Surfaces of Constant Retarded Distance and Radiation Coordinates

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We construct the element of volume vector corresponding to a surface of constant retarded distance around of an arbitrary timelike curve; the method employed is based in the radiation coordinates of Florides-McCrea-Synge for Riemannian 4-spaces. Our results have interest in the study of the electromagnetic Liénard-Wiechert field in curved spacetimes.

Keywords: radiation coordinates, Riemannian 4-spaces, universe function.

1. Introduction

In this work the element of volume vector ds_r is calculated for a surface with a constant retarded distance, which is constructed around of the trajectory of an electric charge with arbitrary motion in a Riemannian space. This is a generalization that was done by Synge [1] in special relativity. The employed method is suggested by the radiation coordinates y^r introduced in [2,3] for the study of gravitational radiation; here they are used in electromagnetic radiation and they are very well adapted for this purpose because with such coordinates the curved space behaves like a "flat space" in some aspects. In other words, the use of y^r implies that what was learned in Minkowski space can be translated naturally to Riemann's spaces. Our expression for ds_r coincides with Villarroel's results obtained in [4] by means of the procedure that DeWitt-Brehme [5] use when constructing a surface with constant instantaneous distance [6,7]. However, we think that our method is more simple and powerful because it turns immediate the results on radiation tensors deduced in [8].

We shall use the "Universe Function" Ω of Ruse [9], which allows having covariant expansions in curved space. This function remained forgotten during a long time and its present relevance may be seen in [4, 5, 8, 10-17].

2. Radiation Coordinates

We assume the Einstein convention for the addition of repeated indices (1, 2, 3, and 1,…,4 for greeks and latin indices, respectively) and that the metric locally takes the diagonal form $(h_{ab}) = (1,1,1,-1)$ at any event. In order to construct the radiation coordinates y^r of [2] we need a timelike curve C (which in this case will be the electron trajectory) with an orthonormal tetrad on it.

$$e^{i'}_{(a)} e_{(b)i'} = h_{ab}, \; e_{(a)i'} e^{(a)}_{j'} = g_{i'j'}, \tag{1}$$

where $e^{i'}_{(4)} = u^{i'} = \frac{dx^{i'}}{ds}$ is the unitary tangent vector to C, and x^r is a totally arbitrary coordinate system with $ds^2 = g_{ij} dx^i dx^j$. The primed indices label points on C.

Now let us see how x^r generates new coordinates. For every P we construct the past sheet of its null cone which intersects to C in P' (retarded point associated to P). We parametrize the null geodesic P P' in the form $x^r(u)$ with $u = u_0$ at P' and $u = u_1 > u_0$ at P with $V^r = \frac{dx^r}{du}$ as its tangent vector satisfying $V^r V_r = 0$. The assigned radiation coordinates to P are given by:

$$y^r = \left[-\Omega_{;j'} + s u_{j'} \right] e^{(r)j'} \tag{2}$$

where $\Omega_{;j'}$ denote the covariant derivative of Ω [14]:

$$\Omega_{;j'} = -(u_1 - u_0) V_{j'}, \ \Omega_{;j'} \Omega^{;j'} = 0 . \tag{3}$$

The expression (2) is equivalent to:

$$y^s = -\Omega_{;j'} e^{(s)j'}, \ y^4 = \Omega_{;j'} u^{j'} + s , \tag{4}$$

which implies that in radiation coordinates the curve C reduces to $y^{s'} = 0, \ y^{4'} = s$. If we introduce the notation:

$$K_{j'} = -\Omega_{;j'} \quad \mathrm{w} = \Omega_{;j'} u^{j'} , \tag{5}$$

then (4) adopts the form of the relation (9.3) of [1] for flat space:

$$y^s = y_s = K_{j'} e^{(s)j'}, \ y^4 = -y_4 = w + s . \tag{6}$$

In this sense the curved space behaves like a Minkowski space, which is very useful.

At P′ the metric tensor can be written in terms of the tetrad as $g_{i'j'} = e^{(s)}_{i'} e_{(s)j'} - u_{i'} u_{j'}$, then from (3) and (5):

$$y^s y_s = K_{i'} K_{j'} \left(g^{i'j'} + u^{i'} u^{j'} \right) = w^2 , \tag{7}$$

thus

$$K_{j'} = y^s e_{(s)j'} + w u_{j'} , \tag{8}$$

therefore $\left(y^r - y^{r'} \right)$ behaves like a null vector because $\left(y^r - y^{r'} \right)\left(y_r - y_{r'} \right) = 0$. Our expressions (7) and (8) coincide with (9.4) and (9.5) of [1].

Following the corresponding procedure in flat space we introduce a new system of coordinates:

$$z^s = y^s , \; z^4 = y^4 - \sqrt{y^s y_s} = s , \tag{9}$$

that is, z^4 remains constant on the null cone with vertex at P' It is clear that the Jacobian of the transformation: $y^r \rightarrow z^r$ is equal to one, therefore:

$$J\left(\frac{z^a}{x^b} \right) = J\left(\frac{y^a}{x^b} \right) \tag{10}$$

Let us calculate (10). If we use that $\Omega_r = \left(u_1 - u_0 \right) V_r$ and $\frac{\partial x^{r'}}{\partial x^r} = u^{r'} s_{,r} = -w^{-1} u^{r'} \Omega_r$, then from (6) and (9) we obtain the partial derivatives:

$$\frac{\partial z^s}{\partial x^i} = -\Omega_{i'i} e^{(s)i'} + w^{-1}\left[\Omega_{j'r'} u^{r'} e^{(s)j'} + \Omega_{r'} \frac{d}{ds} e^{(s)r'}\right]\Omega_i \quad , \qquad (11)$$
$$\frac{\partial z^4}{\partial x^i} = -w^{-1}\Omega_i$$

thus

$$J\left(\frac{z^a}{x^b}\right) \equiv \in^{ijkm} \frac{\partial z^1}{\partial x^i}\frac{\partial z^2}{\partial x^j}\frac{\partial z^3}{\partial x^k}\frac{\partial z^4}{\partial x^m} \quad , \qquad (12)$$
$$= w^{-1}\det\left(-\Omega_{a'b}\right) \in_{j'r't'p'} e^{(1)j'} e^{(2)r'} e^{(3)t'} \Omega^{p'}$$

where we have employed the property $\Omega_m = \Omega_{p'm}\Omega^{p'}$ and the antisymmetry of the Levi-Civita symbol $\in^{ijkm}$. From (3) it is evident that $\Omega^{p'}$ can be expressed in terms of the tetrad as $\Omega^{p'} = b_s e^{(s)p'} + we^{(4)p'}$, then (12) implies the final form:

$$J\left(\frac{z^a}{x^b}\right) = -g^{1/2}(P)\Delta, \qquad (13)$$

such that

$$g(P) = -\det\left(g_{ij}\right), \; g(P') = -\det\left(g_{i'j'}\right) \quad . \qquad (14)$$
$$D = -\det\left(-\Omega_{a'b}\right), \; \Delta = g^{-1/2}(P)\, g^{-1/2}(P')\, D$$

With (13) it is apparent the remark of [5] p. 231 and [12] p. 1251: the geodesics emerging from P begin their intersection when $\Delta^{-1} = 0$, arising the so-called "caustic surface." Therefore we shall accept that P is near to P'.

The relations (9) and (13) permit to consider the volume element of the curved space- time, in fact:

$$d^4x = \left| J\left(\frac{x^b}{z^a}\right)\right| d^4z = g^{-1/2}(P)\Delta^{-1} ds d^3 z \,, \tag{15}$$

but from (6), (7) and (9) it is clear that $z^s z_s = w^2$, thus z^s can be seen as a 3- vector at P' of magnitude w and spherical coordinates q, j with respect to the triad $e^{(s)r'}$, then:

$$d^3 z = w^2 dw dg \ , \ dg = \sin q dq dj \tag{16}$$

being dg the element of solid angle in the rest frame of the charge. In this way (15) turns out to be:

$$d^4 x = g^{-1/2}(P)\Delta^{-1} w^2 ds dw dg \ , \tag{17}$$

which together with (13) represent the generalization to Riemannian spaces from the following results (9.15) and (9.21) of Synge [1] (who uses imaginary coordinates) for Minkowski space:

$$J\left(\frac{z^a}{x^b}\right) = -1,\ d^4 x = w^2 ds dw dg \tag{18}$$

In the next section we will apply (17) to the particular case of the surface $w =$ constant, which is important when studying the electromagnetic radiation.

3. Surface of constant retarded distance

We consider the 3-space $w =$ constant, thus the covariant derivative $w_{;r}$ is orthogonal to that surface. Then it is evident that its vector volume element is given by:

$$ds_r = \left| w_{;a} w^{;a} \right|^{-1/2} w_{;r} ds \ , \tag{19}$$

being ds the 3-element of volume. But when building the shell formed by w, $w + dw$ and the null cones at two points on C, we get for its 4-volume $d^4x = \left|w_{;a}w^{;a}\right|^{-1/2} dwds$ and after comparison with (17) it implies that $\left|w_{,a}w^{,a}\right|^{-1/2} ds = g^{-1/2}(P)\Delta^{-1}w^2 dsdg$, thus (19) adopts the form

$$ds_r = g^{-1/2}(P)\Delta^{-1}w^2 w_{;r} dsdg \ . \tag{20}$$

On the other hand, from (5):

$$w_{;r} = \hat{s}_r - w^{-1}(c + W)\Omega_r \tag{21}$$

with the notation:

$$\hat{s}_r = \Omega_{i'r}u^{i'}, \ c = \Omega_{i'j'}u^{i'}u^{j'}, \tag{22}$$

$$W = \Omega_{i'}\frac{d}{ds}u^{i'} = -K_{i'}a^{i'},$$

where $a^{i'}$ is the acceleration of the charge. Substituting (21) in (20) we find the result (3.35) of Villarroel [4]:

$$ds_r = g^{-1/2}(P)\Delta^{-1}w\left[w\hat{s}_r - (c + W)\Omega_r\right]dsdg \ , \tag{23}$$

which is the generalization to curved spaces of (10.6) of Synge [1].

The deduction of (23) was simple thanks to the radiation coordinates that originated (17). Nevertheless, this is not the end of the usefulness of z^r; in our opinion, its true importance lies on the analogies that we can establish with the Minkowski space, which will be seen more clearly in the next section.

4. Radiation tensors

In the flat space we have the following radiative part of the Maxwell tensor corresponding to the Liénard-Wiechert retarded field $\left(a^2 = a_j a^j\right)$:

$$\underset{R}{T_{rs}} = q^2 w^{-4}\left(a^2 - w^{-2}W^2\right)K_r K_s \tag{24}$$

which is a radiation tensor because it satisfies:

$$\underset{R}{T_{rs}} K^s = 0,\ \underset{R}{T_{rs}}{}^{,s} = 0\,. \tag{25}$$

The continuity equation (25) is consequence of:

$$\begin{aligned} &\left(a^2 w^{-4} K_r K_s\right)^{,s} = 0 \\ &\left(w^{-6} W^2 K_r K_s\right)^{,s} = 0 \end{aligned}, \tag{26}$$

which in turn are particular cases of the identity;

$$\left[f\left(a^2\right) w^{-n} W^m K_r K_s\right]^{,s} = 0,\ -n + m - 4, \tag{27}$$

f being an arbitrary function of a^2.

It is quite natural to ask ourselves if (24) can be extended to the curved space. The answer is affirmative under the two prescriptions: Identify K_r with $-\Omega_r$, see (5).

Multiply (24) by $g^{1/2}(P)\Delta$ due to the fact that d^4x contains the factor $g^{-1/2}(P)\Delta^{-1}$ with respect to the corresponding expression for the flat space, see (17).

Thus:

$$\underset{R}{T_{rs}} = q^2 g^{1/2}(P)\Delta w^{-4}\left(a^2 - w^{-2}W^2\right)\Omega_r \Omega_s, \tag{28}$$

satisfies (25) with covariant derivative, and it is immediate the generalization of (26):

$$\begin{aligned}\left[g^{1/2}(P)\,\Delta a^2 w^{-4}\Omega_r\Omega_s\right]^{;s} &= 0\\ \left[g^{1/2}(P)\,\Delta w^{-6} W^2\Omega_r\Omega_s\right]^{;s} &= 0\end{aligned}\quad. \tag{29}$$

Moreover, from (17) and (28) we have the relation:

$$\underset{R}{T_{rb}}\, d^4x = q^2 w^{-2}\left(a^2 - w^{-2}W^2\right)\Omega_r\,\Omega_b\, ds\, dw\, dg\ , \tag{30}$$

which is important when performing some integration around the world line of q.

We notice that (28) and (29) correspond to the results (2.28),….,(2.31) of Villarroel [8], but in our focusing they emerged naturally through the correspondence with the Minkowski space.

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