a non-inner derivation $\delta$ has $q(v) = 1 - \lambda_\delta(v)$ a quadratic form on $V$, and two quadratic forms on $V$ differ by a linear functional.

If $V$ is infinite-dimensional, something rather interesting occurs. In that case there are linear functionals $\lambda$ on $V$ with kernel not equal to $v^\perp$ for any $v$ of $V$ (as the dual space of $V$ is larger than $V$). If $\delta$ is the derivation with $\lambda_\delta = \lambda$, then $\delta$ is not an inner derivation, yet its restriction to any finitely generated subgroup $H$ of $G$ is an inner derivation. As Derek Holt has pointed out, examples of non-inner derivations which are locally inner can often be built from the local level up, but the discussion here gives a nice global construction of such examples.

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