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Numerical Methods for Open Loop and Closed Loop Optimization of Piecewise Linear Systems

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Abstract—The optimization problem for piecewise linear systems in the class of discrete controls is considered. On the basis of methods for solving linear optimal control problems supplemented with optimization methods with respect to parameters, a dual method is developed for calculating open loop optimal controls. The method is also used in the synthesis of closed loop optimal controls. The results obtained are illustrated by considering problems of optimal excitation and displacement for a one-mass oscillatory system driven by a piecewise elastic force.

1. INTRODUCTION

In the mathematical theory of optimal processes [1], to analyze nonlinear systems of the form

$$\dot{x} = f(x) + bu \quad (x \in \mathbb{R}^n, u \in \mathbb{R}) \tag{1.1}$$

it is often sufficient to use linear approximations

$$\dot{z} = A(t)z + bv, \quad A(t) = \partial f(x(t))/\partial x, \tag{1.2}$$

constructed along certain trajectories $x(t)$ $(t \geq 0)$ of the nonlinear system (1.1).

Linear approximations (1.2) are frequently used to develop approximate methods for solving optimal control problems [2]. It is clear that they can only provide satisfactory descriptions of local behavior of nonlinear systems in the neighborhoods of reference trajectories. Therefore, these approximations have a limited scope.

One natural way to expand the scope of linear optimization methods is to use piecewise linear approximations (first-order splines) of the nonlinear elements of a problem. Even though the model remains nonlinear in this approximation, effective optimization methods can be developed by taking into account specific properties of the piecewise linear model.

Let $X \subset \mathbb{R}^n$ be the domain in the space of states of system (1.1) in which its behavior is investigated. We assume that the closure of X can be represented as the union of polyhedral sets $X_1, \ldots, X_p$ such that $\operatorname{int}X_i \cap \operatorname{int}X_j = \varnothing$ for $i \neq j$. We replace the function $f(x)$ $(x \in X)$ in system (1.1) by a function $\hat{f}(x)$ $(x \in X)$ that is linear on each X_j $(j = \overline{1, p})$. The number $\delta = \max_{x \in \bar{X}} \| f(x) - \hat{f}(x)\| / \| f(x)\|$ is called the approximation error. Choosing sufficiently small values of δ, one can obtain good results for the original nonlinear system by optimizing the piecewise linear system. To develop a more effective optimization method for nonlinear systems, a special procedure was suggested in [3] for correcting a solution to the piecewise linear problem. However, we do not elaborate on details here, since we plan to devote a special paper to optimal control of general nonlinear systems [4].

The aim of this paper is to extend optimization methods for linear systems presented in [5] to the class of piecewise linear systems

$$\dot{x} = \hat{f}(x) + bu, \quad x \in X.$$

This paper contains a detailed description of a module included in the general solution scheme for nonlinear optimal control problems [3]. The paper is organized as follows. In Section 2, the optimization problem for a piecewise linear system is formulated in the class of discrete controls. The choice of this class of feasible controls is dictated by the actual possibilities of solving the difficult problem under consideration.

Discrete open loop controls imply discrete feedback, which makes it easier to analytically determine the trajectories of closed systems. The basic idea of our approach is elaborated in Section 3, where a special linearization of the piecewise linear problem is developed on the basis of a parameterized form of the original problem, and, a scheme for solving the linear problem is described. The key module of the method is described in Sections 4 and 5, where the operations of the open loop optimization method for the linear problem with phase constraints that are necessary to solve the present problem are considered (this method was published in [6]). The second procedure of the method is described in Section 5, where emphasis is placed on the calculation of the gradient of the objective functional with respect to the control switching times. The formulas obtained can be used to develop a variety of parametric optimization methods that are well known from the literature [7–9]. In Section 6, the scheme of a refinement procedure used to construct optimal controls in the class of piecewise continuous functions is described. In Section 7, the main result of this study is presented, namely, the synthesis of closed loop optimal controls. Optimal synthesis is the key problem in control theory since open loop solutions are not used in actual controls; they are required only to reveal the potential capabilities of control systems. The approach described in this paper essentially relies on the dynamic nature of the problem under consideration. The optimal controller generates the values of an optimal feedback in real time. This eliminates the main disadvantage of dynamic programming, i.e., the need to calculate the optimal feedback for all possible states of the system in advance (before the control process begins), which results in the so-called "curse of dimensionality" [10]. The examples considered in Section 8 demonstrate that the algorithm of the optimal controller is efficient and can therefore be implemented on modern computers for relatively complex control systems. As an illustration, we present in Section 8 the numerical results obtained by solving two optimal control problems for a piecewise linear oscillatory system with one degree of freedom. First, the problem of optimal excitation of the system on a finite time interval is solved in both open and closed loop formulations. Then, the problem of optimal transfer of this system to a new state is considered. The calculated results expose the high efficiency of the method in constructing optimal open loop controls and the possibility of implementing closed loop controls on modern computers.

2. STATEMENT OF THE PROBLEM

Let $T = [0, t^*]$ be the control time interval, $h = t^*/N$ be the quantization interval, N be a positive integer, and $T_h = \{0, h, \ldots, t^* - h\}$.

The function $u(t)$ $(t \in T)$ is called the discrete control (with the quantization interval h) if $u(t) = u(kh)$ for $t \in [kh, (k+1)h[$ $(k = \overline{0, N-1})$.

We consider the optimal control problem for the piecewise linear system

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(0) = x_0, \\ Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T = [0, t^*], \end{gathered} \tag{2.1}$$

in the class of discrete controls. Here, $x = x(t)$ is the state vector of the dynamical system at an instant t, $u = u(t)$ is the value of a scalar control, $b \in \mathbb{R}^n$, $g \in \mathbb{R}^{\hat{m}}$, $H \in \mathbb{R}^{\hat{m} \times n}$, and $\operatorname{rank} H = \hat{m} < n$.

The discrete control $u(\cdot) = (u(t), t \in T)$ is called feasible for problem (2.1) if it satisfies the condition $|u(t)| \le 1$ for $t \in T$ and the corresponding trajectory $x(t)$ $(t \in T)$ satisfies the terminal constraint $Hx(t^*) = g$ and crosses the boundaries of domains at discrete times belonging to the set T_h. A feasible control $u^0(\cdot)$ is called the optimal open loop control for problem (2.1) if the corresponding trajectory $x^0(t)$ $(t \in T)$ maximizes the objective functional of problem (2.1).

To introduce the concept of the closed loop optimal control for problem (2.1), we embed problem (2.1) in the family of problems

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \quad \dot{x} = \hat{f}(x) + bu, \quad x(\tau) = z, \\ Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T(\tau) = [\tau, t^*], \end{gathered} \tag{2.2}$$

which depends on a scalar $\tau \in T_h$ and an n-dimensional vector z. Let $u^0(t|\tau, z)$ $(t \in T(\tau))$ be the optimal open loop control for problem (2.2) for (τ, z) and X^τ be the set of states z for which problem (2.2) has a solution. The function

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X^\tau, \quad \tau \in T_h, \tag{2.3}$$

is called the closed loop (discrete) optimal control for problem (2.1).

The trajectory of the control system

$$\dot{x} = \hat{f}(x) + bu^0(t, x) + w(t), \quad x(0) = x_0, \tag{2.4}$$

closed by the optimal feedback (2.3) and subjected to a piecewise continuous perturbation $w(t)$ $(t \in T)$ is defined as the continuous solution to the equation

$$\dot{x} = \hat{f}(x) + bu^*(t) + w(t), \quad x(0) = x_0$$

with the control $u^*(t) = u^0(kh, x(kh))$ $(t \in [kh, (k+1)h[, k = \overline{0, N-1})$. When $\hat{f}(x)$ $(x \in X)$ is a continuous function, there obviously exists a classical solution to this equation.

The purpose of this study is to develop effective algorithms for constructing the optimal open loop control and synthesize the optimal closed loop control for problem (2.1). Since the algorithm is formulated here without using the fact that $\hat{f}(x)$ $(x \in X)$ is independent of time, the results obtained in [6] ensure that our methods can be effectively applied when $\hat{f}(x, t)$ $(x \in X, t \in T)$ is a time-dependent function.

3. A PARAMETERIZED FORM OF THE PROBLEM

Every feasible control $u(t)$ $(t \in T)$ for problem (2.1) generates a trajectory that goes through a (feasible) sequence $X_{i_1}, \ldots, X_{i_k}$ of sets from $X_1, \ldots, X_p$ and crosses the boundaries of adjacent sets at feasible instants $\Theta_{i_1}, \ldots, \Theta_{i_k - 1} \in T_h$. We denote by $X_1^0, \ldots, X_{j^*}^0$ and $\Theta_1^0, \ldots, \Theta_{j^*-1}^0$ the collection of sets and the instants of time at which the optimal trajectory $x^0(t)$ $(t \in T)$ crosses boundaries of sets. (More complex optimal trajectories containing parts of set boundaries will be investigated in another paper.) The sequence of domains $X^0 = \{X_1^0, \ldots, X_{j^*}^0\}$ is called the structure of the optimal trajectory for problem (2.1). We assume that the optimal trajectory $x^0(t)$ crosses the boundaries separating the sets X_j^0 and X_{j+1}^0 $(j = \overline{1, j^*-1})$ in $(n-1)$-dimensional hyperplanes $H_j x = g_j$, where H_j is a 1-by-n matrix and $g_j \in \mathbb{R}$ $(j = \overline{1, j^*-1})$.

The function $\hat{f}(x)$ has the form $\hat{f}(x) = A_j x + a_j$ on the set X_j^0, where $A_j \in \mathbb{R}^{n \times n}$ and $a_j \in \mathbb{R}^n$. We define $m_j = 1$ $(j = \overline{1, j^*-1})$ and $m_{j^*} = \hat{m}$, set $H_{j^*} = H$ and $g_{j^*} = g$, and assume that $m = \sum_{j=1}^{j^*} m_j$.

In many problems having the form of (2.1), the sequence $X_1^0, \ldots, X_{j^*}^0$ can be defined before solving the problem on the basis of a priori information. Then, this problem can be represented as the optimal control problem for the set of linear systems

$$J(\Theta, u) = c'x(\Theta_{j^*}) \longrightarrow \max, \tag{3.1}$$

$$\dot{x}(t) = A_j x(t) + a_j + bu(t), \quad t \in [\Theta_{j-1}, \Theta_j[, \quad j \in J = \{1, 2, \ldots, j^*\}, \quad x(\Theta_0) = x_0, \tag{3.2}$$

$$H_j x(\Theta_j) = g_j, \quad j \in J, \tag{3.3}$$

$$|u(t)| \le 1, \quad t \in T, \quad \Theta_0 < \ldots < \Theta_{j^*-1} < \Theta_{j^*} \tag{3.4}$$

$$(x \in \mathbb{R}^n, u \in \mathbb{R}, g_j \in \mathbb{R}^{m_j}, j = \overline{1, j^*}, \operatorname{rank} H_j = m_j < n, \Theta_j \in T_h, j = \overline{1, j^*-1}, \Theta_0 = 0, \Theta_{j^*} = t^*).$$

In problem (3.1)–(3.4), the instants $\Theta = (\Theta_1, \ldots, \Theta_{j^*-1})$ are calculated along with the control $u(t)$ $(t \in T)$.

If the sequence $X_1^0, \ldots, X_{j^*}^0$ is not known in advance, then problem (2.1) is solved as follows: (1) feasible sequences of sets $X_1, \ldots, X_{j^*}$ and instants Θ are defined; (2) the linear problem (3.1)–(3.4) is solved for these fixed sequences; (3) it is verified that the optimal trajectory of problem (3.1)–(3.4) lies within the domains; (4) if the optimal trajectory corresponds to the selected structure, the instants of transition are corrected; and (5) if the desired structure is violated on the optimal trajectory, new sequences of sets and instants of transition are selected.

In this paper, we describe an algorithm for solving problem (3.1)–(3.4) for a given collection $X_1, \ldots, X_{j^*}$.

The vector Θ and the discrete control $u(\cdot)$ are called an available control for problem (3.1)–(3.4) if they satisfy constraints (3.4). The available control $\{\Theta, u(\cdot)\}$ and the corresponding trajectory $x(t) = x(t|\Theta, u(\cdot))$

$(t \in T)$ for system (3.2) are called feasible if $x(t)$ $(t \in T)$ satisfies constraints (3.3). A feasible control $\{\Theta^0, u^0(\cdot)\}$ is called optimal if it maximizes the objective functional (3.1).

Problem (3.1)–(3.4) is solved in two steps. First, problem (3.1)–(3.4) is linearized along a feasible trajectory, and the linearized problem is solved by the method presented in [6]. Then, the solution to the linearized problem is corrected by choosing optimal instants of transition from one domain of linear solution to another.

At both steps, optimal or suboptimal controls can be constructed by applying the corresponding techniques described in [5, 6].

4. SOLUTION OF THE LINEARIZED PROBLEM

The linearization of problem (3.1)–(3.4) consists in fixing the vector Θ corresponding to a feasible trajectory. The linearized problem is similar to (3.1)–(3.4), but it is assumed that the vector Θ is fixed. Thus, we obtain a linear optimal control problem with intermediate phase constraints.

$$\begin{gathered} c'x(t^*) \longrightarrow \max, \quad \dot{x} = A(t)x + a(t) + bu, \quad x(\Theta_0) = x_0, \\ H(s)x(s) = g(s), \quad s \in S = \{\Theta_1, \dots, \Theta_{j^*}\}, \quad |u(t)| \le 1, \quad t \in T. \end{gathered} \tag{4.1}$$

Here, $A(t) = A_j$, $a(t) = a_j$, and $t \in T_j = [\Theta_{j-1}, \Theta_j[$ for $j = \overline{1, j^*}$; $H(s) = H_j$, $g(s) = g_j$, and $s = \Theta_j$ for $j = \overline{1, j^*}$.

An algorithm for solving this problem is presented in [6]. Here, we describe its basic steps used in solving problem (4.1).

In the functional form, problem (4.1) is written as

$$\sum_{t \in T_h} c(t)u(t) \longrightarrow \max, \quad \sum_{t \in T_h} d(s, t)u(t) = \hat{g}(s), \quad s \in S, \quad |u(t)| \le 1, \quad t \in T_h.$$

Here,

$$c(t) = \int_t^{t+h} \psi_c'(\vartheta) b d\vartheta, \quad d(s, t) = \begin{cases} \int_t^{t+h} G(s, \vartheta) b d\vartheta, & s > t, \\ 0, & s \le t, \end{cases}$$

$$\hat{g}(s) = g(s) - G(s, 0)x_0 - \int_0^s G(s, \vartheta)a(\vartheta)d\vartheta,$$

$\psi_c(t)$ $(t \in T)$ is the solution to the adjoint equation

$$\dot{\psi} = -A'(t)\psi$$

with the initial condition

$$\psi(t^*) = c,$$

and $G(s, t)$ $(t \in T(s) = [0, s])$ is an $[m(s) \times n]$ matrix function that solves the equation

$$\dot{G} = -GA(t), \quad G(s, s) = H(s).$$

Let us take an arbitrary subset $T_w = \{t_l, l = \overline{1, m}\}$ of m elements in the set T_h. The set T_w is called a working basis (or support) of problem (4.1) if the working (or support) matrix

$$D_w = \begin{pmatrix} d(s, t), t \in T_w, \\ s \in S \end{pmatrix}$$

is nonsingular. The pair $\{u(\cdot), T_w\}$ consisting of a feasible control and a working basis is called the working control for problem (4.1).

An implementation of the adaptive method in linear programming proposed in [11, 12] was described in [6]. In the course of an iteration of this method, the stored information about the elements of the control problem corresponding to the current working basis T_w is transformed. Here, we show how the information required for problem (4.1) should be represented taking into account the structure of the matrix function $A(t)$ $(t \in T)$ and the subsequent correction of the time instants $\Theta_1, \ldots, \Theta_{j^*-1}$.

According to [6], the working basis T_w is associated with the following elements.

1. Potentials $\nu = (\nu(s), s \in S)$ provide a solution to the equation $D'_w \nu = c_w$, where $c_w = (c(t), t \in T_w)$.

2. A cocontrol is defined as

$$\Delta(t) = \int_t^{t+h} \left[\psi'_c(\vartheta) - \sum_{s>t} \nu'(s) G(s, \vartheta)\right] b\, d\vartheta, \quad t \in T_h.$$

3. A pseudocontrol $\omega(t)$ $(t \in T_h)$ is defined so that its nonworking values $\omega(t)$ $(t \in T_{nw} = T_h \backslash T_w = \bigcup_{j=1}^{j^*} T_{nw}(j))$ satisfy the relations

$$\omega(t) = \operatorname{sgn}\Delta(t), \text{ if } \Delta(t) \neq 0; \quad \omega(t) \in [-1, 1], \quad \text{if} \quad \Delta(t) = 0, \quad t \in T_{nw}. \tag{4.2}$$

The working values $\omega(t)$ $(t \in T_w)$ of the pseudocontrol are determined from the equation

$$\sum_{t \in T_w} d(s, t)\omega(t) + \sum_{t \in T_{nw}} d(s, t)\omega(t) = \hat{g}(s), \quad s \in S;$$

4. A quasi-control is defined as

$$\tilde{\omega}(t) = \begin{cases} \omega(t), & |\omega(t)| \leq 1, \\ \operatorname{sgn}\omega(t), & |\omega(t)| > 1, \quad t \in T_h; \end{cases}$$

5. The norm of the residual of intermediate constraints on the quasi-control is defined as

$$\tilde{g}(T_w) = \max_{s \in S} \|g(s) - H(s)\tilde{x}(s)\|,$$

where $\tilde{x}(t)$ $(t \in T)$ is the trajectory of system (3.2) corresponding to the quasi-control $\tilde{\omega}(t)$ $(t \in T)$.

For given $\varepsilon \geq 0$ and $\delta \geq 0$, the available control $u(t)$ $(t \in T)$ is called an $\varepsilon\delta$-solution to problem (3.1)–(3.4) if the generated trajectory $x(t)$ $(t \in T)$ satisfies the inequality $c'x^0(t^*) - c'x(t^*) \leq \varepsilon$, and the norm of the residual of intermediate constraints $\tilde{g}(u(\cdot)) = \max_{j \in J} \|g_j - H_j x(\Theta_j)\|$ satisfies the inequality $\tilde{g}(u(\cdot)) \leq \delta$ for the control $u(t)$ $(t \in T)$.

If the pseudocontrol $\omega(t)$ $(t \in T_h)$ constructed on the basis of T_w satisfies the inequality $|\omega(t)| \leq 1$ $(t \in T_w)$, then $\omega(t)$ is an optimal control for problem (3.1)–(3.4). If, for a given $\delta \geq 0$, the inequality $\tilde{g}(T_w) \leq \delta$ is satisfied by the pseudocontrol $\tilde{\omega}(t)$ $(t \in T_h)$ constructed on the basis of T_w, then $\tilde{\omega}(t)$ $(t \in T_h)$ is a 0δ-solution to problem (3.1)–(3.4).

In the course of iteration of the dual adaptive method [6], the working basis is transformed into the optimal working basis T_w.

We now describe the transformation of a working basis T_w into another working basis $\bar{T}_w$. An instant $t^0 \in T_w$ such that $|\omega(t^0)| = \max|\omega(t)| > 1$ $(t \in T_w)$ is eliminated from the working basis. In order to determine the instant of time to be added to the working basis, we calculate a step σ^* along the variation $\Delta\nu$ of potentials that ensures complete relaxation of the dual objective function. In order to calculate σ^*, such a sequence of short steps $\sigma^1, \ldots, \sigma^l$ is executed that a new zero of the varied cocontrol appears at every step. The following data are used in the course of a transformation of the working basis: the set of nonworking zeros of the cocontrol

$$T_{nw0} = \{t \in T_{nw} \backslash S : \Delta(t-h)\Delta(t) < 0\} = \bigcup_{j=1}^{j^*} T_{nw0}(j);$$

the sets $T_{w.nw}(j) = T_w(j) \cup T_{nw0}(j) \cup \{\Theta_{j-1}, \Theta_j - 0\} = \{t_k(j), k \in K(j) \cup k(j) + 1\}$, where $K(j) = \{0, 1, \ldots, k(j)\}$

and $T_{\text{w.nw}} = \bigcup_{j=1}^{j^*} T_{\text{w.nw}}(j)$; the numbers

$$\gamma_j = \begin{cases} \operatorname{sgn}\Delta(\Theta_{j-1}), & \Theta_{j-1} \notin T_{\text{w}}, \\ \operatorname{sgn}\Delta(\Theta_{j-1}+h), & \Theta_{j-1} \in T_{\text{w}}, \quad j \in J, \end{cases}$$

and the vectors

$$r_j = \sum_{t \in T_{\text{nw}}(j)} \int_t^{t+h} F_j(\Theta_j - \vartheta) b\, d\vartheta\, \omega(t) + \int_{\Theta_{j-1}}^{\Theta_j} F_j(\Theta_j - \vartheta) d\vartheta\, a_j, \quad j \in J,$$

where

$$\dot{F}_j = A_j F_j, \quad F_j(0) = E, \quad j \in J.$$

According to [6], the following data are stored and transformed in the course of iteration: (1) the working basis T_{w}; (2) the set T_{nw0} of nonworking zeros; (3) the working matrix D_{w}; (4) the matrices $F_j(\Theta_j - \Theta_{j-1})$ ($j \in J$) and $F_j(\Theta_j - t)$ ($t \in T_{\text{w.nw}}(j), j \in J$); (5) the vectors r_j ($j \in J$); (6) the numbers γ_j ($j \in J$) and the working values of the pseudocontrol $\omega(t)$ ($t \in T_{\text{w}}$); and (7) the potentials ν.

The matrices $G(s, t)$ ($t \in T_{\text{w.nw}}, s \in S, s > t$) and the vectors p and $\psi_c(t)$ ($t \in T_{\text{w.nw}}, s \in S, s > t$) used in [6] are calculated by using the matrices in (4) and the vectors in (5):

$$G(\Theta_k, t) = H_k \Phi_{kj} F_j(\Theta_j - t), \quad \psi_c(t) = c' \Phi_{kj} F_j(\Theta_j - t), \quad k = \overline{j, j^*}, \quad t \in T_{\text{w}}(j),$$

$$p(\Theta_k) = \sum_{j=1}^{k} H_k \Phi_{kj} r_j + G(\Theta_k, 0) x_0, \quad p = \begin{pmatrix} p(\Theta_k), \\ k \in J \end{pmatrix},$$

where

$$\Phi_{kj} = \Phi_{kj}(\Theta) = \begin{cases} \prod_{r=j}^{k-1} F_{r+1}(\Theta_{r+1} - \Theta_r), & k > j, \\ E, & k = j, \\ 0, \quad k < j, \quad j = \overline{0, j^*-1}, & k = \overline{1, j^*}. \end{cases}$$

The operations used to transform $G(s, t)$, $\psi_c(t)$ ($t \in T_{\text{w.nw}}, s \in S, s \geq t$), and p in the course of iteration described in [6] are naturally extended to $F_j(\Theta_j - t)$ ($t \in T_{\text{w.nw}}(j), r_j, j \in J$).

Thus, the algorithm presented in [6] is used to construct an optimal working basis $T_{\text{w}}^0(\Theta)$ for problem (3.1)–(3.4) with a fixed Θ, and the corresponding data (1)–(7) are calculated.

5. OPTIMIZATION OF THE TIME INSTANTS OF TRANSITIONS BETWEEN THE LINEARITY DOMAINS

Solving the linearized problem, we obtain an optimal working basis $T_{\text{w}}^0(\Theta)$ for the fixed vector Θ. If the fixed vector Θ^0 is found, then the corresponding optimal working basis $T_{\text{w}}^0(\Theta^0)$ determines the optimal control for problem (2.1).

The various methods that can be invoked to determine the optimal vector Θ^0 (see [7–9]) are based on the gradient of the objective functional of problem (2.1) with respect to Θ.

Thus, we must calculate the partial derivatives $\partial J(\Theta, u)/\partial \Theta_k$, $k = \overline{1, j^*-1}$ of the objective functional with respect to the instants $\Theta_1, \ldots, \Theta_{j^*-1}$. Denote by $u_{\text{w}}^0(\Theta) = (u^0(t|\Theta), t \in T_{\text{w}}^0(\Theta))$ the working values of the optimal control $u^0(t|\Theta)$ ($t \in T_h$) and by $x(t|\Theta_1, \ldots, \Theta_{j^*-1}, u_{\text{w}}^0(\Theta))$ ($t \in T$)) the corresponding trajectory of system (2.1). For an arbitrary $k \in J \setminus j^*$, consider a small variation $\Delta\Theta_k$ of Θ_k that does not change the working basis $T_w^0(\Theta)$. The optimal control corresponding to the perturbed $\Theta_k + \Delta\Theta_k$ differs from $u^0(t|\Theta)$ ($t \in T_h$) only in the working components $u_w^0(\Theta) + \Delta u_w$. Then,

$$H_j x(\Theta_j|\Theta_1, \ldots, \Theta_k + \Delta\Theta_k, \ldots, \Theta_{j^*-1}, u_w^0(\Theta) + \Delta u_w) - H_j x(\Theta_j|\Theta_1, \ldots, \Theta_{j^*-1}, u_w^0(\Theta))$$
$$= H_j \frac{\partial x(\Theta_j|\Theta, u_w^0(\Theta))}{\partial \Theta_k}\Delta\Theta_k + H_j \frac{\partial x(\Theta_j|\Theta, u_w^0(\Theta))}{\partial u_w}\Delta u_w + o(\Delta\Theta_k\|\Delta u_w\|) = 0, \quad j \in J.$$

Hence,

$$\Delta u_w = -D_w^{-1}(\Theta)\begin{pmatrix} H_j \partial x(\Theta|\Theta, u_w^0(\Theta))/\partial\Theta_k, \\ j \in J \end{pmatrix}.$$

Substitute this expression into the formula for the increment of the objective functional to obtain

$$\frac{\partial J(\Theta, u)}{\partial \Theta_k} = c'\frac{\partial x(t^*|\Theta, u_w^0(\Theta))}{\partial\Theta_k} - \nu'(\Theta)\begin{pmatrix} H_j \partial x(\Theta|\Theta, u_w^0(\Theta))/\partial\Theta_k, \\ j \in J \end{pmatrix}, \quad k = \overline{1, j^*-1}. \tag{5.1}$$

To find the values of (5.1), we calculate $\partial x(\Theta_j|\Theta, u_w^0(\Theta))/\partial\Theta_k$ ($k = \overline{1, j^*-1}$, $j^* \in J$) using the data obtained by solving the linearized problem

$$\partial x(\Theta_k|\Theta, u_w^0(\Theta))/\partial\Theta_k = A_k\Phi_{k0}x_0 + bu^0(\Theta_k - 0|\Theta) + a_k$$
$$+ \sum_{i=1}^{k} A_k\Phi_{ki}\left[r_i + \sum_{t \in T_w(i)} \int_t^{t+h} F_i(\Theta_i - \vartheta)bd\vartheta u^0(t|\Theta)\right],$$

$$\partial x(\Theta_j|\Theta, u_w^0(\Theta))/\partial\Theta_k = \Phi_{jk}(A_k - A_{k+1})\Phi_{k0}x_0 + \Phi_{jk}(a_k - a_{k+1})$$
$$+ \Phi_{jk}b[u^0(\Theta_k - 0|\Theta) - u^0(\Theta_k + 0|\Theta)]$$
$$+ \sum_{i=1}^{k} \Phi_{jk}(A_k - A_{k+1})\Phi_{ki}\left[r_i + \sum_{t \in T_w(i)} \int_t^{t+h} F_i(\Theta_i - \vartheta)d\vartheta u^0(t|\Theta)\right], \quad j > k,$$

$$\partial x(\Theta_j|\Theta, u_w^0(\Theta))/\partial\Theta_k = 0, \quad j < k.$$

In the simplest gradient-based schemes, the direction $\partial J(\Theta, u)/\partial\Theta$ for changing the instants of time is calculated, a step in this direction is chosen by using any of the variety of available methods [7–9], and a new vector $\bar\Theta = (\bar\Theta_1, \ldots, \bar\Theta_{j^*-1})$ is found. The construction of the optimal working basis $T_w^0(\bar\Theta)$ begins with a transformation of the stored data. Namely, we integrate the equations $\dot F_k = A_k F_k$ ($k \in J$) with the initial conditions $F_k(0) = F_k(\Theta_k - \Theta_{k-1})$ from 0 to $\bar\Theta_k - \bar\Theta_{k-1} - \Theta_k + \Theta_{k-1}$ to find $F_k(\bar\Theta_k - \bar\Theta_{k-1})$ ($k \in J$). Then, the matrices $F_k(\bar\Theta_k - t)$ ($t \in T_w(k)$, $k = \overline{1, j^*-1}$)) are determined by solving the equations $F_k(0) = F_k(\Theta_k - t)$ ($t \in T_w(k)$, $k = \overline{1, j^*-1}$) with the initial conditions $F_k(0) = F_k(\Theta_k - t)$ ($t \in T_w(k)$, $k = \overline{1, j^*-1}$)). The vectors $\bar r_j$ ($j \in J$) are calculated by integrating the equations $\dot r_j = A_j r_j$ with the initial conditions $r_j(0) = r_j$ ($j \in J$) from 0 to $\bar\Theta_j - \bar\Theta_{j-1} - \Theta_j + \Theta_{j-1}$ ($j \in J$). Then, we calculate the working matrix $D_w(\bar\Theta)$ corresponding to the working basis $T_w^0(\Theta)$ and to the vector $\bar\Theta$

$$D_w(\bar\Theta) = \begin{pmatrix} d(\bar\Theta_k, t), t \in T_w, \\ k \in J \end{pmatrix},$$

$$d(\bar\Theta_k, t) = H_k\Phi_{kj}(\bar\Theta)\int_t^{t+h} F_k(\bar\Theta_j - \vartheta)bd\vartheta, \quad t \in T_w(j), \quad j \in J, \quad k \in J.$$

As a result, we obtain the data required to solve the linearized problem (3.1)–(3.4) in which both $\bar{\Theta}$ and the starting working basis $T_w(\bar{\Theta}) = T_w^0(\Theta)$ are fixed. The use of the optimal working basis $T_w(\bar{\Theta})$ as a starting working basis with the vector $\bar{\Theta}$ makes it possible to quickly construct a working optimal solution $\{u^0(\cdot|\bar{\Theta}), T_w^0(\bar{\Theta})\}$. The transform $T_w(\Theta) \longrightarrow T_w^0(\Theta)$, $\Theta \longrightarrow \bar{\Theta}$, i.e., the solution of the linearized problem and the determination of $\bar{\Theta}$, will be called the big iteration of the method; the iteration of the dual method used to change the working basis $T_w(\Theta) \longrightarrow \bar{T}_w(\Theta)$ will be referred to as small.

6. REFINEMENT OF DISCRETE OPTIMAL CONTROLS TO MAKE THEM PIECEWISE CONSTANT

In the preceding sections, we described a method for constructing the optimal control for problem (2.1) in the class of discrete available controls. In that method, both transition times and switching points of the control were assumed to belong to the set T_h. Without this assumption, the optimal control in problem (2.1) is a relay function of the form

$$u^0(t) = \operatorname{sgn}\Delta(t), \quad t \in T, \tag{6.1}$$

where $\Delta(t) = \psi'(t)b$ $(t \in T)$ is the cocontrol and $\psi(t)$ $(t \in T)$ is the solution to the adjoint system

$$\psi'(t) = -A_j'\psi(t), \quad t \in T_j,$$

with the initial condition

$$\psi(\Theta_{j^*}) = c - H'v(\Theta_{j^*})$$

and the jump conditions

$$\psi(\Theta_j - 0) = \psi(\Theta_j + 0) - H_j'v(\Theta_j), \quad j \in J\backslash j^*.$$

Thus, the basic elements of optimal control (6.1) are the cocontrol zeros t_i ($\Delta(t_i) = 0$, $i = \overline{1, p}$), the instants Θ_j ($j = \overline{1, j^* - 1}$) at which the dynamic of the system changes, and the potentials $v(\Theta_j)$ ($j \in J$). The basic elements are determined by solving the system of algebraic equations

$$H_j x(\Theta_j) = g_j, \quad j \in J, \quad \Delta(t_i) = 0, \quad i = \overline{1, p},$$
$$\psi'(\Theta_j + 0)[A_{j+1}x(\Theta_j) + a_{j+1} + bu(\Theta_j + 0)] = \psi'(\Theta_j - 0)[A_j x(\Theta_j) + a_j + bu(\Theta_j - 0)], \quad j = \overline{1, j^* - 1}, \tag{6.2}$$

which were called the refinement equations in [13]. Under fairly general conditions (see [13]), the Jacobian matrix of system (6.2) is nonsingular, and the system can be solved by Newton's method. The initial approximations of the basic elements are determined by solving the problem in the class of discrete available controls.

7. THE OPTIMAL CONTROLLER

Following [14], we describe here the algorithm for an optimal controller that implements the optimal feedback (2.3) in problem (2.1) in any particular control process.

The approach developed in [14] is based on the use of an optimal feedback (2.3) in a control process for system (2.1). Assume that the optimal feedback (2.3) has already been determined and the behavior of the closed loop system is described by Eq. (2.4). The function $w(t)$ $(t \in T)$ represents the influence of the perturbations neglected in the mathematical model. When piecewise linear approximations are used to optimize the nonlinear system (1.1), the function $w(t)$ $(t \in T)$ includes, in addition to external perturbations, the deviation of the piecewise linear system (2.1) from the original nonlinear system (1.1).

Assume that a perturbation $w^*(t)$ $(t \in T)$ occurs in a certain control process of system (2.4). Driven by this perturbation and function (2.3), system (2.4) moves along a trajectory $x^*(t)$ $(t \in T)$, while the control $u^*(t) = u^0(t, x^*(t))$ $(t \in T)$ is applied to system (2.4). The function $u^*(t)$ $(t \in T)$ is called the realization of the optimal feedback in a particular control process, and the device that calculates the values of this function in real time is called the optimal controller.

In what follows, we describe the algorithm of the optimal controller.

Assume that the optimal controller algorithm has already been constructed and the controller has operated during the time interval $[0, \tau]$ and produced the controls $u^*(0), \ldots, u^*(\tau)$. At $\tau + h$, information about the state $x^*(\tau + h)$ reached by the system driven by the control and the perturbation $w^*(t)$ $(t \in [0, \tau + h[)$ is fed into the controller. The controller must calculate the control $u^*(\tau + h) = u^0(\tau + h, x^*(\tau + h))$. If the time required for this calculation does not exceed h, we say that the controller operates in real time.

At every $\tau \in T_h$, the controller has to solve problem (2.2) for the state $(\tau, x^*(\tau))$. To reformulate problem (2.2) in a parameterized form, we assume that the system's dynamic can change while its trajectory crosses a certain ε-neighborhood of the hyperplanes $H_j x = g_j$ $(j = \overline{1, j^* - 1})$. The error associated with this assumption can be included in the perturbation $w(t)$ $(t \in T)$. Thus, the parameterized form of problem (2.2) is

$$\begin{gathered} c'x(\Theta_{j^*}) \longrightarrow \max, \quad \dot{x}(t) = A_j x(t) + a_j + bu(t), \quad t \in [\Theta_{j-1}, \Theta_j[, \quad j \in J(\tau) = \{j(\tau), \ldots, j^*\}, \\ x(\Theta_{j(\tau)-1}) = x^*(\tau), \quad |H_j x(\Theta_j) - g_j| \le \varepsilon, \quad j \in J(\tau)\backslash j^*, \quad Hx(\Theta_{j^*}) = g, \\ |u(t)| \le 1, \quad t \in T(\tau), \quad \Theta_{j(\tau)-1} < \Theta_{j(\tau)} < \ldots < \Theta_{j^*-1} < \Theta_{j^*}. \end{gathered} \tag{7.1}$$

We assume that the transition times $\Theta_{j(\tau)}, \ldots, \Theta_{j^*-1}$ lie in the interval $T(\tau)$ and set $\Theta_{j(\tau)-1} = \tau$.

Assume that an optimal trajectory $x^0(t|0, x_0^*)$ $(t \in T)$ characterized by structure $X^0(0) = \{X_j^0(0), j = \overline{1, j^*}\}$ and transition times $\Theta^0(0) = (\Theta_j^0(0), j = \overline{1, j^* - 1})$ is calculated for the initial state $x(0) = x_0^*$ at $\tau = 0$. If $w^*(t) \equiv 0$ $(t \in T)$, then the structure $X^0(\tau) = \{X_j^0(\tau), j = \overline{j(\tau), j^*}\}$ of the optimal trajectory $x^0(t|\tau, x^*(\tau))$ $(t \in T(\tau))$ at any $\tau \in T_h$ is a part of the initial structure $X_j^0(\tau) = X_j^0(\tau)$ $(j = \overline{j(\tau), j^*})$; i.e., $X^0(\tau) \subset X^0(0)$. Moreover, the transition times $\Theta^0(\tau) = (\Theta_j^0(\tau), j = \overline{j(\tau), j^* - 1})$ corresponding to the remaining interval $T(\tau)$ belong to the original $\Theta_j^0(\tau) = \Theta_j^0(0)$ $(j = \overline{j(\tau), j^* - 1})$; i.e., $\Theta^0(\tau) \subset \Theta^0(0)$. A perturbation $w^*(t) \not\equiv 0$ $(t \in T)$ can induce the following changes.

(1) If $X^0(\tau) \subset X^0(0)$, then the transition times change; i.e., $\Theta^0(\tau) \not\subset \Theta^0(0)$.

(2) The optimal trajectory structure changes; i.e., $X^0(\tau) \not\subset X^0(0)$ for a certain $\tau \in T_h$.

Below, we consider case (1).

In case (2), in addition to the algorithm described below, the situations when the optimal trajectory structure changes can be analyzed by examining the behavior of $x^0(t|\tau, x^*(\tau))$ $(t \in T(\tau))$ at the interior points of $T(\tau)$ to find a new region traversed by the optimal trajectory or to reveal that the trajectory crosses a reduced number of hyperplanes. The complexity of the procedures used to detect changes in structure depends on the particular problem, and their description is beyond the scope of this paper.

At the time $\tau \in T_h$, the controller solves problem (7.1); i.e., it determines local maximizing sets $\hat{\Theta}^l(\tau) = (\hat{\Theta}_j^l(\tau), j = \overline{j(\tau), j^* - 1})$ $(l = \overline{1, l^*})$ in problem (7.1) and the corresponding optimal working bases. Then, the controller selects a global maximizing set $\Theta^0(\tau)$ from the sets $\hat{\Theta}^l(\tau)$ $(l = \overline{1, l^*})$. At $\tau + h$, the data calculated at τ are used by the controller to correct the sets $\hat{\Theta}^l(\tau)$ $(l = \overline{1, l^*})$ and the corresponding working bases, to obtain the maximizing sets $\hat{\Theta}^l(\tau + h)$ $(l = \overline{1, l^*})$, and to select an optimal set $\Theta^0(\tau + h)$ from them.

We now describe the correction of the optimal set $\hat{\Theta}(\tau)$ and the corresponding optimal working basis.

If we fix a feasible set $\Theta(\tau)$, then we obtain the linearized problem (7.1). For this problem, we define the concepts of working basis and accompanying elements [6].

Let us select the subsets $S(\tau) = \{\Theta_{j(\tau)}(\tau), \ldots, \Theta_{j^*}\}$ and $T_h(\tau) = \{\tau, \tau + h, \ldots, t^* - h\}$ from the sets $S_w = \{s \in S(\tau)\backslash\Theta_{j^*}, \Theta_{j^*}\}$ and $T_w(\tau) = \{t \in T_h(\tau)\}$, respectively, such that $|S_w(\tau)| = |T_w(\tau)|$. The set $K_w(\tau) = \{S_w(\tau), T_w(\tau)\}$ is called the working basis of the linearized problem (7.1) if the matrix

$$D_w(\tau) = \begin{pmatrix} d(s, t), t \in T_w(\tau), \\ s \in S_w(\tau) \end{pmatrix}$$

is nonsingular. Define $S_{nw}(\tau) = S(\tau)\backslash S_w(\tau)$ and $T_{nw}(\tau) = T_h(\tau)\backslash T_w(\tau)$. The working basis $K_w(\tau)$ is accompanied by the following elements.

1. Potentials $\nu(\tau) = (\nu(s|\tau), s \in S(\tau))$, $\nu_w(\tau) = (\nu_w(s|\tau), s \in S_w(\tau))$, and $\nu_{nw}(\tau) = (\nu_{nw}(s|\tau) = 0, s \in S_{nw}(\tau))$ are such that the basis components $\nu_w(\tau)$ satisfy the equation

$$D'_w(\tau)\nu_w\tau = c_w(\tau),$$

where $c_w(\tau) = c(T_w(\tau))$.

2. The cocontrol is defined as

$$\Delta(t) = \int_t^{t+h}\left[c'\Phi_{kj}F_j(\Theta_j - \vartheta) - \sum_{k\,:\,\Theta_k \in S_w(\tau),\,\Theta_k > t} \nu'(\Theta_k)H_k\Phi_{kj}F_j(\Theta_j - \vartheta)\right]b\,d\vartheta,$$

$$t \in T_{nw}(j|\tau), \quad j \in J(\tau), \quad \Delta(t) = 0, \quad t \in T_w(\tau).$$

3. A pseudocontrol $\omega(\tau) = (\omega(t|\tau), t \in T(\tau))$ and a pseudooutput $\zeta(\tau) = (\zeta(s|\tau), s \in S(\tau))$ are such that the nonworking values $\omega_w(\tau) = (\omega(t|\tau), t \in T_{nw}(\tau))$ of the pseudocontrol are defined by (4.2) and the working values $\zeta_w(\tau) = (\zeta(s|\tau), s \in S_w(\tau))$ of the pseudooutput satisfy the relations

$$\zeta(s|\tau) = g(s) - \varepsilon, \quad \text{if} \quad \nu(s|\tau) < 0; \quad \zeta(s|\tau) = g(s) + \varepsilon, \quad \text{if} \quad \nu(s|\tau) > 0;$$

$$|\zeta(s) - g(s)| \le \varepsilon, \quad \text{if} \quad \nu(s|\tau) = 0, \quad s \in S_w(\tau)\backslash\Theta_{j*}; \quad \zeta(\Theta_{j*}) = g.$$

The working components $\omega_w(\tau) = (\omega(t|\tau), t \in T_w(\tau))$ satisfy the equation

$$\sum_{t \in T_w(\tau)} d(s,t)\omega(t) + \sum_{t \in T_{nw}(\tau)} d(s,t)\omega(t) = \zeta(s) - H_k\Phi_{kj(\tau)}F_{j(\tau)}(\Theta_{j(\tau)} - \tau)x^*(\tau), \quad s = \Theta_k \in S_w(\tau).$$

The nonworking values $\zeta_{nw}(\tau) = (\zeta_{nw}(s|\tau), s \in S_{nw}(\tau))$ of the pseudooutput satisfy the equation

$$\zeta(s) = H_k\Phi_{kj(\tau)}F_{j(\tau)}(\Theta_{j(\tau)} - \tau)x^*(\tau) + \sum_{t \in T_h(\tau)} d(s,t)\omega(t), \quad s = \Theta_k \in S_{nw}(\tau).$$

At the time τ, the following data are calculated and stored for the optimal set $\hat{\Theta}(\tau)$: (1) the optimal working basis $K^0_w(\tau) = K^0_w(\hat{\Theta}(\tau)) = \{S^0_w(\tau), T^0_w(\tau)\}$; (2) the set of nonworking zeros $T_{nw0}(\tau)$; (3) the matrix $D_{|w|}(\tau) = (d(t), t \in T^0_w(\tau))$ and the vector $d(\tau)$, where $d(t) = (d(s,t), s \in S(\tau))$; (4) the matrices $F_j(\hat{\Theta}_j(\tau) - \hat{\Theta}_{j-1}(\tau))$ $(j \in J(\tau))$ and $F_j(\hat{\Theta}_j(\tau) - t)$ $(t \in T_{w.nw}(j|\tau), j \in J(\tau))$; (5) the vectors $r_j(\tau)$ $(j \in J(\tau))$; (6) the numbers

$$\gamma_j(\tau) = \begin{cases} \operatorname{sgn}\Delta(\hat{\Theta}_{j-1}(\tau)), & \hat{\Theta}_{j-1}(\tau) \notin T^0_w(\tau), \\ \operatorname{sgn}\Delta(\hat{\Theta}_{j-1}(\tau) + h), & \hat{\Theta}_{j-1}(\tau) \in T^0_w(\tau), \quad j \in J(\tau); \end{cases}$$

(7) the working values of the pseudooutput $\zeta^0_w(\tau)$; and (8) the working values of the potentials $\nu^0_w(\tau)$.

Two cases are possible at the time τ.

If $\tau \notin T^0_w(\tau)$, then the following initial values are used at $\tau + h$: $K_w(\tau + h) = K^0_w(\tau)$, $\zeta_w(\tau + h) = \zeta^0_w(\tau)$, $D_{|w|}(\tau + h) = D_{|w|}(\tau)$, and $\nu_w(\tau + h) = \nu_w(\tau)$. We also set

$$j(\tau + h) = \begin{cases} j(\tau), & \tau < \hat{\Theta}_{j(\tau)}(\tau) - h, \\ j(\tau) + 1, & \tau \ge \hat{\Theta}_{j(\tau)}(\tau) - h, \end{cases}$$

$$\gamma_{j(\tau+h)}(\tau + h) = \begin{cases} \gamma_{j(\tau+h)}(\tau), & \tau + h \notin T_{w.nw}(\tau), \\ -\gamma_{j(\tau+h)}(\tau), & \tau + h \in T_{w.nw}(\tau), \end{cases}$$

$$\gamma_j(\tau + h) = \gamma_j(\tau), \quad j = \overline{j(\tau + h) + 1, j^*},$$

$$T_{\text{nw}0}(\tau+h) = \begin{cases} T_{\text{nw}0}(\tau), & \tau+h \notin T_{\text{nw}0}(\tau), \\ T_{\text{nw}0}(\tau)\backslash\tau+h, & \tau+h \in T_{\text{nw}0}(\tau), \end{cases}$$

$$T_{\text{w.nw}}(\tau+h) = \begin{cases} T_{\text{w.nw}}(\tau)\backslash\tau, & \tau+h \in T_{\text{w.nw}}(\tau), \\ T_{\text{w.nw}}(\tau)\backslash\tau \cup \tau+h, & \tau+h \notin T_{\text{w.nw}}(\tau), \end{cases}$$

$$r_j(\tau+h) = r_j(\tau), \quad j = \overline{j(\tau+h)+1, j^*},$$

$$r_{j(\tau+h)} = \begin{cases} r_{j(\tau+h)}(\tau) - \int\limits_{\tau}^{\tau+h} F_{j(\tau+h)}(\Theta_{j(\tau+h)}(\tau+h)-\vartheta)bd\vartheta\gamma_{j(\tau)}(\tau) \\ -\int\limits_{\tau}^{\tau+h} F_{j(\tau+h)}(\Theta_{j(\tau+h)}(\tau+h)-\vartheta)d\vartheta a_{j(\tau+h)}, & \tau+h \neq \hat{\Theta}_{j(\tau)}(\tau), \\ r_{j(\tau+h)}(\tau), & \tau+h = \hat{\Theta}_{j(\tau)}(\tau). \end{cases}$$

The initial values of the transition times are set as follows: $\Theta_j(\tau+h) = \hat{\Theta}_j(\tau)$ ($j = \overline{j(\tau+h), j^*}$), $\Theta_{j(\tau+h)-1}(\tau+h) = \tau+h$. The matrices $F_{j(\tau)}(\hat{\Theta}_{j(\tau)} - \hat{\Theta}_{j(\tau)-1})$ are replaced by $F_{j(\tau+h)}(\Theta_{j(\tau+h)} - \tau - h)$.

The working values of the pseudocontrol $\omega_\text{w}(\tau+h)$ are determined by the equation

$$D_\text{w}(\tau+h)\omega_\text{w}(\tau+h) = \zeta_\text{w}(\tau+h) - p_\text{w}(\tau+h),$$

where

$$p_\text{w}(\tau+h) = \begin{pmatrix} p(\Theta_k|\tau+h), \\ \Theta_k \in S_\text{w}(\tau+h) \end{pmatrix},$$

$$p(\Theta_k|\tau+h) = \sum_{j=j(\tau+h)}^{k} H_k\Phi_{kj}r_j(\tau+h) + H_k\Phi_{k\,j(\tau+h)-1}x^*(\tau+h).$$

The nonworking values of the pseudooutput are determined by

$$\zeta(s|\tau+h) = p(s|\tau+h) + \sum_{t \in T_\text{w}(s|\tau+h)} d(s,t)\omega(t), \quad s \in S(\tau+h),$$

where $T_\text{w}(s|\tau+h) = T_\text{w}(\tau+h) \cap [\tau+h, s[$.

These data are used in the big iterations to determine the extremal set $\hat{\Theta}(\tau+h)$ and the corresponding optimal working basis $K^0_\text{w}(\hat{\Theta}(\tau+h))$ of problem (7.1) for the state $(\tau+h, x^*(\tau+h))$. Details pertaining to intermediate inequality constraints are described in [6].

Formula (5.1) takes the form

$$\frac{\partial J(\Theta(\tau+h), u)}{\partial\Theta_k} = c'\frac{\partial x(t^*|\Theta(\tau+h), u^0_\text{w}(\Theta(\tau+h)))}{\partial\Theta_k} - \nu'_\text{w}(\tau+h)\begin{pmatrix} H_j\partial x(\Theta_j|\Theta, u^0_\text{w}(\Theta))/\partial\Theta_k, \\ \Theta_j \in S_\text{w}(\tau+h) \end{pmatrix},$$

$$k = \overline{j(\tau+h), j^*-1}.$$

If the linearized problem (7.1) for the state $(\tau+h, x^*(\tau+h))$ with the initial condition $\Theta(\tau+h) = \hat{\Theta}(\tau)$

has no solutions, then we solve the M-problem

$$J(u, \Theta, \kappa) = c'x(\Theta_{j^*}) - M\left(\sum_{j=j(\tau+h)}^{j^*-1}\kappa_j + \sum_{i=1}^{\hat{m}}\kappa_{j^*(i)}\right) \longrightarrow \max,$$

$$\dot{x}(t) = A_j x(t) + a_j + bu(t), \quad t \in [\Theta_{j-1}, \Theta_j[, \quad j \in J(\tau+h), \quad x(\Theta_{j(\tau+h)-1}) = x^*(\tau+h),$$

$$|H_j x(\Theta_j) - \beta_j\kappa_j - g_j| \le \varepsilon, \quad j \in J(\tau+h)\backslash j^*, \quad H'_{(i)}x(\Theta_{j^*}) - \beta_{(i)}\kappa_{j^*(i)} = g_{(i)}, \quad i = \overline{1, \hat{m}}, \tag{7.2}$$

$$|u(t)| \le 1, \quad t \in T(\tau+h), \quad \Theta_{j(\tau+h)-1} < \Theta_{j(\tau+h)} < \dots < \Theta_{j^*-1} < \Theta_{j^*},$$

$$\kappa_j \ge 0, \quad j \in J(\tau+h)\backslash j^*, \quad \kappa_{j^*(i)} \ge 0, \quad i = \overline{1, \hat{m}},$$

where $\Theta(\tau + h)$ and $K_w(\tau + h)$ are used as the initial conditions. Here,

$$\beta_j = \begin{cases} 1, & H_j\tilde{x}(\Theta_j) > g_j + \varepsilon, \\ 0, & |H_j\tilde{x}(\Theta_j) - g_j| \le \varepsilon, \\ -1, & H_j\tilde{x}(\Theta_j) < g_j - \varepsilon, \quad j \in J(\tau+h)\backslash j^*, \end{cases}$$

$\beta_{(i)} = \operatorname{sgn}(H'_{(i)}\tilde{x}(t^*) - g_{(i)})$, $H_{(i)}$ is the ith row of the matrix H ($i = \overline{1, \hat{m}}$), $\tilde{x}(t) = \tilde{x}(t|\tau+h, x^*(\tau+h))$ ($t \in T(\tau + h)$) is the trajectory of system (7.1) generated by the quasi-control corresponding to the working basis $K_w(\tau + h)$

$$\tilde{x}(\Theta_k) = \Phi_{k\,j(\tau+h)-1}x^*(\tau+h)$$
$$+ \sum_{j=j(\tau+h)}\left[\Phi_{kj}r_j(\tau+h) + \sum_{t \in T_w(j|\tau+h)}\Phi_{kj}\int_t^{t+h}F_j(\Theta_j(\tau+h)-\vartheta)bd\vartheta\operatorname{sgn}\omega(t|\tau+h)\right],$$

and M is an arbitrarily large number.

If the optimal working basis of the M-problem contains the indices of the variables κ, then problem (7.1) has no solutions for the state $(\tau + h, x^*(\tau + h))$. Otherwise, the extremal sets and the corresponding optimal working bases of problems (7.1) and (7.2) are identical.

The controller singles out an optimal set $\Theta^0(\tau + h)$ among the extremal sets $\hat{\Theta}^l(\tau + h)$ ($l = \overline{1, l^*}$) and feeds the control

$$u^*(\tau+h) = \begin{cases} \gamma^0_{j(\tau+h)}(\tau+h), & \tau+h \notin T^0_w(\tau+h), \\ \omega^0(\tau+h), & \tau+h \in T^0_w(\tau+h) \end{cases}$$

as input into system (2.4).

When $\tau \in T^0_w(\tau)$, $\tau = \hat{\Theta}_{j(\tau)} - h$, and $s = \hat{\Theta}_{j(\tau)} \in S_w(\tau)$, we construct the initial working basis $K_w(\tau + h)$ as $T_w(\tau + h) = T^0_w(\tau)\backslash\tau$, $S_w(\tau + h) = S^0_w(\tau)\backslash s_{j(\tau)}$. Then, we calculate $\nu_w(\tau + h)$ and proceed as if $\tau \notin T^0_w(\tau)$.

If $\tau \in T^0_w(\tau)$ and the conditions $\tau = \hat{\Theta}_{j(\tau)} - h$ and $\hat{\Theta}_{j(\tau)} \in S_w(\tau)$ are not satisfied, we construct the initial basis $K_w(\tau + h)$ by eliminating the instant of time τ from $T^0_w(\tau)$ or by solving the auxiliary problem

$$c'x(\Theta_{j^*}) \longrightarrow \max, \quad \dot{x}(t) = A_jx(t) + a_j + bu(t), \quad t \in [\Theta_{j-1}, \Theta_j[, \quad j \in J(\tau+h),$$

$$x(\Theta_{j(\tau+h)-1}) = x^*(\tau+h), \quad |H_jx(\Theta_j) + d(\hat{\Theta}_j(\tau), \tau)u(\tau) - g_j - d(\hat{\Theta}_j(\tau), \tau)u^*(\tau)| \le \varepsilon, \quad j \in J(\tau+h)\backslash j^*,$$

$$Hx(\Theta_{j^*}) + d(\Theta_{j^*}, \tau)u(\tau) = g + d(\Theta_{j^*}, \tau)u^*(\tau), \quad u^*(\tau) \le u(\tau) \le u^*(\tau),$$

$$|u(t)| \le 1, \quad t \in T(\tau+h), \quad \Theta_{j(\tau+h)-1} < \Theta_{j(\tau+h)} < \dots < \Theta_{j^*-1} < \Theta_{j^*}$$

or problem (7.2).

8. EXAMPLES

We illustrate the results obtained by considering two examples of control in a piecewise linear model of an oscillatory system. The first example is a special case of problem (2.1) with a free right end of the trajectory; i.e., $H = 0$ and $g = 0$. For a linear model $f(x) = Ax$, $A \in \mathbb{R}^{n \times n}$, this problem is trivial; indeed, it is sufficient to integrate the adjoint system $\dot\psi = -A'\psi$, $\psi(t^*) = c$ and set $u(t) = \operatorname{sgn}\psi'(t)b$ $(t \in T)$. In the case of a piecewise linear model, it becomes nontrivial.

Example 1. Consider the frictionless motion of a one-mass oscillatory system along a horizontal line. On different parts of the line, the system is driven by forces exerted by different elastic elements (springs) (see Fig. 1). We seek a control that maximizes the velocity gained by the mass in a given time.

The mathematical model of the system is formulated as follows:

$$\begin{gathered}\dot x(t^*) \longrightarrow \max, \quad \ddot x + k_1 x = u, \text{ if } x \ge -\alpha, \text{ and } \ddot x + k_1 x + k_2(x+\alpha) = u, \text{ if } x < -\alpha, \\ x(0) = 0, \quad \dot x(0) = 0, \quad |u(t)| \le 1, \quad t \in T = [0, t^*],\end{gathered} \tag{8.1}$$

where $x = x(t)$ is the deviation of the mass from the equilibrium point $x = 0$ at the instant t, $u = u(t)$ is the control (force), and α is the distance from the equilibrium point to the right end of the second spring.

Changing to the phase variables $x_1 = x$, $x_2 = \dot x$, we obtain problem (2.1) with the parameters $n = 2$, $x = (x_1, x_2)$, $\hat m = 0$, $x_0 = (0, 0)$, and $b = \begin{pmatrix}0\\1\end{pmatrix}$:

$$\hat f(x) = \begin{cases} x_2 \\ -k_1 x_1, \quad x \in X_1 = \{x \in \mathbb{R}^2 : x_1 \ge -\alpha\}; \end{cases}$$

$$\hat f(x) = \begin{cases} x_2 \\ -k_1 x_1 - k_2(x_1 + \alpha), \quad x \in X_2 = \{x \in \mathbb{R}^2 : x_1 \ge -\alpha\}. \end{cases}$$

The problem was solved for $t^* = 4$, $\alpha = 0.5$, $k_1 = 1$, $k_2 = 2$, and $h = 0.04$.

For these values of the parameters, problem (8.1) is stated in parameterized form as

$$\begin{gathered} J(\Theta, u) = x_2(4) \longrightarrow \max, \quad \dot x_1 = x_2, \quad \dot x_2 = -x_1 + u, \quad x_1(0) = x_2(0) = 0, \quad t \in [0, \Theta_1[, \\ x_1(\Theta_1) = -0.5, \quad \dot x_1 = x_2, \quad \dot x_2 = -3x_1 - 1 + u, \quad t \in [\Theta_1, \Theta_2[, \quad x_1(\Theta_2) = -0.5, \\ \dot x_1 = x_2, \quad \dot x_2 = -x_1 + u, \quad t \in [\Theta_2, 4[, \quad |u(t)| \le 1, \quad t \in T = [0, 4]. \end{gathered} \tag{8.2}$$

The optimal values of transition times are $\Theta^0 = (1.76, 3.44)$. The corresponding optimal value of the objective functional is $J^0(\Theta^0) = 2.489546$. To solve problem (8.1), we used the initial value $\Theta = (1.2, 2.4)$, for

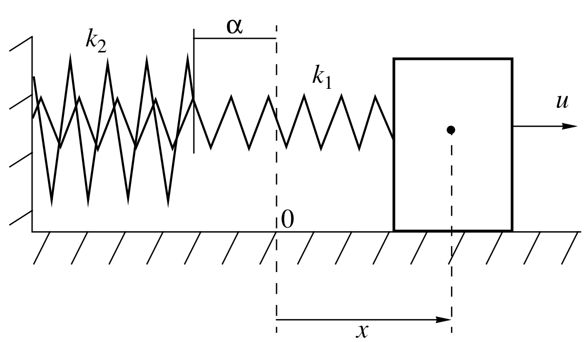

Fig. 1.

which $J^0(\Theta) = 1.508788$. The problem was solved in 41 big iteration steps; at each step, the vector $\bar{\Theta}$ was calculated by the formula $\bar{\Theta} = \Theta + h\Delta\Theta$, where

$$\Delta\Theta = \begin{cases} \begin{pmatrix} \operatorname{sgn}[\partial J^0(\Theta)/\partial\Theta_1] \\ 0 \end{pmatrix}, & |\partial J^0(\Theta)/\partial\Theta_1| > |\partial J^0(\Theta)/\partial\Theta_2| \\ \begin{pmatrix} 0 \\ \operatorname{sgn}[\partial J^0(\Theta)/\partial\Theta_2] \end{pmatrix}, & |\partial J^0(\Theta)/\partial\Theta_1| < |\partial J^0(\Theta)/\partial\Theta_2|. \end{cases}$$

Figure 2 shows the current values of Θ calculated in the course of iterations and the corresponding working instants $T_w^0(\Theta) = \{t_1^0(\Theta), t_2^0(\Theta)\}$. Figure 3a shows the optimal open loop control $u^0(t)$ ($t \in T$) in problem (8.2). (We did not seek the most efficient method of parametric optimization; however, Fig. 2 suggests that the number of big iteration steps can be considerably reduced.)

In [5], a methodology was proposed for estimating the efficiency of solution algorithms for optimal control problems, with the time required to integrate the direct or dual system over the interval T taken as the unit of cost. It was assumed that the number of available processors is sufficient for integrating the required number of equations simultaneously. It was shown in [6] that the cost of the preliminary construction of a working matrix and elements accompanying the working basis equals 2. In terms of the algorithm proposed, the solution to a linear optimal control problem subject to intermediate phase constraints was constructed in a single big iteration step, and the corresponding cost was estimated as the total displacement p of all movable zeros of the cocontrol in the process of small iterations. Thus, the total cost of all iterations required to solve the problem considered in [6] is p. In this study, the cost of the iterations performed to solve problem (2.1) is

$$2k/N + \sum_{i=1}^{k}\left(p^i + \max_{j = \overline{1, j^*-1}}\{\bar{\Theta}_j^i - \Theta_j^i, \bar{\Theta}_j^i - \Theta_j^i - \bar{\Theta}_{j-1}^i + \Theta_{j-1}^i\}/t^*\right),$$

where k is the number of big iteration steps, p^i is the total displacement of all movable zeros at the ith big iteration step, and Θ_j^i and $\bar{\Theta}_j^i$ ($j = \overline{1, j^*-1}$) are the instants of transitions between the domains of linearity at the beginning and at the end of the ith big iteration step. In the example considered here, the cost of the construction of the optimal open loop control is 2.46. Thus, notwithstanding the large number of big iterations, the open loop optimization cost is the time $4 \times 2.46 = 9.84$ required to integrate the oscillatory system

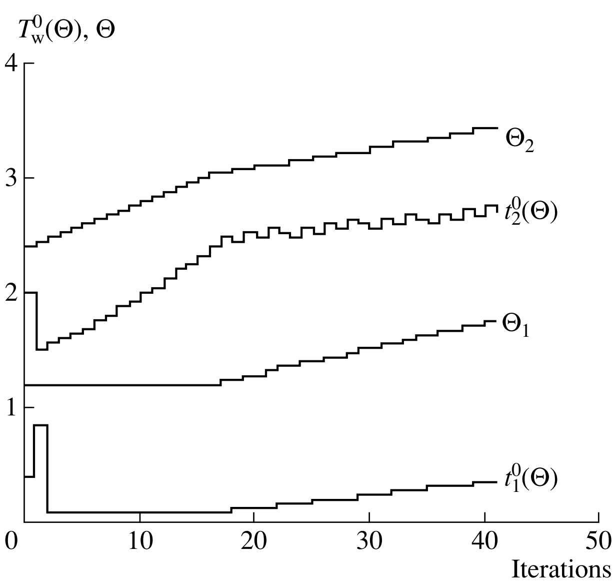

Fig. 2.

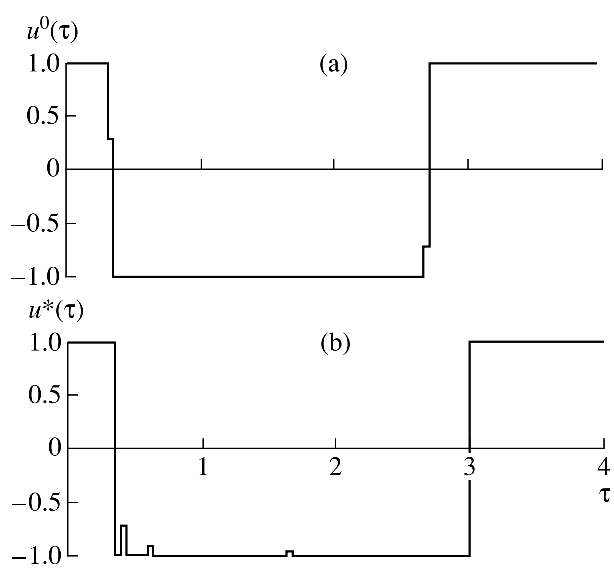

Fig. 3.

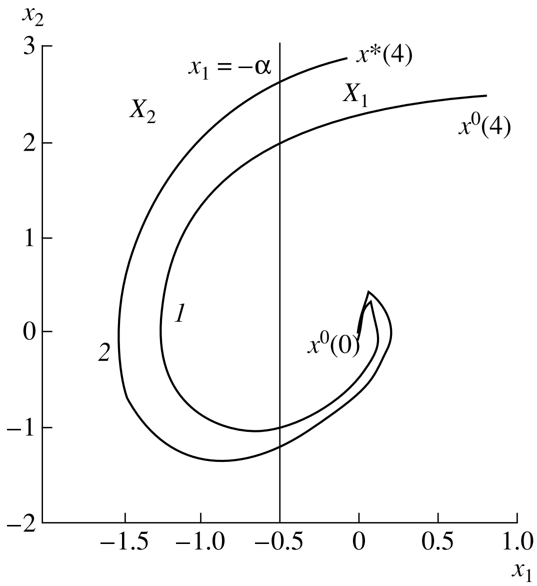

Fig. 4.

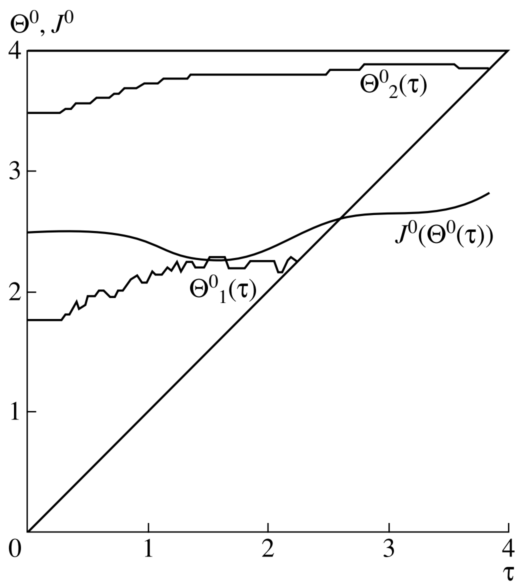

Fig. 5.

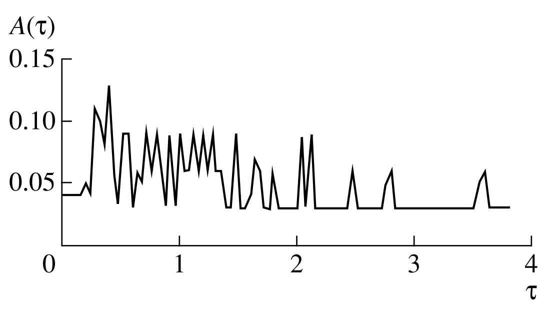

Fig. 6.

over the interval. In other words, the cost of an iteration step in the proposed method is as low as $9.84/41 \approx 0.2$ multiplied by the time required to integrate the system over an interval of length 4.

An optimal closed loop control was constructed in problem (8.2) in the case when the system is subjected to the perturbation $w^*(t) = 0.5\sin(2t)$ $(t \in T)$, which is not perceived by the controller. The dynamics of the system are described by the equations

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \begin{cases} -x_1 + u + w^*(t), & x_1 \geq -0.5, \\ -3x_1 - 1 + u + w^*(t), & x_1 < -0.5. \end{cases} \quad (8.3)$$

The optimal closed loop control $u^*(t)$ $(t \in T)$ produced by the controller is shown in Fig. 3b. Figure 4 shows the phase trajectories of systems (8.2) and (8.3): curve *1* represents the trajectory of system (8.2) generated by the optimal open loop control, and curve *2* is the trajectory of system (8.3) generated by the control $u^*(t)$ $(t \in T)$. Figure 5 illustrates the changes in the optimal transition instants $\Theta^0(\tau)$ $(\tau \in T_h)$ and the corresponding optimal value of the objective functional $J^0(\Theta^0(\tau)) = \max c'x^0(t^*|\tau, x^*(\tau))$ over the interval of control. Figure 6 shows the cost $A(\tau)$ of changing from $\Theta^0(\tau) \longrightarrow \Theta^0(\tau + h)$ and from $K^0_w(\tau)$ to $K^0_w(\tau + h)$. The graph demonstrates that the cost of calculation of the current values of the optimal feedback at any $t \in T_h$ did not exceed 0.13. This means that if the processor time τ^0 required to integrate system (8.1) over an interval length of 4 satisfies the inequality $0.13\tau^0 \leq h = 0.04$ (i.e., $\tau^0 = 0.3$ times the unit of time used in the statement of problem (8.1)), then the optimal controller

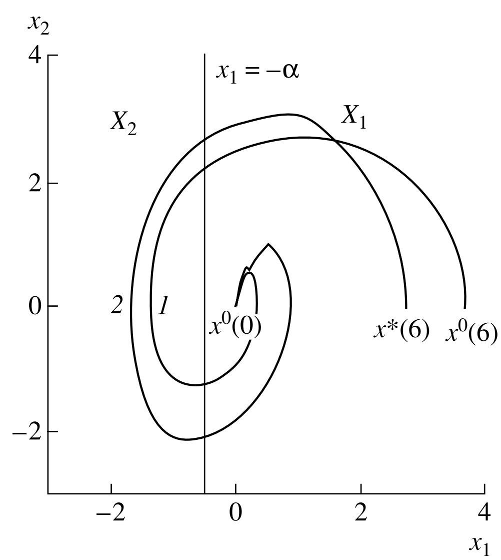

Fig. 7.

algorithm can be implemented. If the time in (8.1) is measured in seconds, then $\tau^0 = 0.3$ s, which is more than sufficient for modern processors to integrate system (8.1) over the time interval [0, 4].

Example 2. Now, we seek a bounded control required to displace the mass in the oscillatory system defined in Example 1 rightwards to a maximal distance and stops it in a given time. Setting $t^* = 6$, $\alpha = 0.5$, $k_1 = 1$, $k_2 = 2$, and $h = 0.06$, we write the problem in parameterized form as

$$\begin{gathered} J(\Theta, u) = x_1(6) \longrightarrow \max, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad x_1(0) = x_2(0) = 0, \quad t \in [0, \Theta_1[, \\ x_1(\Theta_1) = -0.5, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -3x_1 - 1 + u, \quad t \in [\Theta_1, \Theta_2[, \quad x_1(\Theta_2) = -0.5, \\ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad t \in [\Theta_2, 6[, \quad x_2(6) = 0, \quad |u(t)| \le t, \quad t \in T = [0, 6]. \end{gathered} \tag{8.4}$$

To construct an optimal open loop control, we began with $\Theta = (3, 4.5)$, for which $J^0(\Theta) = 2.481625$. Twenty-two big iterations were executed to find the optimal transition instants $\Theta^0 = (2.28, 3.96)$, the optimal working basis $T^0_{\mathrm{w}}(\Theta^0) = \{0.66, 3.24, 5.88\}$, and the optimal value of the objective functional $J^0(\Theta^0) = 3.656788$. Curve *1* in Fig. 7 represents the optimal phase trajectory of problem (8.4). The cost of the iterations executed to construct the optimal open loop control was 1.98.

The realization $u^*(t)$ $(t \in T)$ of the optimal closed loop control in system (8.3) was constructed for

$$w^*(t) = \begin{cases} 0.5\sin(2t), & t \in [0, 3.6[, \\ 0, & t \ge 3.6. \end{cases}$$

The phase trajectory of system (8.3) corresponding to the control $u^*(t)$ $(t \in T)$ is represented by curve *2* in Fig. 7. The cost of the operations executed by the optimal controller at each instant of control did not exceed 0.21.

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