Dynamic Consensus Seeking in Distributed Multi-agent Coordinated Control

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Abstract

Using directed graphs, we consider consensus seeking problem when the information state of each agent is driven by exogenous inputs, random noise, or nonlinear dynamics. We show conditions under which global dynamic consensus can be achieved and provide boundedness analyses for the inconsistency of the information states between agents when communication noise or inconsistent inputs exist. Simulation studies apply the dynamic consensus seeking concept to a multi-agent coordinated control scenario in the context of the distributed virtual leader/virtual structure approach and the behavioral approach respectively.

1 Introduction

Distributed control of multi-agent systems has received significant attention in recent years (c.f. [1, 2, 3, 4]). In some applications of distributed multi-agent systems, shared information plays a central role and facilitates the coordination of the group. As a result, a critical problem for coordinated control is to design appropriate protocols and algorithms such that the group of agents can reach consensus on the shared information in the presence of limited and unreliable information exchange and dynamically changing interaction topologies.

Consensus problems have recently been addressed in [5, 6, 7, 8, 9, 10, 11, 12], to name a few. In [6], sufficient conditions are given for consensus of the heading angles of a group of agents under undirected switching interaction topologies. In [8], average consensus problems are solved for a network of integrators using directed graphs. Using directed graphs, Ref. [11] and [12] show necessary and sufficient conditions for consensus of information under time-invariant and switching interaction topologies respectively.

One common feature of the above research is that the information state on which consensus needs to be reached is assumed to be inherently constant. We will call the case when consensus is sought on a constant state, *static consensus*. However, the information state may be driven by noise due to unreliable information exchange or driven by an exogenous input representing an *a priori* determined feedforward signal. Alternatively, the information state may be dynamically evolving in time according to some inherent dynamics, as happens in formation control problems where the information state may be the leader's states. If the information state is dynamic, we refer to the consensus problem as *dynamic consensus*. The objective of this paper is to consider the dynamic consensus seeking problem for the cases that the information state is driven by (i) a common input, (ii) random noise, and (iii) inconsistent inputs. We show conditions under which dynamic consensus can be achieved with a common input. We also show explicit upper bounds for the inconsistency of the information state between agents when there exists bounded noise or inconsistent inputs using an input-to-state (ISS) analysis. The dynamic consensus seeking concept is then applied to a multi-agent coordinated control scenario in the context of the distributed virtual leader/virtual structure approach and the behavioral approach respectively.

The organization of this paper is as follows. In Section 2 we introduce preliminaries on directed graphs and define the dynamic consensus seeking problem. In Section 3 we show conditions under which global dynamic consensus can be achieved and provide analysis for the inconsistency between agents when there are inconsistent inputs. In Section 4 we apply the concept of dynamic consensus seeking to a multi-agent coordinated control application. Section 5 contains our conclusion.

2 Problem Statement

In this paper, we let $\mathcal{A} = \{A_i | i \in \mathcal{I}\}$ be a set of n agents whose information needs to reach consensus, where $\mathcal{I} = \{1, 2, \cdots, n\}$. A directed graph $\mathcal{G}$ will be used to model the interaction topology among these agents. In $\mathcal{G}$,

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the ith vertex represents the ith agent A_i and a directed edge from A_i to A_j denoted as (A_i, A_j) represents a unidirectional information exchange from A_i to A_j, that is, agent j receives information from agent i, $(i, j) \in \mathcal{I}$. Throughout the paper, we always assume that there is a link from each vertex to itself. A directed path in graph $\mathcal{G}$ is a sequence of edges $(A_{i_1}, A_{i_2}), (A_{i_2}, A_{i_3}), (A_{i_3}, A_{i_4}), \cdots$ in that graph. Graph $\mathcal{G}$ is called strongly connected if there is a directed path from A_i to A_j and A_j to A_i between any pair of distinct vertices A_i and A_j, $\forall (i, j) \in \mathcal{I}$. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A spanning tree of a directed graph is a directed tree formed by graph edges that connect all the vertices of the graph (c.f. [13]). Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, the directed graph of A, denoted by $\Gamma(A)$, is the directed graph on n vertices V_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from V_j to V_i if and only if $a_{ij} \neq 0$ (c.f. [14]).

Let $\xi_i \in \mathbb{R}$, $i \in \mathcal{I}$, represent the ith information state associated with the ith agent. The set of agents $\mathcal{A}$ is said to achieve global consensus asymptotically if for any $\xi_i(0)$, $i \in \mathcal{I}$, $\|\xi_i(t) - \xi_j(t)\| \to 0$ as $t \to \infty$ for each $(i, j) \in \mathcal{I}$ [9]. Unlike the static consensus case where $\xi_i(t)$ converges to a constant, $\xi_i(t)$ may be time-varying as $t \to \infty$ in the dynamic consensus case. However, like the static consensus problem, the inconsistency in information states between agents should approach zero as $t \to \infty$.

Let $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of nonnegative integers and positive integers respectively. Given T as the sampling period, a discrete time consensus scheme is given by

$$\xi_i(k+1) = \frac{1}{\sum_{j=1}^n \alpha_{ij} G_{ij}} \sum_{j=1}^n \alpha_{ij} G_{ij} \xi_j(k) + v_i(k), \quad (1)$$

where $k \in \mathbb{N}$ is the discrete time index, $(i, j) \in \mathcal{I}$, v_i denotes the input at time $t = kT$, α_{ij} are positive constants, $G_{ii} \triangleq 1$, and G_{ij}, $\forall j \neq i$, is 1 if information flows from A_j to A_i and 0 otherwise.

Eq. (1) can be written in matrix form as

$$\xi(k+1) = D\xi(k) + v(k), \quad (2)$$

where $\xi = [\xi_1, \cdots, \xi_n]^T$, $v = [v_1, \cdots, v_n]^T$, $D = [d_{ij}]$, $(i, j) \in \mathcal{I}$, with $d_{ij} = \frac{\alpha_{ij} G_{ij}}{\sum_{j=1}^n \alpha_{ij} G_{ij}}$.

A continuous consensus scheme is given by

$$\dot{\xi}_i(t) = -\sum_{j=1}^n \sigma_{ij} G_{ij} (\xi_i(t) - \xi_j(t)) + v_i(t), \quad (3)$$

where $(i, j) \in \mathcal{I}$, $v_i(t)$ is the input at time t, σ_{ij} are positive constants, $G_{ij}(t)$ is 1 if information flows from A_j to A_i and 0 otherwise.

Eq. (3) can also be written in matrix form as

$$\dot{\xi}(t) = C\xi(t) + v(t), \quad (4)$$

where $C = [c_{ij}]$, $(i, j) \in \mathcal{I}$, with $c_{ii} = -\sum_{j \neq i} (\sigma_{ij} G_{ij})$ and $c_{ij} = \sigma_{ij} G_{ij}$, $j \neq i$.

In the case when the information state is inherently constant, we simply let $v = 0$, which corresponds to the static consensus problem. As a comparison, in the dynamic consensus case, v_i in Eqs. (1) and (3) can represent an exogenous feedforward signal to $\mathcal{A}_i$ denoted as u_i, a disturbance to $\mathcal{A}_i$ due to information exchange noise denoted as w_i, a nonlinear dynamic evolution law denoted as $f(\xi_i, u_i)$, or a combination of the three.

Note that all the results hereafter are still valid in the case of $\xi_i \in \mathbb{R}^m$, $i \in \mathcal{I}$, if we simply replace matrix C in Eq. (2) and matrix D in Eq. (4) by $C \otimes I_m$ and $D \otimes I_m$ respectively, where $\otimes$ denotes the Kronecker product and I_m denotes the $m \times m$ identity matrix.

It has been shown in [12] and [11] that global static consensus can be reached asymptotically using scheme (1) and (3) if and only if graph $\mathcal{G}$ has a spanning tree. In this paper, we will focus on dynamic consensus seeking using a time-invariant interaction topology, i.e., G is constant.

3 Dynamic Consensus Seeking

In this section, we address the dynamic consensus seeking using (1) the discrete time consensus scheme (Eq. (1)) and (2) the continuous time consensus scheme (Eq. (3)) respectively.

Let $\mathbf{1}$ denote an $n \times 1$ column vector with all the entries equal to 1. A matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is nonnegative, denoted as $A \geq 0$, if all its entries are nonnegative. Furthermore, if all its row sums are $+1$, A is said to be a (row) stochastic matrix [14].

3.1 Discrete Time Case

Lemma 3.1 *Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be a stochastic matrix with positive diagonal entries. If the graph associated with A has a spanning tree, then 1 is the unique eigenvalue of A with maximum modulus and $\lim_{m \to \infty} A^m \to \mathbf{1}y^T$, where $m \in \mathbb{N}^+$ and $y = [y_1, \cdots, y_n]^T \geq 0$ satisfies $A^T y = y$ and $\mathbf{1}^T y = 1$.*

Proof: Follows from Corollary 3.2 and Lemma 3.4 in [12]. ∎

Therefore if graph $\mathcal{G}$ has a spanning tree, matrix D in Eq. (2) satisfies the assumptions of Lemma 3.1 and $\lim_{m \to \infty} D^m \to \mathbf{1}\mu^T$, where μ satisfies the properties defined in Lemma 3.1.

Lemma 3.2 *If graph $\mathcal{G}$ has a spanning tree, then $\rho(D - \mathbf{1}\mu^T) < 1$, where $\rho(\cdot)$ denotes the spectral radius of a matrix.*

Proof: Following part (f) of Lemma 8.2.7 in [14], we know that every nonzero eigenvalue of $D - \mathbf{1}\mu^T$ is also an eigenvalue of D. Since 1 is an eigenvalue of D with algebraic multiplicity 1, we know that 1 is not an eigenvalue of $D - \mathbf{1}\mu^T$ following part (g) of Lemma 8.2.7 in [14]. Also note that every eigenvalue of D other than 1 has modulus less than 1. Combining the above arguments gives $\rho(D - \mathbf{1}\mu^T) < 1$. ■

Note that the solution to Eq. (2) is given by $\xi(k) = D^k\xi(0) + \sum_{i=1}^{k} D^{i-1}v(k-i)$, $\forall k \in \mathbb{N}^+$.

Consider the update scheme

$$\vartheta(k+1) = \vartheta(k) + \mu^T v(k), \tag{5}$$

where $\vartheta \in I\!R$ and $\vartheta(0) = \mu^T\xi(0)$. The solution to Eq. (5) is given by $\vartheta(k) = \vartheta(0) + \sum_{i=1}^{k} \mu^T v(k-i)$, $\forall k \in \mathbb{N}$.

Letting $\tilde{\xi} = \xi - \mathbf{1}\vartheta$, gives

$$\tilde{\xi}(k) = (D^k - \mathbf{1}\mu^T)\xi(0) + \sum_{i=1}^{k}(D^{i-1} - \mathbf{1}\mu^T)v(k-i). \tag{6}$$

Note that $\xi_i(k) - \xi_j(k) = \tilde{\xi}_i(k) - \tilde{\xi}_j(k)$. Therefore, if $|\tilde{\xi}_i(k) - \tilde{\xi}_j(k)| \to 0$, $\forall (i,j) \in \mathcal{I}$ as $k \to \infty$, then global consensus is reached asymptotically using the discrete time scheme (1). In the special case that $\tilde{\xi}_i(k) \to 0$, $\forall i \in \mathcal{I}$, as $k \to \infty$, we know that $\xi_i(k) \to \vartheta(k)$, $\forall i \in \mathcal{I}$ as $k \to \infty$.

We have the following theorem regarding $\tilde{\xi}(k)$ as $k \to \infty$.

Theorem 3.1 *Given the discrete time scheme* (1), $\tilde{\xi}(k) \to 0$ *as* $k \to \infty$ *asymptotically if graph* $\mathcal{G}$ *has a spanning tree and* $v_1(k) = v_2(k) = \cdots = v_n(k)$, $\forall k \in \mathbb{N}$, *that is, global consensus is reached asymptotically if each agent is driven by a common input.*

Proof: From Lemma 3.1 we know that $D^k \to \mathbf{1}\mu^T$ as $k \to \infty$ and $\mu^T\mathbf{1} = 1$. Note that $v(k)$ can be represented as $v(k) = \mathbf{1}\eta(k)$, where $\eta(k) = v_i(k)$, $\forall i \in \mathcal{I}$. Also note that $D^i\mathbf{1} = \mathbf{1}$, $\forall i \in \mathbb{N}$ since $\mathbf{1}$ is an eigenvector of D and therefore an eigenvector of D^i, $\forall i \in \mathbb{N}$, associated with eigenvalue 1. Therefore, following Eq. (6), we get that $\tilde{\xi}(k) \to 0$ as $k \to \infty$. ■

Let $|||\cdot|||$ denote a matrix norm and $\|\cdot\|$ denote the induced matrix norm specifically.

Theorem 3.2 *Given the discrete time scheme* (1), *if graph* $\mathcal{G}$ *has a spanning tree, then there exists a matrix norm* $|||\cdot|||_d$ *such that* $|||D - \mathbf{1}\mu^T|||_d \le r < 1$ *and* $\|\cdot\| \le d_M|||\cdot|||_d$, *where* d_M *is a positive constant. Furthermore,* $\left\|\tilde{\xi}(k)\right\|$ *is bounded by* $d_1 + d_2 \sup_{i=0,\cdots,k-1}\|v(i)\|$, *where* $d_1 = d_M\|\xi(0)\|$, $d_2 = \left\|I_n - \mathbf{1}\mu^T\right\| + d_M\frac{r}{1-r}$. *In addition, if* $\|v(k)\|$ *is uniformly bounded by a positive constant* γ, *then* $\left\|\tilde{\xi}(k)\right\|$ *is uniformly bounded by* $d_1 + d_2\gamma$.

Proof: Noting that $\rho(D - \mathbf{1}\mu^T) < 1$ from Lemma 3.2, there exists a matrix norm $|||\cdot|||_d$ such that $|||D - \mathbf{1}\mu^T|||_d \le r < 1$ from Lemma 5.6.10 in [14]. Since every vector norm is equivalent on a finite Euclidean space, there exists a positive constant d_M such that $\|\cdot\| \le d_M|||\cdot|||_d$.

Note that $D^i - \mathbf{1}\mu^T = (D - \mathbf{1}\mu^T)^i$ for $i \in \mathbb{N}^+$ following part (e) in Lemma 8.2.7 in [14]. Letting $p_j = \sum_{i=1}^{j}(D - \mathbf{1}\mu^T)^i v(j-i)$, where $j \in \mathbb{N}^+$, Eq. (6) can be rewritten as

$$\tilde{\xi}(k) = (D - \mathbf{1}\mu^T)^k\xi(0) + (I_n - \mathbf{1}\mu^T)v(k-1) + p_{k-1}.$$

We know that

$$\begin{aligned}\|p_{k-1}\| &\le \sum_{i=1}^{k-1}\left\|(D - \mathbf{1}\mu^T)^i\right\|\|v(k-1-i)\| \\ &\le \sum_{i=1}^{k-1} d_M(\sup_{i=1,\cdots,k-1}\|v(k-1-i)\|)|||(D - \mathbf{1}\mu^T)^i|||_d \\ &\le d_M(\sup_{i=0,\cdots,k-2}\|v(i)\|)\sum_{i=1}^{k-1}|||D - \mathbf{1}\mu^T|||_d^i \\ &\le d_M(\sup_{i=0,\cdots,k-2}\|v(i)\|)\frac{r}{1-r}.\end{aligned}$$

Noting that $|||D - \mathbf{1}\mu^T|||_d < 1$, we get that

$$\begin{aligned}\left\|(D - \mathbf{1}\mu^T)^k\xi(0)\right\| &\le \left\|(D - \mathbf{1}\mu^T)^k\right\|\|\xi(0)\| \\ &\le d_M|||(D - \mathbf{1}\mu^T)^k|||_d\|\xi(0)\| \\ &\le d_M|||D - \mathbf{1}\mu^T|||_d^k\|\xi(0)\| \\ &\le d_M\|\xi(0)\|.\end{aligned}$$

Combining with the fact that $\left\|(I_n - \mathbf{1}\mu^T)v(k-1)\right\| \le \left\|I_n - \mathbf{1}\mu^T\right\|\sup_{i=0,\cdots,k-1}\|v(i)\|$, we show that $\left\|\tilde{\xi}(k)\right\| \le d_1 + d_2\sup_{i=0,\cdots,k-1}\|v(i)\|$.

■

Note that we introduce a general matrix norm in Theorem 3.1 since for general matrices the fact that the spectral radius of a matrix is less than 1 does not necessarily imply that the induced norm of that matrix is less than 1.

Corollary 3.3 *Let* $b = [b_1, \cdots, b_n]^T$, *where* $b_i \in I\!R$, $\forall i \in \mathcal{I}$ *are arbitrary constants. Given the discrete time scheme* (1), *if graph* $\mathcal{G}$ *has a spanning tree and* $v(k) = b$, $\forall k \in \mathbb{N}$, *then* $\left\|\tilde{\xi}(k)\right\|$ *is uniformly bounded and* $\tilde{\xi}(k) \to (I_n - \mathbf{1}\mu^T)b + (D - \mathbf{1}u^T)[I_n - (D - \mathbf{1}\mu^T)]^{-1}$ *as* $k \to \infty$ *asymptotically.*

Proof: The uniform boundedness of $\tilde{\xi}(k)$ follows Theorem 3.2 since $v(k)$ is a constant vector for all $k \in \mathbb{N}$.

Note that Eq. (6) can be rewritten as

$$\tilde{\xi}(k) = (D^k - \mathbf{1}\mu^T)\xi(0) + (I_n - \mathbf{1}\mu^T)b + \sum_{i=1}^{k-1}(D^i - \mathbf{1}\mu^T)b.$$

From Lemma 8.6.1 in [14], we know that $\sum_{i=1}^{k-1}(D^i - \mathbf{1}\mu^T) = (D - \mathbf{1}u^T)(I_n - D^{k-1} + \mathbf{1}\mu^T)[I_n - (D - \mathbf{1}\mu^T)]^{-1}$.

The above limit as $k \to \infty$ then directly follows the fact that $D^k \to \mathbf{1}\mu^T$ as $k \to \infty$. ■

Note that although each $\xi_i(k)$, $i \in \mathcal{I}$ may become unbounded as $k \to \infty$ when driven by an input, their inconsistency is guaranteed to be bounded by the above analyses. If each agent evolves according to some nonlinear dynamics $f(k, \xi_i, u(k))$, where u is the common exogenous input to each agent, then we can let $v_i(k) = f(k, \xi_i, u(k))$ in Eq. (1). In this case, consensus is not guaranteed to be achieved asymptotically in general although a similar analysis to Theorem 3.2 guarantees that $\left\|\tilde{\xi}(k)\right\|$ is uniformly bounded if $\|f(k, \xi_i, u(k))\|$ is uniformly bounded. Consider a dynamic consensus example for five agents with communication topology given by Fig. 1. We assume that $v_i(k) = \sin(2\xi_i(k))$ and $\alpha_{ij} = 1$, where $(i, j) = 1, \cdots, 5$.

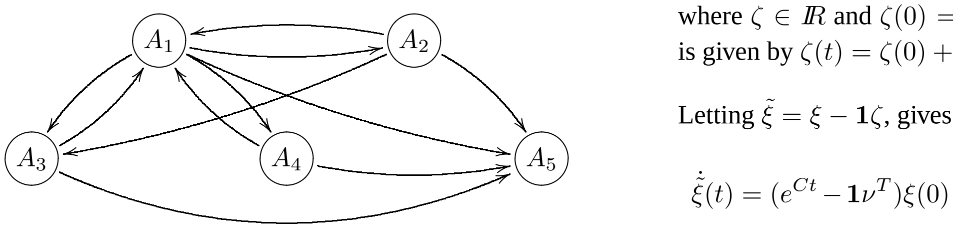

Figure 1: Communication topology with five agents.

Fig. 2 shows the difference between ξ_i and ξ_{i+1}, $i = 1, \cdots, 4$. We can see that consensus is not achieved but the difference of the information state between agents is uniformly bounded.

3.2 Continuous Time Case

Lemma 3.3 *Let C be given in Eq. (4), then e^{Ct}, $\forall t > 0$, is a stochastic matrix with positive diagonal entries. If graph $\mathcal{G}$ has a spanning tree, then 1 is the unique eigenvalue of e^{Ct}, $\forall t > 0$ with maximum modulus and $\lim_{t\to\infty} e^{Ct} \to \mathbf{1}\nu^T$, where $\nu = [\nu_1, \cdots, \nu_n]^T \geq 0$ satisfies $(e^C)^T\nu = \nu$ and $\mathbf{1}^T\nu = 1$.*

Proof: Follows Lemma 1 and Corollary 1 in [11]. ■

Lemma 3.4 *If graph $\mathcal{G}$ has a spanning tree, then $\rho(e^{Ct} - \mathbf{1}\nu^T) < 1$, $\forall t > 0$.*

Proof: Similar to Lemma 3.2 with D replaced by e^{Ct}. ■

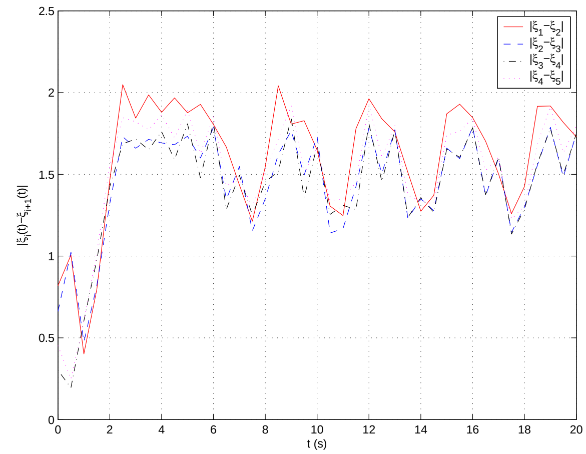

Figure 2: Dynamic consensus with $v_i(k) = \sin(2\xi_i(k))$, $i = 1, \cdots, 5$.

Note that the solution to Eq. (4) is given by $\xi(t) = e^{Ct}\xi(0) + \int_0^t e^{C\tau}v(t-\tau)d\tau$.

Consider the update scheme

$$\dot{\zeta}(t) = \nu^T v(t), \tag{7}$$

where $\zeta \in I\!R$ and $\zeta(0) = \nu^T\xi(0)$. The solution to Eq. (7) is given by $\zeta(t) = \zeta(0) + \int_0^t \nu^T v(t-\tau)d\tau$.

Letting $\tilde{\xi} = \xi - \mathbf{1}\zeta$, gives

$$\dot{\tilde{\xi}}(t) = (e^{Ct} - \mathbf{1}\nu^T)\xi(0) + \int_0^t (e^{c\tau} - \mathbf{1}\nu^T)v(t-\tau)d\tau. \tag{8}$$

Note that if $|\tilde{\xi}_i(t) - \tilde{\xi}_j(t)| \to 0$, $\forall (i, j) \in \mathcal{I}$ as $t \to \infty$, then global consensus is reached asymptotically using continuous time scheme (3). In the special case that $\tilde{\xi}_i(t) \to 0$, $\forall i \in \mathcal{I}$, as $t \to \infty$, we know that $\xi_i(t) \to \vartheta(t)$, $\forall i \in \mathcal{I}$ as $t \to \infty$.

It has been shown in [11] that the continuous time scheme (3) achieves global consensus if graph $\mathcal{G}$ has a spanning tree and $v_1(t) = v_2(t) = \cdots = v_n(t)$, $\forall t \geq T$, where T is any finite positive constant.

Noting that $\rho(e^{Ct} - \mathbf{1}\nu^T) < 1$, $\forall t > 0$, there exists a matrix norm $|||\cdot|||_c$ such that $|||e^{Ct} - \mathbf{1}\nu^T|||_c < 1$, $\forall t > 0$. Let $|||e^C - \mathbf{1}\nu^T|||_c = r$, where $r < 1$ and $\|\cdot\| \leq c_M|||\cdot|||_c$.

Theorem 3.4 *Given the continuous time scheme (3), if graph $\mathcal{G}$ has a spanning tree, then $\left\|\tilde{\xi}(k)\right\|$ is bounded by $c_1 + c_2\sup_{0\leq\tau\leq t}\|v(\tau)\|$, where $c_1 = c_M|||I_n - \mathbf{1}\nu^T|||_c\|\xi(0)\|$, $c_2 = c_M(|||I_n - \mathbf{1}\nu^T|||_c + \frac{r}{1-r})$. Furthermore, if $\|v(t)\|$ is uniformly bounded by a positive constant γ, then $\left\|\tilde{\xi}(t)\right\|$ is uniformly bounded by $c_1 + c_2\gamma$.*

Proof:

Noting that $e^{C\tau}\mathbf{1} = \mathbf{1}$ and $\nu^T e^{C\tau} = \nu^T$, $\forall \tau \geq 0$, we get that

$$\begin{aligned}&(e^t - \mathbf{1}\nu^T)(e^\tau - \mathbf{1}\nu^T)\\ =&e^{t+\tau} - e^t\mathbf{1}\nu^T - \mathbf{1}\nu^T e^\tau + \mathbf{1}\nu^T\\ =&e^{t+\tau} - \mathbf{1}\nu^T.\end{aligned}$$

For any $\tau_2 > \tau_1 >= 0$, noting that $|||e^{C(\tau_2-\tau_1)} - \mathbf{1}\nu^T|||_c < 1$, it can be verified that

$$\begin{aligned}|||e^{C\tau_2} - \mathbf{1}\nu^T|||_c &= |||(e^{C\tau_1} - \mathbf{1}\nu^T)(e^{C(\tau_2-\tau_1)} - \mathbf{1}\nu^T)|||_c\\ &\leq |||(e^{C\tau_1} - \mathbf{1}\nu^T)|||_c.\end{aligned}$$

Letting $\tau_2 = t$ and $\tau_1 = 0$, it can be verified that $\left\|e^{Ct} - \mathbf{1}\nu^T\right\| \leq c_M|||e^{Ct} - \mathbf{1}\nu^T|||_c \leq c_M|||I_n - \mathbf{1}\nu^T|||_c$. Also note that $|||e^{Ck} - \mathbf{1}\nu^T|||_c = |||(e^C - \mathbf{1}\nu^T)^k|||_c \leq |||(e^C - \mathbf{1}\nu^T)|||_c^k \leq r^k$, where $k \in \mathbb{N}^+$.

Let $p_t = \int_0^t (e^{C\tau} - \mathbf{1}\nu^T)v(t-\tau)d\tau$. Let l be the largest integer such that $l \leq t$. We know that

$$\begin{aligned}p_t = &\sum_{i=0}^{l-1}\int_i^{i+1}(e^{C\tau} - \mathbf{1}\nu^T)v(t-\tau)d\tau\\ &+ \int_l^t (e^{C\tau} - \mathbf{1}\nu^T)v(t-\tau)d\tau.\end{aligned}$$

Accordingly, noting that $l \leq t < l+1$, we get that

$$\begin{aligned}\|p_t\| \leq &\sum_{i=0}^{l-1}\int_i^{i+1}\left\|e^{C\tau} - \mathbf{1}\nu^T\right\|\|v(t-\tau)\|\,d\tau\\ &+ \int_l^t \left\|e^{C\tau} - \mathbf{1}\nu^T\right\|\|v(t-\tau)\|\,d\tau\\ &\leq \sup_{0\leq\tau\leq t}\|v(\tau)\|\,(\int_0^1 \left\|e^{C\tau} - \mathbf{1}\nu^T\right\|d\tau\\ &+ \sum_{i=1}^{l-1}\int_i^{i+1}\left\|e^{C\tau} - \mathbf{1}\nu^T\right\|d\tau + \int_l^t \left\|e^{C\tau} - \mathbf{1}\nu^T\right\|d\tau)\\ &\leq \sup_{0\leq\tau\leq t}\|v(\tau)\|\,(c_M|||I_n - \mathbf{1}\nu^T|||_c\\ &+ \sum_{i=1}^{l} c_M|||e^{Ci} - \mathbf{1}\nu^T|||_c)\\ &\leq c_M \sup_{0\leq\tau\leq t}\|v(\tau)\|\,(|||I_n - \mathbf{1}\nu^T|||_c + \sum_{i=1}^{l} r^i)\\ &\leq c_M \sup_{0\leq\tau\leq t}\|v(\tau)\|\,(|||I_n - \mathbf{1}\nu^T|||_c + \frac{r}{1-r}).\end{aligned}$$

Combining the above arguments, we show that $\left\|\tilde{\xi}(t)\right\| \leq c_1 + c_2\sup_{0\leq\tau\leq t}\|v(\tau)\|$. ■

Similar to the discrete time case, if $v_i(t) = f(t, \xi_i, u(t))$, then consensus is not guaranteed to be achieved asymptotically in general. However, in the special case that $f(t, \xi_i, u(t)) = -\gamma\xi_i + g(t, u(t))$, where $\gamma > 0$, dynamic consensus is guaranteed since Eq. (4) can be rewritten as $\dot{\xi} = (C - \gamma I_n)\xi + \mathbf{1}g(t, u(t))$. It is obvious to see that all the eigenvalues of matrix $C - \gamma I_n$ are in the open left half plane. The argument follows a similar analysis to the case where there is a common exogenous input. Also note that if $g(t, u(t)) = 0$, then not only is consensus achieved but each $\xi_i \to 0$ as $t \to \infty$.

4 Application to Cooperative Control

In this section, we apply dynamic consensus seeking to a multi-agent coordinated control scenario where six mobile robots need to preserve a formation shape when performing formation maneuvers. In the following, dynamic consensus seeking will be applied to the distributed virtual leader/virtual structure approach and the behavioral approach respectively.

Let (x_i, y_i), θ_i, and (v_i, ω_i) denote the Cartesian position, orientation, and linear and angular velocity of the ith robot respectively. The kinematic equations for the ith robot are

$$\begin{aligned}\dot{x}_i &= v_i\cos(\theta_i)\\ \dot{y}_i &= v_i\sin(\theta_i)\\ \dot{\theta}_i &= \omega_i.\end{aligned}$$

To focus on the main issue, we consider the simplified dynamics via feedback linearization [15] for a fixed point off the center of the wheel axis denoted as (x_{hi}, y_{hi}), where $x_{hi} = x_i + L\cos(\theta_i)$ and $y_{hi} = y_i + L\cos(\theta_i)$. Given an angle ψ, the rotation matrix is defined as

$$R(\psi) = \begin{bmatrix}\cos(\psi) & \sin(\psi)\\ -\sin(\psi) & \cos(\psi)\end{bmatrix}.$$

Let $z_i = [x_{hi}, y_{hi}]^T$ and $u_i = R(-\theta_i)[v_i, L\omega_i]^T$, where $R(\cdot)$ is the rotation matrix. The simplified kinematic equations can be represented as

$$\dot{z}_i = u_i.$$

The first example will be based on the virtual leader/virtual structure approach (c.f. [16, 17, 18]). We simulate the case where the geometric center of the formation follows a trajectory of a circle with radius 5 meters while the whole group preserves the desired hexagon formation shape with side length equal to 2 meters during the maneuver. Define the states of the formation frame located at the geometric center of the hexagon formation as $r_0(s(t)) = [x_0(s(t)), y_0(s(t)), \theta_0(s(t))]^T$, where $s(t) \in I\!R$ is a parameter, and $(x_0(s), y_0(s))$ and $\theta_0(s)$ denote the position and orientation of the formation frame respectively. In the simulation, we let $x_0(s) = 5\cos(\omega_0 s)$, $y_0(s) = 5\sin(\omega_0 s)$, and $\theta_0(s) = \omega_0 s$, where ω_0 specifies the angular frequency of the desired trajectory for the formation center. Given s, the desired trajectory for each robot can in turn be defined as $z_i^d(s) = [x_0(s), y_0(s)]^T + R(-\theta_0(s))\tilde{z}_{i0}^d(s)$, where $R(\cdot)$ is the rotation matrix and $\tilde{z}_{i0}^d(s)$ is the specified desired deviation of each robot from the formation center [19, 20]. For example, the desired deviation for the first robot is given by $\tilde{z}_{10} = [2\cos(2\pi/3), 2\sin(2\pi/3)]^T$. Note that in this case the parameter s represents the minimum amount of information needed by each robot to coordinate its motion with the group. In a centralized implementation, the parameter s can be implemented at a central location and broadcast to all the robots, which may result in a single point of failure. In this distributed implementation, we will instantiate the parameter s on each robot as s_i, $i = 1, \cdots, 6$, and drive each instantiation via intervehicle communications to achieve consensus.

The discrete consensus scheme is given by

$$s_i(k+1) = \frac{1}{\sum_{j=1}^n \alpha_{ij} G_{ij}} \sum_{j=1}^n \alpha_{ij} G_{ij} s_j(k) + \frac{T_s}{3} + w_i(k),$$

where α_{ij} are chosen as arbitrary positive constants, $T_s = 0.5$ (s) is the sample period, and $w_i(k)$ denotes the communication noise at time $t = kT_s$. Then each robot can track its desired states specified by its parameter instantiation s_i based on a simple tracking law

$$\dot{z}_i = \dot{z}_i^d(s_i) - \gamma(z_i - z_i^d(s_i)),$$

where $\gamma > 0$.

Fig. 3 shows the communication topology for the six robots. Instead of assuming a strongly connected communication network, any communication topology with a spanning tree is sufficient for consensus seeking.

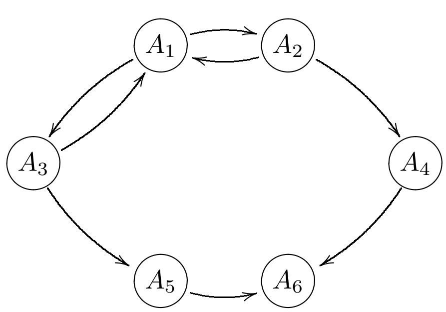

Figure 3: Communication topology with six agents in the distributed virtual leader/virtual structure approach.

Fig. 4 shows the formation maneuvers of the six robots at $t = 0$, 25, 50, 75, and 100 (s) respectively. The green circle represents the desired trajectory of the formation center, the actual formation at each time is represented by polygons with square vertices denoting the actual location of each robot, and the desired formation at each time is represented by polygons with star vertices denoting the desired location of each robot. Fig. 5 shows the consensus of s_i with random communication noise. We can see that the difference between each instantiation is bounded. Fig. 6 shows the tracking errors and formation keeping errors with random communication noise. Here $\mathrm{dist}(a, b)$ is defined as $\|a - b\|$. Note that the desired distance between agents $(1, 6)$, $(2, 5)$, and $(3, 4)$ is 4 meters.

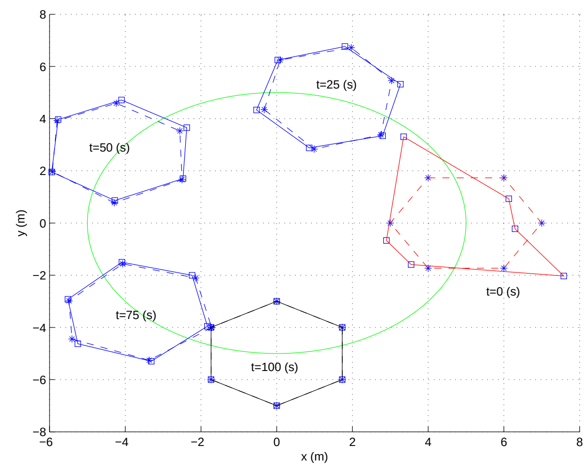

Figure 4: Formation maneuvers of the six robots in the distributed virtual leader/virtual structure approach.

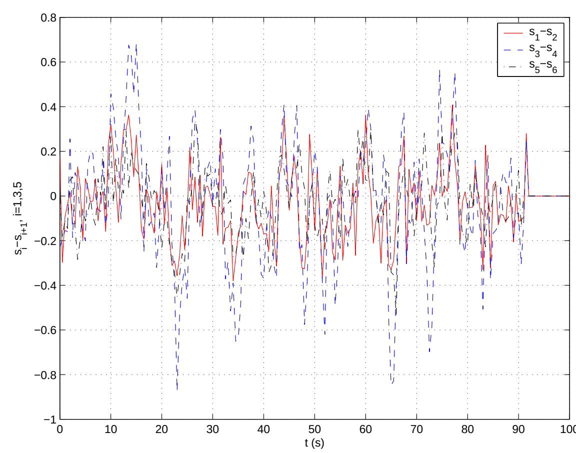

Figure 5: Consensus of s_i with random communication noise.

The second example is based on the behavioral approach introduced in [21]. However, we only assumes that the interaction topology has a spanning tree without requiring the more stringent bidirectional ring topology as in [21]. The six robots will perform a formation maneuver to achieve a defined constant formation pattern as $\{z_1^d, \cdots, z_6^d\}$. In order to achieve the desired goal and preserve the formation shape, we need to guarantee that $z_{ei} = z_i - z_i^d$ approaches zero as well as being in consensus during the maneuver. A

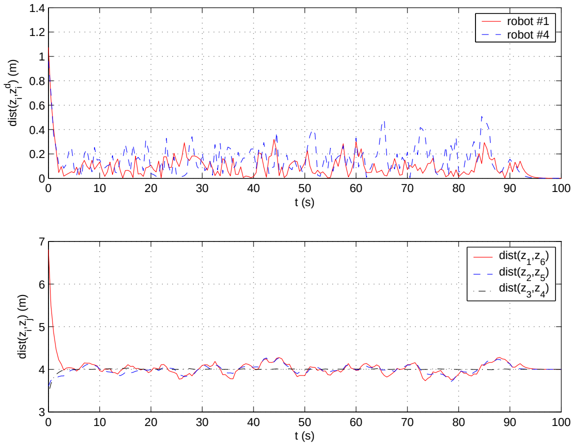

Figure 6: Tracking error and formation keeping error in the distributed virtual leader/virtual structure approach.

simple control law for the ith robot is defined as

$$u_i = -\sum_{j=1}^{n} \sigma_{ij} G_{ij}(z_{ei}(t) - z_{ej}(t)) - \gamma z_{ei}(t) + w_i(t),$$

where $\gamma > 0$, σ_{ij} are arbitrary positive constants, and $w_i(t)$ denotes the measurement noise associated with the ith robot at time t.

After some manipulation, we can show that

$$\dot{z}_e = [(C - \gamma I_6) \otimes I_2] z_e + w(t), \tag{9}$$

where $z_e = [z_{e1}, \cdots, z_{e6}]^T$, C is the 6×6 matrix defined in Eq. (4), $\otimes$ denotes the Kronecker product, and $w = [w_1, \cdots, w_6]^T$. Note that z_{ei} will approach zero if $w(t) = 0$ since all eigenvalues of $C - \gamma I_6$ are on the open left half plane. In addition, the term $(C \otimes I_2) z_e$ guarantees consensus can be achieved for z_{ei}, $i = 1, \cdots, 6$.

Fig. 7 shows the interaction topology for the six robots, where the information flow from A_j to A_i represents that A_i can obtain z_{ej}. Note the existence of a spanning tree in Fig. 7.

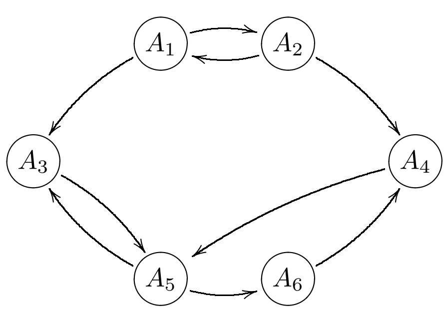

Figure 7: Interaction topology with six agents for the behavioral approach.

Fig. 8 shows the formation maneuvers of the six robots at $t = 0$, 12.5, 25, 37.5 and 50 (s) respectively. The desired formation pattern is represented by the black polygon with star vertices denoting the desired locations for each robot. The actual formation is represented by polygons with square vertices denoting the actual location of each robot. The initial formation is represented by the red polygon with squared vertices. The difference between this case and the previous case is that no trajectory is specified for each robot during the maneuver except that the final desired location for each robot is given. Note that the hexagon formation shape is preserved during the maneuver. Fig. 9 shows the consensus of x_{ei} and y_{ei}, where $x_{ei} = x_{hi} - x_{hi}^d$ and $y_{ei} = y_{hi} - y_{hi}^d$, with random measurement noise. We can see that the difference between each instantiation is bounded. Fig. 10 shows the goal seeking errors and formation keeping errors with random measurement noise. Note that the subplot on the top shows that the robots achieve their goal asymptotically while the subplot on the bottom shows that the desired formation shape is preserved during the maneuver after $t = 10$ (s).

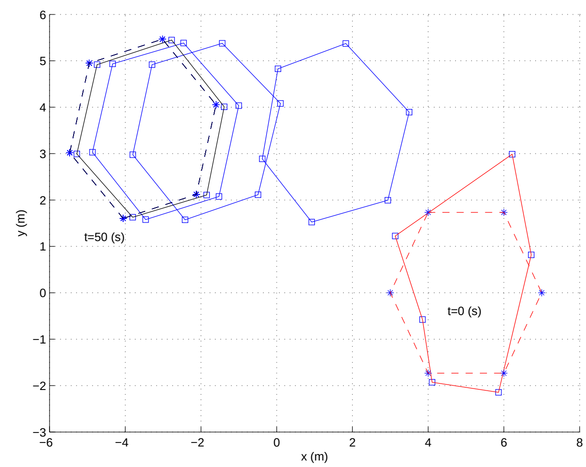

Figure 8: Formation maneuvers of the six robots in the behavioral approach.

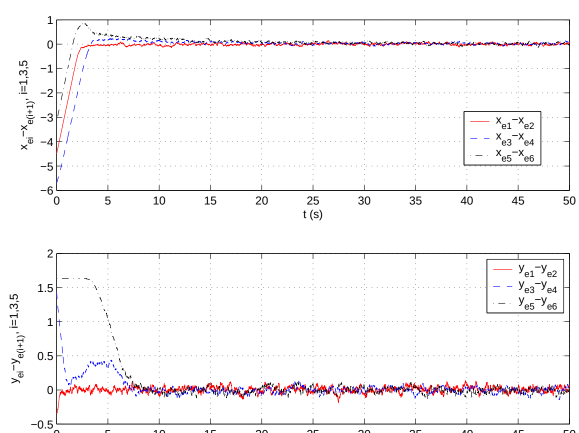

Figure 9: Consensus of x_{ei} and y_{ei} with random measurement noise.

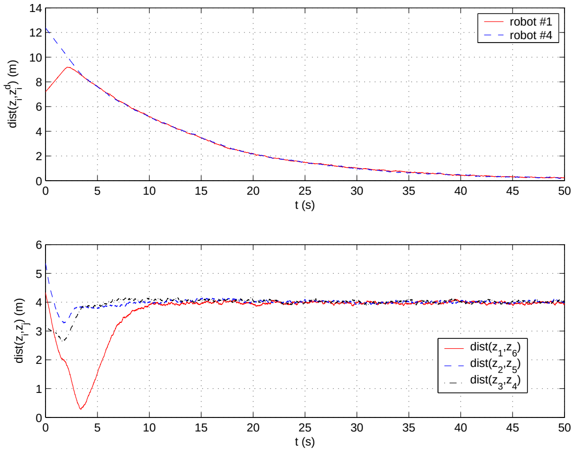

Figure 10: Goal seeking error and formation keeping error in the behavior approach.

5 Conclusion

This paper has considered the problem of dynamic consensus seeking in distributed multi-agent systems. Using both discrete time and continuous time consensus schemes, we showed conditions under which global consensus can be achieved with a common input and gave explicit upper bounds for the inconsistency between agents when information exchange noise exists or there are inconsistent inputs. Applications to cooperative control were presented to show the effectiveness of our results.

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