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Inversion of multidimensional Laplace transforms—analytical and numerical techniques

Vinayagamoorthy, Manickavasagar, Ph.D.

Iowa State University, 1990

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Inversion of multi-dimensional Laplace transforms–analytical and numerical techniques

by

Manickavasagar Vinayagamoorthy

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
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Iowa State University
Ames, Iowa
1990

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GENERAL INTRODUCTION

This dissertation consists of two major parts. The first part deals with theoretical aspects of multi-dimensional Laplace transforms and the second part with numerical aspects. Even though these two parts have overlapping features, they are written in such a manner that they can be read independently. The difficulties in obtaining inverses of Laplace transforms that appear in Physics, Engineering and other applicable sciences lead to continued efforts in expanding the transform tables and in designing algorithms for generating new inverses from the known results. In an attempt to generate new inverses of multi-dimensional Laplace transforms, in Part 1 we use operational techniques to obtain several results. These results can be used to find the inverses of multi-dimensional Laplace transforms from the known one-dimensional results. There are six subdivisions to Part 1. First three are short sections review the literature, state the objective and explain the notations and special functions used of Part 1. In Preliminaries, we describe the definitions, regions of convergence, properties and related theorems in multi-dimensional Laplace transforms. The major results of Part 1 are given in the rest of this part. In the New Results section we obtain several results in multi-dimensional Laplace transforms which are useful in generating new multi-dimensional transform pairs (i.e., forward and inverse transforms as a pair). These results are presented in six theorems. Proofs of the first three theorems are given. The rest of the theorems can be proved in a similar manner. Several examples are given to show how these results can be used to obtain new Laplace transform pairs. In the Further Results section we present two theorems with the proof for one of them. The proof of other theorem is similar and hence it is omitted. In proving these theorems we use mathematical induction along with operational techniques. We also illustrate the procedure of obtaining new inverses of multi-dimensional Laplace transforms by using the above results with some examples.

Part II is devoted to the study of the numerical inversion of Laplace transforms. There are four subdivisions in Part II. The first two subdivisions introduce the related topics to the numerical inversion of Laplace transforms and state the the objective of Part II. The numerical inversion of Laplace transforms is an unstable process. Due to this instability it is impossible to find a universal algorithm to invert the Laplace transform numerically. But there are numerous algorithms of inversion of Laplace transforms, which are efficient with restricted classes of functions, in the existing literature. We will study the major classification of these methods based on theoretical considerations in the section, Methods for Numerically Inverting One-Dimensional Laplace Transforms. We also describe some interesting methods within these classes. Numerical examples are provided to illustrate the accuracy of some of the methods which are of our primary interest. Even though there are many numerical inversion methods only a handful of them deal with multi-dimensional inversion. So we pay more attention to the numerical inversion of multi-dimensional Laplace transforms. In the section, Numerical Inversion of Two-Dimensional Laplace transforms, we develop two new methods to invert two-dimensional Laplace transforms numerically. These methods are the extensions of some of the methods described in one-dimensional case. The first method is based on the expansion of the inverse function in terms of products of Laguerre polynomials. We also describe the generalization the above method by using generalized Laguerre polynomials. The other is based on the Fourier series representation of the inverse integral. Numerical examples are given to illustrate the effectiveness of these methods and the results are compared with the results of Singhal and Vlach method of numerical inversion of multi-dimensional Laplace transforms.

PART I.

INVERSION OF MULTI-DIMENSIONAL LAPLACE TRANSFORMS USING ANALYTICAL TECHNIQUES

LITERATURE REVIEW

The origin of the Laplace transform or Laplace integral goes back to early nineteenth century and it is credited to Poisson. The Laplace integral, like power series and Fourier series was originally investigated in the pursuit of purely mathematical aims and it was subsequently used in several branches of sciences. The clear physical meaning of the Laplace transform contributed to the widespread use of this method.

Enormous amounts of effort have been put into the growth of the theory of Laplace transforms in the last hundred years. During the period from 1892 to 1922, Heaviside contributed numerous papers to this field. Later, the work of Heaviside was named as operational Calculus. His successful application of the Laplace transform to physical problems generated considerable interest among the Physicists and Engineers for this method. Unfortunately, mathematicians of that period did not approve of Heaviside's work due to its lack of mathematical rigor. The credit for pioneering the modern concepts of the Laplace transform is attributed to Bromwich [[2],[3], etc.] and Wagner [49] who constructed the Mathematical foundation and to a certain extent justified Heaviside's work on Operational Calculus. Later writers like Carson, Bush, Humbert, Doetsch, Droste, McLachlan and Widder contributed their share of work to the modern treatment of the Laplace transform.

The natural curiosity of mathematicians to generalize this idea brought the theory of Laplace transforms in two or more variables into existence. According to Ditkin and Prudnikov [20], short notes on the Operational Calculus in two variables based on the two-dimensional Laplace transforms were published during the 1930s. Deleure and Doetsch have successfully applied the methods of Operational Calculus in several variables to the solution of Differential Equations by using the knowledge of Special functions, etc. Indian and European mathematicians and Physicists such as Shastri, Dawan, Varma, Doetsch, Voelker, Humbert, Lubbock, Bansal and many others contributed numerous papers to this topic. Among the recent authors, Dahiya[[9],[10], etc.,] has been one of the major contributor to the field of multi-dimensional Laplace transforms. More recently, many articles appeared in journals of heat and mass transfer using this method.

OBJECTIVE

The objective of this part of the dissertation is to obtain some new results in Multi-Dimensional Laplace transforms, using operational techniques. This part consists of six subdivisions. First three of these subdivisions are short and they review the literature, state the objective and describe the notation of Part I. In 'Preliminaries' we describe the definitions, regions of convergence and some important properties of multi-dimensional Laplace transforms. We also include some important theorems, which are useful for the development of our theory in the rest of Part 1.

In New Results section, we derive some results in multi-dimensional Laplace transforms and use them to obtain new inverses of many functions of Laplace transforms in multi-dimensions. We begin with the one-dimensional Laplace transform with the known inverse function and use operational techniques to obtain multi-dimensional transform pairs. Examples are given to demonstrate the procedure of obtaining multi-dimensional transform pairs using these results.

In Further Results section, we use operational techniques along with mathematical induction to obtain some results in multi-dimensional Laplace transforms. We also present some examples to show the validity of the results.

NOTATIONS AND SPECIAL FUNCTIONS

Let us begin with the description of the notation we will use in this part of the dissertation:

For any real or complex n-dimensional variable $t = (t_1, t_2, \ldots, t_n)$, we denote $t^\nu = (t_1^\nu, t_2^\nu, \ldots, t_n^\nu)$, for any real exponent ν.

Let $p_k(t)$ be the kth symmetric polynomial in the components t_k of t. Then for $t = (t_1, t_2, .., t_n), s = (s_1, s_2, .., s_n)$

(a) $p_1(t^\nu) = t_1^\nu + \ldots + t_n^\nu = \sum_{j=1}^n t_j^\nu$

(b) $p_2(t^\nu) = \sum_{\substack{i,j=1 \\ i<j}}^n t_i^\nu t_j^\nu$

(c) $p_n(t^\nu) = \prod_{j=1}^n t_j^\nu$ and $p_0(t) = 1$, etc.

Also we shall write $st = \sum_{j=1}^n s_j t_j$ and $p_n(dt) = dt_1 dt_2 \ldots dt_n$.

We give a list of all special functions used in Part 1.

Parabolic Cylinder Function:

$$D_\nu(x) = \frac{2^{(\frac{\nu}{2} + \frac{1}{4})}}{\sqrt{x}} W_{\frac{\nu}{2}+\frac{1}{4},\frac{1}{4}}(\frac{x^2}{2})$$

Error Function:

$$Erf(x) = 2\pi^{-\frac{1}{2}} \int_0^x exp(-u^2)\, du$$

Complementary Error Function:

$$Erfc(x) = 2\pi^{-\frac{1}{2}} \int_x^\infty exp(-u^2)\,du$$

Hypergeometric Function:

$$pFq\left[(a);(b);x\right] = \sum_{n=0}^{\infty} \frac{(a_1)n \ldots (a_p)n}{(b_1)n \ldots (b_q)n} \cdot \frac{x^n}{n!}$$

Struve Function:

$$H_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{2})^{\nu+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}$$

Bessel Function:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

Modified Bessel Function:

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

Whittaker Function:

$$M_{\mu,\nu}(x) = x^{\nu+\frac{1}{2}} exp(-\frac{x}{2})\ {}_1F_1\left[\frac{1}{2}+\nu-\mu; 2\nu+1; x\right]$$

Gamma Function:

$$\Gamma(x) = \int_0^\infty u^{x-1} exp(-u)\,du$$

Logarithmic Derivative of the Gamma Function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

PRELIMINARIES

Let R be the region given by

$$R = \{x = (x_1, x_2, \ldots, x_n) | 0 \leq x_1 < \infty, 0 \leq x_2 < \infty, \ldots, 0 \leq x_n < \infty\}.$$

Let $f(x)$ be a function defined on R and integrable in the sense of Lebesque over an arbitrary finite n-dimensional box, $R_X(0 \leq x_1 < X_1, 0 \leq x_2 < X_2, \ldots, 0 \leq x_n < X_n)$.

Consider the n-dimensional integral

$$F(\omega, X) = \int_0^{X_1} \cdots \int_0^{X_n} e^{-\omega x} f(x) p_n(dx), \tag{I.1}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ with $\omega_j = (\sigma_j + i\mu_j)$ for $j = 1, 2, \ldots, n$ and $X = (X_1, X_2, \ldots, X_n)$.

Let S be the class of functions $f(x)$, such that the following conditions are satisfied at least at one point $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$:

1. The integral in (I.1) is bounded at least at the point $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ with respect to the variables $X_1, X_2, \ldots, X_n$. That is,

$$|F(\omega; X)| < M(\omega) \text{ for all } X_1 \geq 0, X_2 \geq 0, \ldots, X_n \geq 0,$$

where $M(\omega)$ is a positive constant independent of $X_1, X_2, \ldots, X_n$.

2. At the point ω,

$$\lim_{X_1, X_2, \ldots, X_n \to \infty} F(\omega; X) = F(\omega)$$

exists and we denote this limit by

$$F(\omega) = L[f(x); \omega] = \int_0^\infty \cdots \int_0^\infty e^{-\omega x} f(x) p_n(dx). \tag{I.2}$$

The integral in (I.2) is called the n-dimensional Laplace transform or n-dimensional Laplace integral of the function $f(x)$, where $x = (x_1, x_2, \ldots, x_n)$. If the conditions 1 and 2 are satisfied simultaneously, then we say that the integral in (I.2) converges boundedly at least for one point $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Thus the class S consists of functions for which the integral in (I.2) converges boundedly at least for one point ω.

Lemma 1 *Let $h = (h_1, h_2, \ldots, h_n)$ be some n-vector with $h_i > 0$ for $i = 1, 2, \ldots, n$ and M be a positive constant. If the function $f(x)$ satisfies the condition*

$|f(x)| \leq Me^{hx}$, where $x = (x_1, x_2, \ldots, x_n)$ then $f(x)$ belongs to the class S at all points $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ for which $\Re\omega_1 > h_1, \Re\omega_2 > h_2, \ldots, \Re\omega_n > h_n$.

Lemma 2 *If the integral in (I.2) converges boundedly at the point $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, then it converges boundedly at all points $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ for which $\Re(\omega_j - \mu_j) > 0$, for $j = 1, 2, \ldots, n$*

The inversion formula for the Laplace transform in n-dimensions can be written as follows:

$$f(x) = \frac{1}{(2\pi i)^n} \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_n - i\infty}^{c_n + i\infty} F(\omega) e^{\omega x} p_n(d\omega), \tag{I.3}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, $x = (x_1, x_2, \ldots, x_n)$ and $-\pi < \arg\omega_1, \arg\omega_2, \ldots, \arg\omega_n < \pi$. The function $F(\omega)$ is assumed to be analytic in the region

$$\{\omega = (\omega_1, \omega_2, \ldots, \omega_n) | \Re\omega_j > \alpha_j \text{ and } c_j > \alpha_j, \text{ for } j = 1, 2, \ldots, n\}.$$

In the contrast to the one-dimensional case, it can be shown that for an arbitrary function $f(x)$ the convergence of the integral in (I.2) does not imply the convergence of the definite integral in (I.1). This will not be the case if the former is assumed to be absolutely convergent. That is, if

$$\lim_{X_1, X_2, \ldots, X_n \to \infty} \int_0^{X_1} \cdots \int_0^{X_n} |e^{-\omega x} f(x)| p_n(dx)$$
$$= \int_0^{\infty} \cdots \int_0^{\infty} e^{-\sigma x} |f(x)| p_n(dx),$$

where $\sigma = (\Re\omega_1, \Re\omega_2, \ldots, \Re\omega_n)$. Therefore by restricting the integral in (I.2) to be absolutely convergent we can assure the convergence of the part integral in (I.1) for all $X_1, X_2, \ldots, X_n \geq 0$. With the above remark, we can write the the following theorems:

Theorem 1 *Let $h = (h_1, h_2, \ldots, h_n)$ be some n-vector with $h_i > 0$ for $for 1, 2, \ldots, n$ and M be a positive constant. If the function $f(x)$ satisfies the condition $|f(x)| \leq Me^{hx}$, where $x = (x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \geq 0$, then the integral in (I.2) converges absolutely at all points $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ for which $\Re\omega_1 > h_1, \Re\omega_2 > h_2, \ldots, \Re\omega_n > h_n$. Furthermore, $|F(\omega)| \leq \frac{M}{p_n(\sigma - h)}$, where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\Re\omega_j = \sigma_j$ for $j = 1, 2, \ldots, n$.*

Theorem 2 *If the integral in (I.2) converges absolutely at the point*

$\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, then it converges absolutely at all points $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ for which $\Re(\omega_j - \mu_j) > 0, for j = 1, 2, \ldots, n$.

NOTE: Absolute convergence of the integral (I.2) at μ implies the bounded convergence at all points for which $\Re(\omega_j - \mu_j) \geq 0$ for $j = 1, 2, \ldots, n$. In proving theorems in Part I, we use the process of the change in the order of integrations of multiple integrals and their conversion to repeated integrals. For the justification of this process we use the following theorem due to Fubini as, given by Royden [38].

Theorem 3 (Fubini) *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces and integrable on $X \times Y$. Then*

i. *for almost all x the function f_x defined by $f_x(y) = f(x, y)$ is an integrable function on Y;*

i'. *for almost all y the function f_y defined by $f_y(x) = f(x, y)$ is an integrable function on X;*

ii. *$\int_Y f(x, y)\, d\nu(y)$ is an integrable function on X;*

ii'. *$\int_X f(x, y)\, d\mu(x)$ is an integrable function of Y;*

iii. *$\int_X [\int_Y f d\nu] d\mu = \int_{X \times Y} f\, d(\mu \times \nu) = \int_Y [\int_X f\, d\mu] d\nu$.*

In the rest of Part I, we will use the following Laplace-Carson transform instead of the Laplace transform. The theory for the Laplace-Carson transform is essentially the same as the Laplace transform (see [20], [46]). Therefore, we will not distinguish one from the other. We write

$$F(\omega) = p_n(\omega) \int_0^\infty \cdots \int_0^\infty e^{-\omega x} f(x) p_n(dx),$$

as the definition of the Laplace-Carson transform and symbolically we denote the pair $F(\omega)$ and $f(x)$ with the relation

$$F(\omega) \stackrel{n}{=} f(x) \;\text{ or }\; f(x) \stackrel{n}{=} F(\omega).$$

For instance, if $n = 1$ then

$$F(\omega) = \omega \int_0^{\infty} e^{-\omega x} f(x)\, dx$$

and symbolically it can be written as

$$F(\omega) \stackrel{1}{=} f(x) \;\text{ or }\; f(x) \stackrel{1}{=} F(\omega).$$

We call $f(x)$ the determining function (or original function) and $F(\omega)$ the generating function(or image function).

SOME NEW RESULTS ON MULTI-DIMENSIONAL LAPLACE TRANSFORMS

The goal of this chapter is to derive some useful results in multi-dimensional Laplace transforms which in turn can be used to obtain inverses of many functions. This is done with a clever manipulation of the results from one-dimensional Laplace transforms. We use several examples to demonstrate ways to obtain new multi-dimensional transform pairs by using the results of this chapter. This chapter consists of six theorems with several examples.

Theorem I.1 Let $n \geq 1$ be the dimension. Let $f(x), F(x), x^{-1}F(x)$ and $x^{\frac{j-2}{2}} f(\sqrt{x})$, for $j = 0,1,2,3,n,n+1,n+2,n+3$, be continuous and absolutely integrable in $(0,\infty)$ (absolute integrability condition may be relaxed). Also, let

$$(i) \quad \varphi(\omega) \stackrel{1}{=} f(x).$$

$$(ii) \quad \varphi(\sqrt{\omega}) \stackrel{1}{=} F(x).$$

$$(iii) \quad \xi(\omega) \stackrel{1}{=} x^{-1} F(x).$$

$$(iv) \quad G_j(\omega) \stackrel{1}{=} x^{\frac{j-2}{2}} f(\sqrt{x}),$$

for $j = 0,1,2,3,n,n+1,n+2$ and $n+3$.

Then the following results hold.

(a)

$$p_n(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \xi(p_1(s^{\frac{1}{2}})^2) \stackrel{n}{=} 4\pi^{-\frac{n}{2}}.$$

$$pn(t^{-\frac{1}{2}})p_1(t^{-1})^{-1}\,G_0\left(\frac{1}{4}\,p_1(t^{-1})\right),$$

(b)

$$pn(s)\,p_1(s^{\frac{1}{2}})^{-2}\,\xi(p_1(s^{\frac{1}{2}})^2) \overset{n}{=} 2^{-n+2}\,\pi^{-\frac{n}{2}}.$$
$$pn(t^{-\frac{3}{2}})\,p_1(t^{-1})^{-1}G_n\left(\frac{1}{4}p_1(t^{-1})\right),$$

(c)

$$s_j^{\frac{1}{2}}\,pn(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})^2\,\xi(p_1(s^{\frac{1}{2}})^2) \overset{n}{=} 2\pi^{-\frac{n}{2}}\,t_j^{-1}.$$
$$pn(t^{-\frac{1}{2}})p_1(t^{-1})^{-1}\,G_1\left(\frac{1}{4}p_1(t^{-1})\right),$$

for $j = 1, 2, \ldots, n$.

(d)

$$s_j\,pn(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})^{-2}\xi(p_1(s^{\frac{1}{2}})^2) \overset{n}{=} \pi^{-\frac{n}{2}}\,pn(t^{-\frac{1}{2}}).$$
$$p_1(t^{-1})^{-1}\,t_j^{-2}\left[G_2\left(\frac{1}{4}p_1(t^{-1})\right) - 2t_j\,G_0\left(\frac{1}{4}p_1(t^{-1})\right)\right],$$

for $j = 1, 2, \ldots, n$.

(e)

$$s_j^{\frac{3}{2}}\,pn(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})^{-2}\xi(p_1(s^{\frac{1}{2}})^2 \overset{n}{=} 2^{-1}\pi^{-\frac{n}{2}}.$$
$$pn(t^{-\frac{1}{2}})\,p_1(t^{-1})^{-1}\,t_j^{-3}\left[G_2\left(\frac{1}{4}p_1(t^{-1})\right) - 2t_jG_0\left(\frac{1}{4}p_1(t^{-1})\right)\right],$$

for $j = 1, 2, \ldots, n$.

(f)

$$s_j^{\frac{1}{2}}\,pn(s)\,p_1(s^{\frac{1}{2}})^{-2}\,\xi(p_1(s^{\frac{1}{2}})^2) \overset{n}{=} 2^{-n+1}\,\pi^{-\frac{n}{2}}\,pn(t^{-\frac{3}{2}}).$$

$$p_1(t^{-1})^{-1}\, t_j^{-2} \left[G_{n+2}\left(\frac{1}{4}p_1(t^{-1})\right) - 2t_j G_n \left(\frac{1}{4}p_1(t^{-1})\right)\right],$$
for $j = 1, 2, \ldots, n$.

(g)

$$s_j p_n(s)\, p_1(s^{\frac{1}{2}})^{-2}\, \xi(p_1(s^{\frac{1}{2}})^2) \overset{n}{=} 2^{-n}\, \pi^{-\frac{n}{2}}\, p_n(t^{-\frac{3}{2}}) \cdot$$
$$p_1(t^{-1})^{-1}\, t_j^{-1} \left[G_{n+3}\left(\frac{1}{4}p_1(t^{-1})\right) - 6t_j\, G_{n+1}\left(\frac{1}{4}p_1(t^{-1})\right)\right].$$

Proof: First we apply the definition of the Laplace-Carson transform to (i) and obtain

$$\varphi(\sqrt{\omega}) = \int_0^\infty \sqrt{\omega}\, e^{-u\sqrt{\omega}}\, f(u)du, \quad \text{with } \Re(\omega) > 0. \tag{I.4}$$

From the hypothesis (ii) and (I.4), we get

$$\int_0^\infty \sqrt{\omega}\, e^{-u\sqrt{\omega}}\, f(u)du \overset{1}{=} F(x).$$

Therefore by using the following result from Roberts and Kaufman [37],

$$\sqrt{\omega}\, e^{-u\sqrt{\omega}} \overset{1}{=} \frac{1}{\sqrt{\pi}}\, x^{-\frac{1}{2}}\, e^{-\frac{u^2}{4x}},$$

we find

$$F(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty x^{-\frac{1}{2}}\, e^{-\frac{u^2}{4x}}\, f(u)du. \tag{I.5}$$

Now we use (iii) and (I.5) to write

$$\frac{\xi(\omega)}{\omega} = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty x^{-\frac{3}{2}} \cdot e^{-\frac{u^2}{4x} - \omega x}\, f(u)du\, dx, \quad \text{where } \Re(\omega) > 0. \tag{I.6}$$

When $\Re(\omega) > 0$, we know that $x^{-\frac{3}{2}}\, e^{-\frac{u^2}{4x} - \omega x}\, f(u)$ is integrable on $(0, \infty) \times (0, \infty)$. Now by applying Fubini's Theorem, we find that the right hand side of (I.6) is absolutely convergent. Therefore, we can interchange the order of integration to get

$$\frac{\xi(\omega)}{\omega} = \int_0^\infty f(u) \left\{ \frac{1}{\sqrt{\pi}} \int_0^\infty x^{-\frac{3}{2}} e^{-\frac{u^2}{4x} - \omega x} \, dx \right\} du. \tag{I.7}$$

By using the result from Roberts and Kaufman [37], we can evaluate the inner integral in (I.7) and obtain

$$\frac{\xi(\omega)}{\omega} = \int_0^\infty f(u) \left[\frac{2}{u} e^{-u\sqrt{\omega}} \right] du. \tag{I.8}$$

Next, we replace ω by $p_1(s^{\frac{1}{2}})^2$. Then (I.8) becomes

$$p_1(s^{\frac{1}{2}})^{-2} \, \xi\left(p_1(s^{\frac{1}{2}})^2\right) = 2\int_0^\infty \exp\left[-u\, p_1(s^{\frac{1}{2}})\right] \cdot u^{-1} f(u) du. \tag{I.9}$$

In the following, we will multiply (I.9) by $s_j^{\mu}\, p_n(s^{\nu})$; $j = 1, 2, \ldots$, for some real values ν and μ and will evaluate the respective inverses of resulting expressions.

(a) We multiply (I.9) by $p_n(s^{\frac{1}{2}})$ and get

$$p_n(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2} \, \xi\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) = 2\int_0^\infty p_n\left(s^{\frac{1}{2}}\right) \exp\left(-u\, p_1\left(s^{\frac{1}{2}}\right)\right) u^{-1} f(u) du \tag{I.10}$$

Again by using Roberts and Kaufman [37], we can find

$$\sqrt{s_j}\, e^{-u\sqrt{s_j}} \stackrel{1}{=} \frac{1}{\sqrt{\pi}} t_j^{-\frac{1}{2}} e^{-\frac{u^2}{4t_j}} \quad \text{for } j = 1, 2, \ldots, n. \tag{I.11}$$

Next by subsituting (I.11) in (I.10), we can derive

$$p_n(s^{\frac{1}{2}})\, p_n(s^{\frac{1}{2}})^{-2} \, \xi\left(p_1\, (s^{\frac{1}{2}})^2\right) \stackrel{n}{=} \tag{I.12}$$

$$2\pi^{-\frac{n}{2}} \int_0^{\infty} p_n(t^{-\frac{1}{2}}) \exp\left[-\frac{u^2}{4} p_1(t^{-1})\right] u^{-1} f(u)du$$

Since the inverse form, which we have obtained above is an integral form, it is not well suited for applications. Hence we simplify it further to obtain a better form.

Now, we substitute $v = u^2$ in (I.12) and get

$$\text{L.H.S.} \overset{n}{=} \pi^{-\frac{n}{2}} p_n(t^{-\frac{1}{2}}) \cdot \int_0^{\infty} exp\left[-\frac{v}{4} p_1(t^{-1})\right] v^{-1} f(\sqrt{v})dv \tag{I.13}$$

If we use (iv) for $j = 0$, then (I.13) becomes

$$\text{L.H.S.} \overset{n}{=} \pi^{-\frac{n}{2}} p_n(t^{-\frac{1}{2}}) G_0\left(\frac{1}{4} p_1(t^{-1})\right) / \left[\frac{1}{4} p_1(t^{-1})\right].$$

Therefore, the final form of the transform pair is given by

$$p_n(s^{\frac{1}{2}}) p_1(s^{\frac{1}{2}})^{-2} \xi\left(p_1(s^{\frac{1}{2}})^2\right) \overset{n}{=} 4\pi^{-\frac{n}{2}} p_n(t^{-\frac{1}{2}}) p_1(t^{-1})^{-1} \cdot G_0\left(\frac{1}{4} p_1(t^{-1})\right)$$

(b) Next, we multiply (I.9) by $p_n(s)$ and get

$$p_n(s) p_1(s^{\frac{1}{2}})^{-2} \xi\left(p_1(s^{\frac{1}{2}})^2\right) = 2 \int_0^{\infty} p_n(s)\, e^{-u\, p_1(s^{\frac{1}{2}})} u^{-1} f(u)du. \tag{I.14}$$

With the help of Roberts and Kaufman [37], we can find

$$s_j\, e^{-u\sqrt{s_j}} \overset{1}{=} \frac{u}{2\sqrt{\pi}} t_j^{-\frac{3}{2}} e^{\frac{-u^2}{4t_j}}, \quad \text{for } j = 1, 2, \ldots, n. \tag{I.15}$$

If we use the result (I.15) in (I.14), we can derive

$$p_n(s)\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) \overset{n}{=} 2^{-n+1}\, \pi^{-\frac{n}{2}}\, . \tag{I.16}$$

$$\int_0^{\infty} p_n(t^{-\frac{3}{2}})\, e^{-\frac{u^2}{4} p_1(t^{-1})}\, u^{n-1}\, f(u)du.$$

Let $v = u^2$ then (I.16) becomes

$$p_n(s)\, p_1(s^{\frac{1}{2}})^{-2}\, \xi(p_1(s^{\frac{1}{2}})^2 \overset{n}{=}$$

$$2^{-n+2}\, \pi^{-\frac{n}{2}}\, p_n(t^{-\frac{3}{2}})\, p_1(t^{-1})^{-1}\, G_n\left(\frac{1}{4} p_1(t^{-1})\right),$$

by using the hypothesis (iv) with $j = n$.

(c) If we multiply (I.9) by $s^{\frac{1}{2}}\, p_n(s^{\frac{1}{2}})$, for $j = 1, 2, \ldots, n$, then we obtain

$$s_j^{\frac{1}{2}}\, p_n(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) = \tag{I.17}$$

$$2\int_0^{\infty} s_j^{\frac{1}{2}}\, p_n(s^{\frac{1}{2}})\, e^{-u\, p_1(s^{\frac{1}{2}})}\, u^{-1}\, f(u)du.$$

$$\text{L.H.S.} = 2\int_0^{\infty} \left(\prod_{k=1,\, k\neq j}^{n} s_k^{\frac{1}{2}}\, e^{-u\sqrt{s_j}}\right) u^{-1}\, f(u)du.$$

By using (I.11) and (I.15), we can derive

$$s_j^{\frac{1}{2}}\, p_n(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) \overset{n}{=} \tag{I.18}$$

$$\pi^{-\frac{n}{2}}\, t_j^{-1}\, p_n(t^{-\frac{1}{2}}) \int_0^{\infty} e^{-\frac{u^2}{4} p_1(t^{-1})}\, f(u)du.$$

Next, by substituting $v = u^2$ in (I.18) and by using (iv) with $j = 1$, we can conclude

$$s_j^{\frac{1}{2}}\, p_n(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) \overset{n}{=} 2\pi^{-\frac{n}{2}}\, t_j^{-1}$$

$$p_n(t^{-\frac{1}{2}})\,p_1(t^{-1})^{-1}\,G_1\left(\frac{1}{4}\,p_1(t^{-1})\right), \quad \text{for } j=1,2,\ldots,n.$$

(d) Next, we multiply (I.9) by $s_j\,p_n(s^{\frac{1}{2}})$, for $j=1,2,\ldots,n$; to get

$$s_j\,p_n(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})\,\xi\left(p_1(s^{\frac{1}{2}})^2\right) = \tag{I.19}$$

$$2\int_0^\infty s_j\,p_n(s^{\frac{1}{2}})\,e^{-up_1(s^{\frac{1}{2}})}\,u^{-1}\,f(u)du.$$

Again by using Roberts and Kaufman [37], we can find

$$s_j^{\frac{3}{2}}\,e^{-u\sqrt{s_j}} \stackrel{1}{=} \frac{1}{4\sqrt{\pi}}\,t_j^{-\frac{5}{2}}\left(u^2-2t_j\right)\,e^{-\frac{u^2}{4t_j}}. \tag{I.20}$$

Now by using (I.20) and (I.11), we can find

$$s_j\,p_n(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})^{-2}\,\xi\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{n}{=} \tag{I.21}$$

$$\frac{\pi^{-\frac{n}{2}}}{2}\left[t_j^{-2}\,p_n(t^{-\frac{1}{2}})\int_0^\infty e^{-\frac{u^2}{4}\,p_1(t^{-1})}\,u\,f(u)du-\right.$$

$$\left.2t_j^{-1}\,p_n(t^{-\frac{1}{2}})\int_0^\infty e^{-\frac{u^2}{4}\,p_1(t^{-1})}\,u^{-1}\,f(u)du\right].$$

Next by substituting $v=u^2$ in (I.21) and by using the hypothesis with $j=0$ and $j=2$, we obtain

$$s_j\,p_n(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})^{-2}\,\xi\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{n}{=} \pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\cdot$$

$$p_1(t^{-1})^{-1}\,t_j^{-2}\left[G_2\left(\frac{1}{4}\,p_1(t^{-1})\right)-2t_j\,G_0\left(\frac{1}{4}\,p_1(t^{-1})\right)\right].$$

(e) If we multiply (I.9) by $s_j^{\frac{3}{2}}\, pn(s^{\frac{1}{2}})$, for $j = 1, 2, \ldots, n$ we obtain

$$s_j^{\frac{3}{2}}\, pn(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) = \tag{I.22}$$

$$2\int_0^{\infty} s_j^{\frac{3}{2}}\, pn(s^{\frac{1}{2}}) e^{-u\, p_1(s^{\frac{1}{2}})}\, u^{-1}\, f(u)du.$$

By using a result from Roberts and Kaufman [37], we get

$$s_j^2\, e^{-u\sqrt{s_j}} \stackrel{1}{=} \frac{u(u^2 - 6t_j)}{8\sqrt{\pi}} \cdot t_j^{-\frac{7}{2}}\, e^{-\frac{u^2}{4t_j}}, \tag{I.23}$$

we use (I.11) and (I.23) to find

$$s_j^{\frac{3}{2}}\, pn(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{n}{=} \tag{I.24}$$

$$\frac{\pi^{-\frac{n}{2}}}{4}\left[t_j^{-3}\, pn(t^{-\frac{1}{2}}) \int_0^{\infty} e^{-\frac{u^2}{4}\, p_1(t^{-1})}\, u^2\, f(u)du - \right.$$

$$\left. 6t_j^{-2}\, pn(t^{-\frac{1}{2}}) \int_0^{\infty} e^{-\frac{u^2}{4}\, p_1(t^{-1})}\, f(u)du\right].$$

Now by substituting $v = u^2$ in (I.24) and by using (iv) with $j = 1$ and $j = 3$, we obtain the final form as

$$s_j^{\frac{3}{2}}\, pn(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{n}{=} \frac{\pi^{-\frac{n}{2}}}{2}\, pn(t^{-\frac{1}{2}}) \cdot$$

$$p_1(t^{-1})^{-1}\, t_j^{-3}\left[G_3\left(\frac{1}{4}\, p_1(t^{-1})\right) - 6t_j\, G_1\left(\frac{1}{4} p_1(t^{-1})\right)\right].$$

(f) Next we multiply (I.9) by $s_j^{\frac{1}{2}}\, pn(s)$, for $j = 1, 2, \ldots, n$, to get

$$s_j^{\frac{1}{2}}\, pn(s)\, p_1(s^{\frac{1}{2}})^{-2}\, \xi\left(p_1(s^{\frac{1}{2}})^2\right) = \tag{I.25}$$

$$2 \int_0^\infty s_j^{\frac{1}{2}} \, pn(s) \, e^{-u \, p_1(s^{\frac{1}{2}})} \, u^{-1} f(u) du.$$

If we use (I.15) and (I.20) in (I.25), we will find

$$s_j^{\frac{1}{2}} \, pn(s) \, p_1(s^{\frac{1}{2}})^{-2} \, \xi \left(p_1(s^{\frac{1}{2}})^2 \right) \overset{n}{=} 2^{-n} \, \pi^{-\frac{n}{2}} \, pn(t^{-\frac{3}{2}}) \cdot \tag{I.26}$$

$$\left[t_j^{-2} \int_0^\infty e^{-\frac{u^2}{4} p_1(t^{-1})} \, u^{n+1} f(u) du - \right.$$

$$\left. 2t_j^{-1} \int_0^\infty e^{-\frac{u^2}{4} p_1(t^{-1})} \, u^{n-1} f(u) du \right].$$

Now we substitute $v = u^2$ and take $j = n$ and $j = n + 2$ in hypothesis (iv) to obtain the following result

$$s_j^{\frac{1}{2}} \, pn(s) \, p_1(s^{\frac{1}{2}})^{-2} \, \xi \left(p_1(s^{\frac{1}{2}})^2 \right) \overset{n}{=} 2^{-n+1} \, \pi^{-\frac{n}{2}} \, .$$

$$pn\left(t^{-\frac{3}{2}} \right) p_1(t^{-1})^{-1} \, t_j^{-2} \left[G_{n+2} \left(\frac{1}{4} p_1(t^{-1}) \right) - \right.$$

$$\left. 2t_j \, G_n \left(\frac{1}{4} p_1(t^{-1}) \right) \right].$$

(g) Finally we multiply (I.9) by $s_j \, pn(s)$, for $j = 1, 2, \ldots, n$, and get

$$s_j \, pn(s) \, p_1(s^{\frac{1}{2}})^{-2} \, \xi \left(p_1(s^{\frac{1}{2}})^2 \right) = \tag{I.27}$$

$$2 \int_0^\infty s_j \, pn(s) \, e^{-u \, p_1(s^{\frac{1}{2}})} \, u^{-1} f(u) du.$$

Now we use (I.15) and (I.23) in (I.27) to obtain

$$s_j \, pn(s) \, p_1(s^{\frac{1}{2}})^{-2} \, \xi \left(p_1(s^{\frac{1}{2}})^2 \right) \overset{n}{=} 2^{-n-1} \, \pi^{-\frac{n}{2}} \, .$$

$$p_n\left(t^{-\frac{3}{2}}\right) p_1(t^{-1})^{-1}\, t_j^{-3}\left[G_{n+3}\left(\frac{1}{4}p_1(t^{-1})\right) -\right.$$
$$\left. 6t_j\, G_{n+1}\left(\frac{1}{4}p_1(t^{-1})\right)\right].$$

Applications of Theorem I.1

Next we provide some examples as application to theorem 1.1. We consider case for n=2.

Example 1.1

Let $f(x) = x^{\nu+1}\, J_\nu\,(ax)$, where $a > 0$ and $\Re\,\nu > -\frac{1}{2}$. Then

$$\varphi(\omega) = \frac{2^{\nu+1}\, a^\nu\, \Gamma\left(\nu+\frac{3}{2}\right)\omega^2}{\sqrt{\pi}\left(\omega^2+a^2\right)^{\nu+\frac{3}{2}}}, \quad \text{with } \Re\,\omega > 0.$$

From which it follows:

$$F(x) = \frac{2^{\nu+1}\, a^\nu}{\sqrt{\pi}}\, x^{\nu+\frac{1}{2}}\, e^{-a^2 x}. \quad \text{Also} \tag{I.28}$$

$$\frac{\xi(\omega)}{\omega} = \frac{2^{\nu+1}\, a^\nu}{\sqrt{\pi}}\,\frac{\Gamma^{\left(\nu+\frac{1}{2}\right)}}{\left(\omega+a^2\right)^{\nu+\frac{1}{2}}} \tag{I.29}$$

and $G_j\,(\omega) = \dfrac{\Gamma\left(\nu+\frac{j+1}{2}\right)a^\nu}{\Gamma(\nu+1)\,2^\nu}\, s^{-\nu-\left(\frac{j+1}{2}\right)}\,{}_1F_1\left[\nu+\frac{j+1}{2};\nu+1;\frac{-a^2}{4s}\right]$.

Then by Theorem (I.4), we obtain the following results:

(a)

$$\frac{\sqrt{s_1\, s_2}}{\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2+a^2\right]^{\nu+\frac{1}{2}}} \stackrel{2}{=} \frac{1}{\sqrt{\pi}\,\Gamma(\nu+1)}\,\frac{(t_1\, t_2)^\nu}{(t_1+t_2)^{\nu+\frac{1}{2}}}.$$

$$ {}_1F_1\left[\nu+\frac{1}{2};\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]. $$

(*b*)

$$ \frac{s_1\,s_2}{\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2+a^2\right]^{\nu+\frac{1}{2}}} \overset{2}{=} \frac{2^{2\nu}\,\Gamma\left(\nu+\frac{3}{2}\right)}{\pi\,\Gamma(\nu+1)}\,\frac{(t_1\,t_2)^{\nu}}{(t_1+t_2)^{\nu+\frac{3}{2}}}\cdot $$

$$ {}_1F_1\left[\nu+\frac{3}{2};\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]. $$

(*c*)

$$ \frac{s_1\,s_2^{\frac{1}{2}}}{\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2+a^2\right]^{\nu+\frac{1}{2}}} \overset{2}{=} \frac{t_1^{-\frac{1}{2}}\,t_2^{\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\,\frac{(t_1\,t_2)^{\nu}}{(t_1+t_2)^{\nu+1}}\cdot $$

$$ {}_1F_1\left[\nu+1;\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]. $$

(*d*)

$$ \frac{s_1^{\frac{3}{2}}\,s_2^{\frac{1}{2}}}{\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2+a^2\right]^{\nu+\frac{1}{2}}} \overset{2}{=} \frac{2^{2\nu-1}\,t_1^{-\frac{3}{2}}\,t_2^{-\frac{1}{2}}}{\Gamma(2\nu+1)}\left(\frac{t_1\,t_2}{(t_1+t_2)}\right)^{\nu+\frac{1}{2}}\cdot $$

$$ \left\{2t_2\,\Gamma\left(\nu+\frac{3}{2}\right)\,{}_1F_1\left[\nu+\frac{3}{2};\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]-\right. $$

$$ \left.(t_1+t_2)\,\Gamma\left(\nu+\frac{1}{2}\right)\,{}_1F_1\left[\nu+\frac{1}{2};\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]\right\}. $$

(*e*)

$$ \frac{s_1^2\,s_2^{\frac{1}{2}}}{\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2+a^2\right]^{\nu+\frac{1}{2}}} \overset{2}{=} \frac{t_1^{-3}\,t_2^{-1}}{2\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\,\frac{(t_1\,t_2)^{\nu+\frac{3}{2}}}{(t_1+t_2)^{\nu+2}}\cdot $$

$$\left\{2t_2\,\frac{\Gamma(\nu+2)}{\Gamma(\nu+1)}\,{}_1F_1\left[\nu+2;\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]-\right.$$
$$\left.3(t_1+t_2)\,{}_1F_1\left[\nu+1;\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]\right\}.$$

(*f*)

$$\frac{s_1^{\frac{3}{2}}\,s_2}{\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2+a^2\right]^{\nu+\frac{1}{2}}} \stackrel{2}{=} \frac{2^{2\nu}\,t_1^{-2}\,t_2^{-1}}{\pi\,\Gamma(2\nu+1)}\,\frac{(t_1\,t_2)^{\nu+1}}{(t_1+t_2)^{\nu+\frac{5}{2}}}\,.$$
$$\left\{2t_2\,\Gamma\left(\nu+\frac{5}{2}\right)\,{}_1F_1\left[\nu+\frac{5}{2};\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]-\right.$$
$$\left.(t_1+t_2)\,\Gamma\left(\nu+\frac{3}{2}\right)\,{}_1F_1\left[\nu+\frac{3}{2};\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]\right\}.$$

(*g*)

$$\frac{s_1^2\,s_2}{\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2+a^2\right]^{\nu+\frac{1}{2}}} \stackrel{2}{=} \frac{2^{2\nu}}{\pi\,\Gamma(2\nu+1)}\,\frac{(t_1\,t_2)^{\nu+\frac{1}{2}}}{(t_1+t_2)^{\nu+3}}\,.$$
$$\left\{2t_2\,\Gamma(\nu+3)\,{}_1F_1\left[\nu+3;\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]-\right.$$
$$\left.3(t_1+t_2)\,\Gamma(\nu+2)\,{}_1F_1\left[\nu+2;\nu+1;-\frac{a^2\,t_1\,t_2}{t_1+t_2}\right]\right\}.$$

Example 1.2

Let $f(x)=x^{2\nu}\log x, \Re\,\nu>0$, then $\varphi(\omega)=\Gamma\,(\nu+1)\,\omega^{-2\nu}\,\left[\psi(2\nu+1)-\log\omega\right]$, provided $\Re\,\omega>0$. Also

$$F(x)=\frac{\Gamma(2\nu+1)}{2\Gamma(\nu+1)}\,t^{\nu}\,\left[2\psi(2\nu+1)-\psi(\nu+1)+\log x\right].$$

Hence $\xi(\omega) = \Gamma(2\nu)\,\omega^{-\nu+1}\,[2\psi(2\nu+1) - \psi(\nu+1) + \psi(\nu) - \log\omega]$. And $G_j(\omega) = \frac{1}{2}\Gamma\left(\nu + \frac{j}{2}\right)\omega^{-\nu-\frac{1}{2}+1}\left[\psi\left(\nu+\frac{j}{2}\right) - \log\omega\right]$, for $j = 0, 1, \ldots, 5$.

Now by using the theorem (I.1), we can obtain the following results. First we find $\left(\sqrt{s_1}+\sqrt{s_2}\right)^{-2}\xi\left(\left(\sqrt{s_1}+\sqrt{s_2}\right)^2\right) = \left(\sqrt{s_1}+\sqrt{s_2}\right)^{-2}\cdot$
$\Gamma(2\nu)\left(\sqrt{s_1}+\sqrt{s_2}\right)^{-2\nu+2}\left[2\psi(2\nu+1) - \psi(\nu+1) + \psi(\nu) - 2\log\left(\sqrt{s_1}+\sqrt{s_2}\right)\right]$. Denote $\left(\sqrt{s_1}+\sqrt{s_2}\right)^{-2}\xi\left(\left(\sqrt{s_1}+\sqrt{s_2}\right)^2\right)/\Gamma(2\nu)$ by $\eta(s_1,s_2)$. That is, $\eta(s_1,s_2) =$ $\left(\sqrt{s_1}+\sqrt{s_2}\right)^{-2\nu}\,[2\psi(2\nu+1) - \psi(\nu+1) + \psi(\nu)$
$-2\log\left(\sqrt{s_1}+\sqrt{s_2}\right)]$. Then

(a)

$$\sqrt{s_1, s_2}\,\eta(s_1,s_2) \overset{2}{=} 2\frac{(t_1 t_2)^{-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{t_1 t_2}{t_1+t_2}\right)^{\nu}\cdot \left[\psi(\nu) - \log\left(\frac{t_1+t_2}{4t_1 t_2}\right)\right].$$

(b)

$$s_1 s_2\,\eta(s_1,s_2) \overset{2}{=} \frac{2^{2\nu-1}\,\Gamma(\nu+1)}{\pi\,\Gamma(2\nu)}\sqrt{t_1 t_2}\left(\frac{t_1 t_2}{t_1+t_2}\right)^{\nu-1}\cdot \left[\psi(\nu+1) - \log\left(\frac{t_1+t_2}{4t_1 t_2}\right)\right].$$

(c)

$$s_1 s_2^{\frac{1}{2}}\,\eta(s_1,s_2) \overset{2}{=} \frac{1}{\sqrt{\pi}\,\Gamma(\nu)}t_1^{-1}(t_1+t_2)^{-\frac{1}{2}}\left(\frac{t_1 t_2}{t_1+t_2}\right)^{\nu}\cdot \left[\psi(\nu+1) - \log\left(\frac{t_1+t_2}{4t_1 t_2}\right)\right].$$

(d)

$$s_1^{\frac{3}{2}} s_2^{\frac{1}{2}} \eta(s_1, s_2) \stackrel{2}{=} \frac{t_1^{-\frac{3}{2}} t_2^{-\frac{1}{2}}}{4\pi\, \Gamma(2\nu)\, (t_1 + t_2)} \left(\frac{4t_1\, t_2}{t_1 + t_2}\right)^2 \cdot$$
$$\left\{2t_2\, \Gamma(\nu+1) \left[\psi(\nu+1) - \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right] - \right.$$
$$\left. (t_1 + t_2)\, \Gamma(\nu) \left[\psi(\nu) - \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right]\right\}.$$

(e)

$$s_1^2 s_2^{\frac{1}{2}} \eta(s_1, s_2) \stackrel{2}{=} \frac{t_1^{-\frac{5}{2}} t_2^{-\frac{1}{2}}}{8\pi\, \Gamma(2\nu)\, (t_1 + t_2)} \left(\frac{4t_1\, t_2}{t_1 + t_2}\right)^{\nu+\frac{1}{2}} \cdot$$
$$\left\{2t_2\, \Gamma\left(\nu + \frac{3}{2}\right) \left[\psi\left(\nu + \frac{3}{2}\right) - \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right] - \right.$$
$$\left. 3(t_1 + t_2)\, \Gamma\left(\nu + \frac{1}{2}\right) \left[\psi\left(\nu + \frac{1}{2}\right) - \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right]\right\}.$$

(f)

$$s_1^{\frac{3}{2}} s_2\, \eta(s_1, s_2) \stackrel{2}{=} \frac{t_1^{-\frac{5}{2}} t_2^{-\frac{3}{2}}}{8\pi\, \Gamma(2\nu)\, (t_1 + t_2)} \left(\frac{4t_1\, t_2}{t_1 + t_2}\right)^{\nu+1} \cdot$$
$$\left\{2t_2\, \Gamma(\nu+2) \left[\psi(\nu+2) - \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right] - \right.$$
$$\left. (t_1 + t_2)\, \Gamma(\nu+1) \left[\psi(\nu+1) \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right]\right\}.$$

(g)

$$s_1^2 s_2\, \eta(s_1, s_2) \stackrel{2}{=} \frac{t_1^{-\frac{3}{2}} t_2^{-\frac{3}{2}}}{16\pi\, \Gamma(2\nu)\, (t_1 + t_2)} \left(\frac{4t_1\, t_2}{t_1 + t_2}\right)^{\nu+\frac{3}{2}} \cdot$$
$$\left\{2t_2\, \Gamma\left(\nu + \frac{5}{2}\right) \left[\psi\left(\nu + \frac{5}{2}\right) - \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right] - \right.$$
$$\left. 3(t_1 + t_2)\, \Gamma\left(\nu + \frac{3}{2}\right) \left[\psi\left(\nu + \frac{3}{2}\right) - \log\left(\frac{t_1 + t_2}{4t_1\, t_2}\right)\right]\right\}.$$

Theorem I.2: Suppose that $f(x), F(x), x\,F(x), x^{\frac{j}{2}} f(\sqrt{x})$, for $j = 0,1,2,3,n-1,n,n+1,n+2$ are continuous and absolutely integrable in $(0,\infty)$ and that

(i) $\varphi(\omega) \stackrel{1}{=} f(x)$.

(ii) $\sqrt{\omega}\,\varphi(\sqrt{\omega}) \stackrel{1}{=} F(x)$.

(iii) $\xi(\omega) \stackrel{1}{=} x\,F(x)$.

(iv) $G_j(\omega) \stackrel{1}{=} x^{\frac{j}{2}} f(\sqrt{x})$, $j = 0,1,2,3,n-1,n,n+1,n+2$.

Then

(a)

$$p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \stackrel{n}{=} \pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\cdot p_1(t^{-1})^{-1}\,G_0\left(\frac{1}{4}p_1(t^{-1})\right),$$

(b)

$$p_n(s)\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \stackrel{n}{=} 2^{-n}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{3}{2}})\,p_1(t^{-1})^{-1}\cdot G_n\left(\frac{1}{4}p_1(t^{-1})\right),$$

(c)

$$\sqrt{s_j}\,p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \stackrel{n}{=} 2^{-1}\,\pi^{-\frac{n}{2}}\,t_j^{-1}\cdot p_n(t^{-\frac{1}{2}})p_1(t^{-1})^{-1}\,G_1\left(\frac{1}{4}p_1(t^{-1})\right),$$

(d)

$$s_j\,p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\xi(p_1(\sqrt{s})^2) \stackrel{n}{=} 2^{-2}\,\pi^{-\frac{n}{2}}\cdot p_1(t^{-1})^{-1}\,p_n(t^{-\frac{1}{2}})\,t_j^{-2}\left\{G_2\left(\frac{1}{4}p_1(t^{-1})\right) - 2t_j\,G_0\left(\frac{1}{4}p_1(t^{-1})\right)\right\},$$

(e)

$$s_j^{\frac{3}{2}}\, p_n(\sqrt{s})\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) \stackrel{n}{=} 2^{-3}\, \pi^{-\frac{n}{2}}\, p_1(t^{-1})^{-1} \cdot$$
$$p_n(t^{-\frac{1}{2}})\, t_j^{-3} \left\{ G_3\left(\frac{1}{4} p_1(t^{-1})\right) - 6t_j G_1\left(\frac{1}{4} p_1(t^{-1})\right)\right\}.$$

(f)

$$\sqrt{s_j}\, p_n(s)\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) \stackrel{n}{=} 2^{-n-1}\, \pi^{-\frac{n}{2}}\, p_n(t^{-\frac{3}{2}}) \cdot$$
$$p_1(t^{-1})^{-1}\, t_j^{-1} \left\{ G_{n+1}\left(\frac{1}{4} p_1(t^{-1})\right) - 2t_j G_{n-1}\left(\frac{1}{4} p_1(t^{-1})\right)\right\},$$

(g)

$$s_j p_n(s)\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) \stackrel{n}{=} 2^{-n-2}\, \pi^{-\frac{n}{2}}\, p_n(t^{-\frac{3}{2}}) \cdot$$
$$p_1(t^{-1})^{-1}\, t_j^{-2} \left\{ G_{n+2}\left(\frac{1}{4} p_1(t^{-1})\right) - 6t_j\, G_n\left(\frac{1}{4} p_1(t^{-1})\right)\right\},$$

for $j = 0, 1, 2, 3$ and $n, n+1, n+2$.

Proof: Let us start by using the explicit expression of (i) to write

$$\sqrt{\omega}\, \varphi(\sqrt{\omega}) = \int_0^\infty \omega\, e^{-u\sqrt{\omega}}\, du, \quad \text{where } \Re(s) > 0. \tag{I.30}$$

Now by using Roberts and Kaufman [37], we can find

$$\omega\, e^{-u\sqrt{\omega}} \stackrel{1}{=} 2^{-1}\, \pi^{\frac{1}{2}}\, u\, x^{-\frac{3}{2}}\, e^{-\frac{u^2}{4x}}. \tag{I.31}$$

Since $\sqrt{\omega}\, \varphi(\sqrt{\omega}) \stackrel{1}{=} F(t)$, from (I.31), we obtain

$$F(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty x^{-\frac{3}{2}}\, e^{-\frac{u^2}{4x}}\, u\, f(u)\, du. \tag{I.32}$$

Next, we use (I.32) and the hypothesis (*iii*) to write

$$\frac{\xi(\omega)}{\omega} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \int_0^\infty x^{-\frac{1}{2}} \cdot \tag{I.33}$$
$$e^{-\frac{u^2}{4x} - \omega x}\, u\, f(u)\, du\, dx, \text{ with } \Re(\omega) > 0.$$

Since $x^{-\frac{1}{2}} e^{-\frac{u^2}{4x} - \omega x} u\, f(u)$ is integrable on $(0,\infty) \times (0,\infty)$ for $\Re(\omega) > 0$. Therefore, by Fubini's Theorem, we can write

$$\frac{\xi(\omega)}{\omega} = \frac{1}{2} \int_0^\infty u\, f(u) \cdot \tag{I.34}$$
$$\left\{ \int_0^\infty x^{-\frac{1}{2}} e^{-\frac{u^2}{4x} - \omega x}\, dx \right\} du, \text{ where } \Re(\omega) > 0.$$

Now we can use the table by Roberts and Kaufman [37], to evaluate the inner integral of equation (I.34) and obtain

$$\frac{1}{\sqrt{\pi}} \int_0^\infty x^{-\frac{1}{2}} e^{-\frac{u^2}{4x} - \omega x}\, dx = \omega^{-\frac{1}{2}} e^{-u\sqrt{\omega}}, \text{ with } \Re(\omega) > 0. \tag{I.35}$$

By using (I.35), we can write (I.34) as

$$\frac{\xi(\omega)}{\omega} = \frac{1}{2} \int_0^\infty \omega^{-\frac{1}{2}} e^{-u\sqrt{\omega}}\, u f(u)\, du, \text{ where } \Re(\omega) > 0. \tag{I.36}$$

If we replace ω by $p_1(\sqrt{s})^2$ then we can write equation (I.36) as

$$p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) = \frac{1}{2} \int_0^\infty \exp\left[-u\, p_1(\sqrt{s})\right]\, u\, f(u)\, du. \tag{I.37}$$

From the equation (I.37), we will obtain the results $(a), (b), (c), (d), (e), (f)$, and (g).

In the proofs to follow, the left hand side of preceding equation is denoted by L.H.S.

(a) We multiply the equation (I.37) by $p_n(\sqrt{s})$ and get

$$p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) = \frac{1}{2}\int_0^\infty p_n(\sqrt{s})\,\exp\left[-u\,p_1(\sqrt{s})\right]u\,f(u)du. \tag{I.38}$$

Next, we use the result from Roberts and Kaufman [37], stated as equation (I.11) in (I.38) to arrive at

$$p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^{-1}\pi^{-\frac{n}{2}}\cdot \int_0^\infty p_n(t^{-\frac{1}{2}})\,\exp\left(\frac{u^2}{4}p_1(t^{-1})\right)u\,f(u)\,du. \tag{I.39}$$

The form of inverse, we obtained above, is not an applicable one. Therefore we manipulate the equation (I.40) further to obtain a better form. To this end let $v = u^2$, then equation (I.39) becomes

$$\text{L.H.S.} \overset{n}{=} 2^{-2}\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\cdot \int_0^\infty \exp\left(-\frac{v}{4}p_1(t^{-1})\right)f(\sqrt{v})\,dv. \tag{I.40}$$

Since $G_0(\omega) \overset{1}{=} f(\sqrt{x})$, the equation (I.40) becomes

$$\text{L.H.S.} \overset{n}{=} 2^{-2}\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\,G_0\left(\frac{1}{4}p_1(t^{-1})\right)/\frac{1}{4}p_1(t^{-1}).$$

Therefore, we have the final form as

$$p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} \pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\,p_1(t^{-1})\cdot G_0\left(\frac{1}{4}p_1(t^{-1})\right).$$

(b) If we multiply (I.37) by $p_n(s)$ then we will get

$$p_n(s)\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) = \frac{1}{2}\int_0^\infty p_n(s)\,\exp\left[-u\,p_1(\sqrt{s})\right]u\,f(u)\,du. \tag{I.41}$$

Now we use the result given by equation (I.15) in (I.41) and derive

$$p_n(s)\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^{-n-1}\,\pi^{-\frac{n}{2}} \cdot \int_0^\infty p_n(t^{-\frac{3}{2}})\,\exp\left(-\frac{u^2}{4}\,p_1(t^{-1})\right) u^{n+1}\,f(u)\,du. \tag{I.42}$$

Next, we substitute $v = u^2$ in (I.42) to get

$$\text{L.H.S.} \overset{n}{=} 2^{-n-2}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{3}{2}}) \int_0^\infty \exp\left[-\frac{v}{4}\,p_1(t^{-1})\right] v^{\frac{n}{2}}\,f(\sqrt{v})\,dv.$$

Hence the final form is given as

$$p_n(s)\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \;\overset{n}{=}\; 2^{-n}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{3}{2}})\,p_1(t^{-1})^{-1} \cdot G_n\left(\frac{1}{4}\,p_1(t^{-1})\right).$$

(c) Next, we multiply (I.37) by $\sqrt{s_j}\,p_n(\sqrt{s})$, for $j = 1, 2, \ldots, n$, to get

$$\sqrt{s_j}\,p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) = \frac{1}{2}\int_0^\infty \sqrt{s_j}\,p_n(\sqrt{s})\,\exp\left[-u\,p_1(\sqrt{s})\right]\,u\,f(u)du. \tag{I.43}$$

By using (I.11) and (I.15), we can derive

$$\sqrt{s_j}\,p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^{-2}\,\pi^{-\frac{n}{2}} \cdot \int_0^\infty p_n(t^{-\frac{1}{2}})\,\exp\left[-\frac{u^2}{4}\,p_1(t^{-1})\right] u^2\,f(u)\,du. \tag{I.44}$$

Now we substitute $v = u^2$ in (I.43) and use the fact $G_1(\omega) \overset{1}{=} x^{\frac{1}{2}}\,f(\sqrt{x})$ to get

$$\sqrt{s_j}\,p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^{-1}\,\pi^{\frac{n}{2}} \cdot t_j^{-1}\,p_n(t^{-\frac{1}{2}})\,p_1(t^{-1})^{-1}\,G_1\left(\frac{1}{4}\,p_1(t^{-1})\right).$$

(d) We multiply the equation (I.37) by $s_j\, pn(\sqrt{s})$ to get

$$s_j\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) = \frac{1}{2}\int_0^\infty s_j\, pn(\sqrt{s})\, \exp(-up_1(\sqrt{s}))\, u\, f(u)\, du. \tag{I.45}$$

By using results from Roberts and Kaufman [37] stated in the equations (I.11) and (I.20), we obtain

$$\begin{aligned} \text{L.H.S.} \ &\overset{n}{=}\ 2^{-3}\pi^{-\frac{n}{2}} \int_0^\infty t_j^{-2}\, pn(t^{-\frac{1}{2}}(u^2 - 2t_j)\, \cdot \\ &\qquad \exp\left[-\frac{u}{4} p_1(t^{-1})\right] f(u)\, du. \\ &\overset{n}{=}\ 2^{-3}\pi^{-\frac{n}{2}} \left\{ t_j^{-2}\, pn(t^{\frac{1}{2}}) \int_0^\infty \exp\left[\frac{u^2}{4} p_1(t^{-1})\right] \cdot \right. \\ &\qquad (u^3 f(u)\, du - 2t_j^{-1}\, pn(t^{-\frac{1}{2}})\, \cdot \\ &\qquad \left. \int_0^\infty \exp\left[-\frac{u^2}{4} p_1(t^{-1})\right] u f(u)\, du \right\}. \end{aligned} \tag{I.46}$$

Next we substitude $v = u^2$ in (I.46) and use the facts

$G_2(\omega) \overset{1}{=} x\, f(\sqrt{s})$ and $G_0(\omega) \overset{1}{=} f(\sqrt{x})$ to get the final form:

$$\begin{aligned} s_j\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) &\overset{n}{=} 2^{-2}\pi^{-\frac{n}{2}}\, p_1(t^{-1})^{-1}\, \cdot \\ pn(t^{-\frac{1}{2}})\, t_j^{-2} &\left\{ G_2\left(\frac{1}{4}p_1(t^{-1})\right) - 2t_j\, G_0\left(\frac{1}{4}p_1(t^{-1})\right)\right\}. \end{aligned}$$

(e) Multiplying the equation (I.37) by $s_j^{\frac{3}{2}}\, pn(\sqrt{s})$ yields

$$s_j^{\frac{3}{2}}\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) = \frac{1}{2}\int_0^\infty s_j^{\frac{3}{2}}\, pn(\sqrt{s})\, \exp(-up_1(\sqrt{s})\, u\, f(u)\, du. \tag{I.47}$$

By using the results of equations (I.11) and (I.23), we can arrive at

$$\text{L.H.S.} \stackrel{n}{=} 2^{-4}\pi^{-\frac{n}{2}} \int_0^\infty t_j^{-3}\, pn(t^{-\frac{1}{2}})\, u(u^2-6t_j) \cdot \exp\left[-\frac{u^2}{4}p_1(\sqrt{s})\right] u\, f(u)\, du. \tag{I.48}$$

Next by substituting $v = u^2$ in (I.48), we obtain

$$\text{L.H.S.} \stackrel{n}{=} 2^{-5}\pi^{-\frac{n}{2}} t_j^{-3}\, pn(t^{-\frac{1}{2}}) \left\{ \int_0^\infty \exp\left[-\frac{u^2}{4}p_1(t^{-1})\right] \cdot v^{\frac{3}{2}} f(\sqrt{v})\, dv - 6t_j \int_0^\infty \exp\left[-\frac{v}{4}p_1(t^{-1})\right] v^{\frac{1}{2}} f(\sqrt{v})\, dv \right\}.$$

Since $G_3(\omega) \stackrel{1}{=} x^{\frac{3}{2}} f(\sqrt{x})$ and $G_1(\omega) \stackrel{1}{=} x^{\frac{1}{2}} f(\sqrt{x})$, we find

$$s_j^{\frac{3}{2}}\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) \stackrel{n}{=} 2^{-3}\pi^{-\frac{n}{2}} t_j^{-3} \cdot pn(t^{-\frac{1}{2}})\, p_1(t^{-1})^{-1} \left\{ G_3\left(\frac{1}{4}p_1(t^{-1})\right) - 6t_j\, G_1\left(\frac{1}{4}p_1(t^{-1})\right)\right\}.$$

(f) Next we consider

$$\sqrt{s_j}\, pn(s)\, p_1(\sqrt{s})^{-1}\, \xi(p_1(\sqrt{s})^2) = \frac{1}{2}\int_0^\infty s_j^{\frac{1}{2}}\, pn(s)\, \exp\left[-up_1(\sqrt{s})\right] u\, f(u)\, du. \tag{I.49}$$

Now we use equations (I.15) and (I.20) and arrive at

$$\text{L.H.S.} \stackrel{n}{=} 2^{-n-2}\pi^{-\frac{n}{2}} \int_0^\infty t_j^{-1}\, pn(t^{-\frac{3}{2}} \cdot u^{n-1}(u^2-2t_j) \exp\left[-\frac{u^2}{4}p_1(\sqrt{s})\right] uf(u)\, du. \tag{I.50}$$

Again we substitute $v = u^2$ and obtain

$$\text{L.H.S.} \ \overset{n}{=} \ 2^{-n-3}\,\pi^{-\frac{n}{2}}\,t_j^{-1}\,p_n(t^{-\frac{3}{2}}) \left\{ \int_0^\infty v^{\frac{n+1}{2}} \exp\left[-\tfrac{v}{4} p_1(t^{-1})\right] \cdot \right.$$
$$\left. f(\sqrt{v})\,dv - 2t_j \int_0^\infty v^{\frac{n-1}{2}} \exp\left[-\tfrac{v}{4} p_1(t^{-1})\right] f(\sqrt{v})\,dv \right\}.$$

From the hypothesis (iv), we have $G_{n+1}(\omega) \overset{1}{=} x^{\frac{n+1}{2}} f(\sqrt{x})$ and $G_{n-1}(\omega) \overset{1}{=} x^{\frac{n-1}{2}} f(\sqrt{x})$.

Therefore, we obtain the following transform pair:

$$\sqrt{s_j}\,p_n(s)\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s}))^2 \overset{n}{=} 2^{-n-1}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{3}{2}}) \cdot$$
$$p_1(t^{-1})^{-1}\,t_j^{-1} \left\{ G_{n+1}\left(\tfrac{1}{4} p_1(t^{-1})\right) - 2t_j\,G_{n-1}\left(\tfrac{1}{4} p_1(t^{-1})\right) \right\}.$$

(g) Finally, we multiply (I.37) by $s_j\,p_n(s)$ and get

$$s_j\,p_n(s)\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) = \qquad \text{(I.51)}$$
$$\frac{1}{2} \int_0^\infty s_j\,p_n(s) \exp\left[-u p_1(\sqrt{s})\right] u\,f(u)\,du.$$

By using the results (I.15) and (I.23), we derive

$$\text{L.H.S.} \overset{n}{=} 2^{-n-3}\,\pi^{-\frac{n}{2}} \int_0^\infty t_j^{-2}\,p_n(t^{-\frac{3}{2}}) \cdot \qquad \text{(I.52)}$$
$$u^n(u^2 - 6t_j) \exp\left(-\frac{u^2}{4} p_1(t^{-1})\right) u f(u)\,du.$$

Next we substitute $v = u^2$ and take $G_{n+2}(\omega) \overset{1}{=} x^{\frac{n+2}{2}} f(\sqrt{x})$ and $G_n(\omega) \overset{1}{=} x^{\frac{n}{2}} f(\sqrt{s})$ to obtain

$$s_j\,p_n(s)\,p_1(\sqrt{s})^{-1}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^{-n-2}\,\pi^{-\frac{n}{2}}\,t_j^{-2} \cdot$$
$$p_n(t^{-\frac{3}{2}})\,p_1(t^{-1})^{-1} \left\{ G_{n+2}\left(\tfrac{1}{4} p_1(t^{-1})\right) - 6t_j\,G_n\left(\tfrac{1}{4} p_1(t^{-1})\right) \right\}.$$

Example: (Two-dimensions)

Let $f(t) = t^{2c+1}\ {}_nF_m\left[\begin{matrix}(a); k^2 t^2\\ (b);\end{matrix}\right]$, $\Re c \geq 0$. Then

$$\varphi(s) = \frac{\Gamma(2c+2)}{s^{2c+1}} \cdot$$

$$_{n+2}F_m\left[\begin{matrix}(a), c+1, \frac{2c+3}{2}; & \frac{4k^2}{s^2}\\ (b) & ;\end{matrix}\right], \quad \begin{matrix} n \leq m-1\\ \Re\, s > 0 \text{ if } n+2 \leq m\\ \Re\, s > 2|\Re\, k| \text{ if } n = m-1.\end{matrix}$$

$$\sqrt{s}\,\varphi(\sqrt{s}) = \frac{\Gamma(2c+2)}{s^c}\ {}_{n+2}F_m\left[\begin{matrix}(a), c+1, \frac{2c+3}{2}; & \frac{4k^2}{s}\\ (b) & ;\end{matrix}\right].$$

But $\sqrt{s}\,\varphi(\sqrt{s}) \stackrel{1}{=} F(t)$, we have

$$F(t) = \Gamma(2c+2)\frac{t^c}{\Gamma(c+1)} \cdot$$

$$_{n+2}F_{m+1}\left[\begin{matrix}(a), c+1, \frac{2c+3}{2}; & 4k^2 t\\ (b), c+1 & ;\end{matrix}\right], \quad \begin{matrix} n \leq m-1\\ \Re\, s > 0 \text{ if } n < m-1\\ \Re\, s > \Re\, k \text{ if } n = m-1.\end{matrix}$$

Since $\psi(s) \stackrel{1}{=} t\,F(t)$, we obtain

$$\frac{\psi(s)}{s} = \frac{\Gamma(2c+2)}{\Gamma(c+1)}\frac{\Gamma(c+2)}{s^{c+1}} \cdot$$

$$_{n+3}F_{m+1}\left[\begin{matrix}(a), c+1, \frac{2c+3}{2}, c+2; & \frac{4k^2}{s}\\ (b), c+1 & ;\end{matrix}\right],$$

$$n \leq m-1$$

$$\Re\, s > 0 \text{ if } n < m-1$$

$$\Re\, s > \Re\, k \text{ if } n = m-1$$

$$= \frac{2^{2c+1}\,\Gamma(c+\frac{3}{2})\,\Gamma(c+2)}{\sqrt{\pi}}\, s^{-c-1}\,.$$

$${}_{n+3}F_{m+1}\left[\begin{array}{l}(a), c+1, \frac{2c+3}{2}, c+2;\ \frac{4k^2}{s}\\ (b), c+1 \qquad\qquad\qquad ;\end{array}\right].$$

By Duplication Theorem:

$$= \frac{1}{2}\,\frac{\Gamma(2c+3)}{s^{c+1}}\,.$$

$${}_{n+3}F_{m+1}\left[\begin{array}{l}(a), c+1, \frac{2c+3}{2}, c+2;\ \frac{4k^2}{s}\\ (b), c+1 \qquad\qquad\qquad ;\end{array}\right].$$

Again by Duplication Theorem.

Therefore

$$\frac{\varphi\left[(\sqrt{p}+\sqrt{q})^2\right]}{(\sqrt{p}+\sqrt{q})} = \frac{\Gamma(2c+3)}{2(\sqrt{p}+\sqrt{q})^{2c+1}}\,.$$

$${}_{n+3}F_{m+1}\left[\begin{array}{l}(a), c+1, \frac{2c+3}{2}, c+2;\ \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2}\\ (b), c+1 \qquad\qquad\qquad ;\end{array}\right].$$

By using the results of Theorem I.2, we form the following table.

$$\frac{(pq)^{\frac{1}{2}}}{(\sqrt{p}+\sqrt{q})^{2c+1}}\ {}_{n+3}F_{m+1}\left[\begin{array}{l}(a), c+1, \frac{2c+3}{2}, c+2;\ \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2}\\ (b), c+1 \qquad\qquad\qquad ;\end{array}\right] \underline{\underline{2}}$$

$$\frac{1}{\sqrt{\pi}\,\Gamma(c+2)}\,\frac{(xy)^{c+1}}{(x+y)^{c+\frac{3}{2}}}\ {}_{n+1}F_{m}\left[\begin{array}{l}(a), c+\frac{3}{2};\ \frac{4k^2xy}{x+y}\\ (b) \qquad\quad ;\end{array}\right]$$

$$\frac{pq}{(\sqrt{p}+\sqrt{q})^{2c+1}}\ {}_{n+3}F_{m+1}\left[\begin{array}{l}(a), c+1, \frac{2c+3}{2}, c+2;\ \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2}\\ (b), c+1 \qquad\qquad\qquad ;\end{array}\right] \underline{\underline{2}}$$

$$\frac{\Gamma(c+\frac{5}{2})}{\pi\Gamma(2c+3)}\frac{(4xy)^{c+1}}{(x+y)^{c+\frac{5}{2}}}\ {}_{n+1}F_m\left[\begin{array}{ll}(a),c+\frac{5}{2}; & \frac{4k^2xy}{(x+y)}\\ (b) & ;\end{array}\right]$$

$$\frac{p^{\frac{1}{2}}\,q)}{(\sqrt{p}+\sqrt{q})^{2c+1}}\ {}_{n+3}F_{m+1}\left[\begin{array}{ll}(a),c+1,\frac{2c+3}{2},c+2; & \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2}\\ (b),c+1 & ;\end{array}\right]\ \underline{\underline{2}}$$

$$\frac{1}{\sqrt{\pi}\,\Gamma\left(c+\frac{3}{2}\right)}\left(\frac{x}{y}\right)^{\frac{1}{2}}\frac{(xy)^{c+1}}{(x+y)^{c+2}}\ {}_{n+1}F_m\left[\begin{array}{ll}(a),c+2; & \frac{4k^2xy}{(x+y}\\ (b) & ;\end{array}\right]$$

$$\frac{p^{\frac{1}{2}}\,q^{\frac{3}{2}}}{(\sqrt{p}+\sqrt{q})^{2c+1}}\ {}_{n+3}F_{m+1}\left[\begin{array}{ll}(a),c+1,\frac{2c+3}{2},c+2; & \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2}\\ (b),c+1 & ;\end{array}\right]\ \underline{\underline{2}}$$

$$\frac{x}{\pi\Gamma(2c+3)}\frac{(4xy)^{c}}{(x+y)^{c+1}}\left\{\Gamma\left(c+\frac{5}{2}\right)\left(\frac{4xy}{x+y}\right)\ {}_{n+1}F_m\left[\begin{array}{ll}(a),c+\frac{5}{2}; & \frac{4k^2xy}{(x+y}\\ (b) & ;\end{array}\right]-\right.$$
$$\left.2\Gamma\left(c+\frac{3}{2}\right)y\ {}_{n+1}F_m\left[\begin{array}{ll}(a),c+\frac{3}{2}; & \frac{4k^2xy}{x+y}\\ (b) & ;\end{array}\right]\right\}$$

$$\frac{pq^{\frac{3}{2}}}{(\sqrt{p}+\sqrt{q})^{2c+1}}\ {}_{n+3}F_{m+1}\left[\begin{array}{ll}(a),c+1,\frac{2c+3}{2},c+2; & \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2}\\ (b),c+1 & ;\end{array}\right]\ \underline{\underline{2}}$$

$$\frac{x^{-\frac{1}{2}}y^{-\frac{3}{2}}}{16\pi\Gamma(2c+3)}\frac{(4xy)^{c+1}}{(x+y)^{c+2}}\left\{4\Gamma(c+3)\left(\frac{xy}{x+y}\right)\ {}_{n+1}F_m\left[\begin{array}{ll}(a),c+3; & \frac{4k^2xy}{(x+y}\\ (b) & ;\end{array}\right]-\right.$$

$$2\Gamma(c+2)y \; {}_{n+1}F_m \left[\begin{matrix} (a), c+3; & \frac{4k^2xy}{x+y} \\ (b) \quad ; & \end{matrix} \right] \Bigg\}$$

$$\frac{p^{\frac{1}{2}} q^2}{(\sqrt{p}+\sqrt{q})^{2c+1}} \; {}_{n+3}F_{m+1} \left[\begin{matrix} (a), c+1, \frac{2c+3}{2}, c+2; & \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2} \\ (b), c+1 \quad ; & \end{matrix} \right] \overset{2}{=}$$

$$\frac{x^{\frac{1}{2}} y^{-\frac{5}{2}}}{4\pi\Gamma(2c+3)} \frac{(4xy)^{c+1}}{(x+y)^{c+2}} \left\{ 4\Gamma(c+3) \left(\frac{xy}{x+y}\right) \; {}_{n+1}F_m \left[\begin{matrix} (a), c+3; & \frac{4k^2xy}{(x+y} \\ (b) \quad ; & \end{matrix} \right] - \right.$$

$$\left. 6\Gamma(c+2)y \; {}_{n+1}F_m \left[\begin{matrix} (a), c+3; & \frac{4k^2xy}{x+y} \\ (b) \quad ; & \end{matrix} \right] \right\}$$

$$\frac{pq^2}{(\sqrt{p}+\sqrt{q})^{2c+1}} \; {}_{n+3}F_{m+1} \left[\begin{matrix} (a), c+1, \frac{2c+3}{2}, c+2; & \frac{4k^2}{(\sqrt{p}+\sqrt{q})^2} \\ (b), c+1 \quad ; & \end{matrix} \right] \overset{2}{=}$$

$$\frac{2x}{\pi\Gamma(2c+3)} \frac{(4xy)^c}{(x+y)^{c+\frac{5}{2}}} \left\{ 4\Gamma\left(c+\frac{7}{3}\right) \left(\frac{xy}{x+y}\right) \cdot \; {}_{n+1}F_m \left[\begin{matrix} (a), c+3; & \frac{4k^2xy}{(x+y} \\ (b) \quad ; & \end{matrix} \right] - \right.$$

$$\left. 6\Gamma\left(c+\frac{5}{2}\right) y \; {}_{n+1}F_m \left[\begin{matrix} (a), c+\frac{5}{2}; & \frac{4k^2xy}{x+y} \\ (b) \quad ; & \end{matrix} \right] \right\}$$

Theorem I.3:

Let us assume that

$$(i)\quad \varphi(\omega) \stackrel{1}{=} f(x).$$
$$(ii)\quad \omega\varphi(\sqrt{\omega}) \stackrel{1}{=} F(x).$$
$$(iii)\quad \xi(\omega) \stackrel{1}{=} x^{-1} F(x).$$
$$(iv)\quad G_j(\omega) \stackrel{1}{=} x^{\frac{j}{2}} f(\sqrt{x}),$$
$$j = 0, 1, 2, 3, n-1, n, n+1, n+2,$$

and that $f(x), F(x), x\,F(x), x^{\frac{j}{2}} f(\sqrt{x})$, for $j = 0, 1, 2, 3, n-1, n, n+1, n+2$ be continuous and absolutely integrable in $(0, \infty)$. Then

(a)

$$p_n(\sqrt{s})\,p_1(\sqrt{s})^{-2}\left[2\xi(p_1(\sqrt{s})^2 + \varphi(p_1(\sqrt{s}))\right] \stackrel{n}{=}$$
$$2\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})p_1(t^{-1})^{-1}\,G_0\left(\frac{1}{4}p_1(t^{-1})\right).$$

(b)

$$p_n(s)\,p_1(\sqrt{s})^2\left[2\xi(p_1(\sqrt{s})^2 + \varphi(p_1(\sqrt{s}))\right] \stackrel{n}{=}$$
$$2^{-n+1}\pi^{-\frac{n}{2}}p_n(t^{-\frac{3}{2}})\,p_1(t^{-1})^{-1}\,G_n\left(\frac{1}{4}p_1(t^{-1})\right).$$

(c)

$$\sqrt{s_j}\,p_n(\sqrt{s})\,p_1(\sqrt{s})^{-2}\left[2\xi(p_1(\sqrt{s})^2) + \varphi(p_1(\sqrt{s})\right] \stackrel{n}{=}$$
$$\pi^{-\frac{n}{2}}\,t_j^{-1}\,p_n(t^{-\frac{1}{2}})p_1(t^{-1})^{-1}\,G_1\left(\frac{1}{4}p_1(t^{-1})\right), \text{for } j = 1, 2, \ldots, n.$$

(d)

$$s_j\, p_n(\sqrt{s})\, p_1(\sqrt{s})^{-2}\left[2\xi(p_1(\sqrt{s})^2)+\varphi(p_1(\sqrt{s}))\right] \overset{n}{=} \pi^{-\frac{n}{2}}\, t_j^{-2}\,.$$
$$p_n(t^{-\frac{1}{2}})\, p_1(t^{-1})^{-1}\left[G_2\left(\frac{1}{4}p_1(t^{-1})\right)-2t_j\, G_0\left(\frac{1}{4}p_1(t^{-1})\right)\right],$$
for $j=1,2,\ldots,n$.

(e)

$$s_j^{\frac{3}{2}}\, p_n(\sqrt{s})\, p_1(\sqrt{s})^{-2}\left[2\xi(p_1(\sqrt{s})^2+\varphi(p_1(\sqrt{s}))\right] \overset{n}{=} 2^{-1}\,\pi^{-\frac{n}{2}}\, t_j^{-3}\,.$$
$$p_n(t^{-\frac{1}{2}})\, p_1(t^{-1})^{-1}\left[G_3\left(\frac{1}{4}p_1(t^{-1})\right)-2t_j G_1\left(\frac{1}{4}p_1(t^{-1})\right)\right],$$
for $j=1,2,\ldots,n$.

(f)

$$\sqrt{s_j}\, p_n(s)\, p_1(\sqrt{s})^{-2}\left[2\xi(p_1(\sqrt{s})^{-2}+\varphi(\sqrt{s})\right] \overset{n}{=} 2^{-n}\,\pi^{-\frac{n}{2}}\, t_j^{-1}\,.$$
$$p_n(t^{-\frac{3}{2}})\, p_1(t^{-1})^{-1}\left[G_{n+1}\left(\frac{1}{4}p_1(t^{-1})\right)-2t_j G_{n-1}\left(\frac{1}{4}p_1(t^{-1})\right)\right],$$
for $j=1,2,\ldots,n$.

(g)

$$s_j p_n(s)\, p_1(\sqrt{s})^{-2}\left[2\xi(p_1(\sqrt{s})^2)+\varphi(\sqrt{s})\right] \overset{n}{=} 2^{-n-1}\,\pi^{-\frac{n}{2}}\, t_j^{-2}\,.$$
$$p_n(t^{-\frac{3}{2}})\, p_1(t^{-1})^{-1}\left[G_{n+2}\left(\frac{1}{4}p_1(t^{-1})\right)-6t_j\, G_n\left(\frac{1}{4}p_1(t^{-1})\right)\right],$$
for $j=1,2,\ldots,n$.

Proof: From the hypothesis (i), we can find

$$\varphi(\sqrt{\omega})=\int_0^\infty \sqrt{\omega}\, f(u)\, e^{-u\sqrt{\omega}}, \quad \text{where } \Re\,\omega>0 \tag{I.53}$$

Since $\omega\,\varphi(\sqrt{\omega}) \stackrel{1}{=} F(x)$ and

$$\omega^{\frac{3}{2}}\, e^{-u\sqrt{\omega}} \stackrel{1}{=} \frac{(u^2-2x)}{4\sqrt{\pi}}\, x^{-\frac{5}{2}}\, e^{-\frac{u^2}{4x}},$$

we can write

$$F(x) = \int_0^{\infty} \frac{(u^2-2x)}{4\sqrt{\pi}}\, x^{-\frac{5}{2}}\, e^{-\frac{u^2}{4x}}\, f(u)du. \tag{I.54}$$

Now by using (I.54) and explicit form of (iii), we obtain

$$\frac{\xi(\omega)}{\omega} = \frac{1}{4\sqrt{\pi}} \int_0^{\infty}\int_0^{\infty} (u^2-2x)\, x^{-\frac{3}{2}} \cdot e^{-\frac{u^2}{4x}-\omega x}\, f(u)du\,dt \quad \text{with } \Re\,\omega > 0. \tag{I.55}$$

We know that $x^{-\frac{3}{2}}\,(u^2-2x)\,e^{-\frac{u^2}{4x}-\omega x}\, f(u)$ is integrable in $(0,\infty)\times(0,\infty)$ for $\Re\,\omega > 0$. Therefore we can apply Fubini's Theorem in order to interchange the order of integration. Now we can write (I.55) as

$$\frac{\xi(\omega)}{\omega} = \int_0^{\infty} f(u)\left\{\frac{1}{4\sqrt{\pi}} \cdot \int_0^{\infty} (u^2-2x)\, x^{-\frac{3}{2}}\, e^{-\frac{u^2}{4x}-\omega x}\, dx\right\} du, \quad \text{where } \Re\,\omega > 0. \tag{I.56}$$

Next we can use results from Roberts and Kaufman [37] to evaluate the inner integral to get

$$\frac{1}{4\sqrt{\pi}} \int_0^{\infty} \left(u^2 x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}}\right) e^{-\frac{u^2}{4x}-\omega x}\, dx = \frac{1}{2} e^{-u\sqrt{\omega}} \left(u - \omega^{-\frac{1}{2}}\right). \tag{I.57}$$

Hence we can write (I.56) as

$$\frac{\xi(\omega)}{\sqrt{\omega}} = \frac{\sqrt{\omega}}{2} \int_0^{\infty} u\, f(u)\, e^{-u\sqrt{\omega}}\, du - \frac{1}{2}\int_0^{\infty} f(u)\, e^{-u\sqrt{\omega}}\, du. \tag{I.58}$$

But from hypothesis (i), it follows

$$\int_0^\infty f(u)\, e^{-u\sqrt{\omega}}\, du = \omega^{-\frac{1}{2}}\, \varphi(\omega). \tag{I.59}$$

By using (I.59) we can write (I.58) as

$$\omega^{-1}\,(2\,\xi(\omega) + \varphi(\sqrt{\omega})) = \int_0^\infty u\, f(u)\, e^{-u\sqrt{\omega}}\, du. \tag{I.60}$$

Next, we replace ω by $p_1(\sqrt{s})^2$ and obtain

$$p_1(\sqrt{s})^{-2}\,\left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi\left(p_1(\sqrt{s})\right)\right] = \int_0^\infty u\, f(u)\, e^{-up_1(\sqrt{s})}\, du. \tag{I.61}$$

To prove (a).

From (I.61), it follows

$$p_n(\sqrt{s})\, p_1(\sqrt{s})^{-2}\,\left[2\xi(p_1(\sqrt{s})^2) + \varphi(p_1(\sqrt{s}))\right] = \int_0^\infty p_n(\sqrt{s})\, u\, f(u)\, e^{-up_1(\sqrt{s})}\, du. \tag{I.62}$$

By using (I.11), we can arrive at

$$\text{L.H.S} \overset{n}{=} 2^{-1}\, \pi^{-\frac{n}{2}} \int_0^\infty p_n(t^{-\frac{1}{2}}) \cdot \exp\left[-\frac{u^2}{4}\, p_1(t^{-1})\right]\, u\, f(u) du. \tag{I.63}$$

Next, we substitute $v = u^2$ in (I.63) to get

$$\text{L.H.S} \overset{n}{=} 2^{-1}\, \pi^{-\frac{n}{2}}\, p_n(t^{-\frac{1}{2}}) \cdot \int_0^\infty \exp\left[-\frac{v}{4}\, p_1(t^{-1})\right]\, f(\sqrt{v}) dv.$$

By taking $j = 0$ in (iv), we obtain the final form as

$$p_n(\sqrt{s})\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi\left(p_1(\sqrt{s})\right)\right] \overset{n}{=} 2\pi^{-\frac{n}{2}}\, p_n(t^{-\frac{1}{2}})\, p_1(t^{-1})^{-1}\, G_0\left(\frac{1}{4}p_1(t^{-1})\right).$$

To prove (b).

From (I.61), we have

$$p_n(s)\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right)\right] = \int_0^\infty p_n(s) \exp\left[-u p_1(\sqrt{s})\right] u\, f(u) du. \tag{I.64}$$

From (I.64), with the use of equation (I.15), we can find

$$\text{L.H.S} \overset{n}{=} 2^{-n}\pi^{-\frac{n}{2}} \int_0^\infty p_n\left(t^{-\frac{3}{2}}\right) u^{n+1} f(u) \cdot \exp\left[-u p_1(t^{-1})\right] du. \tag{I.65}$$

Now by substituting $v = \sqrt{v}$ and by taking $G_n(\omega) \overset{1}{=} x^{\frac{n}{2}} f(\sqrt{x})$, we can get the final form

$$p_n(s)\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi(p_1(\sqrt{s}))\right] \overset{n}{=} 2^{-n+1}\pi^{-\frac{n}{2}}\, p_n\left(t^{-\frac{3}{2}}\right) p_1(t^{-1})^{-1}\, G_n\left(\frac{1}{4}p_1\left(t^{-1}\right)\right).$$

To prove (c).

Multiplying the equation (I.61) by $\sqrt{s_j}\, p_n(\sqrt{s})$, for $j = 1, 2, \ldots, n$, yields

$$\sqrt{s_j}\, p_n(\sqrt{s})\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi(p_1(\sqrt{s})) = \right] \int_0^\infty \sqrt{s_j}\, p_n(\sqrt{s}) \cdot \exp\left[-u\, p_1(\sqrt{s})\right] u\, f(u) du. \tag{I.66}$$

By using equations (I.11) and (I.15), we arrive at

$$\text{L.H.S} \overset{n}{=} 2^{-1}\pi^{-\frac{n}{2}} \int_0^\infty t_j^{-1}\, pn\left(t^{-\frac{1}{2}}\right) \cdot \tag{I.67}$$
$$\exp\left[-\frac{u^2}{4}\, p_1(t^{-1})\right] u^2 f(u)du.$$

After substituting $v = u^2$ and taking $G_1(\omega) \overset{1}{=} \sqrt{x}\, f(\sqrt{x})$, we conclude

$$\sqrt{s_j}\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi\left(p_1(\sqrt{s})\right)\right] \overset{n}{=}$$
$$\pi^{-\frac{n}{2}}\, pn(t^{-\frac{1}{2}})\, p_1(t^{-1})^{-1}\, t_j^{-1}\, G_1\left(\frac{1}{4}p_1(t^{-1})\right), \quad \text{for } j = 1, 2, \ldots, n.$$

To prove (d).

We multiply the equation (I.61) by $s_j\, pn(\sqrt{s})$, for $j = 1, 2, \ldots, n$, and use facts given by equations (I.11) and (I.20) to get

$$s_j\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi(\sqrt{s})\right] \overset{n}{=} 2^{-2}\pi^{-\frac{n}{2}} \cdot \tag{I.68}$$
$$\int_0^\infty t_j^{-2}\, pn(t^{-\frac{1}{2}}) \exp\left[-\frac{u^2}{4}\, p_1(t^{-1})\right] (u^2 - 2t_j)\, u\, f(u)du.$$

Now we substitute $v = u^2$ and take $G_2(\omega) \overset{1}{=} x\, f(\sqrt{x}), G_0(\omega) = f(\sqrt{x})$ to get

$$s_j\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2 + \varphi(\sqrt{s})\right]\right. \overset{n}{=} \pi^{-\frac{n}{2}}\, t_j^{-2}\, pn(t^{-\frac{1}{2}})\, p_1(t^{-1})^{-1} \cdot$$
$$\left[G_2\left(\frac{1}{4}p_1(t^{-1})\right) - 2t_j\, G_0\left(\frac{1}{4}p_1(t^{-1})\right)\right], \text{ for } j = 1, 2, \ldots, n.$$

To prove (e).

Next we multiply the equation (I.61) by $s_j^{\frac{3}{2}}\, pn(\sqrt{s})$ for $j = 1, 2, \ldots, n$, and use the results of equation (I.11) and (I.23) to get

$$s_j^{\frac{3}{2}}\, pn(\sqrt{s})\, p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi(\sqrt{s})\right] \overset{n}{=} 2^{-3}\pi^{-\frac{n}{2}} \cdot \tag{I.69}$$
$$\int_0^\infty t_j^{-3}\, pn(t^{-\frac{1}{2}}) \exp\left[-\frac{u^2}{4}\, p_1(t^{-1})\right] u^2(u^2 - 2t_j)\, f(u)du.$$

Now we take $v = u^2$ and $G_3(\omega) \overset{1}{=} x^{\frac{3}{2}} f(\sqrt{x}), G_1(\omega) = x^{\frac{1}{2}} f(\sqrt{x})$ to arrive at

$$s_j^{\frac{3}{2}} p_n(\sqrt{s}) p_1(\sqrt{s})^{-2} \left[2\xi\left(p_1(\sqrt{s})^2\right) + \varphi(\sqrt{s})\right] \overset{n}{=} 2^{-1} \pi^{-\frac{n}{2}} t_j^{-3} p_n(t^{-\frac{1}{2}}) \cdot$$
$$p_1(t^{-1})^{-1} \left[G_3\left(\frac{1}{4} p_1(t^{-1})\right) - 2t_j G_1\left(\frac{1}{4} p_1(t^{-1})\right)\right], \text{ for } j = 1, 2, \ldots, n.$$

To prove (f) and (g). We can prove these two cases by following the same lines as in the above cases, with the use of equations (I.15), (I.20) and (I.23) and by taking $G_{n+1}(\omega) \overset{1}{=} x^{\frac{n+2}{2}} f(\sqrt{x})$, $G_{n-1}(\omega) \overset{1}{=} x^{\frac{n-1}{2}} f(\sqrt{x})$, $G_{n+2}(\omega) \overset{1}{=} x^{\frac{n+2}{2}} f(\sqrt{x})$ and $G_n(\omega) \overset{1}{=} x^{\frac{n}{2}} f(\sqrt{x})$.

Example: (Two-dimensions)

Let $f(t) = e^{at} - 1, \Re\, a < 0$, so that $\varphi(s) = \frac{a}{s-a}$, $\Re\, s > 0$, $\Re\, a^2$. But $s\,\varphi(\sqrt{s}) = \frac{a\,s}{\sqrt{s}-a} \overset{1}{=} F(t)$.

Therefore $F(t) = \dfrac{a\, e^{a^2 t_2} D_{-2}(-a\sqrt{2t})}{\sqrt{\pi t}}$.

Since $\psi(s) \overset{1}{=} t\, F(t)$, we have

$$\frac{\psi(s)}{s} = \frac{a}{\sqrt{\pi}} \cdot \left(\frac{\pi}{s}\right)^{\frac{1}{2}} \left(-\sqrt{2}\,a + \sqrt{2s}\right)^{-2} = \frac{a}{2\sqrt{s}\,(\sqrt{s}-a)^2}.$$

Replacing s by $(\sqrt{p}+\sqrt{q})^2$, we obtain

$$\frac{\psi\left[(\sqrt{p}+\sqrt{q})^2\right]}{(\sqrt{p}+\sqrt{q})^2} = \frac{a}{2(\sqrt{p}+\sqrt{q})(\sqrt{p}+\sqrt{q}-a)^2}.$$

Also we can find

$$\frac{\varphi(\sqrt{p}+\sqrt{q})}{(\sqrt{p}+\sqrt{q})^2} = \frac{a}{(\sqrt{p}+\sqrt{q}-a)(\sqrt{p}+\sqrt{q})^2}.$$

Next we apply Theorem I.3 to this example and obtain following Table of Inverse Laplace Carson Transform pairs.

$$\frac{\sqrt{pq}\,(2(\sqrt{p}+\sqrt{q})-a)}{(\sqrt{p}+\sqrt{q})^2\,(\sqrt{p}+\sqrt{q}-a)^2} \overset{2}{=} \frac{2xy}{\sqrt{\pi}\,(x+y)^{\frac{3}{2}}}\, e^{\frac{a^2\,xy}{x+y}}\, Erfc\left(a\sqrt{\frac{xy}{x+y}}\right).$$

$$\frac{pq\,(2(\sqrt{p}+\sqrt{q})-a)}{(\sqrt{p}+\sqrt{q})^2\,(\sqrt{p}+\sqrt{q}-a)^2} \overset{2}{=} \frac{8\Gamma(2)\,(xy)^{\frac{1}{2}}}{\pi a(x+y)^2} \left\{3e^{\frac{a^2\,xy}{x+y}}\, D_{-4}\left[\left(\frac{2a^2xy}{x+y}\right)^2\right] - 1\right\}.$$

$$\frac{p^{\frac{1}{2}}\,q\,(2(\sqrt{p}+\sqrt{q})-a)}{(\sqrt{p}+\sqrt{q})^2\,(\sqrt{p}+\sqrt{q}-a)^2} \overset{2}{=} \frac{x^{\frac{1}{2}}\,y^{-\frac{1}{2}}}{\pi a\,(x+y)}\,\cdot$$

$$\left\{\frac{\sqrt{\pi\,xy}}{(x+y)^{\frac{3}{2}}}\left(2a^2xy+x+y\right)e^{\frac{a^2xy}{x+y}}\, Erfc\left(a\sqrt{\frac{xy}{x+y}}\right) - \frac{2axy}{x+y} - \frac{2\sqrt{xy}\,\Gamma\left(\frac{3}{2}\right)}{\sqrt{x+y}}\right\}.$$

$$\frac{p^{\frac{1}{2}}\,q^2\,(2(\sqrt{p}+\sqrt{q})-a)}{(\sqrt{p}+\sqrt{q})^2\,(\sqrt{p}+\sqrt{q}-a)^2} \overset{2}{=} \frac{xy^{-1}}{\pi\,a(x+y^{\frac{5}{2}}}\,\cdot$$

$$\left\{\sqrt{2}\,\Gamma(s)\,x\,e^{\frac{a^2xy}{2(x+y)}}\, D_{-5}\left[\left(\frac{2a^2xy}{x+y}\right)^{\frac{1}{2}}\right] - 4x\Gamma\left(\frac{5}{2}\right) - \sqrt{\pi}\left(2a^2xy+x+y\right)\cdot\right.$$

$$\left. e^{\frac{a^2xy}{x+y}}\, Erfc\left(a\sqrt{\frac{xy}{x+y}}\right) - 3a(x+y) + 6\Gamma\left(\frac{3}{2}\right)(x+y)\right\}.$$

$$\frac{p^{\frac{1}{2}}\,q^{\frac{3}{2}}\,(2(\sqrt{p}+\sqrt{q})-a)}{(\sqrt{p}+\sqrt{q})^2\,(\sqrt{p}+\sqrt{q}-a)^2} \overset{2}{=} \frac{2x^{\frac{3}{2}}\,y^{-\frac{1}{2}}}{4\pi a(x+y)^2}\,\cdot$$

$$\left\{3\Gamma(2)\,e^{\frac{a^2xy}{2(x+y)}}\, D_{-4}\left[\left(\frac{2a^2xy}{x+y}\right)^{\frac{1}{2}}\right] - \Gamma(2) - a\sqrt{\pi}\,y\left(\frac{x+y}{4xy}\right)^{\frac{1}{2}}\cdot\right.$$

$$\left. e^{\frac{a^2xy}{x+y}}\, Erfc\left(a\sqrt{\frac{xy}{x+y}}\right)\right\}.$$

$$\frac{p q^{\frac{3}{2}} (2(\sqrt{p}+\sqrt{q})-a)}{(\sqrt{p}+\sqrt{q})^2 (\sqrt{p}+\sqrt{q}-a)^2} \stackrel{2}{=} \frac{1}{2\pi a(x+y)^{\frac{5}{2}}} \cdot$$

$$\Bigg\{ \sqrt{2}\,\Gamma(5) x\, e^{\frac{a^2xy}{2(x+y)}} D_{-5}\left[\left(\frac{2a^2xy}{x+y}\right)^{\frac{1}{2}}\right] - \sqrt{\pi}\,(2a^2xy+x+y)\cdot$$

$$e^{\frac{a^2xy}{x+y}}\, Erfc\Big(a\sqrt{\tfrac{xy}{x+y}}\Big) + a(x+y) + 2\Gamma\left(\tfrac{3}{2}\right)(y-2x)\Bigg\}.$$

$$\frac{pq^2 (2(\sqrt{p}+\sqrt{q})-a)}{(\sqrt{p}+\sqrt{q})^2 \sqrt{p}+\sqrt{q}-a)^2} \stackrel{2}{=} \frac{\Gamma(2)\, x^{\frac{1}{2}}\, y^{-\frac{3}{2}}}{\pi(x+y)^3} \Bigg\{(3x+3y-4xy) + e^{\frac{a^2xy}{x+y}} \cdot$$

$$\left[60xyD_{-6}\left[\left(\frac{2a^2xy}{x+y}\right)^{\frac{1}{2}}\right] - \left(xy+y^2\right) D_{-4}\left[\left(\frac{2a^2xy}{x+y}\right)^{\frac{1}{2}}\right]\right]\Bigg\}.$$

Proofs of the theorems to follow can be done by using similar techniques as in Theorem I.3 and the previous theorems with some modifications. Therefore, in what follows only the results are presented with some examples next.

Theorem I.4:

Assume that

(i) $\varphi(\omega) \stackrel{1}{=} f(x)$.

(ii) $\varphi(\sqrt{\omega}) \stackrel{1}{=} F(x)$.

(iii) $\xi(\omega) \stackrel{1}{=} x\,F(x)$.

(iv) $G_j(\omega) \stackrel{1}{=} x^{\frac{j-1}{2}} f(\sqrt{x})$,

for $j = 0, 1, 2, 3, n$ and $n+2$.

and that $f(x), F(x)$ and $x\,F(x), x^j\,f(\sqrt{x})$, for $j = 0, 1, 2, 3, n$ and $n+2$ be continuous and absolutely integrable in $(0, \infty)$. Then

(a)

$$p_n(\sqrt{s})\,p_1(\sqrt{s})^{-1}\,\xi[p_1(\sqrt{s})^2] \overset{n}{=} 2^{-1}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}}) \cdot$$
$$p_1(t^{-1})^{-1}\left[2G_0\left(\frac{1}{4}p_1(t^{-1})\right)+p_1(T^{-1})\,G_2\left(\frac{1}{4}p_1(t^{-1})\right)\right].$$

(b)

$$p_n(s)\,p_1(\sqrt{s})\,\xi[p_1(\sqrt{s})^2] \overset{n}{=} 2^{-n-1}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\,p_1(t^{-1})^{-1}.$$
$$\left[2(n+1)\,G_n\left(\frac{1}{4}p_1(t^{-1})\right)+p_1(t^{-1})\,G_{n+2}\left(\frac{1}{4}p_1(t^{-1})\right)\right].$$

(c)

$$\sqrt{s_j}\,p_n(\sqrt{s})\,p_1(\sqrt{s})\,\xi[p_1(\sqrt{s})^2] \overset{n}{=} 2^{-n}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}}) \cdot$$
$$p_1(t^{-1})^{-1}\,t_j^{-1}\left[4G_1\left(\frac{1}{4}p(t^{-1})\right)+p_1(t^{-1})\,G_3\left(\frac{1}{4}p_1(t^{-1})\right)\right],$$

for $j = 1, 2, \ldots, n$.

Example: (Two-dimensions)

Let $f(t) = t\,I_1(t), \Re s > 0$, then

$$\varphi(s) = \frac{s}{(s^2-1)^{3/2}} \text{ and} \varphi\sqrt{s} = \frac{\sqrt{s}}{(s-1)^{3/2}}$$

Since $F(t) \overset{1}{=} \varphi\sqrt{s}$, we find $F(t) = t\;\, {}_1F_1[3/2; 2; t]$. Also

$$\begin{aligned}\psi(s) &= 2s^{-2}\,{}_1F_1[3/2; 3; s^{-1}]\\ G_j(s) &= 2\Gamma(\frac{j+3}{2})s^{\frac{-j+1}{2}}e^{1/8s}M_{\frac{-j-1}{2},\frac{1}{2}}(s^{-1}),\end{aligned}$$

for $j = 1, 2, 3, 4$.

Now by using Theorem I.4, we can obtain the following pairs:

$$1.\sqrt{s_1 s_2}(\sqrt{s_1}+\sqrt{s_2})^{-3}{}_2F_1[3/2,3;2;(\sqrt{s_1}+\sqrt{s_2})^{-2}]$$
$$\overset{2}{=} \frac{1}{4\sqrt{\pi(t_1+t_2)}}e^{\frac{t_1t_2}{2(t_1+t_2)}}$$
$$[M_{-1/2,1/2}(\frac{4t_1t_2}{t_1+t_2})+3M_{-3/2,1/2}(\frac{4t_1t_2}{t_1+t_2})]$$

$$2.\frac{s_1s_2}{(\sqrt{s_1}+\sqrt{s_2})^{-3}}{}_2F_1[3/2,3;2;(\sqrt{s_1}+\sqrt{s_2})^{-2}]$$
$$\overset{2}{=} \frac{3t_1t_2}{8\sqrt{\pi(t_1+t_2)^3}}e^{\frac{t_1t_2}{2(t_1+t_2)}}$$
$$[3M_{-3/2,1/2}(\frac{4t_1t_2}{t_1+t_2})+5M_{-5/2,1/2}(\frac{4t_1t_2}{t_1+t_2})]$$

$$3.{s_1}^{3/2}s_2(\sqrt{s_1}+\sqrt{s_2})^{-3}{}_2F_1[3/2,3;2;(\sqrt{s_1}+\sqrt{s_2})^{-2}]$$
$$\overset{2}{=} -\frac{t_1^{-1/2}t_2^{1/2}}{\pi(t_1+t_2)e^{\frac{t_1t_2}{2(t_1+t_2)}}}$$
$$[M_{0,1/2}(\frac{4t_1t_2}{t_1+t_2})+M_{-2,1/2}(\frac{4t_1t_2}{t_1+t_2})]$$

Theorem I.5:

Let (i) $\varphi(\omega)\overset{1}{=}f(x)$.

(ii) $\varphi(\sqrt{\omega})\overset{1}{=}F(x)$.

(iii) $\xi(\omega)\overset{1}{=}x^{-2}F(x)$.

(iv) $G_j(\omega)\overset{1}{=}x^{\frac{j-5}{2}}f(\sqrt{x})$,

for $j=1,2,3,4,n+1$ and $n+3$.

Also let $f(x), F(x), x^{-2}\,F(x), x^{\frac{j-5}{2}}$, for $j = 1,2,3,4, n+1$ and $n+3$ be continuous and absolutely integrable in $(0,\infty)$. Then the following results hold:

(a)

$$p_n(\sqrt{s})\,p_1(\sqrt{s})^{-2}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^2\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\cdot$$
$$p_1(t^{-1})^{-1}\left[p_1(t^{-1})\,G_3\left(\frac{1}{4}\,p_1(t^{-1})\right) + 2G_1\left(\frac{1}{4}\,p_1(t^{-1})\right)\right].$$

(b)

$$p_n(s)\,p_1(\sqrt{s})^{-2}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^{-n+2}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{3}{2}})\cdot$$
$$p_1(t^{-1})^{-1}\left[p_1(t^{-1})\,G_{n+3}\left(\frac{1}{4}\,p_1(t^{-1})\right)\,p_1(t^{-1}) +\right.$$
$$\left. 2(n+1)\,G_{n+1}\left(\frac{1}{4}p_1(t^{-1})\right)\right].$$

(c)

$$\sqrt{s_j}\,p_n(\sqrt{s})\,p_1(\sqrt{s})^{-2}\,\xi(p_1(\sqrt{s})^2) \overset{n}{=} 2^{-n+3}\,\pi^{-\frac{n}{2}}\,p_n(t^{-\frac{1}{2}})\cdot$$
$$p_1(t^{-1})^{-1}\,t_j^{-1}\left[p_1(t^{-1})\,G_4\left(\frac{1}{4}p_1(t^{-1})\right) + G_2\left(\frac{1}{4}p(t^{-1})\right)\right],$$

for $j = 1,2,\ldots,n$.

Example: (Two-dimensions)

Let $f(t) = t^{2c}\;{}_nF_m\left[\begin{matrix}(a);t^2\\(b);\end{matrix}\right]$,

$Rec \geq 3$. Then

$$\varphi(s) = \frac{\Gamma(2c+1)}{s^{2c}}\cdot$$
$${}_{n+1}F_m\left[\begin{matrix}(a),2c+1;\; s^{-2}\\(b)\qquad\quad ;\end{matrix}\right],\quad \begin{matrix}n \leq m\\ \Re\, s > 0\end{matrix}$$

Which implies

$$\frac{\varphi(\sqrt{s})}{s} = \frac{\Gamma(2c+1)}{s^{c+1}} \; {}_{n+1}F_m \left[\begin{matrix} (a), 2c+1; \; s^{-1} \\ (b) \qquad ; \end{matrix} \right].$$

But $\frac{\varphi(\sqrt{s})}{s} \stackrel{1}{=} F(t)$. Therefore

$$F(t) = \Gamma(2c+1) \frac{t^c}{\Gamma(c+1)} \cdot$$
$${}_{n+1}F_{m+1} \left[\begin{matrix} (a), 2c+1; \; t \\ (b), c+1; \end{matrix} \right],$$

From which we find,

$$\begin{aligned} \psi(s) &= 2^{2c} \frac{\Gamma(c+1/2)}{\Gamma(c-1)} s^{c-2} \, . \\ & \quad {}_{n+2}F_{m+1} \left[\begin{matrix} (a), 2c+1, c-1; \; s^{-1} \\ (b), c+1 \qquad ; \end{matrix} \right], \\ K_j(s) &= \Gamma((2c+j-3)/2) s^{(-2c+j-3)/2} \, . \\ & \quad {}_{n+1}F_m \left[\begin{matrix} (a), \frac{2c+j-3}{2}; \; s^{-1} \\ (b) \qquad ; \end{matrix} \right], \end{aligned}$$

for $j = 1, 2, \ldots, 5$.

Now by using the results from Theorem I.5, we can obtain the following pairs:

1. $\sqrt{s_1 s_2}(\sqrt{s_1} + \sqrt{s_2})^{c-2}$

$${}_{n+2}F_{m+1} \left[\begin{matrix} (a), 2c+1, c-1; \; (\sqrt{s_1} + \sqrt{s_2})^{-2} \\ (b), c+1 \qquad ; \end{matrix} \right]$$

$$\stackrel{2}{=} \frac{2(t_1t_2)^{c-1/2}}{\Gamma(c+1/2)\Gamma(c-1)(t_1+t_2)^c} \cdot$$

$$\{2\Gamma(c)\ {}_{n+1}F_m \left[\begin{array}{l} (a), c;\ \frac{t_1+t_2}{4t_1t_2} \\ (b) \qquad ; \end{array} \right]$$

$$+\ \Gamma(c-1)\ {}_{n+1}F_m \left[\begin{array}{l} (a), c-1;\ \frac{t_1+t_2}{4t_1t_2} \\ (b) \qquad ; \end{array} \right] \}.$$

2. $s_1s_2(\sqrt{s_1}+\sqrt{s_2})^{c-2}$

$${}_{n+2}F_{m+1} \left[\begin{array}{l} (a), 2c+1, c-1;\ (\sqrt{s_1}+\sqrt{s_2})^{-2} \\ (b), c+1 \qquad ; \end{array} \right]$$

$$\stackrel{2}{=} \frac{2(t_1t_2)^{c-3/2}}{\Gamma(c+1/2)\Gamma(c-1)(t_1+t_2)^c} \cdot$$

$$\{2\Gamma(c+1)\ {}_{n+1}F_m \left[\begin{array}{l} (a), c+1;\ \frac{t_1+t_2}{4t_1t_2} \\ (b) \qquad ; \end{array} \right]$$

$$+\ 3\Gamma(c)\ {}_{n+1}F_m \left[\begin{array}{l} (a), c-1;\ \frac{t_1+t_2}{4t_1t_2} \\ (b) \qquad ; \end{array} \right] \}.$$

3. $s_1s_2^{1/2}(\sqrt{s_1}+\sqrt{s_2})^{c-2}$

$${}_{n+2}F_{m+1} \left[\begin{array}{l} (a), 2c+1, c-1;\ (\sqrt{s_1}+\sqrt{s_2})^{-2} \\ (b), c+1 \qquad ; \end{array} \right]$$

$$\stackrel{2}{=} \frac{2t_1-1(t_1t_2)^{c-1}}{\Gamma(c+1/2)\Gamma(c-1)(t_1+t_2)^{c-1/2}} \cdot$$

$$\{4\Gamma(c+1/2)\ {}_{n+1}F_m \left[\begin{array}{l} (a), c+1/2;\ \frac{t_1+t_2}{4t_1t_2} \\ (b) \qquad ; \end{array} \right]$$

$$+ \quad \Gamma(c-1/2) \ {}_{n+1}F_m \left[\begin{array}{l} (a), c-1/2; \ \frac{t_1+t_2}{4t_1t_2} \\ (b) \qquad ; \end{array} \right] \Big\}.$$

FURTHER RESULTS ON MULTI-DIMENSIONAL LAPLACE TRANSFORMS

We have used the operational techniques to obtain some results in the previous chapter. In this chapter, we use Mathematical induction with operational methods to derive some more results in multi-dimensional Laplace transforms.

Theorem I.6:

Suppose that, for $\Re(\omega) > 0$ and $0 < x < \infty$

$$(i) \quad \varphi(\omega) \stackrel{1}{=} f(x)$$

$$(ii) \quad \varphi_1 \stackrel{1}{=} (\pi x)^{-\frac{1}{2}} \varphi(x^{-1})$$

and $\varphi_j(\omega) \stackrel{1}{=} 2\pi^{-\frac{1}{2}} x^{\frac{1}{2}} \varphi_{j-1}\left((2x)^{-2}\right)$ for $j = 2, 3, \ldots$ and that $f(x^{2^n})$, for $n = 0, 1, 2, \ldots$ and $x^{-\frac{1}{2}} \varphi(x^{-1}), x^{\frac{1}{2}} \varphi_j(2x)^{-2})$ for $j = 1, 2, \ldots$ are continuous and absolutely integrable in $(0, \infty)$. Then

$$(*) \quad \varphi_n(\omega) = 2\omega \int_0^\infty e^{-2\sqrt{\omega}\, u} f(u^{2^n})\, du, \quad \text{for } n = 1, 2, \ldots.$$

Moreover, if we assume that for $k = 1, 2, 3, N, N+1, N+2, N+3 and n = 1, 2, \ldots$, $x^{\frac{k-2}{2}} f(x^{2^{n-1}})$ are continuous and absolutely integrable in $(0, \infty)$ then by taking $\omega = p_1 \left(s^{\frac{1}{2}}\right)^2$ we can obtain the following results:

(a)

$$p_N\left(s^{\frac{1}{2}}\right)p_1\left(s^{\frac{1}{2}}\right)^{-2}\varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right)\stackrel{N}{=}\pi^{-\frac{N}{2}}p_N\left(t^{-\frac{1}{2}}\right)\cdot$$

$$p_1(t^{-1})^{-1}A_n\left(p_1(t^{-1})\right),\ \text{where } A_n(\omega)\stackrel{1}{=}x^{-\frac{1}{2}}f\left(x^{2^{n-1}}\right).$$

(b)

$$s_j^{\frac{1}{2}}p_N\left(s^{\frac{1}{2}}\right)p_1\left(s^{\frac{1}{2}}\right)^{-2}\varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right)\stackrel{N}{=}$$

$$\pi^{-\frac{N}{2}}p_N\left(t^{-\frac{1}{2}}\right)t_j^{-1}p_1(t^{-1})^{-1}B_n\left(p_1(t^{-1})\right)\ \text{ for } j=1,2,\ldots,N;$$

$$\text{where } B_n(\omega)\stackrel{1}{=}f\left(x^{2^{n-1}}\right).$$

(c)

$$p_N(s)p_1\left(s^{-\frac{1}{2}}\right)^{-2}\varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right)\stackrel{N}{=}\pi^{-\frac{N}{2}}p_N\left(t^{-\frac{3}{2}}\right)\cdot$$

$$p_1(t^{-1})^{-1}C_n\left(p_1(t^{-1})\right),\ \text{where } C_n(\omega)\stackrel{1}{=}x^{\frac{N}{2}-\frac{1}{2}}f\left(x^{2^{n-1}}\right).$$

(d)

$$s_j p_N\left(s^{\frac{1}{2}}\right)p_1\left(s^{\frac{1}{2}}\right)^{-2}\varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right)\stackrel{N}{=}2^{-1}\pi^{-\frac{N}{2}}\cdot$$

$$p_N\left(t^{-\frac{1}{2}}\right)p_1(t^{-1})^{-1}\left[2D_n\left(p_1(t^{-1})\right)-t_j A_n\left(p_1(t^{-1})\right)\right],$$

for $j=1,2,\ldots,n;$ where $D_n(\omega)\stackrel{1}{=}x^{\frac{1}{2}}f\left(x^{2^{n-1}}\right)$

and $A_n(\omega)$ is defined as in (a).

(e)

$$s_j^{\frac{1}{2}}\, p_N(s)\, p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi_n\left(\left(p_1(s^{\frac{1}{2}})\right)^2\right) \stackrel{N}{=} 2^{-1}\pi^{-\frac{N}{2}}\, t_j^{-1}\cdot$$

$$p_N\left(t^{-\frac{3}{2}}\right) p_1(t^{-1})^{-1}\left[2E_n\left(p_1(t^{-1})\right) - t_j\, F_n\left(p_1(t^{-1})\right)\right],$$

for $j = 1, 2, \ldots, n$; where $E_n(\omega) \stackrel{1}{=} x^{\frac{N}{2}}\, f\left(x^{2^{n-1}}\right)$

and $F_n(\omega) \stackrel{1}{=} x^{\frac{N}{2}-1}\, f\left(x^{2^{n-1}}\right)$.

(f)

$$s_j\, p_N(s)\, p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi_n\left(\left(p_1(s^{\frac{1}{2}})\right)^2\right) \stackrel{N}{=} 2^{-1}\pi^{-\frac{N}{2}}\, t_j^{-2}\cdot$$

$$p_N\left(t^{-\frac{3}{2}}\right) p_1(t^{-1})^{-1}\left[G_n(p_1(t^{-1})) - 3t_j\, C_n(p_1(t^{-1}))\right],$$

for $j = 1, 2, \ldots, n$; where $G_n(\omega) \stackrel{1}{=} x^{\frac{N}{2}+\frac{1}{2}}\, f\left(x^{2^{n-1}}\right)$

and $C_n(\omega)$ is defined as in (c).

(g)

$$s_j^{\frac{3}{2}}\, p_N\left(s^{\frac{1}{2}}\right) p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi_n\left((p_1(s^{\frac{1}{2}})^2\right) \stackrel{N}{=} 2^{-1}\pi^{-\frac{N}{2}}\, t_j^{-3}\cdot$$

$$p_N\left(t^{-\frac{1}{2}}\right) p_1(t^{-1})^{-1}\left[D_n(p_1(t^{-1})) - 3t_j\, H_n(p_1(t^{-1}))\right],$$

for $j = 1, 2, \ldots, n$; where $H_n(\omega) \stackrel{1}{=} x\, f(x^{2^{n-1}})$

and $H_n(\cdot)$ is defined as in (b)

Proof: We will use mathematical induction to prove the first part of the theorem.

Since $\varphi(\omega) \stackrel{1}{=} f(x)$, we can write (ii) as

$$\varphi_1(\omega) = \pi^{-\frac{1}{2}}\omega \int_0^\infty \int_0^\infty x^{-\frac{3}{2}} e^{-\frac{v}{x}-\omega x} f(v)dv\,dx. \tag{I.70}$$

The integrand $x^{-\frac{3}{2}} e^{-\frac{v}{x}-\omega x} f(v)$ is absolutely integrable in $(0,\infty)\times(0,\infty)$, for $\Re(\omega) > 0$. Therefore, by Fubini's Theorem we can interchange the order of integration and obtain

$$\varphi_1(\omega) = \omega \int_0^\infty f(v) \left\{ \pi^{-\frac{1}{2}} \int_0^\infty x^{-\frac{3}{2}} e^{-\frac{v}{x}-\omega x} dx \right\} dv. \tag{I.71}$$

But the inner integral in (I.71) can be evaluated with the use of Roberts and Kaufman [37]. Hence we have

$$\varphi_1(\omega) = \omega \int_0^\infty e^{-2\sqrt{v\,\omega}}\, v^{-\frac{1}{2}} f(v)dv. \tag{I.72}$$

With the substitution $u = \sqrt{v}$ in (I.72), we obtain the following result

$$\varphi_1(\omega) = 2\omega \int_0^\infty e^{-2u\sqrt{\omega}} f(u^2)du.$$

Therefore, the result (*) is true for $n = 1$. Next we consider the case for $n = 2$.

We are given that

$$\begin{aligned} \varphi_2(\omega) \quad &\stackrel{1}{=} \quad 2\pi^{-\frac{1}{2}} x^{\frac{1}{2}} \varphi_1\left((2x)^{-2}\right), \\ \text{i.e., } \varphi_2(\omega) \quad &= \quad \pi^{-\frac{1}{2}}\omega \int_0^\infty \int_0^\infty x^{-\frac{3}{2}} e^{-\frac{v}{x}-\omega x} f(v^2)dv\,dx. \end{aligned} \tag{I.73}$$

Since $x^{-\frac{3}{2}} e^{-\frac{v}{x}-\omega x} f(v^2)$ is absolutely integrable in $(0,\infty)\times(0,\infty)$, for $\Re(\omega) > 0$, we can apply Fubini's theorem and interchange the order of integration. Therefore, we have

$$\varphi_2(\omega) = \omega \int_0^\infty f(v^2) \left\{ \pi^{-\frac{1}{2}} \int_0^\infty x^{-\frac{3}{2}} e^{-\frac{v}{x}-\omega x} dx \right\} dv. \tag{I.74}$$

The inner integral in (I.74) is the same as the inner integral in the equation (I.71). From which, it follows

$$\varphi_2(\omega) = \omega \int_0^\infty e^{-2\sqrt{v\,\omega}}\, v^{-\frac{1}{2}}\, f(v^2)\, dv. \tag{I.75}$$

Now by substituting $u = \sqrt{v}$ in (I.75), we can find

$$\varphi_2(\omega) = 2\omega \int_0^\infty e^{-2\sqrt{\omega}\,u}\, f(u^4) du.$$

Hence the result is true for $n = 2$, too.

Now suppose that the result is true for $n = m$.

That is

$$\varphi_m(\omega) = 2\omega \int_0^\infty e^{-2\sqrt{\omega}\,u}\, f(u^{2^m})\, du. \tag{I.76}$$

But $\varphi_{m+1}(\omega) \stackrel{1}{=} 2\pi^{-\frac{1}{2}}\, x^{\frac{1}{2}}\, \varphi_m\left((2x)^{-2}\right)$. Therefore

$$\varphi_{m+1}(\omega) = \pi^{-\frac{1}{2}}\, \omega \int_0^\infty \int_0^\infty x^{-\frac{3}{2}}\, e^{-\frac{v}{x} - \omega\, x}\, f(v^{2^m})\, dv\, dx.$$

which is essentially same as in (I.70), except for argument of the function f. By following the same steps, we get

$$\varphi_{m+1}(\omega) = \omega \int_0^\infty e^{-2\sqrt{v\,\omega}}\, v^{-\frac{1}{2}}\, f(v^{2^m})\, dv.$$

Again with the substitution $u = \sqrt{v}$, we find

$$\varphi_{m+1}(\omega) = 2\omega \int_0^\infty e^{-2u\sqrt{\omega}}\, f\left(u^{2^{m+1}}\right)\, du.$$

Therefore the result is true for $n = m + 1$. But we have that the result is true for $n = 1$. Hence it is true for all n, i.e.,

$$\varphi_n(\omega) = 2\omega \int_0^\infty e^{-2u\sqrt{\omega}}\, f(u^{2^n})\, du. \tag{I.77}$$

Next we replace ω by $p_1\left(s^{\frac{1}{2}}\right)^2$ in (I.77) to get

$$\varphi n(p_1(\sqrt{s})^2) = 2p_1\left(s^{\frac{1}{2}}\right)^2 \int_0^\infty e^{-2up_1\left(s^{\frac{1}{2}}\right)} f(u^{2^n})\, du. \tag{I.78}$$

<u>To prove (a):</u>

We multiply both sides of (I.78) by $p_N(s^{\frac{1}{2}})$ and find

$$p_N\left(s^{\frac{1}{2}}\right) p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) = \tag{I.79}$$

$$2\int_0^\infty p_N\left(s^{\frac{1}{2}}\right) e^{-2up_1(s^{\frac{1}{2}})} f(u^{2^n})du.$$

By Roberts and Kaufman [37], we find

$$p_N\left(s^{\frac{1}{2}}\right) e^{-2up_1\left(s^{\frac{1}{2}}\right)} \stackrel{N}{=} \pi^{-\frac{N}{2}} p_N\left(t^{-\frac{1}{2}}\right) e^{-u^2 p_1(t^{-1})}. \tag{I.80}$$

From the equations (I.79) and (I.80), we obtain

$$p_N\left(s^{\frac{1}{2}}\right) p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{N}{=} \tag{I.81}$$

$$2\pi^{-\frac{N}{2}} p_N(t^{-\frac{1}{2}}) \int_0^\infty f(u^{2^n})\, e^{-u^2 p_1(t^{-1})} du.$$

With the substitution $v = u^2$in (I.81), we find

$$p_N\left(s^{\frac{1}{2}}\right) p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) \stackrel{N}{=} \pi^{-\frac{N}{2}} p_N\left(t^{-\frac{1}{2}}\right) \cdot$$

$$p_1(t^{-1})^{-1} A_n(p_1(t^{-1})) \text{ where } A_n(\omega) \stackrel{1}{=} x^{-\frac{1}{2}} f\left(x^{2^{n-1}}\right).$$

<u>To prove (b):</u>

Next, we multiply (I.78) by $s_j^{\frac{1}{2}} p_N\left(s^{\frac{1}{2}}\right)$ and obtain

$$s_j^{\frac{1}{2}} p_N\left(s^{\frac{1}{2}}\right) p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) = 2\int_0^\infty s_j^{\frac{1}{2}} \cdot \tag{I.82}$$

$$p_N\left(s^{\frac{1}{2}}\right) e^{-2up_1\left(s^{\frac{1}{2}}\right)} f\left(u^{2^n}\right)\, du \quad \text{for } j = 1, 2, \ldots, N.$$

Again by using Roberts and Kaufman [37], we have

$$s_j^{\frac{1}{2}} p_N\left(s^{\frac{1}{2}}\right) e^{-2up_1\left(s^{\frac{1}{2}}\right)} \overset{N}{=} \pi^{-\frac{N}{2}} t_j^{-1} u\, p_N\left(t^{-\frac{1}{2}}\right) e^{-u^2 p_1(t^{-1})}. \tag{I.83}$$

From (I.82) and (I.83), it follows

$$s_j^{\frac{1}{2}} p_N\left(s^{\frac{1}{2}}\right) p_1\left(s^{\frac{1}{2}}\right)^{-2} \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) \overset{N}{=} 2\pi^{-\frac{N}{2}} t_j^{-1} \cdot \tag{I.84}$$

$$p_N\left(t^{-\frac{1}{2}}\right) \int_0^\infty e^{-u^2 p_1(t^{-1})} u f\left(u^{2^n}\right) du.$$

Now, we substitute $v = u^2$ in (I.84) to obtain

$$s_j^{\frac{1}{2}} p_N(s^{\frac{1}{2}}) p_1(s^{\frac{1}{2}})^{-2} \varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \overset{N}{=} \pi^{-\frac{N}{2}} t_j^{-1} \cdot$$

$$p_N(t^{-\frac{1}{2}}) p_1(t^{-1})^{-1} B_n(\omega) \quad \text{where } B_n(\omega) \overset{1}{=} f(x^{2^{n-1}})$$

$$\text{for } j = 1, 2, \ldots, N.$$

<u>To prove (c):</u>

From the equation (I.78), we obtain

$$p_N(s)\,p_1(s^{\frac{1}{2}})^{-2}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) = 2\int_0^{\infty} p_N(s)\cdot \exp\left(-2up_1(s^{\frac{1}{2}})\right) f(u^{2^n})\,du. \tag{I.85}$$

By using Roberts and Kaufman [37], we find

$$p_N(s)\exp\left(-2up_1(s^{\frac{1}{2}})\right) \stackrel{N}{=} \pi^{-\frac{N}{2}}\,u^N\,p_N\left(t^{-\frac{3}{2}}\right)\exp(-u^2\,p_1(t^{-1})).$$

Hence from equation (I.85), we obtain

$$p_N(s)\,p_1(s^{\frac{1}{2}})^{-2}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{N}{=} 2\pi^{-\frac{N}{2}}\,p_N(t^{-\frac{3}{2}})\cdot \int_0^{\infty} u^N\,f(u^{2^n})\exp(-u^2\,p_1(t^{-1}))\,du. \tag{I.86}$$

With the substitution of $v = u^2$ in (I.86), we get

$$p_N(s)\,p_1(s^{\frac{1}{2}})^{-2}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{N}{=} \pi^{-\frac{N}{2}}\,p_N(t^{-\frac{3}{2}})\cdot \int_0^{\infty} v^{\frac{N}{2}-\frac{1}{2}}\,f(v^{2^{n-1}})\exp(-v\,p_1(t^{-1}))\,dv.$$

If we take $C_n(\omega) = x^{\frac{N}{2}-\frac{1}{2}}\,f(x^{2^{n-1}})$, then

$$p_N(s)\,p_1(s^{\frac{1}{2}})^{-2}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{N}{=} \pi^{-\frac{N}{2}}\,p_N(t^{-\frac{3}{2}})\cdot \left[p_1(t^{-1})\right]^{-1} C_n\left(p_1(t^{-1})^{-1}\right).$$

<u>To prove (d):</u>

Now, we multiply (I.78) by $s_j\, p_N(s^{\frac{1}{2}})$ to obtain

$$s_j\, p_N(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) = 2\int_0^\infty s_j\, p_N(s^{\frac{1}{2}})\cdot \tag{I.87}$$
$$\exp\left(-2up_1(s^{\frac{1}{2}})\right) f(u^{2^n})\, du, \;\text{ for } j = 1, 2, \ldots, N.$$

We know that

$$s_j^{\frac{3}{2}} \exp(-2us_j^{\frac{1}{2}}) \stackrel{1}{=} \left(4\pi^{\frac{1}{2}}\right)^{-1} t_j^{-\frac{5}{2}} (4u^2 - 2t_j)\cdot \tag{I.88}$$
$$\exp\left(-u^2 t_j^{-1}\right), \;\text{ for } j = 1, 2, \ldots, n.$$

$$s_k^{\frac{1}{2}} \exp(-2us_k^{\frac{1}{2}}) \stackrel{1}{=} (\pi t_k)^{-\frac{1}{2}} \exp(-u^2 t_k^{-1}), \;\text{ for } k = 1, 2, \ldots, n.$$

By combining (I.87) and (I.88), we get

$$s_j\, p_n(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) \stackrel{N}{=} \pi^{-\frac{N}{2}}\, t_j^{-2}\,. \tag{I.89}$$
$$p_N\left(t^{-\frac{1}{2}}\right) \int_0^\infty (2u^2 - t_j)\, f(u^{2^n})\, \exp\left(-u^2 p_1(t^{-1})\right) du.$$

Next, we substitute $v = u^2$ in the equation (I.89) and obtain

$$s_j\, p_N(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \varphi_n\left(p_1(s^{\frac{1}{2}})^2\right)$$
$$\stackrel{N}{=} \pi^{-\frac{N}{2}}\, p_N\left(t^{-\frac{1}{2}}\right) \left[t_j^{-2} \int_0^\infty v^{\frac{1}{2}} f\left(v^{2^{n-1}}\right) \exp(-v\, p_1(t^{-1}))\, dv - \right.$$
$$\left. 2^{-1} t_j \int_0^\infty v^{-\frac{1}{2}} f\left(v^{2^{n-1}}\right) \exp(-v\, p_1(t^{-1}))\, dv\right]$$
$$\stackrel{N}{=} 2^{-1}\pi^{-\frac{N}{2}}\, p_N\left(t^{-\frac{1}{2}}\right) \left[p_1(t^{-1})\right]^{-1} \left[2D_n\left(p_1(t^{-1})\right) - \right.$$

$$t_j\, A_n\,(p_1(t^{-1}))\Big], \text{ where } D_n(\omega) \overset{1}{=} x^{\frac{1}{2}} f\left(x^{2^{n-1}}\right) \text{ and}$$

$$A_n(\omega) \overset{1}{=} x^{-\frac{1}{2}} f\left(x^{2^{n-1}}\right), \text{ for } j = 1, 2, \ldots, N.$$

To prove (e):

From the equation (I.78), we have

$$s_j^{\frac{1}{2}}\, p_N(s)\, p_1(s^{\frac{1}{2}})^{-2} \varphi\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) = 2\int_0^\infty s_j^{\frac{1}{2}}\, p_N(s)\cdot \tag{I.90}$$

$$\exp\left(-2u\,p_1\left(s^{\frac{1}{2}}\right)\right) f(u^{2^n})\,du, \text{ for } j = 1, 2, \ldots, N.$$

Since

$$s_j^{\frac{3}{2}} \exp\left(-2us_j^{\frac{1}{2}}\right) \overset{1}{=} \left(4\pi^{\frac{1}{2}}\right)^{-1} t_j^{-\frac{5}{2}} (4u^2 - 2t_j) \exp(-u^2 t_j^{-1})$$

and

$$s_k \exp\left(-2us_k^{\frac{1}{2}}\right) \overset{1}{=} \pi^{-\frac{1}{2}} t_k^{-\frac{3}{2}} u \exp(-u^2 t_k^{-1}) \text{ for } k = 1, 2, \ldots, N.$$

Therefore, from the equation (I.90), we obtain

$$s_j^{\frac{1}{2}}\, p_N(s)\, p_1(s^{\frac{1}{2}})^{-2} \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) \overset{N}{=} \pi^{-\frac{N}{2}} t_j^{-1} p_N\left(t^{-\frac{3}{2}}\right) \cdot \tag{I.91}$$

$$\int_0^\infty u^{N-1} (2u^2 - t_j) f(u^{2^n}) \exp\left(-u^2 p_1(t^{-1})\right) du.$$

Again by substituting $v = u^2$ in (I.91), we find

$$s_j^{\frac{1}{2}}\, p_N(s)\, p_1(s^{\frac{1}{2}})^{-2} \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) \overset{N}{=} 2^{-1} \pi^{-\frac{N}{2}} t_j^{-1}\cdot$$

$$p_N\left(t^{-\frac{3}{2}}\right) p_1(t^{-1})^{-1} \left[2E_n\left(p_1(t^{-1})\right) - t_j F_n(p_1(t^{-1}))\right],$$

$$\text{where } E_n(\omega) \overset{1}{=} x^{\frac{N}{2}} f\left(x^{2^{n-1}}\right) \text{ and } F_n(\omega) = x^{\frac{N}{2}-1} f\left(x^{2^{n-1}}\right).$$

To prove (f):

It follows from the equation (I.78) that

$$s_j\, p_N(s)\, p_1(s^{\frac{1}{2}})^{-2}\, \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) = 2\int_0^\infty s_j\, p_N(s)\cdot \tag{I.92}$$

$$\exp\left(-2u\, p_1\left(s^{\frac{1}{2}}\right)\right) f\left(u^{2^n}\right) du, \quad \text{for } j = 1, 2, \ldots, N.$$

Now by using the tables from Roberts and Kaufman [37], we find

$$s_j^2 \exp\left(-2us_j^{\frac{1}{2}}\right) \stackrel{1}{=} \left(2\pi^{\frac{1}{2}}\right)^{-1} t_j^{-\frac{n}{2}}\, u(u^2 - 3t_j) \exp\left(-u^2\, t_j^{-1}\right)$$

and

$$s_k \exp\left(-2us_k^{\frac{1}{2}}\right) \stackrel{1}{=} \pi^{-\frac{1}{2}} ut_k^{-\frac{3}{2}} \exp\left(-u^2\, t_k^{-1}\right), \text{ for } j, k = 1, 2, \ldots, N. \tag{I.93}$$

By combining equations (I.92) and (I.93), we obtain

$$s_j\, p_N(s)\, p_1(s^{\frac{1}{2}})^{-2}\, \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)\right) \stackrel{N}{=} \pi^{-\frac{N}{2}}\, t_j^{-2}\cdot \tag{I.94}$$

$$p_N(t^{-\frac{3}{2}}) \int_0^\infty u^N\, (u^2 - 3t_j)\, f(u^{2^n}) \exp(-u^2\, p_1(t^{-1})\, du.$$

Next by substituting $v = u^2$ in (I.94) and by taking $G_n(\omega) \stackrel{1}{=} x^{\frac{N}{2}+\frac{1}{2}}\cdot$ $f(x^{2^{n-1}})$ and $C_n(\omega) \stackrel{1}{=} x^{\frac{N}{2}-\frac{1}{2}}\, f(x^{2^{n-1}})$, we can obtain the required result.

To prove (g):

We multiply the equation (I.78) by $s_j^{\frac{3}{2}}\, p_N(s^{\frac{1}{2}})$ to obtain

$$s_j^{\frac{3}{2}}\, p_N(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) = 2\int_0^\infty s_j^{\frac{3}{2}}\, p_N(s^{\frac{1}{2}})\cdot \tag{I.95}$$

$$\exp\left(-2up_1\left(s^{\frac{1}{2}}\right)\right) f(u^{2^n})\, du, \quad \text{for } j = 1, 2, \ldots, N.$$

Again by using the results from Roberts and Kaufman [37], we find

$$s_j^{\frac{3}{2}}\, p_N(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-2}\, \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) \stackrel{N}{=} \pi^{-\frac{N}{2}}\, t_j^{-3}. \tag{I.96}$$

$$p_N\left(t^{-\frac{1}{2}}\right) \int_0^\infty u(u^2 - 3t_j)\, f(u^{2^n}) \exp(-u^2 p_1(t^{-1}))\, du.$$

Now by substituting $v = u^2$ in (I.96) and by using

$$\begin{aligned} H_n(\omega) &\stackrel{1}{=} x f\left(x^{2^{n-1}}\right) \quad \text{and} \\ B_n(\omega) &\stackrel{1}{=} x f\left(x^{2^{n-1}}\right), \end{aligned}$$

as defined before, we can obtain the required result.

Examples based on theorem I.8:

We consider the Case $N = 2$ in the following examples.

Example 1.1:

Let $f(x) = x \;\; {}_pF_q[(a);(b);x]$, with $p + 2^n \le q$, then

$\varphi(w) = \omega^{-1} \;\; {}_{p+1}F_1[(a),2;(b);\omega^{-1}]$.

Also from the theorem

$$\begin{aligned} \varphi_n(\omega) \;=\;& 2^{-2^n}\, \omega^{-2^{n-1}+\frac{1}{2}}\, \Gamma(2^n+1) \;\; {}_{p+2^n}F_q \cdot \\ & \left[(a), \frac{2^n+1}{2^n}, \frac{2^n+2}{2^n}, \ldots, \frac{2^n+2^n}{2^n}; (b); \left(\frac{2^{n-1}}{\sqrt{\omega}}\right)^{2^n}\right], \; p + 2^n \le q. \end{aligned}$$

If we take $s = (s_1, s_2)$, that is $N = 2$ in theorem I.1, we find

$$\frac{\varphi_n\left((\sqrt{s_1}+\sqrt{s_2})^2\right)}{(\sqrt{s_1}+\sqrt{s_2})^2} = 2^{-2^n}\, (\sqrt{s_1}+\sqrt{s_2})^{-2^n-1}\, \Gamma(2^n+1) \cdot$$

$${}_{p+2^n}F_q\left[(a), 1 + 2^{-n}, 1 + 2.2^{-n}, \ldots, 2; (b); \left(\frac{2^{n-1}}{\sqrt{s_1}+\sqrt{s_2}}\right)^{2^n}\right].$$

Let us denote

$$\eta(s_1,s_2)=\frac{2^{2^n}}{\Gamma(2^n+1)}\,\frac{\varphi n\left((\sqrt{s_1}+\sqrt{s_2})^2\right)}{(\sqrt{s_1}+\sqrt{s_2})^2}.$$

That is

$$\eta(s_1,s_2)=(\sqrt{s_1}+\sqrt{s_2})^{-2^n-1}\,.$$
$$_{p+2^n}F_q\left[(a),1+2^{-n},\ldots,2;(b);\left(\frac{2^{n-1}}{\sqrt{s_1}+\sqrt{s_2}}\right)^{2^n}\right].$$

Now by using the results of theorem I.6, we can obtain the following transform pairs: In the following, we take $p+2^n\le q$.

(a)

$$\sqrt{s_1s_2}\,\eta(s_1,s_2)\overset{2}{=}\frac{1}{\sqrt{\pi}\,\Gamma\left(2^{n-1}+1\right)}(t_1t_2)^{-\frac{1}{2}}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n+1}+\frac{1}{2}}.$$
$$_{n+2^{n-1}}F_1\left[(a),1+2^{-n+2},1+3.2^{-n+2},\ldots,2-2^{-n+2};(b);\right.$$
$$\left.\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\right],\ \text{where}\quad p+2^n\le q.$$

(b)

$$s_1s_2^{\frac{1}{2}}\,\eta(s_1,s_2)\ \overset{2}{=}\ \frac{2^{-2^n}}{\pi}\,\frac{\Gamma(2^{n-1}+1)}{\Gamma(2^n+1)}\,t_1^{-\frac{3}{2}}t_2^{-\frac{1}{2}}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n-1}+1}.$$
$$_{p+2^{n-1}}F_q\left[(a),1+2^{-n+1},1+2.2^{-n+1},\ldots,2;(b);\right.$$
$$\left.\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\right]$$
$$\overset{2}{=}\ \frac{1}{\sqrt{\pi}\,\Gamma\left(2^{n-1}+\frac{1}{2}\right)}\,t_1^{-\frac{3}{2}}t_2^{-\frac{1}{2}}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n-1}+1}.$$

$$
{}_{p+2^{n-1}}F_q\left[(a),1+2^{-n+1},\ldots,2;(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\right].
$$

(c)

$$
s_1s_2\,\eta(s_1,s_2)\overset{2}{=}\frac{2^{2^n}}{\pi}\frac{\Gamma(2^{n-1}+\frac{3}{2})}{\Gamma(2^n+1)}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n-1}+\frac{3}{2}}.
$$
$$
{}_{p+2^{n-1}}F_q\left[(a),1+3.2^{-n+1},1+5.2^{-n+1},\ldots,2+2^{-n+2};(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\right].
$$

(d)

$$
s_1^{\frac{3}{2}}s_2^{\frac{1}{2}}\,\eta(s_1,s_2)\overset{2}{=}\frac{t_1^{-\frac{5}{2}}t_2^{-\frac{1}{2}}}{2\sqrt{\pi}\,\Gamma(2^{n-1}+1)}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n-1}+\frac{3}{2}}.
$$
$$
\Bigg\{(2^n+1)\ {}_{p+2^{n-1}}F_q\left[(a),1+3.2^{-n+2},1+5.2^{-n+2},\ldots,2+2^{-n+2};(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\right]-
$$
$$
\left(\frac{t_1t_2}{t_2}\right)\ {}_{p+2^{n-1}}F_q\left[(a),1+2^{-n+2},1+3.2^{-n+2},\ldots,2-2^{-n+2};(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\right]\Bigg\}.
$$

(e)

$$
s_1^{\frac{3}{2}}s_2\,\eta(s_1,s_2)\overset{2}{=}\frac{t_1^{-\frac{5}{2}}t_2^{-\frac{3}{2}}}{2\sqrt{\pi}\,\Gamma(2^{n-1}+\frac{1}{2})}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n-1}+2}.
$$

$$\Big\{(2^n+2)\ {}_{p+2^{n-1}}F_q\Big[(a),1+2.2^{-n+1},1+3.2^{-n+1},\dots,$$

$$2+2^{-n+1};(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\Big]-$$

$$\left(\frac{t_1t_2}{t_2}\right)\ {}_{p+2^{n-1}}F_q\Big[(a),1+2^{-n+1},1+2.2^{-n+1},\dots,2;$$

$$(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\Big]\Big\}.$$

(f)

$$s_1^2\,s_2\,\eta(s_1,s_2)\overset{2}{=}\frac{2^{2^n-1}t_1^{-\frac{7}{2}}t_2^{-\frac{3}{2}}}{\pi\,\Gamma(2^n+1)}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n-1}+\frac{5}{2}}.$$

$$\Big\{\Gamma\left(2^{n-1}+\frac{5}{3}\right)\ {}_{p+2^{n-1}}F_q\Big[(a),1+5.2^{-n+2},1+7.2^{-n+2},\dots,$$

$$2+3.2^{-n+2};(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\Big]-3\Gamma\left(2^{n-1}+\frac{3}{2}\right)\left(\frac{t_1+t_2}{t_2}\right).$$

$${}_{p+2^{n-1}}F_q\Big[(a),1+3.2^{-n+2},1+5.2^{-n+2},2+2^{-n+2};(b);$$

$$\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\Big]\Big\}.$$

(g)

$$s_1^2\,s_2^{\frac{1}{2}}\,\eta(s_1,s_2)\overset{2}{=}\frac{t_1^{-\frac{7}{2}}t_2^{-\frac{1}{2}}}{2\sqrt{\pi}\,\Gamma(2^{n-1}+1)}\left(\frac{t_1t_2}{t_1+t_2}\right)^{2^{n-1}+\frac{3}{2}}.$$

$$\Big\{\left(2^{n-1}+\frac{1}{2}\right)\ {}_{p+2^{n-1}}F_q\Big[(a),1+3.2^{-n+2},1+4.2^{-n+2},\dots,$$

$$2+2^{-n+2};(b);\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\Big]-3\left(\frac{t_1+t_2}{t_2}\right).$$

$$
{}_{p+2^{n-1}}F_q\left[(a), 1+2^{-n+2}, 1+3.2^{-n+2}, \ldots, 2-2^{-n+2}; (b);\right.
$$
$$
\left.\left.\left(\frac{2^{n-1}t_1t_2}{t_1+t_2}\right)^{2^{n-1}}\right]\right\}.
$$

Example 1.2: $n = 2$.

$$
\begin{aligned}
\text{Let } f(x) &= e^{-\sqrt{x}},\ x > 0. \\
\varphi_2(\omega) &= 2\omega \int_0^\infty e^{-2u\sqrt{\omega}}\, e^{-u^2}\, du. \\
&= 2\omega \cdot \frac{1}{2}\sqrt{\pi}\, e^{\omega} \operatorname{Erfc}(\sqrt{\omega}). \\
&= \sqrt{\pi}\, \omega\, e^{\omega} \operatorname{Erfc}(\sqrt{\omega}).
\end{aligned}
$$

By using Theorem I.6, we can find the following transform pairs:

(a)

$$
\sqrt{s_1s_2}\, e^{(\sqrt{s_1}+\sqrt{s_2})^2} \operatorname{Erfc}(\sqrt{s_1}+\sqrt{s_2}) \overset{2}{=} \pi^{-\frac{3}{2}} (t_1t_2)^{-\frac{1}{2}} \cdot
$$
$$
A_2\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_1}+\frac{1}{t_2}\right)^{-1}, \quad \text{where } A_2(\omega) \overset{1}{=} x^{-\frac{1}{2}}\, e^{-x}
$$
$$
\text{i.e.,} \qquad A_2(\omega) = \frac{\omega\, \Gamma(\frac{1}{2})}{(\omega+1)^{\frac{1}{2}}}.
$$

That is

$$
\sqrt{s_1s_2}\, e^{(\sqrt{s_1}+\sqrt{s_2})^2} \operatorname{Erfc}(\sqrt{s_1}+\sqrt{s_2}) \overset{2}{=} \pi^{-1}(t_1+t_2+t_1t_2)^{-\frac{1}{2}}.
$$

(b)

$$
s_1 s_2^{\frac{1}{2}}\, e^{(\sqrt{s_1}+\sqrt{s_2})^2} \operatorname{Erfc}(\sqrt{s_1}+\sqrt{s_2}) \overset{2}{=} \pi^{-\frac{3}{2}}\, t_1^{-\frac{3}{2}}\, t_2^{-\frac{1}{2}} \cdot
$$
$$
B_2\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_1}+\frac{1}{t_2}\right)^{-1}, \quad \text{where } B_2(\omega) = \frac{\omega}{(\omega+1)}.
$$

That is,

$$s_1 s_2^{\frac{1}{2}}\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\, \text{Erfc}\,\left(\sqrt{s_1}+\sqrt{s_2}\right) \overset{2}{=} \pi^{-\frac{3}{2}}\, \frac{t_1^{-\frac{1}{2}}}{t_1+t_2+t_1 t_2}.$$

(c)

$$s_1 s_2\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\, \text{Erfc}\,\left(\sqrt{s_1}+\sqrt{s_2}\right) \overset{2}{=} \pi^{-\frac{3}{2}}\, (t_1\, t_2)^{-\frac{3}{2}} \left(\frac{1}{t_1}+\frac{1}{t_2}\right)^{-1} \cdot$$

$$C_2\left(\frac{1}{t_1}+\frac{1}{t_2}\right), \text{ where}$$

$$C_2(\omega) = \frac{\omega\, \Gamma(\frac{3}{2})}{(\omega+1)^{\frac{3}{2}}}.$$

Therefore,

$$s_1 s_2\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\, \text{Erfc}\,\left(\sqrt{s_1}+\sqrt{s_2}\right) \overset{2}{=} (2\pi)^{-1}\, (t_1+t_2+t_1 t_2)^{-\frac{3}{2}}.$$

(d)

$$s_1^{\frac{3}{2}}\, s_2^{\frac{1}{2}}\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\, \text{Erfc}\,\left(\sqrt{s_1}\,\sqrt{s_2}\right) \overset{2}{=} \frac{1}{2}\pi^{-\frac{3}{2}} \cdot$$

$$(t_1 t_2)^{-\frac{1}{2}} \left(\frac{1}{t_1}+\frac{1}{t_2}\right)^{-1} \left[2 D_2\left(\frac{1}{t_1}+\frac{1}{t_2}\right) - t_1\, A_2\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right],$$

where $D_2(\omega) = C_2(\omega)$, which is defined in (c).

$$s_1^{\frac{3}{2}}\, s_2^{\frac{1}{2}}\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\, \text{Erfc}\,\left(\sqrt{s_1}\,\sqrt{s_2}\right)$$

$$\overset{2}{=} \frac{1}{2}\pi^{-\frac{3}{2}}\, (t_1 t_2)^{-\frac{1}{2}} \left[2\, \frac{\Gamma(\frac{3}{2})}{\left(t_1^{-1}+t_2^{-1}+1\right)^{-\frac{3}{2}}} - \Gamma(\frac{1}{2}) \frac{t_1}{\left(t_1^{-1}+t_2^{-1}+1\right)^{\frac{1}{2}}}\right],$$

$$\overset{2}{=} (2\pi)^{-1}\, (t_1+t_2+t_1 t_2)^{-\frac{3}{2}}\, \left[t_1 t_2 - t_1(t_1+t_2+t_1 t_2)\right].$$

That is,

$$s_1^{\frac{3}{2}}\, s_2^{\frac{1}{2}}\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\,\mathrm{Erfc}\,\left(\sqrt{s_1}+\sqrt{s_2}\right) \stackrel{2}{=} \frac{t_1^2(1+t_2)}{2\pi(t_1+t_2+t_1t_2)^{\frac{3}{2}}}.$$

(*e*)

$$s_1^{\frac{3}{2}}\, s_2\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\,\mathrm{Erfc}\,\left(\sqrt{s_1}\,\sqrt{s_2}\right)$$

$$\stackrel{2}{=} \frac{1}{2}\pi^{-\frac{3}{2}}\, t_1^{-\frac{5}{2}}\, t_2^{-\frac{3}{2}} \left(t_1^{-1}\, t_2^{-1}\right)^{-1} \left[2E_2\left(\frac{1}{t_1}+\frac{1}{t_2}\right) - t_1 F_2\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right],$$

where $E_2(\omega) = \dfrac{\omega}{(\omega+1)^2}$ and $F_2(\omega) = B_2(\omega) = \dfrac{\omega}{\omega+1}$.

$$\stackrel{2}{=} \frac{1}{2}\,\pi^{-\frac{3}{2}}\, t_1^{-\frac{3}{2}}\, t_2^{-\frac{1}{2}}\,(t_1+t_2+t_1t_2)^{-2}\,[2t_1t_2 - t_1(t_1+t_2+t_1t_2)].$$

That is,

$$s_1^{\frac{3}{2}}\, s_2\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\,\mathrm{Erfc}\,\left(\sqrt{s_1}+\sqrt{s_2}\right) \stackrel{2}{=}$$

$$2^{-1}\,\pi^{-1}\,(t_1t_2)^{-\frac{1}{2}}\,(t_1+t_2+t_1t_2)^{-2}\,(t_2-t_1-t_1t_2).$$

(*f*)

$$s_1^2\, s_2\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\,\mathrm{Erfc}\,\left(\sqrt{s_1}\,\sqrt{s_2}\right) \stackrel{2}{=} 2^{-1}\,\pi^{-\frac{3}{2}}\, t_1^{-\frac{7}{2}}\, t_2^{-\frac{3}{2}}\,\cdot$$

$$\left(t_1^{-1}+t_2^{-1}\right)^{-1}\left[G_2\left(t_1^{-1}+t_2^{-1}\right) - 3t_1\, C_2\left(t_1^{-1}+t_2^{-1}\right)\right],$$

where $G_2(\omega) = \dfrac{\omega\,\Gamma(\frac{5}{2})}{(\omega+1)^{\frac{5}{2}}}$ and $C_2(\omega)$ is as in (*c*).

That is,

$$s_1^2\, s_2\, e^{\left(\sqrt{s_1}+\sqrt{s_2}\right)^2}\,\mathrm{Erfc}\,\left(\sqrt{s_1}+\sqrt{s_2}\right) \stackrel{2}{=}$$

$$-2^{-3}\,\pi^{-1}\,(t_1^{-1}\,(t_1+t_2+t_1t_2)^{-\frac{5}{2}}\,(12t_1+9t_2+12t_1t_2).$$

(g)

$$s_1^2 s_2^{\frac{1}{2}} e^{(\sqrt{s_1}+\sqrt{s_2})^2} \operatorname{Erfc}(\sqrt{s_1}\sqrt{s_2}) \stackrel{2}{=} 2^{-1} \pi^{-\frac{3}{2}} t_1^{-\frac{7}{2}} t_2^{-\frac{1}{2}} \cdot$$
$$\left(t_1^{-1}+t_2^{-1}\right)^{-1} \left[H_2\left(t_1^{-1}+t_2^{-1}\right) - 3t_1 B_2\left(t_1^{-1}+t_2^{-1}\right)\right],$$

where $H_2(\omega) = E_2(\omega)$.

Therefore,

$$s_1^2 s_2^{\frac{1}{2}} e^{(\sqrt{s_1}+\sqrt{s_2})^2} \operatorname{Erfc}(\sqrt{s_1}+\sqrt{s_2}) \stackrel{2}{=}$$
$$-2^{-1} \pi^{-\frac{3}{2}} t_1^{-\frac{3}{2}} t_2^{\frac{1}{2}} (t_1+t_2+t_1t_2)^{-2} (3t_1+2t_2+3t_1t_2).$$

Theorem I.7:

Suppose that, for $\Re(\omega) > 0$ and $0 < x < \infty$

$$\begin{array}{llll} (i) & \varphi(\omega) & \stackrel{1}{=} & f(x) \\ (ii) & \varphi_1(\omega) & \stackrel{1}{=} & \sqrt{\frac{x}{\pi}}\,\varphi\left(\frac{1}{x}\right) \\ (iii) & \varphi_2(\omega) & \stackrel{1}{=} & \sqrt{\frac{x}{\pi}}\,\varphi_1\left(\frac{1}{4x^2}\right) \\ \vdots & \vdots & & \vdots \\ (n) & \varphi_n(\omega) & \stackrel{1}{=} & \sqrt{\frac{x}{\pi}}\,\varphi_{n-1}\left(\frac{1}{4x^2}\right). \end{array}$$

Then

$$\varphi_n(\omega) = 2\sqrt{\omega}\int_0^\infty e^{-2u\sqrt{\omega}}(u)^{(2^n-1)} f(u^{2^n})du,$$

for $n = 1, 2, \ldots$.

Furthermore, if we take $\omega = p_1(s^{\frac{1}{2}})^2$, the following results can be obtained:

(a)

$$p_N(s^{\frac{1}{2}})\, p_1(s^{\frac{1}{2}})^{-1} \varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \stackrel{N}{=} \pi^{-\frac{N}{2}} p_N(t^{-\frac{1}{2}}) \cdot$$
$$p_1(t^{-1})^{-1} T_n\left(p_1(t^{-1})\right) \text{ where } T_n(\omega) \stackrel{1}{=} x^{2^{n-1}-1} f(x^{2^{n-1}}).$$

(b)

$$p_N(s)\,p_1(s^{\frac{1}{2}})^{-1}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \overset{N}{=} \pi^{-\frac{N}{2}}\,p_N\left(t^{-\frac{3}{2}}\right)p_1(t^{-1})^{-1}\,.$$
$$K_n(p_1(t^{-1})) \quad \text{where } K_n(\omega) \overset{1}{=} x^{2^{n-1}+\frac{N}{2}-1}\,f(x^{2^{n-1}}).$$

(c)

$$s_j^{\frac{1}{2}}\,p_N(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})^{-1}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \overset{N}{=} \pi^{-\frac{N}{2}}\,t_j^{-1}\,p_N(t^{-\frac{1}{2}})\,.$$
$$p_1(t^{-1})^{-1}\,L_n(p_1(t^{-1})) \quad \text{where } L_n(\omega) \overset{1}{=} x^{2^{n-1}-\frac{1}{2}}\,f(x^{2^{n-1}})$$
for $j = 1, 2, \ldots, N$.

(d)

$$s_j\,p_N(s^{\frac{1}{2}})\,p_1(s^{\frac{1}{2}})^{-1}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \overset{N}{=} 2^{-1}\,\pi^{-\frac{N}{2}}\,t_j^{-2}\,.$$
$$p_N(t^{-\frac{1}{2}})\,p_1(t^{-1})^{-1}\left[2M_n(p_1(t^{-1})) - t_j\,T_n(p_1(t^{-1}))\right],$$
for $j = 1, 2, \ldots, N$; where $M_n(\omega) \overset{1}{=} x^{2^{n-1}}\,f\left(x^{2^{n-1}}\right)$

and $T_n(\omega)$ is defined as in (a).

(e)

$$s_j^{\frac{1}{2}}\,p_N(s)\,p_1(s^{\frac{1}{2}})^{-1}\,\varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \overset{N}{=} 2^{-1}\,\pi^{-\frac{N}{2}}\,t_j^{-1}\,.$$
$$p_N(t^{-\frac{3}{2}})\,p_1(t^{-1})^{-1}\left[2Q_n\left(p_1(t^{-1})\right) - t_j\,R_n(p_1(t^{-1}))\right],$$
for $j = 1, 2, \ldots, N$; where $Q_n(\omega) \overset{1}{=} x^{2^{n-1}+\frac{N}{2}-\frac{1}{2}}\,f\left(x^{2^{n-1}}\right)$

and $R_n(\omega) \overset{1}{=} x^{2^{n-1}+\frac{N}{2}-\frac{3}{2}}\,f\left(x^{2^{n-1}}\right)$.

(f)

$$s_j\, p_N(s)\, p_1(s^{\frac{1}{2}})^{-1}\, \varphi_n\left(p_1(s^{\frac{1}{2}})^2\right) \overset{N}{=} 2^{-1}\, \pi^{-\frac{N}{2}}\, t_j^{-2}\,.$$

$$p_N\left(t^{-\frac{3}{2}}\right) p_1(t^{-1})^{-1}\left[Y_n\left(p_1(t^{-1})\right) - 3t_j\, Q_n(p_1(t^{-1}))\right],$$

for $j = 1, 2, \ldots, N$; where $Y_n(\omega) \overset{1}{=} x^{2^{n-1}+\frac{N}{2}+\frac{1}{2}}\, f(x^{2^{n-1}})$

and $Q_n(\cdot)$ is defined as in (e).

(g)

$$s_j^{\frac{3}{2}}\, p_N\left(s^{\frac{1}{2}}\right) p_1\left(s^{\frac{1}{2}}\right)^{-1} \varphi_n\left(p_1\left(s^{\frac{1}{2}}\right)^2\right) \overset{N}{=} 2^{-1}\, \pi^{-\frac{N}{2}}\, t_j^{-3}\,.$$

$$p_N\left(t^{-\frac{1}{2}}\right) p_1(t^{-1})^{-1}\left[Z_n\left(p_1(t^{-1})\right) - 3t_j\, L_n\left(p_1(t^{-1})\right)\right],$$

for $j = 1, 2, \ldots, N$; where $Z_n(\omega) \overset{1}{=} x^{2^{n-1}+\frac{1}{2}}\, f\left(x^{2^{n-1}}\right)$

and $L_n(\cdot)$ is defined as in (c).

The proof of this theorem is similar to that of theorem I.6, hence it has been omitted.

Example: $n = 2$, $N = 2$.

$$\begin{aligned}
\text{Let } f(x) &= {}_pF_q\left[(a)_p; (b)_q; x\right] \qquad x > 0, \quad p + 4 \leq q \\
\varphi_2(\omega) &= 2\sqrt{\omega}\int_0^\infty e^{-2u\sqrt{\omega}}\, u^3\, {}_pF_q\left[(a)_p; (b)_q; u^4\right] du \\
&= 2\sqrt{\omega}\,\frac{\Gamma(4)}{(2\sqrt{\omega})^4}\; {}_{p+4}F_q\left[(a)_p, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; (b)_q; \left(\frac{4}{2\sqrt{\omega}}\right)^4\right] \\
&= \frac{3}{4}\omega^{-\frac{3}{2}}\; {}_{p+4}F_q\left[(a)_p, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; (b)_q;\, 16\omega^{-2}\right].
\end{aligned}$$

From Theorem I.7, we can obtain the following results:

First we denote, $\frac{4}{3}\left(\sqrt{s_1}+\sqrt{s_2}\right)^{-1}\varphi_2\left[\left(\sqrt{s_1}+\sqrt{s_2}\right)^2\right]$ by $\eta(s_1,s_2)$. That is,

$$\eta(s_1,s_2)=\left(\sqrt{s_1}+\sqrt{s_2}\right)^{-4}\cdot {}_{p+4}F_q\left[(a)_p,1,\frac{5}{4},\frac{3}{2},\frac{7}{4};(b)_q;\frac{16}{\left(\sqrt{s_1}+\sqrt{s_2}\right)^4}\right].$$

(a) $\sqrt{s_1s_2}\,\eta(s_1,s_2)\overset{2}{=}\frac{4}{3}\pi^{-1}(t_1t_2)^{-\frac{1}{2}}\left(t_1^{-1}+t_2^{-1}\right)^{-1}\cdot T_2\left(t_1^{-1}+t_2^{-1}\right)$, where

$$\begin{aligned} T_2(\omega) &\overset{1}{=} x\,{}_pF_q\left[(a)_p;(b)_q;x^2\right] \\ \text{i.e.,}\qquad T_2(\omega) &= \omega^{-1}\,{}_{p+2}F_q\left[(a),1,\frac{3}{2};(b);(\frac{2}{\omega})^2\right]. \end{aligned}$$

Therefore,

$$\sqrt{s_1s_2}\,\eta(s_1,s_2)\overset{2}{=}\frac{4}{3}\pi^{-1}(t_1t_2)^{-\frac{3}{2}}(t_1+t_2)^{-2}\cdot {}_{p+2}F_q\left[(a),1,\frac{3}{2};(b);\left(\frac{2t_1t_2}{t_1+t_2}\right)^2\right].$$

(b) $s_1s_2\,\eta(s_1,s_2)\overset{2}{=}\frac{4}{3}\pi^{-1}(t_1t_2)^{-\frac{3}{2}}\left(t_1^{-1}+t_2^{-1}\right)^{-1}\cdot K_2\left(t_1^{-1}+t_2^{-1}\right)$, where

$$\begin{aligned} K_2(\omega) &\overset{1}{=} x^2\,{}_pF_q\left[(a)_p;(b)_p;x^2\right] \\ \text{i.e.,}\qquad K_2(\omega) &= \omega^{-2}\,{}_{p+2}F_q\left[(a),1,\frac{3}{2},2;(b);\left(\frac{2}{\omega}\right)^2\right]\Gamma(3). \end{aligned}$$

Therefore,

$$s_1s_2\eta(s_1,s_2)\overset{2}{=}\frac{8}{3}\pi^{-1}(t_1t_2)^{-\frac{3}{2}}(t_1+t_2)^{-3}\cdot {}_{p+2}F_q\left[(a),\frac{3}{2},2;(b);\left(\frac{2t_1t_2}{t_1+t_2}\right)^2\right].$$

(c) $s_1 s_2^{\frac{1}{2}} \eta(s_1, s_2) \overset{2}{=} \frac{4}{3}\pi^{-1} t_1^{-\frac{3}{2}} t_2^{-\frac{1}{2}} \cdot \left(t_1^{-1} + t_2^{-1}\right)^{-1} L_2\left(t_1^{-1} + t_2^{-1}\right),$

where

$$\begin{aligned} L_2(\omega) &\overset{1}{=} x^{\frac{3}{2}}\, {}_pF_q\left[(a);(b);x^2\right] \\ \text{i.e.,} \qquad L_2(\omega) &= \Gamma(\frac{5}{2})\,\omega^{-\frac{3}{2}}\ {}_{p+2}F_q\left[(a),\frac{5}{4},\frac{7}{4};(b);(\frac{2}{\omega})^2\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} s_1 s_2^{\frac{1}{2}}\, \eta(s_1, s_2) &\overset{2}{=} \pi^{-\frac{1}{2}} t_1 t_2^{2} (t_1 + t_2)^{-\frac{5}{2}} \cdot \\ &{}_{p+2}F_q\left[(a),\tfrac{5}{4},\tfrac{7}{4};(b);\left(\frac{2t_1t_2}{t_1+t_2}\right)^2\right]. \end{aligned}$$

(d) $s_1^{\frac{3}{2}} s_2^{\frac{1}{2}} \eta(s_1, s_2) \overset{2}{=} \frac{2}{3}\pi^{-1} t_1^{-\frac{5}{2}} t_2^{-\frac{1}{2}} \left(t_1^{-1} + t_2^{-1}\right)^{-1} \left\{2M_2\left(t_1^{-1} + t_2^{-1}\right) - t_1 T_2\left(t_1^{-1} + t_2^{-1}\right)\right\}$, where

$$M_2(\omega) \overset{1}{=} x^2\ {}_pF_q\left[(a);(b);x^2\right]$$

$$\begin{aligned} \text{i.e.,} \qquad M_2(\omega) &= 2\omega^{-2}\ {}_{p+2}F_q\left[(a),\frac{3}{2},2;(b);(\frac{2}{\omega})^2\right]. \\ \text{and} \qquad T_2(\omega) &= \omega^{-1}\ {}_{p+2}F_q\left[(a),1,\frac{3}{2};(b);(\frac{2}{\omega})^2\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} s_1^{\frac{3}{2}} s_2^{\frac{1}{2}}\, \eta(s_1, s_2) &\overset{2}{=} \frac{2}{3}\pi^{-1} t_1^{\frac{1}{2}} t_2^{\frac{5}{2}} (t_1 + t_2)^3 \cdot \\ &\left\{4\ {}_{p+2}F_q\left[(a),\frac{3}{2},2;(b);\left(\frac{2t_1t_2}{t_1+t_2}\right)^2\right]\right. \\ &\left. - \left(1 + \frac{t_1}{t_2}\right)\ {}_{p+2}F_q\left[(a),1,\frac{3}{2};(b);\left(\frac{2t_1t_2}{t_1+t_2}\right)^2\right]\right\}. \end{aligned}$$

(e) $s_1^{\frac{3}{2}} s_2\, \eta(s_1, s_2) \overset{2}{=} \frac{2}{3}\pi^{-1}\, t_1^{-\frac{5}{2}}\, t_2^{-\frac{3}{2}}\, (t_1^{-1} + t_2^{-1})^{-1} \left\{2Q_2(t_1^{-1} + t_2^{-1}) - t_1 R_2\left(t_1^{-1} + t_2^{-1}\right)\right\}$, where

$$\begin{aligned} Q_2(\omega) &\overset{1}{=} x^{\frac{5}{2}}\, {}_pF_q\left[(a);(b);x^2\right] \\ \text{i.e.,}\quad Q_2(\omega) &= \Gamma(\tfrac{7}{2})\,\omega^{-\frac{5}{2}}\ {}_{p+2}F_q\left[(a),\frac{7}{4},\frac{9}{4};(b);(\frac{2}{\omega})^2\right] \\ \text{and}\quad R_2(\omega) &\overset{1}{=} x^{\frac{3}{2}}\ {}_pF_q\left[(a);(b);x^2\right] \\ \text{i.e.,}\quad R_2(\omega) &= \Gamma(\tfrac{5}{2})\,\omega^{-\frac{3}{2}}\ {}_{p+2}F_q\left[(a);\frac{5}{4},\frac{7}{4};\left(\frac{2}{\omega}\right)^2\right]. \end{aligned}$$

(f) $s_1^2 s_2\, \eta(s_1, s_2) \overset{2}{=} \frac{2}{3}\pi^{-1}\, t_1^{-\frac{7}{2}}\, t_2^{-\frac{3}{2}}\, (t_1^{-1} + t_2^{-1})^{-1} \left\{Y_2(t_1^{-1} + t_2^{-1}) \cdot 3t_1\, Q_2(t_1^{-1} + t_2^{-1})\right\}$, where

$$\begin{aligned} Y_2(\omega) &\overset{1}{=} x^{\frac{7}{2}}\, {}_pF_q\left[(a);(b);x^2\right] \\ \text{i.e.,}\quad Y_2(\omega) &= \Gamma(\tfrac{9}{2})\,\omega^{-\frac{7}{2}}\ {}_{p+2}F_q\left[(a),\frac{9}{4},\frac{11}{4};(b);(\frac{2}{\omega})^2\right] \\ \text{and}\quad Q_2(\omega) &= \Gamma(\tfrac{7}{2})\,\omega^{-\frac{5}{2}}\ {}_{p+2}F_q\left[(a);\frac{7}{4},\frac{9}{4};(b);\left(\frac{2}{\omega}\right)^2\right]. \end{aligned}$$

(g) $s_1^2 s_2^{\frac{1}{2}}\, \eta(s_1, s_2) \overset{2}{=} \frac{2}{3}\pi^{-1}\, t_1^{-\frac{7}{2}}\, t_2^{-\frac{1}{2}}\, \left(t_1^{-1} + t_2^{-1}\right) \left\{Z_2\left(t_1^{-1} + t_2^{-1}\right) - 3t_1\, L_2\left(t_1^{-1} + t_2^{-1}\right)\right\}$, where

$$\begin{aligned} Z_2(\omega) &\overset{1}{=} x^{\frac{5}{2}}\, {}_pF_q\left[(a);(b);x^2\right] \\ \text{i.e.,}\quad Z_2(\omega) &= \Gamma(\tfrac{7}{2})\,\omega^{-\frac{5}{2}}\ {}_{p+2}F_q\left[(a),\frac{7}{4},\frac{9}{4};(b);(\frac{2}{\omega})^2\right] \\ \text{and}\quad L_2(\omega) &= \Gamma(\tfrac{5}{2})\,\omega^{-\frac{3}{2}}\ {}_{p+2}F_q\left[(a);\frac{5}{4},\frac{7}{4};(b);\left(\frac{2}{\omega}\right)^2\right]. \end{aligned}$$

PART II.

NUMERICAL INVERSION OF LAPLACE TRANSFORMS

INTRODUCTION

The use of the method of Laplace transforms to solve physical problems is very limited without the use of numerical procedures. The reason for this is that in many cases the transform can be found but is too complicated to apply any analytical technique to invert it. Several numerical methods have been developed to invert the Laplace transform over the past half century. Due to the increase in computational facilities, the last three decades have seen numerous methods offering efficient inversion techniques. Interested readers are referred to Piessens [33] and Piessens and Dang [34] for the bibliography for the period from 1934 to 1976. Also, for an extensive survey on the algorithms for the numerical inversion of Laplace transforms for one-dimensional problems, the reader should consult Davies and Martin [18].

It is well-known that unlike the forward Laplace transform the inverse Laplace transform is not stable under small perturbations. Since the inversion process is unstable, it is impossible to find a universal algorithm to invert the Laplace transform numerically. However, the methods using orthogonal polynomials and Fourier series have performed well with many different types of problems. Among the methods described in the literature, the methods which use Laguerre polynomials by Weeks [50] and by Piessens and Branders [32] and the methods using Fourier series by Dubner and Abate [22] and by Crump [7] are noted for their accuracy with several different types of functions. For this same reason we have sought to extend these methods to two-dimensions. Though there are many methods of inversion of Laplace transforms in existence, only a handful of them deal with the multi-dimensional case (e.g., [40], [42]). Our main attention in this part of the dissertation goes toward developing numerical techniques to invert multi-dimensional problems. Let us start with some useful definitions which we have already introduced in Part I.

The Laplace transform of a function $f(t)$ is generally defined by the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad (l_1)$$

and its inverse is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds, \qquad (i_1)$$

where $c > d$ for some number d and F is analytic for $\Re s > d$. The second integral above is a contour integral along the vertical line $\Re s = c$. The extensions of (l_1) and (i_1) to two-dimensions are given by the equations (l_2) and (i_2) below. As in the one-dimensional case, the last integral below is also a contour integral along vertical line segments $\Re s_1 = c_1$ and $\Re s_2 = c_2$. The extension to higher dimensions is straight-forward from two-dimensions.

$$F(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-s_1 t_1 - s_2 t_2} f(t_1, t_2)\, dt_1 dt_2 \qquad (l_2)$$

and

$$f(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} e^{s_1 t_1 + s_2 t_2} F(s_1, s_2)\, ds_1 ds_2, \ (i_2)$$

where $c_1 > d_1, c_2 > d_2$ and $F(s_1, s_2)$ is analytic for $\Re s_1 > d_1$ and $\Re s_2 > d_2$ (i.e., $\Re s_1 > d_1$ and $\Re s_2 > d_2$ specifies the region in which F is analytic).

OBJECTIVE

The objective of this part of the dissertation is to develop some numerical methods to invert multi-dimensional Laplace transforms. Among the existing methods for multi-dimensional inversion of Laplace transforms developed in the last two decades, only two of them present numerical results. One of the methods is developed by Singhal, Vlach and Vlach [42] and the other one is by Shih, Shen and Shiau [40]. The former one is an extension of the one-dimensional algorithm based on the evaluation of the inverse integral using residue Calculus after replacing the exponential function by its Padé approximate, and the latter is based on the expansion of $F(s)$ in terms of shifted Legendre polynomials. The second method is similar to the one-dimensional method proposed by Bellman et al. [1]. We will compare our results with the first method.

Here we propose two different two-dimensional inversion techniques, which are the extensions of the one-dimensional methods based on the expansion of the inverse function by Laguerre series, and Fourier series representations of the forward transform. An extensive survey conducted by Davies and Martin [18] on one-dimensional numerical inversion algorithms for the Laplace transform suggests that the methods based on expanding $f(t)$ in a Laguerre series are the best general purpose methods and that the method based on Fourier series representations is the next amongst the existing techniques.

Under the section 'Methods for Numerical Inversion of One-Dimensional Transforms', we look at some numerical techniques for inverting the one-dimensional Laplace transforms. Initially we consider the major classification of these methods based on theoretical considerations and then we discuss some interesting methods within these classes. We emphasize those methods which are of primary interest here by presenting numerical examples which illustrate their accuracy.

Numerical Inversion of Two-Dimensional Laplace transforms section is devoted to the development of two new two-dimensional numerical techniques to invert the Laplace transforms with the use of Laguerre series and Fourier series. The method based on an expansion of the inverse function in a series of Laguerre polynomials, in the case of one-dimension, has been the most successful one. Especially, the papers presented by Weeks and by Piessens and Branders are highly recommended by many authors. Naturally we are interested in extending this method to higher dimensions. Method 1 describes the numerical procedure using Laguerre polynomials to invert the Laplace transform in two-dimensions. Examples are provided to illustrate the effectiveness of the method and the results are compared with the results of Singhal, Vlach and Vlach method. Error analysis and the selection of parameters are discussed in detail. We also give a generalization of this method. Method 2 uses the Fourier series of the forward transform in approximating the inverse function. This method is an extension of the one-dimensional method developed by Dubner and Abate and later improved by Crump. Several numerical examples are presented and a comparison is made with the results of the Singhal, Vlach and Vlach method. We also study the procedures for error control and the selection of parameters. Finally, we discuss the implementation of the methods as a Fortran program.

METHODS FOR NUMERICALLY INVERTING ONE-DIMENSIONAL LAPLACE TRANSFORMS

Here we study the methods of numerical inversion of the Laplace transform in one-dimension. We describe some major types of the existing techniques classified from a theoretical standpoint. Under each class we look at some well-known methods which are of special interest. We will develop some multi-dimensional methods based on two of the methods. Descriptions of the methods which are directly related to our multi-dimensional methods are given in detail. For an extensive survey on the numerical methods of inverse Laplace transform, the reader should consult Davies and Martin [18].

Of the four classes we consider in this chapter, methods using Fourier series expansions and the methods using orthogonal polynomials are highly regarded for their applicability to a wide range of functions. Especially, methods by Dubner and Abate [22] (and the improved version by Crump [7]) based on Fourier series representations, and that of Weeks [50] (and the modified version by Piessens and Branders [32]) based on the expansion in terms of Laguerre polynomials are recommended. These methods are described in detail. Also, we briefly describe the method by Singhal and Vlach [41], which has been extended to multi-dimensional problems. These methods play an important role in the development and implementation of our multi-dimensional methods which are discussed later.

1. Methods which use the exponential substitution in the forward integral Many authors like Bellman et al. [1] have developed methods based on using substitution $x = e^{-\nu t}$, $\nu > 0$ in the forward integral

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Here we will describe the method proposed by Bellman, Kalaba, and Lockett [1]. The general case is described in Luke [29].

Let $x = e^{-t}$ in the above integral to obtain

$$F(s) = \int_0^1 x^{s-1} f(-\ln x)\,dx.$$

By applying a quadrature formula with weights $\{w_i\}$ and the nodes $\{x_i\}$ to be subsequently determined, and taking n different values for s, say $s = 1, 2, \ldots, n$, one gets

$$\sum_{i=1}^{n} x_i^k y_i = a_k, k = 0, 1, \ldots, n-1, \tag{II.1}$$

where $y_i = w_i f(-\ln x_i)$ and $a_k = F(k+1)$. Multiply both sides of (II.1) by q_{kj} also to be determined, and sum over k to obtain

$$\sum_{i=1}^{n} y_i g_j(x_i) = \sum_{k=0}^{n-1} a_k q_{kj}, \tag{II.2}$$

where $g_j(x) = \sum_{k=0}^{n-1} q_{kj} x^k$.

Let us take

$$g_j(x) = \frac{P_n^*(x)}{(x - x_j) P_n^{*'}(x_j)},$$

where $P_n^*(x) = P_n(1-2x)$ is the shifted Legendre polynomial. For the zeros x_i of $P_n^*(x)$, we have

$$g_j(x_i) = \delta_{ij}.$$

Therefore, with this choice of nodes, equation (II.2) takes the form

$$y_i = \sum_{k=0}^{n-1} a_k q_{kj},$$

for $i = 1, 2, \ldots, n$, where q_{kj}'s are the coefficients of the polynomial $g_j(x)$ which is of degree $n-1$. Once the y_i's are known, $f(.)$ can be evaluated at the points $t_i = -\ln x_i$ from the equation

$$y_i = w_i f(-\ln x_i).$$

A major disadvantage of this method is that it gives values for $f(t)$ only at a restricted set of non-equidistant points as determined by the zeros of the Legendre polynomials. This method works well, if the inverse function is of a decaying exponential type.

2. Methods using Gaussian quadrature formulas for approximating the inversion integral Several papers have been written which use Gaussian quadrature formulas. Among these are papers by Salzer [39] and Piessens and Branders [32]. The method is given below.

First the inversion integral is written in the following form:

$$f(t) = \frac{1}{2\pi i t}\int_{ct-i\infty}^{ct+i\infty} F(z/t)e^z\,dz. \tag{II.3}$$

Let $F(z/t) = z^{-r}\,G(z)$, where r is a positive real number that must be chosen so that $s^r F(s)$ is analytic and has no branch point at infinity, thus

$$s^r F(s) = \sum_{j=0}^{\infty} a_j s^{-j}.$$

Therefore (II.3) becomes

$$\begin{aligned} t\,f(t) &= \int_{ct-i\infty}^{ct+i\infty} e^z z^{-r} G(z)\,dz \\ &\approx \sum_{j=1}^{N} w_j G(z_j), \end{aligned} \tag{II.4}$$

$\{w_j\}$ and $\{z_j\}$ are determined such that the integration is exact whenever $G(z)$ is a polynomial in u^{-1}, of degree less than or equal to $2N-1$. Then we have a Gaussian integration formula, that has degree of precision $2N-1$; see Piessens [36] for details.

We will present another general method similar in nature which yields the above approximations when $r = 0$. This method was proposed by Singhal and Vlach [41] and made use of Padé approximations. The same authors extended this method to multi-dimensional problems. We will compare this method with our two-dimensional methods later.

Replace e^z in the equation (II.3) by its Padé approximation

$$\psi_{N,M}(z) = \frac{\sum_{k=0}^{M} (M+N-k)! \binom{M}{k} z^k}{\sum_{k=0}^{N} (M+N-k)! \binom{N}{k} (-z)^k}, \tag{II.5}$$

with $M < N$. All the poles of (II.5) are simple and, for M not differing considerably from N, all are in the right half plane. $\psi_{M,N}(z)$ has the property that the first $M+N+1$ terms of its Taylor expansion equal those of e^z. Then the new integral

$$f(t) = \frac{1}{2\pi i t} \int_{ct-i\infty}^{ct+i\infty} F(z/t)\psi_{M,N}(z)\,dz \tag{II.6}$$

can be evaluated by residue Calculus by closing the path of integration around the poles of $\psi_{M,N}$ in the right half plane. M and N are chosen such that the function

$F(z/t)\psi_{M,N}$ has at least two more finite poles than zeros to avoid the contribution of the path along the infinite arc to the integral. Also it has been shown that the the method is exact for $F(s) = s^{-k}, k = 0, 1, 2, \ldots, M + N + 1$. In the above method $M = N - 2$ gives the Gaussian method for $r = 0$.

3. Methods representing $f(t)$ as a Fourier series We write the inverse integral as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ct} (\cos \omega t + i \sin \omega t) [\Re F(s) + i \Im F(s)] \, d\omega, \tag{II.7}$$

where $s = c + i\omega$ and is larger than the real part of all singularities of $F(s)$. Because $f(t)$ is real, the imaginary part of the right hand side of the equation (II.7) equals zero. By the Schwarz Reflection Principle $\Re F(s)$ is an even function of ω and $\Im F(s)$ is an odd function of ω. Therefore

$$f(t) = \frac{e^{ct}}{\pi} \int_{0}^{\infty} [\Re F(s) \cos \omega t - \Im F(s) \sin \omega t] \, d\omega. \tag{II.8}$$

For $t < 0, f(t) = 0$ which implies

$$f(t) = \frac{2e^{ct}}{\pi} \int_{0}^{\infty} [\Re F(s) \cos \omega t] \, d\omega \tag{II.9}$$

or

$$f(t) = \frac{-2e^{ct}}{\pi} \int_{0}^{\infty} [\Im F(s) \sin \omega t] \, d\omega. \tag{II.10}$$

The first paper using this idea to numerically invert the Laplace transform by using Fourier representation was written by Dubner and Abate [22]. Their method essentially approximated the integral in (II.9) by the trapezoidal rule,

$$f(t) \approx \frac{2e^{ct}}{T} [1/2 \Re\{F(c)\} + \sum_{k=1}^{\infty} \Re\{F(c + i\frac{\pi k}{T}\} \cos \frac{k\pi t}{T}], \tag{II.11}$$

where T is a parameter. Dubner and Abate derived this formula by considering the expansion of $f(t)$, $0 \le t < T$, as a Fourier cosine series.

The major source of error in this method (see [22] for derivation) is the discretization error associated with the trapezoidal rule, which is given by

$$E_d = \sum_{k=1}^{\infty} e^{-2ckT}[f(2nT+t) + e^{2ct} f(2nT-t)] \qquad \text{(II.12)}$$

Suppose $F(s) = \mathcal{L}[f(t)]$ is analytic for $\Re s > c_0$. Then it was shown in [22] that there are choices of $c \ge c_0$ and $T > 0$ such that E_d can be arbitrarily small for $0 \le t < T/2$.

There are many improved versions of Dubner and Abate's method which have appeared in the literature. Durbin [23], Crump [7] and Honig and Hirdes [26] used the integral in the equation(II.8) instead of (II.9) along with a process to accelerate the convergence of the series to obtain better results.

Here we will give a brief description of Crump's method and will demonstrate its accuracy with some examples. The method is essentially the same as Dubner and Abate's. The epsilon algorithm of Macdonald [30], which is a general purpose algorithm, is used to accelerate the convergence of the approximating series.

Suppose we want to approximate $\sum_{k=1}^{\infty} a_k$, using $2N+1$ partial sums

$$s_m = \sum_{k=1}^{m} a_k, \; m = 1, 2, \ldots, 2N+1.$$

Define

$$\begin{aligned} \varepsilon_{p+1}^{m} &= \varepsilon_{p-1}^{m+1} + \{\varepsilon_p^{m+1} - \varepsilon_p^{m}\}^{-1}, \\ & \quad \text{for } m = 1, 2, \ldots, 2N+1 \\ & \quad \text{and } p = 0, 1, \ldots, 2N+1, \end{aligned}$$

$$\text{with } \varepsilon_0^m = \varepsilon_1^m = 0. \tag{II.13}$$

Then the sequence

$$\varepsilon_1^1, \varepsilon_3^1, \ldots, \varepsilon_{2N+1}^1$$

is a sequence of approximations to the sum of the series that will often better approximate the sum than $s_1, s_2, \ldots, s_{2N+1}$.

The approximation to the integral in (II.8) is given by the trapezoidal rule,

$$f(t) \approx \frac{2e^{ct}}{T}[1/2\Re\{F(c)\} + \sum_{k=1}^{\infty} \Re\{F(c+ik\pi)\}\cos\frac{k\pi t}{T} \tag{II.14}$$

$$- \Im F(c+ik\pi)\}\sin\frac{k\pi s)}{T}], \tag{II.15}$$

where T is a parameter. The discretization error is given by

$$E_d = \sum_{k=1}^{\infty} e^{-2akT} f(2kT+t).$$

This method can be used to invert the function $f(t)$ which is a piecewise continuous function satisfying the condition

$$|f(t)| < Me^{\alpha t}, \text{ for real } \alpha \text{ and positive } M.$$

For such functions it was shown in [7] that for $0 < t < 2T$ the error can be made arbitrarily small by suitably choosing the parameters a and T. We have written a fortran program for Crump's method (see the Appendix A) and tested it on the following examples.

Example 1:

$$F(s) = \frac{s-1}{(s-1)^2+1} - 1/s$$

$$f(t) = e^t \cos t - 1$$

Example 2:

$$F(s) = \frac{e^{-5s}}{s}$$

$$f(t) = U(t-5),$$

where U(t) denotes Heaviside's function.

The output for the above examples are shown in Table II.1 and Table II.2. These results show that a few terms in the series are enough to approximate the function with high accuracy, at least in some cases. The survey on numerical inversion of the Laplace transform done by Davies and Martin [18] ranked Crump's method as one of the highly recommended methods for its accuracy with many different functions. The other two top ranked methods suggested by this survey are by Weeks [50] and by Piessens and Branders [32], which are described in the next class.

4. Methods using the expansion of $f(t)$ in terms of orthogonal polynomials Many methods which use orthogonal polynomials can be found in the existing literature. Methods based on the expansion of $f(t)$ in terms of the Laguerre polynomials are of our main interest. The notion of expanding inverse functions of Laplace transforms in a series of Laguerre polynomials have been around since 1935 (see Widder [52], Tricomi [45]). It was in 1956 that the numerical procedures using Laguerre polynomials were introduced by Lanczos [28] and Papoulis [31]. Later

Table II.1: Output for Crump's Method: Example 1

$$F(s) = \frac{s-1}{(s-1)^2+1} - \frac{1}{s}$$

$$f(t) = e^t \cos t - 1$$

$a = 2.771 \qquad T = 7.5 \; n = 59$

time	approx value	exact value	abs-error	rel-error
1.0	0.46869	0.46869	0.46022E-06	0.98193E-06
2.0	-4.07493	-4.07493	0.11973E-09	0.29382E-10
3.0	-20.88453	-20.88453	0.38399E-10	0.18386E-11
4.0	-36.68773	-36.68773	0.33482E-09	0.91262E-11
5.0	41.09920	41.09920	0.18082E-09	0.43997E-11
6.0	386.36034	386.36034	0.65553E-09	0.16967E-11
7.0	825.75421	825.75421	0.52858E-08	0.64012E-11
8.0	-434.72949	-434.72949	0.18797E-07	0.43237E-10
9.0	-7383.96498	-7383.96498	0.11344E-05	0.15363E-09
10.0	-18482.78033	-18482.78033	0.10204E-07	0.55206E-12

Table II.2: Output for Crump's Method: Example 2

$$F(s) = \frac{2}{s} - \frac{1}{(s+1)}e^{-5s}$$

$$f(t) = 0 \qquad \text{if } t < 5$$
$$= 2 - e^{(t-5)} \qquad \text{if } t > 5$$

$a = 1.308$ $T = 8.0$ $n = 63$

time	approx val	exact val	abs-error	rel-error
1.0	0.000000	0.000000	0.20000E-09	0.20000E-09
2.0	0.000000	0.000000	0.20000E-09	0.20000E-09
3.0	0.000000	0.000000	0.19952E-09	0.19952E-09
4.0	0.000000	0.000000	0.14622E-07	0.14622E-07
6.0	1.632119	1.632121	0.11914E-05	0.72999E-06
7.0	1.864665	1.864665	0.17197E-09	0.92224E-10
8.0	1.950213	1.950213	0.19993E-09	0.10252E-09
9.0	1.981684	1.981684	0.20001E-09	0.10093E-09
10.0	1.993262	1.993262	0.20001E-09	0.10034E-09

Weeks [50] presented an efficient method, which is still considered to be one of the best methods for inverting Laplace transforms numerically. We describe this method next.

Let $f(t)$ be approximated by the finite sum

$$f_N(t) = e^{ct} \sum_{k=0}^{N} a_k \psi_k(t/T), \tag{II.16}$$

where $\phi_k(x) = e^{-x/2} L_k(x)$ with $L_k(x)$ denoting the Laguerre polynomial of degree k. By orthogonality the coefficients $\{a_k\}$ are given by

$$a_k = 1/T \int_0^\infty e^{-st} f(t) \phi_k(t/T)\, dt. \tag{II.17}$$

Let

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

and

$$F_N(s) = \int_0^\infty e^{-ct} f_N(t)\, dt.$$

The Rodrique's formula for the Laguerre polynomial of order k is given by

$$L_k(t) = \frac{1}{k!} e^t \frac{d^k}{dt^k} (e^{-t} t^k)$$

or

$$L_k(t) = \sum_{i=1}^{k} \binom{k}{k-i} \frac{(-t)^i}{\nu!}.$$

Also the Laplace transform of $L_k(t)$ is given by $\frac{(s-1)^k}{s^{k+1}}$. Therefore by taking the Laplace transform of both sides of equation (II.16) we find

$$F_N(s) = \sum_{k=0}^{N} a_k \frac{(s - c - 1/2T)^k}{(s - c + 1/2T)^{k+1}}. \tag{II.18}$$

Let $s = c + i\omega$. If $f(t)$ satisfies the conditions

$$\int_0^\infty e^{-ct} |f(t)|\, dt \quad < \quad \infty$$

and

$$\int_0^\infty e^{-2ct} |f(t)|^2\, dt \quad < \quad \infty,$$

whenever $c \geq c_0$, for some number c_0, then $F_N(c+i\omega)$ converges to $F(c+i\omega)$ in the mean for $\omega \in (-\infty, \infty)$.

Set $\omega = (1/2T)\cot\theta/2$ for$0 < \theta < \pi$, in (II.18), then it follows from the above fact that

$$\frac{1}{2T}(1 + i\cot\theta/2)F(c + \frac{i}{2T}\cot\frac{\theta}{2}) \approx \sum_{k=0}^{N} a_k e^{ik\theta}. \tag{II.19}$$

Since f is real, $\{a_k\}$ are real too. Therefore, by considering the real part of the terms on both sides of equation (II.19), one can obtain

$$h(\theta) \quad \approx \quad \sum_{k=0}^{N} a_k \cos k\theta,$$

where

$$h(\theta) \quad = \quad \Re\{(1/2T + i/2T\cot\theta/2)F(c + i/2T\cot\theta/2)\}.$$

The coefficients $\{a_k\}$ can be approximated by trigonometric interpolation formulas. The Laguerre polynomials can be evaluated by using the following recursive relations:

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= 1 - x, \\ xL_n(x) &= (2n - 1 - x)L_{n-1}(x) - (n-1)L_{n-2}(x); n > 1. \end{aligned} \tag{II.20}$$

In this method the parameters c and T are introduced to accelerate the convergence of the series in (II.16).

Piessens and Branders [32] replaced the sum in equation (II.16) by $t^a e^{ct} \sum_{k=0}^{N} a_k L_k^a(bt)$, where $L_k^a(x)$ a generalized Laguerre polynomial of degree k, and a, b, c are parameters introduced to smooth out any irregularities of f and to accelerate the convergence of the resulting series. This method is an extension of the above method since it can be used to invert a wider range of problems.

Table II.3 and Table II.4 show the numerical results of Piessens and Branders method applied to the following problems:

Example 3:

$$F(s) = \frac{1}{\sqrt{1+s^2}}$$

$$f(t) = J_0(t)$$

Example 4:

$$F(s) = \frac{e^{-1/s}}{\sqrt{s}}$$

$$f(t) = \frac{\cos\sqrt{t}}{\pi t}$$

The output in Table II.3 and Table II.4 illustrates the exceptional accuracy of Piessens and Branders method. Computer programs for this method are given in the Appendix B.

Table II.3: Output of Piessens and Branders Method : Example 3

$$F(s) = \frac{1}{\sqrt{s^2+1}}$$
$$f(t) = J_0(t) \text{ (Bessel function)}$$
$$a = 0.0 \qquad b = 1.0 \; c = 0.0$$
$$n = 100 \qquad m = 100$$

time	exact value	approx value	abs-error	rel-error
1.0	0.765197686558	0.765197686558	0.12351E-14	0.16141E-14
2.0	0.223890779141	0.223890779141	0.16653E-14	0.74382E-14
3.0	-0.260051954902	-0.260051954902	0.42327E-14	0.16276E-13
4.0	-0.397149809864	-0.397149809864	0.81463E-14	0.20512E-13
5.0	-0.177596771314	-0.177596771314	0.53846E-14	0.30319E-13
6.0	0.150645257251	0.150645257251	0.62311E-14	0.41363E-13
7.0	0.300079270520	0.300079270520	0.15987E-13	0.53277E-13
8.0	0.171650807138	0.171650807138	0.53013E-14	0.30884E-13
9.0	-0.090333611183	-0.090333611183	0.28658E-13	0.31724E-12
10.0	-0.245935764451	-0.245935764451	0.66475E-14	0.27029E-13

Root- Mean Sq. Error: 0.11426E-13

Root- Mean Sq.(with wt. exp(-t)): 0.31541E-14

Table II.4: Output of Piessens and Branders Method : Example 4

$$F(s) = \frac{e^{-1/s}}{\sqrt{s}}$$

$$f(t) = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$$

$a = -0.5 \quad b = 1.0 \; c = 0.0$

$n = 30 \quad m = 30$

time	exact value	approx value	abs-error	rel-error
1.0	-0.2347857104	-0.2347857104	0.58287E-15	0.24825E-14
2.0	-0.3795389758	-0.3795389758	0.49960E-15	0.13163E-14
3.0	-0.3089411519	-0.3089411519	0.20817E-15	0.67381E-15
4.0	-0.1843894611	-0.1843894611	0.23592E-15	0.12795E-14
5.0	-0.0600375326	-0.0600375326	0.52909E-16	0.88127E-15
6.0	0.0427283361	0.0427283361	0.43108E-15	0.10089E-13
7.0	0.1167044615	0.1167044615	0.44409E-15	0.38052E-14
8.0	0.1616082470	0.1616082470	0.58287E-15	0.36067E-14
9.0	0.1805726914	0.1805726914	0.13184E-14	0.73012E-14
10.0	0.1782597589	0.1782597589	0.11102E-14	0.62281E-14

Root- Mean Sq. Error: 0.66217E-15

Root- Mean Sq.(with wt. exp(-t)): 0.66547E-15

NUMERICAL INVERSION OF TWO-DIMENSIONAL LAPLACE TRANSFORMS

In the following we describe two new methods to invert the Laplace transform in two dimensions. The first method is based on the expansion of the inverse function in terms of Laguerre polynomials. The other method is based on the representations of the inverse function in terms of Fourier series.

Method 1: A Method Based on Expansion of the Inverse Function in a Laguerre Series

We describe a method to invert Laplace transforms in two-dimensions based on expanding the inverse function in a series of products of (Generalized) Laguerre polynomials. The success of the methods based on expanding the inverse function in a series of Laguerre polynomials in one-dimension has motivated us to work on a two-dimensional method of the same kind. The method we present in this section is an extension of the methods presented by Weeks [50] and by Piessens and Branders [32], and the method suggested by Luke [29].

In the following we assume $F(s_1, s_2)$ and $f(t_1, t_2)$ are the two-dimensional Laplace transform and its inverse respectively. The definitions of F and f are given in (l_2) and (i_2).

Description of the method Let us assume that we can write

$$f(t_1,t_2) = e^{c_1t_1+c_2t_2} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}L_j(bt_1)L_k(bt_2), \text{ for } 0 < t_1,t_2 < \infty,$$

where $b > 0$ and $L_n(.)$ is the Laguerre polynomial of degree n. Let $F(s_1,s_2)$ be the Laplace transform of $f(t_1,t_2)$, with $\Re s_1 > c_1$ and $\Re s_2 > c_2$. Then by formally interchanging the integrals and the infinite sums, we obtain

$$F(s_1,s_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}\frac{(s_1-c_1-b)^j}{(s_1-c_1)^{j+1}} \frac{(s_2-c_2-b)^k}{(s_2-c_2)^{k+1}} \tag{II.21}$$

Our goal is to approximate $f(t_1,t_2)$ by the finite sum,

$$f_N(t_1,t_2) = e^{c_1t_1+c_2t_2} \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} a_{jk}L_j(bt_1)L_k(bt_2). \tag{II.22}$$

To achieve this goal, we have to find suitable choice of parameters and algorithms to evaluate a_{jk}, $L_j(.)$ and $L_k(.)$, for $j,k = 0,1,\cdots,N-1$. Strategies for parameter selection will be explained later.

We use the theory of Complex analysis for several variables to find the coefficients $\{a_{jk}\}$. Let us begin by introducing the following bilinear transformations. We set

$$z_1 = \frac{(s_1-c_1-b)}{(s_1-c_1)} \text{ and } z_2 = \frac{(s_2-c_2-b)}{(s_2-c_2)}. \tag{II.23}$$

Under these transformations the quarter space

$$\Re s_1 > c_1 + \frac{b}{2}, \Re s_2 > c_2 + \frac{b}{2} \tag{II.24}$$

is mapped into a unit polydisc

$$D = \{(z_1,z_2); |z_1| < 1, |z_2| < 1\}.$$

Then, from equation (II.21) we find

$$\begin{aligned}
\chi(z_1, z_2) &= \frac{b^2}{(1-z_1)(1-z_2)} F\left(\frac{b}{1-z_1} + c_1, \frac{b}{1-z_2} + c_2\right) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} z_1^j z_2^k, \qquad \text{(II.25)}
\end{aligned}$$

where the first equality defines χ. Since $F(s_1, s_2)$ is analytic for $\Re s_1 > c_1 + \frac{b}{2}, \Re s_2 > c_2 + \frac{b}{2}$, $\chi(z_1, z_2)$ is analytic in D. In fact, χ is analytic in a bigger region containing D, since $b > 0$ and $F(s_1, s_2)$ is analytic for $\Re s_1 > c_1$ and $\Re s_2 > c_2$ for the class of functions (see Parameter selection, etc.) for which this method can be efficiently applied.

Next we restrict χ to the boundary of the unit polydisc by setting

$$\begin{aligned}
z_1 &= e^{i\theta_1}, \quad -\pi \le \theta_1 \le \pi \\
z_2 &= e^{i\theta_2}, \quad -\pi \le \theta_2 \le \pi \qquad \text{(II.26)}
\end{aligned}$$

in the equation (II.25) to get

$$\begin{aligned}
\psi(\theta_1, \theta_2) &= \chi(e^{i\theta_1}, e^{i\theta_2}) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} e^{i(j\theta_1 + k\theta_2)}. \qquad \text{(II.27)}
\end{aligned}$$

Since f is real, for $j, k = 0, 1, \ldots,$ $\{a_{jk}\}$ are real. Therefore

$$\begin{aligned}
\Re\{\psi(\theta_1, \theta_2)\} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} \cos(j\theta_1 + k\theta_2), \text{ and} \\
\Im\{\psi(\theta_1, \theta_2)\} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} \sin(j\theta_1 + k\theta_2). \qquad \text{(II.28)}
\end{aligned}$$

Then the coefficients $\{a_{jk}\}$ are given by

$$\begin{aligned} a_{00} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Re\{\psi(\theta_1,\theta_2)\}\, d\theta_1 d\theta_2 \text{and} \\ a_{jk} &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Re\{\psi(\theta_1,\theta_2)\}\cos(j\theta_1 + k\theta_2)\, d\theta_1 d\theta_2 \\ &\text{or} \\ a_{jk} &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Im\{\psi(\theta_1,\theta_2)\}\sin(j\theta_1 + k\theta_2)\, d\theta_1 d\theta_2, \end{aligned} \tag{II.29}$$

where j and k are not zero simultaneously. Again by using the fact that a_{jk} are all real, we get the following:

$$\begin{aligned} \Re\{\psi(-\theta_1,-\theta_2)\} &= \Re\{\psi(\theta_1,\theta_2)\}, \\ \Re\{\psi(-\theta_1,\theta_2)\} &= \Re\{\psi(\theta_1,-\theta_2)\}, \text{and} \\ \Im\{\psi(-\theta_1,\theta_2)\} &= -\Im\{\psi(\theta_1,-\theta_2)\}. \end{aligned} \tag{II.30}$$

Now by using the equation (II.30), the coefficients in (II.29) can be written as

$$\begin{aligned} a_{00} = \frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi} & [\Re\{\psi(\theta_1,\theta_2)\} + \Re\{\psi(\theta_1,-\theta_2)\}]\, d\theta_1 d\theta_2, \text{and} \\ a_{jk} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} & [\Re\{\psi(\theta_1,\theta_2)\}\cos(j\theta_1 + k\theta_2) \\ + & \Re\{\psi(\theta_1,-\theta_2)\} cos(j\theta_1 - k\theta_2)]\, d\theta_1 d\theta_2 \\ & \text{or} \\ a_{jk} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} & [\Im\{\psi(\theta_1,\theta_2)\} sin(j\theta_1 + k\theta_2) \\ + & \Im\{\psi(\theta_1,-\theta_2)\}\sin(j\theta_1 - k\theta_2)]\, d\theta_1 d\theta_2, \end{aligned} \tag{II.31}$$

for $j, k = 0, 1, \ldots$, but j and k are not zero simultaneously.

Let us define

$$\chi_1(\theta_1,\theta_2) = \Re\{\psi(\theta_1,\theta_2)\} + \Re\{\psi(\theta_1,-\theta_2)\}$$

$$\begin{aligned}
\chi_2(\theta_1,\theta_2) &= \Re\{\psi(\theta_1,\theta_2)\} - \Re\{\psi(\theta_1,-\theta_2)\} \\
\eta_1(\theta_1,\theta_2) &= \Im\{\psi(\theta_1,\theta_2)\} + \Im\{\psi(\theta_1,-\theta_2)\} \\
\eta_2(\theta_1,\theta_2) &= \Im\{\psi(\theta_1,\theta_2)\} - \Im\{\psi(\theta_1,-\theta_2)\}.
\end{aligned} \tag{II.32}$$

Then the coefficients $\{a_{jk}\}$ can be written in a simplified form as

$$\begin{aligned}
a_{00} = \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi & \quad \chi_1(\theta_1,\theta_2)\,d\theta_1 d\theta_2 \\
a_{jk} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi & \quad [\chi_1(\theta_1,\theta_2)\cos(j\theta_1)\cos(k\theta_2) \\
& - \chi_2(\theta_1,\theta_2)\sin(j\theta_1)\sin(k\theta_2)]\,d\theta_1 d\theta_2
\end{aligned} \tag{II.33}$$

or

$$\begin{aligned}
a_{jk} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi & \quad [\eta_1(\theta_1,\theta_2)\sin(j\theta_1)\cos(k\theta_2) \\
& + \eta_2(\theta_1,\theta_2)\cos(j\theta_1)\sin(k\theta_2)]\,d\theta_1 d\theta_2,
\end{aligned}$$

again for $j,k = 0,1,\ldots,$ but j and k are not zero simultaneously.

For j,k=0,1,...,N-1 the coefficients $\{a_{jk}\}$ can be approximated by a quadrature formula. The midpoint rule gives

$$\begin{aligned}
a_{00} = \frac{1}{2M^2}\sum_{l=1}^{M}\sum_{m=1}^{M} & \quad \chi_1(\alpha_l,\beta_m) \\
a_{jk} = \frac{1}{M^2}\sum_{l=1}^{M}\sum_{m=1}^{M} & \quad \chi_1(\alpha_l,\beta_m)\cos(j\alpha_l)\cos(k\beta_m) \\
& - \chi_2(\alpha_l,\beta_m)\sin(j\alpha_l)\sin(k\beta_m)
\end{aligned} \tag{II.34}$$

or

$$\begin{aligned}
a_{jk} = \frac{1}{M^2}\sum_{l=1}^{M}\sum_{m=1}^{M} & \quad \eta_1(\alpha_l,\beta_m)\sin(j\alpha_l)\cos(k\beta_m) \\
& - \eta_2(\alpha_l,\beta_m)\cos(j\alpha_l)\sin(k\beta_m),
\end{aligned}$$

where $j, k = 0, 1, \ldots, N-1$ but j and k are not zero simultaneously and

$$\begin{aligned} \alpha_l &= \frac{(2l-1)\pi}{2M} \text{ for } l = 1, 2, \ldots, M \\ \beta_m &= \frac{(2m-1)\pi}{2M} \text{ for } m = 1, 2, \ldots, M. \end{aligned} \qquad \text{(II.35)}$$

An algorithm for summing the truncated series The next major task is to find an algorithm to sum the series, once we have evaluated the coefficients $\{a_{jk}\}$. F. J. Smith [43] has proposed an efficient algorithm for summing a truncated series of orthogonal polynomials. By using Smith's algorithm, Piessens and Branders have obtained excellent results for their one-dimensional inversion method. Next we describe a modification of this algorithm which suits our needs.

Suppose that we have already evaluated the coefficients a_{jk}, for j,k=0,1,...,N-1. Then an algorithm for computing the sum of the series in (II.22) is as follows.

$$B_{N+1,k} = B_{N,k} = 0, \text{ for } k = 0, 1, \ldots, N-1$$

$$\begin{aligned} B_{j,k} &= a_{jk} + \left(2 - \frac{bt_1 + 1}{j+1}\right)B_{j+1,k} - \left(1 - \frac{1}{j+2}\right)B_{j+2,k}, \\ &\quad \text{for } j = N-1, N-2, \ldots, 0. \end{aligned} \qquad \text{(II.36)}$$

Then

$$\sum_{j=0}^{N-1} a_{jk} L_j(bt_1) = B_{0,k}, \text{ for } k = 0, 1, \ldots, N-1. \qquad \text{(II.37)}$$

Next, we start with

$$\begin{aligned} A_{N+1} = &\quad A_N = 0, \text{ and then} \\ A_k = &\; B_{0,k} + \left(2 - \tfrac{bt_2+1}{k+1}\right)A_{k+1} - \left(1 - \tfrac{1}{k+2}\right)A_{k+2}, \\ &\quad \text{for } k = N-1, N-2, \ldots, 0. \end{aligned} \qquad \text{(II.38)}$$

This implies

$$\sum_{k=0}^{N-1} B_{0,k} L_k(bt_2) = A_0.$$

Therefore, the sum in (II.2) is given by

$$\sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_{jk} L_j(bt_1) L_k(bt_2) = A_0.$$

In summary, to get the sum in (II.22) we have used the the theory of Complex Analysis to obtain an analytic function χ,defined in (II.25) in a polydisc, from the original Laplace transform F. Equations (II.26) through (II.33) describe the way we have obtained the actual coefficients. Next we have used the midpoint rule to find the approximate values of these coefficients. The evaluation process is shown in (II.34) through (II.35). Finally, we have used the modified form of F. J. Smith's algorithm to sum the truncated series. In the later part of this section we will study strategies to choose the parameters and the error control.

A generalization We describe a generalization of the above method. Let us use the generalized Laguerre polynomials in the series expansion of $f(t_1, t_2)$ instead of ordinary Laguerre polynomials. This was suggested by Luke [29] and implemented by Piessens and Branders [32] for the one-dimensional case. The resulting method will be an improved version of the method described above, and it can be used to invert a wider class of functions efficiently. Since the procedure is essentially the same as the above method, only a brief description is given below.

Let us assume that

$$\begin{aligned} f(t_1, t_2) &= {t_1}^{a_1}\, {t_2}^{a_2}\, e^{c_1 t_1 + c_2 t_2}. \\ & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} L_j^{a_1}(bt_1) L_k^{a_2}(bt_2), \end{aligned} \tag{II.39}$$

where $-1 < a_1, a_2 < 1$ and $b > 0$. We give the important equations of this method next. By taking the Laplace transform of (II.39) and setting

$$z_j = \frac{s_j - c_j - b_j}{s_j - c_j} \text{ with } \Re(s_j) \geq c_j + \frac{b}{2} \text{ for } j = 1, 2., \tag{II.40}$$

we find

$$\begin{aligned} \chi(z_1, z_2) &= (\frac{b}{1 - z_1})^{a_1+1}(\frac{b}{1 - z_2})^{a_2+1} \\ & \quad F(\frac{b}{1 - z_1} + c_1, \frac{b}{1 - z_2} + c_2), \end{aligned} \tag{II.41}$$

where $F(s_1, s_2) = \mathcal{L}[f(t_1, t_2)]$ ($\mathcal{L}$ denotes the Laplace transform).

Again, $\chi(z_1, z_2)$ is analytic in D, where

$$D = \{(z_1, z_2); |z_1| \leq 1, |z_2| \leq 1\}^1.$$

$$\begin{aligned} \chi(z_1, z_2) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} \frac{\Gamma(j + a_1 + 1)\Gamma(k + a_2 + 1)}{j!k!} z_1^j z_2^k \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_{jk} z_1^j z_2^k. \end{aligned} \tag{II.42}$$

Comparing (II.25) we see that $\{A_{jk}\}$ in (II.42) takes the place of $\{a_{jk}\}$ in (II.25). Therefore we can follow the analogous argument as in (II.26) through (II.35). Smith's algorithm is also applicable to this method with minor changes.

1

$$\begin{aligned} \lim_{z_1, z_2 \to 1} \chi(z_1, z_2) = & \quad \lim_{s_1, s_2 \to \infty} s_1^{a_1+1} s_2^{a_2+1} F(s_1, s_2) = \\ & \lim_{s_1, s_2 \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_{jk} s_1^{-j} s_2^{-k} = d_{00}. \end{aligned}$$

Therefore, $\chi(z_1, z_2)$ is analytic at $(1, 1)$. Also $F(s_1, s_2)$ is analytic for $\Re s_1 > c_1, \Re s_2 > c_2$. Thus, χ is analytic in a bigger region which contains the unit polydisc; we will use this fact in analyzing error.

Parameter selection and error analysis Parameters b, c_1 and c_2 are introduced to improve the speed of convergence of the approximating series. In the expansion with generalized Laguerre polynomials the parameters a_1 and a_2 are introduced to smooth out irregularities in f. Thus the inversion process becomes more robust in the sense that it can be used to invert a wider class of functions efficiently. The selection of these parameters is explained below.

Since $F(s_1, s_2)$ is analytic for $\Re s_1 > c_1$ and $\Re s_2 > c_2$, and $b > 0$ it follows that $F(s_1, s_2)$ is analytic for $\Re s_1 \geq c_1 + \frac{b}{2}$ and $\Re s_2 \geq c_2 + \frac{b}{2}$, for any choice of $b > 0$. For practical purposes, we need a good choice of b. To pick a better choice for b, let us study the series in (II.21). It involves products of Laguerre polynomials. It is shown in Szegö [44] that the generalized Laguerre polynomial $L_n^{\alpha}(x)$ has n positive zeros and the largest zero x_n satisfies the inequality

$$x_n < 2n + \alpha + 1 + [(2n + \alpha + 1)^2 + 1/4 - \alpha] \approx 4n.$$

It is also known that $L_n^{\alpha}(x)$ increases monotonically for $x \geq 4n$ and oscillates in the interval $0 < x < 4n$. From these facts we can conveniently say that the product $L_n^{\alpha}(x)L_m^{\alpha}(y)$ will be oscillatory in the interval $(0, 4\min(m, n)) \times (0, 4\min(m, n))$ and monotonically increasing for $x, y > 4\max(m, n)$. We know that the series representation of an arbitrary function strongly depends on the oscillatory behavior of the function. We will use all these facts to find a bound for b.

Let $0 < t_1, t_2 < t_{max}$, where t_{max} denotes the maximum value of t_1 and t_2 in the numerical approximation. Now let us consider the approximate sum of f in (II.22). The last term in the finite sum is the product of Laguerre polynomials of order $N-1$. Therefore from the above remark we find

$$b\,t_{max} < 4N-4.$$

Since the singularities of $F(s_1,s_2)$ are away from the lines $\Re s_1 = c_1 + \frac{b}{2}$ and $\Re s_2 = c_2 + \frac{b}{2}$, it is evident that larger b values may improve the convergence of the series in (II.22). But then we need more terms in the series approximation of f (see error analysis). Otherwise the approximation intervals for t_1 and t_2 will be smaller.

Next we consider the parameters c_1 and c_2. These parameters can be chosen as any arbitrary pair of real numbers such that (c_1,c_2) is larger than the real part of all singularities of $F(s_1,s_2)$. If possible we choose c_1 and c_2 to satisfy

$$\begin{aligned} c_1 &= \max\{\Re s_1^* | F(s_1^*,s_2) \text{ is singular, for any } s_2\} \\ c_2 &= \max\{\Re s_2^* | F(s_1,s_2^*) \text{ is singular, for any } s_1\}. \end{aligned}$$

(see example 1 for illustration).

Let us look at the parameters a_1 and a_2 in the case of the method based on the generalized polynomials. The introduction of parameters a_1 and a_2 will enable us to invert a wider class of functions efficiently. This method works extremely well if a_1 and a_2 can be chosen such a that $t_1^{-a_1}t_2^{-a_2}f(t_1,t_2)$ can be approximated by a polynomial in t_1 and t_2. In other words, this technique is efficient if we can find values for a_1 and a_2 such that $s_1^{a_1+1}s_2^{a_2+1}F(s_1,s_2)$ is analytic at ∞ and thus it can be expanded as

$$s_1^{a_1+1}s_2^{a_2+1}F(s_1,s_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}s_1^{-j}s_2^{-k}.$$

It is not always possible to find such values for a_1 and a_2 to satisfy the above requirement.

Note: If we can find the parameters a_1, a_2 to satisfying the above condition then

$$\lim_{z_1,z_2\to 1} \chi(z_1,z_2) = \lim_{s_1,s_2\to\infty} s_1^{a_1+1} s_2^{a_2+1} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk} s_1^{-j} s_2^{-k} = c_{00}.$$

Therefore χ is analytic at $(1,1)$. Hence χ is analytic in a bigger region containing D defined by (II.25) and (II.42).

Now we analyze the truncation error for the method based on the expansion of $f(t_1,t_2)$ in a series of generalized Laguerre polynomials. Let us denote the truncation error by E_t. E_t is given by

$$\begin{aligned} E_t &= f(t_1,t_2) - e^{c_1t_1+c_2t_2} t_1{}^{a_1} t_2{}^{a_2} \\ &\quad \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} a_{jk} L_j(bt_1) L_k(bt_2), \end{aligned} \tag{II.43}$$

where $-1 < a_1, a_2 < 1$. If we take

$$T = e^{-c_1t_1-c_2t_2} t_1^{-a_1} t_2^{-a_2} E_t, \text{ then}$$

$$\begin{aligned} T &= \sum_{j=0}^{\infty}\sum_{k=N}^{\infty} a_{jk} L_j^{a_1}(bt_1) L_k^{a_2}(bt_2) \\ &\quad + \sum_{j=N}^{\infty}\sum_{k=0}^{N-1} a_{jk} L_j^{a_1}(bt_1) L_k^{a_2}(bt_2). \end{aligned} \tag{II.44}$$

We will use the following Lemma from Erdelyi et al. [24], which gives the bounds for Laguerre polynomials.

Lemma 1 *Let $\alpha > -1$ be real. Then*

$$\begin{aligned} |L_n^{\alpha}(x)| &\le (\alpha+1)_n (n!)^{-1} e^{x/2}, && \textit{if } \alpha \ge 0 \\ |L_n^{\alpha}(x)| &\le [2 - (\alpha+1)_n (n!)^{-1} e^{x/2}] e^{x/2}, && \textit{if } -1 < \alpha < 0, \end{aligned}$$

where $(\alpha+1)_n = (\alpha+1)(\alpha+2)\ldots(\alpha+n)$.

In our discussion we will use only the first case (i.e., $\alpha \geq 0$) of Lemma 1. The second case can be done with a similar technique.

The following Lemma is taken out of the textbook by Hörmander [27] and it will be used in finding a bound for the coefficients a_{jk}.

Lemma 2 (Cauchy's inequalities) *If χ is analytic and $|\chi| \leq M$, where M is a positive constant, in a polydisc*

$$D = \{z : |z_j| < R_j, j = 1, 2\},$$

then

$$|\partial^\alpha \chi(0)| \leq M\alpha! R^{-\alpha},$$

where

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2), \\ \partial^\alpha &= (\frac{\partial}{\partial z_1})^{\alpha_1}(\frac{\partial}{\partial z_2})^{\alpha_2} \\ \alpha! &= \alpha_1!\alpha_2!. \end{aligned} \tag{II.45}$$

Next we apply Lemma 1 and Lemma 2 to find a bound for T. Let us start with the first term on the right hand side of the equation (II.44). χ is given by (II.42) and it is analytic in region containing the unit polydisc D. Therefore for some $R_1, R_2 > 1$,

$$\begin{aligned} &|\sum_{j=0}^{\infty}\sum_{k=N}^{\infty} a_{jk} L_j^{a_1}(bt_1) L_k^{a_2}(bt_2)| \\ &\leq \sum_{j=0}^{\infty}\sum_{k=N}^{\infty} |a_{jk}||L_j^{a_1}(bt_1)||L_k^{a_2}(bt_2)| \\ &\leq M \sum_{j=0}^{\infty}\sum_{k=N}^{\infty} \frac{j!k!}{\Gamma(a_1+j+1)\Gamma(a_2+k+1)} R_1^{-j} R_2^{-k} \end{aligned}$$

$$
\begin{aligned}
&\frac{(a_1+1)_j(a_2+1)_k}{j!k!} e^{\frac{(bt_1+bt_2)}{2}} \\
&\leq M \frac{e^{\frac{(bt_1+bt_2)}{2}}}{\Gamma(a_1+1)\Gamma(a_2+1)} \sum_{j=0}^{\infty} \sum_{k=N}^{\infty} R_1^{-j} R_2^{-k},
\end{aligned}
$$

since $\Gamma(a+n) = \Gamma(a)(a)_n$ if $a > 0$. That is, for $R_1, R_2 > 1$,

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{k=N}^{\infty} |a_{jk} L_j^{a_1}(bt_1) L_k^{a_2}(bt_2)| \\
&\leq M \frac{e^{\frac{(bt_1+bt_2)}{2}}}{\Gamma(a_1+1)\Gamma(a_2+1)} \frac{R_2^{-N}}{(1-R_1^{-1})(1-R_2^{-1})}.
\end{aligned} \tag{II.46}
$$

Similarly, if we consider the second term on the right hand side of the equation (II.44), we can find

$$
\begin{aligned}
&|\sum_{j=N}^{\infty} \sum_{k=0}^{N-1} a_{jk} L_j^{a_1}(bt_1) L_k^{a_2}(bt_2)| \\
&\leq M \frac{e^{\frac{(bt_1+bt_2)}{2}}}{\Gamma(a_1+1)\Gamma(a_2+1)} \frac{R_1^{-N}(1-R_2^{-N})}{(1-R_1^{-1})(1-R_2^{-1})}.
\end{aligned} \tag{II.47}
$$

By combining the two bounds above we obtain the following bound for E_t

$$
\begin{aligned}
E_t \;\leq\; & M t_1^{a_1} t_2^{a_2} \frac{e^{(c_1+b/2)t_1+(c_2+b/2)t_2}}{\Gamma(a_1+1)\Gamma(a_2+1)} \\
& \frac{(R_1^{-N} + R_2^{-N} - R_1^{-N} R_2^{-N})}{(1-R_1^{-1})(1-R_2^{-1})}.
\end{aligned} \tag{II.48}
$$

This bound tells us that by increasing the number of terms N in the truncated series we can control the error considerably. Since we do not have a proper strategy to pick b besides the bound given under the Parameters Selection, this bound can be used to get an estimate for b. This bound suggests that we need large N for bigger values of b.

Numerical examples and comments We use the following examples to test the validity of the above algorithm. In what follows, $F(s_1, s_2)$ and $f(t_1, t_2)$ denote the Laplace transform and its inversion respectively, as defined in (l_2) and (i_2).

Examples:

$$1.\ F(s_1, s_2) = \frac{1}{(s_1+1)(s_2+2)}$$

$$f(t_1, t_2) = e^{-t_1-2t_2}$$

$$2.\ F(s_1, s_2) = \frac{1}{s_1\sqrt{(s_1 s_2+1)}}$$

$$f(t_1, t_2) = \frac{\sin(2\sqrt{t_1 t_2})}{\pi t_2}$$

$$3.\ F(s_1, s_2) = \frac{1}{(s_1-1)(s_2-1)(s_1+s_2-1)}$$

$$\begin{aligned} f(t_1, t_2) &= e^{t_1}(e^{t_2}-1) \text{ for } t_1 > t_2 \\ &= e^{t_2}(e^{t_1}-1) \text{ for } t_2 > t_1 \end{aligned}$$

All calculations for these examples were carried out in double precision. The parameters and notations will closely follow the ones used in the text. We have used the method based on the expansion by the generalized Laguerre polynomials.

Example 1:

The numerical results of the Example 1 are given in Table II.5. Here we consider the function $e^{-t_1-2t_2}$, a decreasing exponential function. We obtained excellent accuracy with $N = 8$ and $M = 8$. The comparison of the results from the present method to the Padé approximation method by Singhal, Vlach and Vlach [42] is shown in the chart below.

The values shown in the chart are the absolute error and the relative error in approximating $f(t_1,t_2)$ at $(t_1,t_2) = (1,1)$. Singhal et al. presented their results for the point (1,1). We therefore chose the same point (1,1) for the purposes of comparison.

Method	Absolute Error	Relative Error
Singhal et al. :	1.338×10^{-8}	2.687×10^{-7}
Present Method :	5.117×10^{-17}	1.028×10^{-15}

The values of the parameters a_1, a_2, c_1, c_2, and b in Table II.5 are chosen by the strategies described in the text. The description is given next.

The transform function $F(s_1,s_2) = \frac{1}{(s_1+1)(s_2+2)}$ does not have any singularities for (s_1,s_2) such that $\Re s_1 > -1$ and $\Re s_2 > -2$, but it does have singularities at $(-1,s_2)$ and at $(s_1,-2)$. We therefore picked $c_1 = -1$ and $c_2 = -2$, as described in the section of Parameters Selection.

Since $s_1 s_2\, F(s_1,s_2) = \frac{1}{(1+\frac{1}{s_1})(1+\frac{2}{s_2})}$ is analytic at ∞, it can be expanded as a series in s_1^{-1} and s_2^{-1} for large values of $\Re s_1$ and $\Re s_2$. Therefore we picked $a_1 = a_2 = 0$ as explained in the strategy. Finally, b is chosen in an ad hoc manner.

Example 2:

The inverse function in this example,

$$f(t_1, t_2) = \frac{\sin(2\sqrt{t_1 t_2})}{(\pi t_2)},$$

is an oscillatory type function. Table II.6 shows the error in the numerical approximation of this function using the present method for different values of the pair (N, M). It is evident from the table that the accuracy can be improved by increasing the number of terms (i.e., N and M) in the series involved until the round off error becomes dominant. If we study the last two columns (i.e., $(n = m = 24)$ and $(n = m = 32)$), in Table II.6, we observe that the relative error has not changed much. Therefore we can not improve the results much only by taking large values for m and n. In general the round off error increases with n and m. Due to this we can not improve the accuracy only by increasing m or n after certain values.

As in the case of example 1, the method performs well here since we could determine the optimal values of the parameters. Since $F(s_1, s_2) = \frac{1}{(s_1\sqrt{s_1 s_2 + 1})}$, $s_1 = 0$ is a singularity of F. The other singularities are given by the equation $s_1 s_2 = -1$. So our choice of c_1 has to be greater than or equal to zero. We have chosen the values for c_1 and c_2 by trial and error. The pair $c_1 = 0.0, c_2 = 0.0$, produced better results. In selecting a_1 and a_2 we used the strategy given in the algorithm. That is, $F(s_1, s_2) = s_1^{-3/2} s_2^{-1/2} \frac{1}{(1 + \frac{1}{s_1 s_2})}$. It can be expanded in terms of s_1^{-1} and s_2^{-1}, if we pick $a_1 = 1/2$ and $a_2 = -1/2$.

Table II.5: Output of Laguerre Series Method: Example 1

$$F(s_1,s_2) = \frac{1}{(s_1+1)(s_2+2)}$$
$$f(t_1,t_2) = e^{-t_1-2t_2}$$
$a_1 = 0.0$ $b = 1.0$ $c_1 = -1.0$
$a_2 = 0.0$ $c_2 = -2.0$ $m = n = 8$

t1	t2	exact val	approx val	abs-error	rel-error
1.0	1.0	0.0497871	0.0497871	0.51174E-16	0.10279E-14
1.0	2.0	0.0067379	0.0067379	0.86736E-18	0.12873E-15
1.0	3.0	0.0009119	0.0009119	0.97578E-18	0.10701E-14
1.0	4.0	0.0001234	0.0001234	0.18974E-18	0.15374E-14
1.0	5.0	0.0000167	0.0000167	0.37269E-19	0.22315E-14
2.0	1.0	0.0183156	0.0183156	0.19949E-16	0.10892E-14
2.0	2.0	0.0024788	0.0024788	0.65052E-18	0.26244E-15
2.0	3.0	0.0003355	0.0003355	0.32526E-18	0.96959E-15
2.0	4.0	0.0000454	0.0000454	0.67763E-19	0.14926E-14
2.0	5.0	0.0000061	0.0000061	0.12494E-19	0.20334E-14
3.0	1.0	0.0067379	0.0067379	0.95410E-17	0.14160E-14
3.0	2.0	0.0009119	0.0009119	0.10842E-18	0.11890E-15
3.0	3.0	0.0001234	0.0001234	0.57598E-19	0.46672E-15
3.0	4.0	0.0000167	0.0000167	0.20329E-19	0.12172E-14
3.0	5.0	0.0000023	0.0000023	0.48704E-20	0.21547E-14
4.0	1.0	0.0024788	0.0024788	0.33068E-17	0.13341E-14
4.0	2.0	0.0003355	0.0003355	0.54210E-19	0.16160E-15
4.0	3.0	0.0000454	0.0000454	0.27105E-19	0.59703E-15
4.0	4.0	0.0000061	0.0000061	0.10588E-19	0.17232E-14
4.0	5.0	0.0000008	0.0000008	0.20514E-20	0.24670E-14
5.0	1.0	0.0009119	0.0009119	0.11384E-17	0.12484E-14
5.0	2.0	0.0001234	0.0001234	0.30493E-19	0.24709E-15
5.0	3.0	0.0000167	0.0000167	0.23717E-19	0.14200E-14
5.0	4.0	0.0000023	0.0000023	0.38116E-20	0.16863E-14
5.0	5.0	0.0000003	0.0000003	0.62204E-21	0.20335E-14

Example 3: The inverse function of the transform function

$$F(s_1, s_2) = [(s_1 - 1)(s_2 - 1)(s_1 + s_2 - 1)]^{-1},$$

is given by

$$\begin{aligned} f(t_1, t_2) &= e^{t_1}(e^{t_2} - 1) \text{ if } t_1 > t_2 \\ &= e^{t_2}(e^{t_1} - 1) \text{ if } t_2 > t_1, \end{aligned}$$

which is neither smooth nor bounded function. In Table II.7, we present the results for parameter values $a_1 = a_2 = 0.0, c_1 = c_2 = 1.0, b = 5.0$ and $n = m = 8$. The results we obtain here are not as good as in Table II.5 and Table II.6. The main reason for this is that we are unable to find the optimal values for a_1 and a_2 to apply the present method efficiently.

Though the numerical results are not that impressive, they are still comparable to the results obtained by Singhal, Vlach and Vlach [42]. As a comparison we look at our results and Singhal, Vlach and Vlach results for the above function calculated at $(1, 1)$.

Method	Absolute Error	Relative Error
Singhal et al. :	5.32×10^{-2}	1.139×10^{-2}
Present Method :	3.74×10^{-2}	8.01×10^{-3}

Method 2: Fourier Series Representations

Let $f(t_1, t_2)$ be a real valued function of t_1 and t_2 and $f(t_1, t_2) = 0$, for t_1 or $t_2 < 0$. As we defined before the Laplace transform and its inverse are given by

$$F(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-s_1 t_1 - s_2 t_2} f(t_1, t_2)\, dt_1 dt_2 \text{ and}$$

Table II.6: Output for Laguerre Series Method: Example 2

$$F(s_1,s_2) = \frac{1}{(s_1\sqrt{s_1 s_2+1})},\ f(t_1,t_2) = \frac{\sin(2\sqrt{t_1 t_2})}{\pi t_2}$$

$a_1 = 0.5$ $\quad b = 2.0$ $c_1 = 0.0$

$a_2 = -0.5$ $\quad c_2 = 0.0$

Relative Error for Different Values of n and m

t1	t2	n = m = 16	n = m= 24	n = m = 32
1.0	1.0	0.49961E-13	0.82661E-13	0.12457E-12
1.0	2.0	0.26590E-11	0.73311E-12	0.97928E-12
1.0	3.0	0.62112E-11	0.16646E-11	0.16580E-11
1.0	4.0	0.52850E-10	0.10749E-11	0.10398E-11
1.0	5.0	0.96762E-09	0.14113E-11	0.47230E-13
2.0	1.0	0.14803E-12	0.40942E-12	0.58703E-12
2.0	2.0	0.15725E-11	0.38840E-12	0.47919E-12
2.0	3.0	0.23498E-11	0.61867E-12	0.61921E-12
2.0	4.0	0.22570E-09	0.15403E-11	0.14334E-11
2.0	5.0	0.89203E-07	0.36273E-10	0.15393E-10
3.0	1.0	0.93855E-12	0.55422E-12	0.71543E-12
3.0	2.0	0.13844E-11	0.32149E-12	0.33818E-12
3.0	3.0	0.27127E-10	0.21746E-11	0.18331E-11
3.0	4.0	0.60657E-09	0.13860E-11	0.11927E-11
3.0	5.0	0.11732E-07	0.13948E-11	0.17149E-11
4.0	1.0	0.17042E-10	0.27877E-12	0.33361E-12
4.0	2.0	0.93976E-10	0.55798E-12	0.37814E-12
4.0	3.0	0.12097E-09	0.88696E-12	0.43271E-12
4.0	4.0	0.14105E-08	0.66418E-12	0.55296E-12
4.0	5.0	0.63445E-07	0.18039E-11	0.71664E-11
5.0	1.0	0.43073E-09	0.25357E-12	0.25716E-12
5.0	2.0	0.39463E-07	0.82594E-11	0.14981E-11
5.0	3.0	0.22939E-08	0.34482E-12	0.58545E-13
5.0	4.0	0.17701E-07	0.11878E-11	0.81004E-12
5.0	5.0	0.24544E-06	0.61946E-12	0.10778E-10

Table II.7: Output for Laguerre Series Method: Example 3

$$
\begin{aligned}
F(s_1, s_2) &= \frac{1}{(s_1 - 1)(s_2 - 1)(s_1 + s_2 - 1)} \\
f(t_1, t_2) &= e^{t_1}(e^{t_2} - 1) \text{ if } t_1 \geq t_2 \\
&= e^{t_2}(e^{t_1} - 1) \text{ if } t_1 < t_2
\end{aligned}
$$

$a_1 = 0.0$ $b = 5.0$ $c1 = 1.0$

$a_2 = 0.0$ $c_2 = 1.0$ $m = n = 8$

t1	t2	exact val	approx val	abs-error	rel-error
0.2	0.2	0.2704219	0.2428847	0.27537E-01	0.10183E+00
0.2	0.4	0.3302941	0.3343463	0.40522E-02	0.12269E-01
0.2	0.6	0.4034221	0.3989854	0.44368E-02	0.10998E-01
0.2	0.0	0.4927409	0.4942253	0.14844E-02	0.30126E-02
0.2	1.0	0.6018351	0.6126281	0.10793E-01	0.17934E-01
0.4	0.2	0.3302941	0.3343463	0.40522E-02	0.12269E-01
0.4	0.4	0.7337162	0.7037167	0.30000E-01	0.40887E-01
0.4	0.6	0.8961630	0.9130254	0.16862E-01	0.18816E-01
0.4	0.8	1.0945760	1.0415947	0.52981E-01	0.48403E-01
0.4	1.0	1.3369181	1.2822907	0.54627E-01	0.40861E-01
0.6	0.2	0.4034221	0.3989854	0.44368E-02	0.10998E-01
0.6	0.4	0.8961630	0.9130254	0.16862E-01	0.18816E-01
0.6	0.6	1.4979981	1.4281112	0.69887E-01	0.46654E-01
0.6	0.8	1.8296590	1.8678172	0.38158E-01	0.20855E-01
0.6	1.0	2.2347506	2.2372663	0.25157E-02	0.11257E-02
0.8	0.2	0.4927409	0.4942253	0.14844E-02	0.30126E-02
0.8	0.4	1.0945760	1.0415947	0.52981E-01	0.48403E-01
0.8	0.6	1.8296590	1.8678172	0.38158E-01	0.20855E-01
0.8	0.8	2.7274915	2.7684462	0.40955E-01	0.15016E-01
0.8	1.0	3.3313656	3.4245307	0.93165E-01	0.27966E-01
1.0	0.2	0.6018351	0.6126281	0.10793E-01	0.17934E-01
1.0	0.4	1.3369181	1.2822907	0.54627E-01	0.40861E-01
1.0	0.6	2.2347506	2.2372663	0.25157E-02	0.11257E-02
1.0	0.8	3.3313656	3.4245307	0.93165E-01	0.27966E-01
1.0	1.0	4.6707743	4.6333697	0.37405E-01	0.80082E-02

$$f(t_1,t_2) = \frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} e^{s_1t_1+s_2t_2}F(s_1,s_2)\,ds_1ds_2, \quad \text{(II.49)}$$

with c_1, c_2 such that $F(s_1,s_2)$ does not have any singularities for $\Re s_1 > \alpha_1$ and $c_1 > \alpha_1$ and $\Re s_2 > \alpha_2$ and $c_2 > \alpha_2$. The second integral above is a contour integral along vertical line segments $\Re s_1 = c_1$ and $\Re s_2 = c_2$.

Also we shall assume that

$$|f(t_1,t_2)| \leq Me^{\alpha_1t_1+\alpha_2t_2},$$

where α_1, α_2 are real and M is a positive constant, to assure the existence of the Laplace integral defining $F(s_1,s_2)$ for $\Re s_1 > \alpha_1$ and $\Re s_2 > \alpha_2$. Therefore our choices of c_1, c_2 have to satisfy the condition

$$c_1 > \alpha_1 \text{ and } c_2 > \alpha_2.$$

The flexibility in choosing the parameters c_1, c_2 will provide us with a better approximation to the integral in (II.49).

The following two-dimensional inverse Laplace transforms technique is an extension of the one-dimensional methods based on the representation of the inverse function in terms of Fourier series. In particular, it extends the one-dimensional methods of Dubner and Abate [22] and Crump [7].

Before we describe the method, let us look at some preliminaries. Since $s_1 = c_1 + i\omega_1$ and $s_2 = c_2 + i\omega_2$, the equation (II.49) can be written in the form:

$$\begin{aligned} f(t_1,t_2) &= \frac{e^{c_1t_1+c_2t_2}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega_1t_1+\omega_2t_2)} \\ &\quad F(c_1+i\omega_1, c_2+i\omega_2)\,d\omega_1d\omega_2. \end{aligned} \quad \text{(II.50)}$$

By using the fact that $f(t_1,t_2)$ is real, we can find

$$\begin{aligned} f(t_1,t_2) &= \frac{e^{c_1t_1+c_2t_2}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Re\{F(c_1+i\omega_1,c_2+i\omega_2)\} \\ &\quad \cos(\omega_1t_1+\omega_2t_2)-\Im\{F(c_1+i\omega_1,c_2+i\omega_2)\} \\ &\quad \sin(\omega_1t_1+\omega_1t_2)]\,d\omega_1d\omega_2. \end{aligned} \tag{II.51}$$

We will simplify the equation (II.51) further to get a better form. To this end, let

$$\begin{aligned} \Phi(\omega_1,\omega_2) &= \Re\{F(c_1+i\omega_1,c_2+i\omega_2)\}\cos(\omega_1t_1+\omega_2t_2) \\ &- \Im\{F(c_1+i\omega_1,c_2+i\omega_2)\}\sin(\omega_1t_1+\omega_1t_2). \end{aligned} \tag{II.52}$$

Since $f(t_1,t_2)$ is real, we have

$$F(\overline{s_1},\overline{s_2})=\overline{F(s_1,s_2)}.$$

Which implies

$$\begin{aligned} \Re F(\overline{s_1},\overline{s_2}) &= \Re\overline{F(s_1,s_2)} \\ &= \Re F(s_1,s_2) \end{aligned}$$

and

$$\begin{aligned} \Im F(\overline{s_1},\overline{s_2}) &= \Im\overline{F(s_1,s_2)} \\ &= -\Im F(s_1,s_2). \end{aligned}$$

Thus,

$$\begin{aligned} \Re\{\Phi(-\omega_1,-\omega_2)\} &= \Re\{\Phi(\omega_1,\omega_2)\} \\ \Im\{\Phi(\omega_1,-\omega_2)\} &= -\Im\{\Phi(-\omega_1,\omega_2)\}. \end{aligned}$$

Therefore the equation (II.51) can be written as

$$\begin{aligned}
f(t_1,t_2) &= \frac{e^{c_1t_1+c_2t_2}}{2\pi^2}\int_0^\infty\int_0^\infty [\Re\{F(c_1+i\omega_1,c_2+i\omega_2)\}\cos(\omega_1t_1+\omega_2t_2) \\
&- \Im\{F(c_1+i\omega_1,c_2+i\omega_2)\}\sin(\omega_1t_1+\omega_1t_2)]\,d\omega_1 d\omega_2 \\
&+ \int_0^\infty\int_0^\infty [\Re\{F(c_1+i\omega_1,c_2-i\omega_2)\}\cos(\omega_1t_1-\omega_2t_2) \\
&- \Im\{F(c_1+i\omega_1,c_2-i\omega_2)\}\sin(\omega_1t_1-\omega_1t_2)]\,d\omega_1 d\omega_2. \qquad \text{(II.53)}
\end{aligned}$$

Description of the method We obtain the Fourier series for a function $g^{00}(t_1,t_2)$ that is periodic with period $2T$ in t_1 and t_2, and equal to

$$e^{c_1t_1+c_2t_2}f(t_1,t_2) \text{ on } (0,2T)\times(0,2T).$$

It turns out that the Fourier coefficients of the series may be approximated using $F(s_1,s_2)$. For $j,k=0,1,\ldots$, define $g^{jk}(t_1,t_2)=e^{-(c_1t_1+c_2t_2)}f(t_1,t_2)$ in $(2jT,2(j+1)T)\times(2kT,2(k+1)T)$, with g^{jk} specified elsewhere from the condition that it be periodic with period $2T$ in t_1 and t_2. The Fourier series representation for g^{jk} is given by

$$\begin{aligned}
g^{jk}(t_1,t_2) &= 1/4a_{00}^{jk} + 1/2\sum_{m=1}^{\infty}(a_{0m}^{jk}\cos my + b_{0m}^{jk}\sin my) \\
&+ 1/2\sum_{n=1}^{\infty}(a_{n0}^{jk}\cos nx + c_{n0}^{jk}\sin nx) \\
&+ 1/2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}^{jk}\cos nx\cos my + b_{nm}^{jk}\cos nx\sin my \\
&+ c_{nm}^{jk}\sin nx\cos my + d_{nm}^{jk}\sin nx\sin my), \qquad \text{(II.54)}
\end{aligned}$$

where $x=\frac{\pi t_1}{T}$ and $y=\frac{\pi t_1}{T}$. Also the coefficients are given by

$$a_{nm}^{jk} = \frac{1}{T^2}\int_{2jT}^{2(j+1)T}\int_{2kT}^{2(k+1)T} f(u,v)\cos(n\pi u/T)$$

$$\cos(m\pi v/T)e^{-c_1u-c_2v}\,dudv$$

$$b_{nm}^{jk} = \frac{1}{T^2}\int_{2jT}^{2(j+1)T}\int_{2kT}^{2(k+1)T} f(u,v)\cos(n\pi u/T)$$
$$\sin(m\pi v/T)e^{-c_1u-c_2v}\,dudv$$

$$c_{nm}^{jk} = \frac{1}{T^2}\int_{2jT}^{2(j+1)T}\int_{2kT}^{2(k+1)T} f(u,v)\sin(n\pi u/T)$$
$$\cos(m\pi v/T)e^{-c_1u-c_2v}\,dudv$$

$$d_{nm}^{jk} = \frac{1}{T^2}\int_{2jT}^{2(j+1)T}\int_{2kT}^{2(k+1)T} f(u,v)\sin(n\pi u/T)$$
$$\sin(m\pi v/T)e^{-c_1u-c_2v}\,dudv. \tag{II.55}$$

Next, we substitute the integrals in (II.55) for the coefficients in the equation (II.54) and take the sum of the resulting series over$j$ and k. By formally interchanging order of sums, we obtain

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{T^2}\int_{2jT}^{2(j+1)T}\int_{2kT}^{2(k+1)T} g^{jk}(t_1,t_2)\,dt_1dt_2$$
$$= (1/2T^2)\{1/2F(c_1,c_2)$$
$$+ \sum_{n=1}^{\infty}[\Re\{F(c_1,c_2+im\pi)\}(\cos m\pi t_2/T)$$
$$- \Im\{F(c_1,c_2+im\pi)\}\sin(n\pi t_1/T)]$$
$$+ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}[\Re\{F(c_1+in\pi/T,c_2+im\pi/T)\}\cos(n\pi t_1/T+m\pi t_2/T)$$
$$+ \Re\{F(c_1+in\pi/T,c_2-im\pi/T)\}\cos(n\pi t_1/T-m\pi t_2/T)$$
$$- \Im\{F(c_1+in\pi/T,c_2+im\pi/T)\}\sin(n\pi t_1/T+m\pi t_2/T)$$
$$- \Im\{F(c_1+in\pi/T,c_2-im\pi/T)\}\sin(n\pi t_1/T-m\pi t_2/T)]\}. \tag{II.56}$$

Let us denote the sum on the right hand side of (II.56) by $g(t_1,t_2)$. As we defined before $g^{00}(t_1,t_2) = e^{-a_1t_1-a_2t_2}f(t_1,t_2)$, we have

$$f(t_1,t_2) = e^{c_1t_1+c_2t_2}g^{00}(t_1,t_2) \text{ on } (0,2T)\times(0,2T).$$

So from (II.56), we find that the approximate value of $f(t_1,t_2)$ is

$$\bar{f}(t_1,t_2) = e^{c_1t_1+c_2t_2}g(t_1,t_2)$$

i.e.,

$$f(t_1,t_2) = \bar{f}(t_1,t_2) + E_d,$$

where

$$E_d = -[e^{c_1t_1+c_2t_2}\{\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} g^{jk}(t_1,t_2) + \sum_{j=1}^{\infty} g^{j0}(t_1,t_2)\}]. \tag{II.57}$$

That is,

$$\begin{aligned} E_d &= -e^{c_1t_1+c_2t_2}\{\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} e^{-c_1(2jT+t_1)-c_2(2kT+t_2)} \\ &\quad f(2jT+t_1,2kT+t_2) + \sum_{j=1}^{\infty} e^{-c_1(2jT+t_1)-c_2t_2} \\ &\quad f(2jT+t_1,t_2)\} \\ &= \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} e^{-2c_1jT-2c_2kT}f(2jT+t_1,2kT+t_2) \\ &+ \sum_{j=1}^{\infty} e^{-2jc_1T}f(2jT+t_1,t_2) \end{aligned} \tag{II.58}$$

Error analysis and parameter selection There are two major sources of error in the approximation of $f(t_1,t_2)$ besides the round off error. One of them is discretization error given by the E_d in the equation (II.58) and the other is the truncation error, say E_t. We first look at the bound for E_d.

Since

$$|f(t_1,t_2)| \le Me^{\alpha_1 t_1+\alpha_2 t_2},$$

we have

$$\begin{aligned}
E_d &\le \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} e^{-2c_1 jT-2c_2 kT} Me^{\alpha_1(t_1+2jT)+\alpha_2(t_2+2kT)} \\
&- \sum_{j=1}^{\infty} e^{-2c_1 jT} Me^{\alpha_1(t_1+2jT)+\alpha_2 t_2} \\
&= Me^{\alpha_1 t_1+\alpha_2 t_2}[\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} e^{-2jT(c_1-\alpha_1)-2kT(c_2-\alpha_2)} \\
&+ \sum_{j=1}^{\infty} e^{-2jT(c_1-\alpha_1)}] \\
&= Me^{\alpha_1 t_1+\alpha_2 t_2}[\frac{e^{2T(c_1-\alpha_1)}+e^{2T(c_2-\alpha_2)}-1}{(e^{2T(c_1-\alpha_1)}-1)(e^{2T(c_2-\alpha_2)}-1)}]. \qquad \text{(II.59)}
\end{aligned}$$

This suggests us by taking c_1 and c_2 larger than α_1 and α_2 respectively, the error E_d can be made arbitrarily small. We will use this fact to choose the value of parameters c_1 and c_2. Suppose we want the relative error to be less than ϵ, then we may expect

$$\frac{E_d}{Me^{\alpha_1 t_1+\alpha_2 t_2}} < \epsilon$$

That is,

$$\frac{e^{2T(c_1-\alpha_1)}+e^{2T(c_2-\alpha_2)}-1}{(e^{2T(c_1-\alpha_1)}-1)(e^{2T(c_2-\alpha_2)}-1)} < \epsilon. \qquad \text{(II.60)}$$

Since we choose $c_1 > \alpha_1$ and $c_2 > \alpha_2$ and $T > 0$, we can replace the denominator in (II.60) by $e^{2T(c_1-\alpha_1)+2T(c_2-\alpha_2)}$ and write

$$e^{-2T(c_1-\alpha_1)}+e^{-2T(c_2-\alpha_2)}-e^{-2T(c_1-\alpha_1)-2T(c_2-\alpha_2)} < \epsilon. \qquad \text{(II.61)}$$

To find the parameters c_1 and c_2, we first pick a value for c_1 such that $c_1 > \alpha_1$. Then we can find c_2, roughly, by using the following procedure. Let $x_1 = e^{-2T(c_1-\alpha_1)}$, then

$$c_2 = \alpha_2 - 1/2T \ln(\frac{\epsilon - x_1}{1 - x_1}). \tag{II.62}$$

The value of c_1 is arbitrary except for $c_1 > \alpha_1$, so by changing the value of c_1 in (II.62), we can judge which pair of values of c_1 and c_2 will be suitable for our computation.

The parameter T is chosen from trying several values for T. The only requirement is that $t_{max} < 2T$. By experimenting we find that when $0.5t_{max} \leq T \leq 0.8t_{max}$, this method gives better results.

Next we consider the truncation error E_t. Let us assume that $f(t_1, t_2)$ is approximated by $f_N(t_1, t_2)$, where

$$\begin{aligned} f_N(t_1,t_2) = \quad & (1/2T^2)\{1/2F(c_1,c_2) \\ + & \sum_{m=1}^{N} [\Re\{F(c_1, c_2 + im\pi/T)\} \cos m\pi t_2/T) \\ - & \Im\{F(c_1, c_2 + im\pi) \sin m\pi t_1/T)] \\ + & \sum_{m=1}^{N} \sum_{n=1}^{N} [\Re\{F(c_1 + in\pi/T, c_2 + im\pi/T)\} \cos(n\pi t_1/T + m\pi t_2/T) \\ + & \Re\{F(c_1 + in\pi/T, c_2 - im\pi/T)\} \cos(n\pi t_1/T - m\pi t_2/T) \\ - & \Im\{F(c_1 + in\pi/T, c_2 + im\pi/T)\} \sin(n\pi t_1/T + m\pi t_2/T) \\ - & \Im\{F(c_1 + in\pi/T, c_2 - im\pi/T)\} \sin(n\pi t_1/T - m\pi t_2/T)]\}. \quad \text{(II.63)} \end{aligned}$$

To control the truncation error, we evaluate $f_{N+1}(t_1, t_2)$ and $f_{N+N/4}(t_1, t_2)$, and pick N for which the difference between f_{N+1} and $f_{N+N/4}$ is negligible. By imposing stronger conditions on $f(t_1, t_2)$, we can find a bound for the truncation error. The interested reader is encouraged to refer to Weinberger [51].

Examples and comments We have tried this method on several functions. We present some examples which are representative of the problems used to test this method. The first two examples are chosen because they are used by Singhal, Vlach and Vlach [42] to illustrate their method in inverting the two-dimensional Laplace transforms. Hence we can use these two examples to compare our results to Singhal and Vlach method. The other example is chosen to test the performance of the above method on oscillatory type problems.

All calculations of the following examples are carried out in double precision. The Fortran program for this method is given in the Appendix D. All computations are performed on a VAX 11/785 Central Processing Unit.

The results are presented in tables on next few pages. The parameters shown in the tables closely follow the description of the parameters in the text.

Examples:

$$1.\ F(s_1, s_2) = \frac{1}{(s_1+1)(s_2+2)}$$

$$f(t_1, t_2) = e^{-t_1 - 2t_2}$$

$$2.\ F(s_1, s_2) = \frac{1}{(s_1-1)(s_2-1)(s_1+s_2-1)}$$

$$f(t_1, t_2) = e^{t_1}(e^{t_2} - 1) \text{ for } t_1 \geq t_2$$

$$= e^{t_2}(e^{t_1} - 1) \text{ for } t_2 > t_1$$

$$3.\ F(s_1, s_2) = \frac{1}{s_1\sqrt{(s_1 s_2 + 1)}}$$

$$f(t_1, t_2) = \frac{\sin(2\sqrt{t_1 t_2})}{(\pi t_2)}$$

$$4.\ F(s_1, s_2) = \frac{1}{\sqrt{s_1 s_2 + 1}}$$

$$f(t_1, t_2) = \frac{\cos(2\sqrt{t_1 t_2})}{\pi\sqrt{t_1 t_2}}$$

Example 1:

Since the inverse function is a smooth and decaying exponential function, we expect the above method to perform well. In fact we get a good approximation to the inverse function though it is not as good as that of method 1. The results are still comparable to the results of Singhal et al. (see the chart below). By increasing the number of terms (n) in the approximating series we could improve the accuracy, but then the Epsilon algorithm fails to perform well with large values of n. In Table II.8 we show the results for $n = 12$ and $n = 24$. We also present a comparison chart for method 1, Singhal et al. method and the present method below.

The values shown in the chart below are the absolute error and the relative error in approximating $f(t_1, t_2)$ at $(t_1, t_2) = (1, 1)$. (The point (1,1) is chosen for the same reason given in Section 1.)

Method	Absolute Error	Relative Error
Singhal et al. :	1.338×10^{-8}	2.687×10^{-7}
Present Method :	4.257×10^{-7}	8.55×10^{-6}
Method 1 :	5.117×10^{-17}	1.028×10^{-15}

Example 2:

Like Method 1 and Singhal et al. method, this method performs poorly with this function. This may be due to the nonsmoothness of the function. Still the present method gives better results than the other two. Comparison chart is given below. Also the output for this example is given in Table II.9.

Again the values shown in the table below are the absolute error and the relative error in approximating $f(t_1, t_2)$ at $(t_1, t_2) = (1, 1)$.

Method	Absolute Error	Relative Error
Singhal et al. :	5.32×10^{-2}	1.139×10^{-2}
Method 1 :	3.74×10^{-2}	8.01×10^{-3}
Present Method :	1.33×10^{-2}	2.85×10^{-3}

Examples 3 and 4:

For these examples both inverse functions are oscillatory in nature. They are also bounded away from zero. Even though they are similar, the same parameter values will not work for both functions. To get better results we have to pick $T = 3.7, a_1 = a_2 = 2.0$ for Example 3. Output for Example 3 is given in Table II.10. The above choices did not produce good results for Example 4. Table II.11 shows the relative errors for two different pairs of values of a_1 and a_2. From the table we can see that $a_1 = a_2 = 1.5$ and $T = 3.25$ gives better results than the one for $a_1 = a_2 = 2.0$ and

Table II.8: Output for Fourier Series Method: Example 1

$$F(s_1, s_2) = \frac{1}{(s_1+1)(s_2+1)}$$
$$f(t_1, t_2) = e^{-t_1-2t_2}$$
$$c_1 = 1.0 \qquad c_2 = 0.0$$
$$T = 3.5 \qquad n = 24$$

t1	t2	aprox val	exact val	abs-error	rel-error
1.0	1.0	0.04979	0.04979	0.42569E-06	0.85502E-05
2.0	1.0	0.01832	0.01832	0.40152E-07	0.21922E-05
3.0	1.0	0.00674	0.00674	0.14791E-07	0.21951E-05
4.0	1.0	0.00248	0.00248	0.54501E-08	0.21987E-05
5.0	1.0	0.00091	0.00091	0.20221E-08	0.22175E-05
1.0	2.0	0.00674	0.00674	0.14997E-07	0.22258E-05
2.0	2.0	0.00248	0.00248	0.41007E-08	0.16543E-05
3.0	2.0	0.00091	0.00091	0.15125E-08	0.16587E-05
4.0	2.0	0.00034	0.00034	0.55643E-09	0.16587E-05
5.0	2.0	0.00012	0.00012	0.20340E-09	0.16482E-05
1.0	3.0	0.00091	0.00091	0.20518E-08	0.22501E-05
2.0	3.0	0.00034	0.00034	0.55643E-09	0.16587E-05
3.0	3.0	0.00012	0.00012	0.20524E-09	0.16631E-05
4.0	3.0	0.00005	0.00005	0.75503E-10	0.16631E-05
5.0	3.0	0.00002	0.00002	0.27601E-10	0.16526E-05
1.0	4.0	0.00012	0.00012	0.46606E-09	0.37765E-05
2.0	4.0	0.00005	0.00005	0.75304E-10	0.16587E-05
3.0	4.0	0.00002	0.00002	0.27776E-10	0.16631E-05
4.0	4.0	0.00001	0.00001	0.10218E-10	0.16631E-05
5.0	4.0	0.00000	0.00000	0.37354E-11	0.16526E-05
1.0	5.0	0.00002	0.00002	0.66914E-10	0.40064E-05
2.0	5.0	0.00001	0.00001	0.10127E-10	0.16482E-05
3.0	5.0	0.00000	0.00000	0.37353E-11	0.16526E-05
4.0	5.0	0.00000	0.00000	0.13742E-11	0.16526E-05
5.0	5.0	0.00000	0.00000	0.50296E-12	0.16442E-05

Table II.9: Output for Fourier Series Method: Example 2

$$
\begin{aligned}
F(s_1, s_2) &= \frac{1}{(s_1 - 1)(s_2 - 1)(s_1 + s_2 - 1)} \\
f(t_1, t_2) &= e^{t}1(e^{t}2 - 1) \text{ if } t_1 \geq t_2 \\
&= e^{t}2(e^{t}1 - 1) \text{ if } t_1 < t_2 \\
c_1 = 15.0 &\quad c_2 = 10.21 \\
T = 1.0 &\quad n = 24
\end{aligned}
$$

t1	t2	aprox val	exact val	abs-error	rel-error
0.2	0.2	0.28585	0.27042	0.154304E-01	0.570605E-01
0.4	0.2	0.32965	0.33029	0.648771E-03	0.196422E-02
0.6	0.2	0.40349	0.40342	0.639672E-04	0.158562E-03
0.8	0.2	0.49290	0.49274	0.160616E-03	0.325965E-03
1.0	0.2	0.60179	0.60184	0.468106E-04	0.777798E-04
0.2	0.4	0.31797	0.33029	0.123230E-01	0.373092E-01
0.4	0.4	0.72788	0.73372	0.583399E-02	0.795129E-02
0.6	0.4	0.89636	0.89616	0.196890E-03	0.219703E-03
0.8	0.4	1.09446	1.09458	0.120949E-03	0.110498E-03
1.0	0.4	1.33693	1.33692	0.105417E-04	0.788511E-05
0.2	0.6	0.40655	0.40342	0.312589E-02	0.774843E-02
0.4	0.6	0.90075	0.89616	0.458228E-02	0.511322E-02
0.6	0.6	1.48705	1.49800	0.109432E-01	0.730520E-02
0.8	0.6	1.83060	1.82966	0.944372E-03	0.516146E-03
1.0	0.6	2.23431	2.23475	0.440661E-03	0.197186E-03
0.2	0.8	0.54711	0.49274	0.543651E-01	0.110332E+00
0.4	0.8	1.06332	1.09458	0.312565E-01	0.285558E-01
0.6	0.8	1.79472	1.82966	0.349377E-01	0.190952E-01
0.8	0.8	2.70345	2.72749	0.240410E-01	0.881431E-02
1.0	0.8	3.33313	3.33137	0.176271E-02	0.529125E-03
0.2	1.0	0.72431	0.60184	0.122473E+00	0.203499E+00
0.4	1.0	1.46197	1.33692	0.125053E+00	0.935385E-01
0.6	1.0	2.22309	2.23475	0.116597E-01	0.521744E-02
0.8	1.0	3.30274	3.33137	0.286270E-01	0.859318E-02
1.0	1.0	4.65747	4.67077	0.133070E-01	0.284898E-02

$T = 3.25$. These examples demonstrate the importance in selecting the parameters for better results.

Table II.10: Output for Fourier Series Method: Example 3

$$F(s_1, s_2) = \frac{1}{(s_1\sqrt{s_1 s_2 + 1})}, \qquad f(t_1, t_2) = \frac{\sin(2\sqrt{t_1 t_2})}{\pi t_2}$$

$$c_1 = 2.0 \qquad c_2 = 2.0$$

$$T = 3.7 \qquad n = 24$$

t1	t2	aprox val	exact val	abs-error	rel-error
1.00	1.00	0.28952	0.28944	0.778E-04	0.269E-03
2.00	1.00	0.09805	0.09806	0.120E-04	0.123E-03
3.00	1.00	-0.10088	-0.10089	0.652E-05	0.646E-04
4.00	1.00	-0.24093	-0.24090	0.290E-04	0.121E-03
5.00	1.00	-0.30929	-0.30917	0.120E-03	0.387E-03
1.00	2.00	0.04903	0.04903	0.160E-05	0.326E-04
2.00	2.00	-0.12045	-0.12045	0.138E-06	0.114E-05
3.00	2.00	-0.15639	-0.15639	0.629E-07	0.402E-06
4.00	2.00	-0.09329	-0.09329	0.240E-07	0.257E-06
5.00	2.00	0.00658	0.00658	0.344E-06	0.522E-04
1.00	3.00	-0.03345	-0.03363	0.177E-03	0.526E-02
2.00	3.00	-0.10426	-0.10426	0.708E-07	0.679E-06
3.00	3.00	-0.02965	-0.02965	0.115E-06	0.386E-05
4.00	3.00	0.06379	0.06379	0.755E-07	0.118E-05
5.00	3.00	0.10549	0.10548	0.367E-06	0.348E-05
1.00	4.00	-0.05998	-0.06022	0.240E-03	0.398E-02
2.00	4.00	-0.04665	-0.04665	0.131E-06	0.280E-05
3.00	4.00	0.04784	0.04784	0.155E-07	0.324E-06
4.00	4.00	0.07873	0.07873	0.581E-07	0.738E-06
5.00	4.00	0.03679	0.03678	0.115E-04	0.312E-03
1.00	5.00	-0.06168	-0.06183	0.153E-03	0.248E-02
2.00	5.00	0.00263	0.00263	0.717E-06	0.272E-03
3.00	5.00	0.06329	0.06329	0.334E-07	0.528E-06
4.00	5.00	0.02943	0.02943	0.917E-07	0.312E-05
5.00	5.00	-0.03449	-0.03463	0.140E-03	0.404E-02

Table II.11: Output for Fourier Series Method: Example 4

$$F(s_1,s_2) = \frac{1}{\sqrt{s_1 s_2 + 1}}, \quad f(t_1,t_2) = \frac{\cos(2\sqrt{t_1 t_2})}{\pi\sqrt{t_1 t_2}}$$

$T = 3.25$ $n = 24$

t1	t2	exact val	rel-error a1=a2=1.5	rel-error a1=a2=2.0
1.00	1.00	-0.13246	0.447E-04	0.154E-04
2.00	1.00	-0.21413	0.322E-04	0.222E-03
3.00	1.00	-0.17430	0.281E-04	0.123E-03
4.00	1.00	-0.10403	0.109E-03	0.977E-03
5.00	1.00	-0.03387	0.305E-04	0.467E-03
1.00	2.00	-0.21413	0.287E-04	0.203E-04
2.00	2.00	-0.10403	0.331E-04	0.127E-05
3.00	2.00	0.02411	0.253E-03	0.981E-05
4.00	2.00	0.09118	0.213E-04	0.844E-06
5.00	2.00	0.10057	0.925E-05	0.276E-04
1.00	3.00	-0.17430	0.200E-04	0.817E-03
2.00	3.00	0.02411	0.253E-03	0.949E-05
3.00	3.00	0.10188	0.214E-04	0.828E-06
4.00	3.00	0.07343	0.501E-04	0.195E-05
5.00	3.00	0.00886	0.424E-03	0.257E-03
1.00	4.00	-0.10403	0.133E-03	0.404E-02
2.00	4.00	0.09118	0.213E-04	0.157E-06
3.00	4.00	0.07343	0.501E-04	0.194E-05
4.00	4.00	-0.01158	0.456E-03	0.216E-04
5.00	4.00	-0.06312	0.966E-04	0.377E-02
1.00	5.00	-0.03387	0.500E-02	0.635E-01
2.00	5.00	0.10057	0.107E-04	0.530E-05
3.00	5.00	0.00886	0.348E-03	0.960E-04
4.00	5.00	-0.06312	0.763E-05	0.194E-04
5.00	5.00	-0.05342	0.358E-03	0.150E-02

CONCLUSIONS

This thesis is primarily devoted to developing theoretical methods as well as the numerical methods of inversion of the multi-dimensional Laplace transforms. For the theoretical development we have used operational techniques. We have obtained obtained several results in multi-dimensional Laplace transforms, which are useful in deriving inverses of many functions. We have successfully applied these results to the known one-dimensional Laplace transform pairs and obtained many new multi-dimensional Laplace transform pairs. These results on the inversion of multi-dimensional Laplace transforms may be useful for solving problems in various disciplines, such as Physics and Engineering. Availability of these results will also further enhance the use of the multi-dimensional Laplace transforms. Moreover, these results may be helpful in the future development of algorithms for generating new inverses of the Laplace transforms.

As for numerical inversion processes, we have developed two numerical techniques to invert two-dimensional Laplace transforms. One of the methods is an extension of the one-dimensional method based on expanding the inverse function in terms of Laguerre polynomials. The other method is two-dimensional extension of the Fourier series representations of the inverse function integral. Among these two methods, the former one performed very well with smooth functions. When compared to the former method, the latter method did not do well but with nonsmooth functions its performance is considerably better. We compared these two methods to Singhal and Vlach method. For the functions we tested Method 1 outperformed Singhal et al. method and Method 2. Method 2 is comparable in accuracy to Singhal et al. method.

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APPENDIX A. FORTRAN PROGRAM FOR CRUMP METHOD

```
c*******************************************************************
c   This program uses the method described by Kenny Crump in
c   his paper entitled "Numerical Inversion of the Laplace
c   Transforms Using Fourier Series Approximation'. This
c   paper appeared on 'Journal of the Association for
c   Computing Machinery', 23 (1976) no. 1, pages 89-96.
c   This contains two subroutines with the main program.
c   One of the subroutine is a function 'cmpfn' evaluate
c   the function (Laplace transform) to be inverted.
c   Other one is a subroutine 'epal'  which is used to
c   accelerate the convergence of series involved
c   in inverting the Laplace transform.
c*******************************************************************
c
      program crump
c
      implicit double precision (a,b,d-h,p-y)
      implicit double complex  c,z
```

```
c
      external cmpfn
      double precision  a, factor, tcapt, step, delta, t, alpha
      double precision  err, relerr, exactf, aproxf, pi, tmax
      double precision  fr(0:101), fim(101), epsum
      double complex cs, zs, za, zfa
      double complex cmpfn
      integer m, n
      logical flag
c
      parameter (pi=3.14159265358979323846d0, tmax=10.0d0)
c
      common t, tcapt, epsum, n, fr, fim, flag
c
      open(unit=1, access='append', status='old')
c
c
c
c
c
c
c
      n=31
      m=10
```

```
      alpha=1.0d0
      factor=0.7d0
c
c
c     End of data set
c
      step=tmax/dble(m)
      tcapt=factor*tmax
 999  a=alpha+10.0d0*dlog(10.0d0)/(2.0d0*tcapt)
      za=dcmplx(a,0.0d0)
      zfa=cmpfn(za)
      fr(0)=dreal(zfa)
      do 10 k=1,n
         delta=pi*dble(k)/tcapt
         cs=dcmplx(a, delta)
         zs=cmpfn(cs)
         fr(k)=dreal(zs)
         fim(k)=dimag(zs)
 10   continue
c
c     End of loading f
c
      write(1,1000)
      write(1,2000) a, n, tcapt
```

```
      write(1,3000)
      do 20 i=1,m
         t=step*dble(i)
         exactf=dexp(t)*dcos(t)-1.0d0
         ordsum=fr(0)/2.0d0
         do 30 k1=1,n
            x=dble(k1)*pi*t/tcapt
            ordsum=ordsum+fr(k1)*dcos(x)-fim(k1)*dsin(x)
30       continue
         fbar=(dexp(a*t)/tcapt)*ordsum
         orderr=dabs(exactf-fbar)
         flag=.true.
         call epal
         if (flag) then
            write(1,4000) t, exactf
         else
            aproxf=(dexp(a*t)/tcapt)*epsum
            eperr=dabs(exactf-aproxf)
            if (exactf .ne. 0.0d0) then
              ordrelerr=orderr/dabs(exactf)
              eprelerr=eperr/dabs(exactf)
            else
               ordrelerr=orderr
               eprelerr=eperr
```

```
                endif
                write(1,5000) t, exactf, ordrelerr, eprelerr
            endif
 20      continue
         tcapt=tcapt+0.2d0
         if (tcapt .le. 9.5d0) go to 999
c
c      Format statements
c
 1000    format('1',' F(s)=((s-1)/((s-1)*(s-1)+1))-1/s')
 2000    format(2x,' a= ',f6.3,2x,' n= ',i3,2x,'tcapt= ',f6.3,/)
 3000    format(1x,'  time    exact value   relerror  eprelerr ')
 4000    format(1x,f6.3,4x,'epal failure',2x,f15.7)
 5000    format(1x,f6.3,2x,f15.6,2(1x,e12.5))
c
c      Format ends
c
         stop
         end
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c      End of the main program
c cccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
```

```
c=================================================================
c    The following function evaluates the Laplace transform,
c    a generic complex function of single complex variable z1,
c    which is to be inverted numerically using the ongoing
c    algorithm.
c=================================================================
c
      double complex function cmpfn(z)
c
      implicit double precision (a,b,d-h,p-y)
      implicit double complex  c, z
c
      double complex z, z1, zone
c
      zone=dcmplx(1.0d0, 0.0d0)
      z1=z-zone
      cmpfn=z1/(z1*z1+zone)-zone/z
      return
      end
c
c*****************************************************************
c    End of complex function 'cmpfn'
c*****************************************************************
c=================================================================
```

```
c  The subroutine EPAL is introduced to increase the speed of
c  of the convergence of the series involved in Crump's method.
c  EPAL stands for Epsilon algorithm for convergence of the
c  series.This algorithm was originally used by Macdonald, J. R.
c  in his paper "Accelerated convergence, divergence, iteration
c  and curve fitting; J. Appl. Physics 10(1964), 3034-3041.
c==================================================================
c
          subroutine epal
c
        implicit double precision (a,b,d-h,p-y)
        implicit double complex  c, z
c
        integer n
        double precision  denom, t, sum, x, tcapt, pi, tmax, epsum
        double precision  eps(0:2,101), fr(0:101), fim(101)
        logical flag
c
        parameter (pi=3.14159265358979323850d0, tmax=10.0d0)
c
        common t, tcapt, epsum, n, fr, fim, flag
c
        do 105 i1=1,n
           eps(0,i1)=0.0d0
```

```
105   continue
      sum=fr(0)/2.0d0
      do 5 k1=1,n
         x=dble(k1)*pi*t/tcapt
         sum=sum+fr(k1)*dcos(x)-fim(k1)*dsin(x)
         eps(1,k1)=sum
5     continue
      n1=n-1
9     do 15 j1=1,n1
         denom=eps(1,j1+1)-eps(1,j1)
         flag=(dabs(denom) .le. 1.d-36)
         if (.not. flag) then
            eps(2,j1)=eps(0,j1+1)+1.0d0/denom
         else
            go to 19
         endif
15    continue
      n2=n1+1
      do 35 k2=1,n2
         eps(0,k2)=eps(1,k2)
35    continue
      do 45 k3=1,n1
         eps(1,k3)=eps(2,k3)
45    continue
```

```
      n1=n1-1
      if (n1 .ge. 1) go to 9
      epsum=eps(1,1)
 19   return
      end
c
c*************************************************************
c     End of subroutine 'epal'
c*************************************************************
```

APPENDIX B. FORTRAN PROGRAM FOR PIESSENS AND BRANDERS METHOD

```
c*******************************************************************
c   This program implements the algorithm described by Piessens   *
c   and Branders in their paper published in October 1971 on      *
c   the journal PROC. IEE, Vol. 118, No. 10, entitled             *
c   'Numerical Inversion of the Laplace Transform Using           *
c    Generalized Laguerre Polynomials'. This program consists     *
c    of four subroutines and functions, namely, 'coef', 'phi'     *
c    'capf' and 'factor'.                                         *
c*******************************************************************
c
      integer n, m, nostep
c
      parameter (n=100, m=100, nostep=10)
c
      double precision  a, b, c, delta, t, p, q, denom, aux, wtsum
      double precision  pi, tmax, exactf, aproxf, abserr, relerr
      double precision  br(0:2), ak(0:n), wterr, rterr, errsum
```

```
      double precision  dbsj0
c
      parameter (pi=3.14159265358979323850d0, tmax=10.d0)
c
      common a, b, c, ak
c
*     open(unit=9, access='append', status='old')
c
c**********************************************************************
c tmax= Maximum value of t(time),n= Number of terms in the sum       *
c of Fourier series approximation of the function f,                 *
c t= time variable,  ak= coefficients of the series,                  *
c nostep=Number of subdivision of the t interval [0, tmax],          *
c m= Number of terms term used evaluate the function 'phi',          *
c delta= step-size, capf= The Laplce transform of the                *
c function f, abserr=absolute error,relerr=relative error
c aproxf= numerical approx of f, exactf= exact value of function f *
c**********************************************************************
c
c*******************
c     Data begins *
c*******************
c
c     a=0.d0
```

```
c        b=1.0d0
c        c=0.d0
         a=-0.5d0
         b=1.0d0
         c=0.d0
c
c*******************
c      End of data *
c*******************
c
c
         write(6,1000)
         write(6,1001)
         write(6,2000) a, b, c, n, m
         write(6,3000)
c
         call coef
         delta=tmax/dble(nostep)
         errsum=0.d0
         wtsum=0.d0
         denom=0.d0
         do 10 j=1,nostep
           t=dble(j)*delta
           br(1)=0.d0
```

```
      br(2)=0.d0
      do 20 k=n,0,-1
         br(0)=ak(k)+(2.d0-(b*t+1.d0-a)/dble(k+1))*br(1)
     *            -(1.d0-(1.d0-a)/dble(k+2))*br(2)
         aux=br(1)
         br(1)=br(0)
         br(2)=aux
 20   continue
      aproxf=dexp(-c*t+a*dlog(t))*br(0)
c     exactf=dbsj0(t)
      exactf=dcos(2.d0*dsqrt(t))/dsqrt(pi*t)
      abserr=dabs(exactf-aproxf)
      if (exactf .ne. 0.d0) then
         relerr=abserr/dabs(exactf)
      else
         relerr=abserr
      endif
c
      write(6,4000) t, exactf, aproxf, abserr, relerr
c
c
c     ---------------------------------------------------
c     To evaluate the root mean squared error and weighted root
c     mean squared error
```

```
c       -----------------------------------------------------------
c
             errsum=errsum+(exactf-aproxf)*(exactf-aproxf)
             wtsum=wtsum+errsum*dexp(-t)
             denom=denom+dexp(-t)
 10       continue
          rterr=dsqrt(errsum/dble(nostep))
          wterr=dsqrt(wtsum/denom)
c
          write(6,5000) rterr
          write(6,6000) wterr
c
c*******************
c     Format begins *
c*******************
c
 1000   format(2x,'Transform:        F(s)=exp(-1/s)/sqrt(s)',/)
 1001   format(2x,'Inverse function: f(t)=cos(2*sqrt(t))/sqrt(pi*t)
     *  ',/)
 2000   format(2x,'a=', f6.3,1x,'b=',f6.3,1x,'c=',f6.3,' n= ',i3,
     *          1x,'m= ',i3,/)
 3000   format(1x,' time    exact value     approx value   ',
     *             '  abs-error     rel-error',/)
 4000   format(1x,f5.2,2(2x,f15.12),2(2x,e11.5),/)
```

```
 5000   format(2x,'Root- Mean Sq. Error: ',e11.5)
 6000   format(2x,'Root- Mean Sq.(with wt. exp(-t)): ', e11.5),/)
c
c********************
c      Format ends   *
c *******************
c
        stop
        end
c
c******************************
c      End of main program    *
c******************************
c
c***************************************************************
c   Subroutine COEF evaluates the coefficients of the series    *
c   used to approximate the function f(t), which is the inverse *
c   function of the given function F(s).(in this program it is  *
c   defined as CAPF(.,. )                                       *
c***************************************************************
c
        subroutine coef
c
        integer n, m, nostep
```

```
c
      parameter (n=100, m=100, nostep=10)
c
      double precision y(0:m), sai(0:m), ak(0:n)
      double precision sum, sum1, smult, x1, factor, a, b, c, pi
      complex*16 phiy
      complex*16 phi
c
      parameter (pi=3.1415926535897932385d0)
c
      common a, b, c, ak
c
      do 100 l=0,m
         y(l)=pi*dble(2*l+1)/(2.d0*dble(m+1))
         phiy=phi(y(l))
         sai(l)=dreal(phiy)
 100  continue
      sum1=0.d0
      do 130 l0=0,m
        sum1=sum1+sai(l0)
 130  continue
      ak(0)=factor(0,a)*sum1/dble(m+1)
      do 110 k=1,n
        smult=2.d0*factor(k,a)/dble(m+1)
```

```
          sum=0.d0
          do 120 l1=0,m
             x1=y(l1)*dble(k)
             sum=sum+sai(l1)*dcos(x1)
 120      continue
          ak(k)=smult*sum
 110     continue
        return
        end
c*************************************************************
c                                                            *
c     The function 'PHI' is defined as                       *
c                                                            *
c           ((1/1-z)**(a+1))*F(1/1-z)                        *
c                                                            *
c   This function will be used to evaluate the coefficients *
c   of the approximating series.                             *
c                                                            *
c*************************************************************
c
        function phi(x2)
c
        double precision  x2, a, b, c
        complex*16 za, zb, zc, zx, wf
```

```
      complex*16 capf
c
      common a, b, c
c
c
      zb=dcmplx(b/2.d0,b/(2.d0*dtan(x2/2.d0)))
      za=cdexp((a+1.d0)*cdlog(zb))
      zc=zb-dcmplx(c,0.d0)
      wf=capf(zc)
      phi=za*wf
      return
      end
c*******************************************************************
c                                                                  *
c     The following function evaluates the Laplace transform,      *
c     a user defined generic complex function, whose inverse       *
c     to be evaluated using this program                           *
c                                                                  *
c*******************************************************************
      function capf(z)
      complex*16 z, one, alpha
c
      one=dcmplx(1.d0,0.d0)
      alpha=z*z+one
```

```
        capf= cdexp(-one/z)/cdsqrt(z)
c        capf=one/cdsqrt(alpha)
        return
        end
c
c=================================================================
c      End of complex function 'capf'
c
c=================================================================
c
c*****************************************************************
c                                                                *
c  The following function evaluates the value of                 *
c                                                                *
c         Factorial(k)/Gamma(a+k+1)                              *
c                                                                *
c   for the given value of k and a. This will be used to evaluate *
c   the coefficients of the series approximation.                *
c                                                                *
c*****************************************************************
       double precision function factor(i,a2)
c
       double precision  a2, a3, produt
       double precision  dgamma
```

```
      integer  j1, i
c
      a3=a2+1.d0
      if (i .eq. 0) then
         produt=1.d0/(dgamma(a3))
      else
         produt=1.d0/(dgamma(a3))
         do 200 j1=1,i
            produt=dble(j1)/(dble(j1)+a2)*produt
 200     continue
      endif
      factor=produt
      return
      end
c*********************************************
c    End of function 'factor'                *
c*********************************************
```

APPENDIX C. FORTRAN PROGRAM FOR LAGUERRE SERIES METHOD (2 DIM.)

```
cccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c    The purpose of this program "doublap" is to find the
c    inverse of two-dimensional Laplace transform of a
c    user-provided  analytic function in two (complex)
c    variables. The program uses the series of products of
c    generalized Laguerre polynomials as the approximation
c    to the inverse function. It consists of subroutine
c    'coef' and real functions 'laguer' and 'factor' and complex
c     functions 'zfn' and 'zphi'.
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
      program doublap
c
*       implicit double precision(a-h, p-r, t-y)
*       implicit double complex  z, o
```

```
c
      integer n, m, nostep
c
      parameter (n=24, m=24, nostep=5)
c
c
      real*8  a1, a2, b, c1, c2, t1max, t2max, delta1, delta2
      real*8  err, relerr, exactf, aproxf, pi, sum, t1, t2
      real*8  an(0:n), ak(0:n,0:n), v(0:n), aux, out
      real*8  dbsj1
c
      parameter(pi=3.1415926535897932385d0,t1max=5.d0,t2max=5.d0)
c
      common a1, a2, b, c1, c2, ak
c
*     open( unit=2, access='append',status= 'old')
c
      delta1=t1max/dble(nostep)
      delta2=t2max/dble(nostep)
c
c
c
      a1=0.0d0
      a2=0.0d0
```

```
      c1=0.0d0
      c2=0.0d0
      b=1.0d0
c
c********************************************************************
c     Write statements for data and titles
c********************************************************************
c
      write(6,1000)
      write(6,1001)
      write(6,2000) a1, b, c1, n, m
      write(6,3000) a2, c2
      write(6,4000)
c
c
      call coef
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c     The routine 'coef' is called to evaluate the coefficients
c     of the series, which approximates the inverse function
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
      do 10 j1=1,nostep
         t1=dble(j1)*delta1
         do 20 n2=0,n
```

```
            do 30 n1=0,n
               an(n1)=ak(n1,n2)
 30         continue
c
c*****************************************************************
c     Routine is called to evaluate the Laguerre polynomial
c*****************************************************************
c
            call laguer(t1, an, n, a1, b, v(n2))
 20      continue
         do 40 j2=1,nostep
            t2=dble(j2)*delta2
            call laguer(t2, v, n, a2, b, sum)
            aproxf=dexp(-c1*t1-c2*t2+a1*dlog(t1)+a2*dlog(t2))*sum
            exactf=dbsj1(2.0d0*dsqrt(t1+t2))/dsqrt(t1+t2)
            err=dabs(exactf-aproxf)
            if (exactf .ne. 0.d0) then
              relerr=err/exactf
            else
              relerr=err
            endif
            write(6,5000) t1, t2, exactf, aproxf, err, relerr
40       continue
10    continue
```

```
c
c****************************************************************
c       Format statements
c****************************************************************
c
 1000   format('1','   F(s1,s2)=(exp(-1/s1)-exp(-1/s2))/(s1-s2)',/)
 1001   format(2x,'  f(t1,t2)= J1(2*sqrt(t1+t2))/sqrt(t1+t2)',/)
 2000   format(3x,' a1=',f6.3,1x,'b=',f6.3,1x,'c1=',f6.3,
     *          1x,'n= ',i4,1x,'m= ',i4)
 3000   format(3x,' a2=',f6.3, 1x,'c2=',f6.3,/)
 4000   format(1x,'     t1      t2      exact value   approx value',
     *         '   error      relerror')
 5000   format(2x,2(2x,f5.2),2(2x,f12.7),2(2x,e11.5))
c
c************End of format statements*************************
c
        stop
        end
c

ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c************End of main*****************************************
c
```

```
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c****************************************************************
c   Subroutine COEF evaluates the coefficients of the series    *
c   used to approximate the function f(t1,t2), which is the     *
c   inverse function of the given Laplace transform F(s1,s2).   *
c   (in this program it is given by the function CAPF(.,. )     *
c                                                               *
c****************************************************************
c
c
        subroutine coef
c
c
*        implicit double precision(a-h, p-r, t-y)
*        implicit double complex s, z, o
c
        integer n, m, nostep
c
        parameter (n=24, m=24, nostep=5)
c
        real*8 y1(0:m), y2(0:m), ak(0:n,0:n)
        real*8 x1(0:m,0:n), x2(0:m,0:n), a1, a2, b, c1, c2
        real*8 sumn1, sumn2, sum0, sum1, factor, pi, t1max, t2max
```

```
      real*8 smul, sai1(0:m,0:m), sai2(0:m,0:m)
      complex*16 zphiy1, zphiy2
      complex*16 zphi
c
      parameter (pi=3.1415926535897932385d0,t1max=5.d0,t2max=5.d0)
c
      common a1, a2, b, c1, c2, ak
c
      do 200 l1=0,m
        y1(l1)=pi*dble(2*l1+1)/(2.d0*dble(m+1))
 200  continue
c
      do 210 l2=0,m
         y2(l2)=pi*dble(2*l2+1)/(2.d0*dble(m+1))
 210  continue
c
      do 220 l2=0,m
         do 220 l1=0,m
            zphiy1=zphi(y1(l1),y2(l2))+zphi(y1(l1),-y2(l2))
            zphiy2=zphi(y1(l1),y2(l2))-zphi(y1(l1),-y2(l2))
            sai1(l1,l2)=dreal(zphiy1)
            sai2(l1,l2)=dreal(zphiy2)
 220  continue
c
```

```
      do 230 n1=0,n
         do 230 l1=0,m
            x1(l1,n1)=y1(l1)*dble(n1)
 230  continue
c
      do 240 n2=0,n
         do 240 l2=0,m
            x2(l2,n2)=y2(l2)*dble(n2)
 240  continue
c
c
      do 295 n2=0,n
         do 295 n1=0,n
            smul=factor(n1,a1)*factor(n2,a2)/dble((m+1)*(m+1))
            sum1=0.d0
            do 290 l2=0,m
               do 290 l1=0,m

              sum1=sum1+sai1(l1,l2)*dcos(x1(l1,n1))*
     *                dcos(x2(l2,n2))-sai2(l1,l2)*
     *                dsin(x1(l1,n1))*dsin(x2(l2,n2))
 290        continue
            if ((n1 .eq. 0) .and. (n2 .eq. 0)) then
               ak(n1,n2)=sum1*smul/2.d0
```

```
              else
                 ak(n1,n2)=smul*sum1
              endif
 295      continue
c
          return
          end
c
*********************
c  End of 'coef '
c
*********************
c
c*************************************************************
c    Subroutine 'laguer'  is Smith algorithm for summing
c    series of orthogonal polynomials. Here we use this
c    algorithm to sum the series of (generalized) Laguerre
c    polynomials.
c*************************************************************
c
        subroutine laguer(t, as, nn, ap, bp, out1)
c
c
*        implicit double precision(a-h, p-r, t-y)
```

```
*        implicit double complex s, z, o
c
       real*8 t, ap, bp, out1, as(0:nn)
       real*8 aux, br(0:2)
c
       br(1)=0.d0
       br(2)=0.d0
       do 300 k=nn,0,-1
          br(0)=as(k)+(2.d0-(bp*t+1.d0-ap)/dble(k+1))*br(1)
     *          -(1.d0-(1.d0-ap)/dble(k+2))*br(2)
          aux=br(1)
          br(1)=br(0)
          br(2)=aux
 300   continue
       out1=br(0)
c
       return
       end
c
c***********************
c   End of 'laguer'.
c***********************
c
c*****************************************************************
```

```
c*************************************************************
c                                                            *
c     The function 'ZPHI' is defined as                      *
c                                                            *
c     ((1/1-z1)**(a1+1)*(1-z2)**(a2+1))*F(1/1-z1,1/1-z2).     *
c                                                            *
c   This function will be used to evaluate the coefficients *
c   of the approximating series.                             *
c                                                            *
c*************************************************************
c
      function zphi(xx1, xx2)
c
c
*      implicit double precision(a-h, p-r, t-y)
*      implicit double complex s, z, o
c
      external zfn
      real*8 xx1, xx2, a1, a2, b, c1, c2
      complex*16 za1, za2, zb1, zb2, zc1, zc2, zx1
      complex*16 zx2, zwf
      complex*16 zfn
c
      common a1, a2, b, c1, c2
```

```
c
c
      zx1=dcmplx(1.d0,1.d0/dtan(xx1/2.d0))
      zx2=dcmplx(1.d0,1.d0/dtan(xx2/2.d0))
      zb1=(b/2.d0)*zx1
      zb2=(b/2.d0)*zx2
      za1=cdexp((a1+1.d0)*cdlog(zb1))
      za2=cdexp((a2+1.d0)*cdlog(zb2))
      zc1=zb1-dcmplx(c1,0.d0)
      zc2=zb2-dcmplx(c2,0.d0)
      zwf=zfn(zc1,zc2)
      zphi=za1*za2*zwf
c
      return
      end
c
c*******************************
c    End of the function 'zphi'
c*******************************
c
c*****************************************************************
c                                                                *
c    The following function 'zfn' evaluates the Laplace          *
c    transform, a user defined generic complex function,         *
```

```
c     whose inverse to be evaluated using this program.                *
c*********************************************************************
c
        function zfn(z1,z2)
c
c
*        implicit double precision(a-h, p-r, t-y)
*        implicit double complex s, z, o
c
        complex*16 z1, z2, one, alpha
c
        one=dcmplx(1.d0,0.d0)
        zfn=(cdexp(-one/z1)-cdexp(-one/z2))/(z1-z2)
        return
        end
c
c*********************************************************************
c                                                                     *
c  The following function evaluates the value of                      *
c                                                                     *
c     j! k!/[Gamma(a1+j+1)*Gamma(a2+k+1)]                             *
c                                                                     *
c   for the given value of j, k, a1 and a2. This will be used to      *
c   evaluate the coefficients of the series approximation.            *
```

```
c                                                                     *
c*********************************************************************
c
      function factor(i,aa)
c
c
*      implicit double precision(a-h, p-r, t-y)
*      implicit double complex s, z, o
c
      real*8 aa, produc
      real*8 dgamma
      integer j1, i
c
c
      if (i .eq. 0) then
         produc=1.d0/dgamma(1.d0+aa)
      else
         produc=1.d0/dgamma(1.d0+aa)
         do 400 j1=1,i
            produc=dble(j1)/(dble(j1)+aa)*produc
 400     continue
      endif
      factor=produc
c
```

```
      return
      end
c
*****************************************************************
*    End of function 'factor'
*
*****************************************************************
c
```

APPENDIX D. FORTRAN PROGRAM FOR FOURIER SERIES METHOD(2 DIM.)

```
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c                                                            c
c The purpose of this program is to evaluate the inversion of the c
c Laplace transform in two dimension of a user-provided analytic  c
c function. The method is an extension of the one dimensional     c
c inversion technique described by Dubner and Abate in their      c
c 1968 paper on J. ACM 15 and Crump's paper on J. ACM 23 (1976).  c
c This program consists of a subroutine called EPAL, which is     c
c the abbreviated form of Epal algorithm and a complex function    c
c ZFN.                                                            c
c                                                                 c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c      **** Theoretical restrictions ****
c
c     maximum value of t1 & t2 'tmax' must be smaller than period
c     2 times 'capt' and the value of factor cannot exceed 1.
```

```
c
c
c
c     **** Input and output parameters ****
c
c  1.  zfn    :          name of complex function provided by user
c  2.  epsl   :          accuracy of relative error
c  3.  t1, t2:           independent variables of inverse function
c  4.  tmax   :          maximum value of t1 and t2
c  5.  capt   :          half period of the inverse function
c  6.  n      :   number of terms in the series approximation
c  7.  exactf:   exact value of inverse function
c  8.  aproxf:   numerical approximation of inverse function
c  9.  abserr :   absolute abserr in approximation
c  10. relerr:   absolute relative error
c  11. flag1 :   to signal the failure of epsilon algorithm
c  12. flag2 :        -do-
c
c******************************************************************
c
      program dcrump
c
c
      implicit double precision (a-h, p-r, t-y)
```

```
      implicit double complex s, z, o
c
      real*8 a1, a2, capt, step, epsl, beta, tnu
      real*8 alfa1, alfa2, factor, pi, aproxf
      real*8 tmax, t1, t2, abserr, relerr, exactf
      real*8 epsum, x, y, f(100), parsum(100)
      real*8 totsum, delta(0:100)
c
      double complex s1(0:100), s2(0:100), sz1, sz2
      double complex zfn
c
      logical flag1, flag2
c
      integer m,. n
      integer i1, j1, j2, jj, k, kk, nn
c
c
      parameter (pi=3.14159265358979323850d0, tmax=1.d0)
c
      open (unit=9, access='append', status='old')
c
c
c     ****** input data ******
c
```

```
      n=24
      factor= 0.75d0
      m=5
      alfa1=-1.0d0
      alfa2=-2.0d0
      a1=1.0d0
      a2=0.0d0
c
c     ***** end of data *****
c
      capt=factor*tmax
c     epsl=dexp(-8.0d0*dlog(10.0d0))
c 101  tnu=dexp(-2.0d0*capt*(a1-alfa1))
c      beta=dlog((1.d0-tnu)/(epsl-tnu))
c      a2=beta/(2.d0*capt)+alfa2
      step=tmax/dble(m)
c
 101  write(9,1000)
      write(9,1001)
      write(9,2000) a1, a2
      write(9,2001) capt, n
      write(9, 3000)
c
      do 105 i1=0,n
```

```
          delta(i1)=pi*dble(i1)/capt
          s1(i1)=dcmplx(a1, delta(i1))
 105      s2(i1)=dcmplx(a2, delta(i1))
 115    continue
c
c
        do 125 j=1,m
          t2=dble(j)*step
          do 135 i=1,m
            t1=dble(i)*step
            do 145 jj=0,n
              y=delta(jj)*t2
              do 155 ii=0,n
               x=delta(ii)*t1
               sz1=zfn(s1(ii),s2(jj))
               sz2=zfn(s1(ii),dconjg(s2(jj)))
               if ((ii .eq. 0) .and. (jj .eq. 0))
     *          f(ii+1)=0.5d0*dreal(zfn(s1(ii),s2(jj)))
               if ((ii .ne. 0) .and. (jj .eq. 0))
     *          f(ii+1)=dreal(sz1)*dcos(x)-dimag(sz1)*dsin(x)
               if ((ii .eq. 0) .and. (jj .ne. 0))
     *          f(ii+1)=dreal(sz1)*dcos(y)-dimag(sz1)*dsin(y)
               if ((ii .ne. 0) .and. (jj .ne. 0))
     *          f(ii+1)=dreal(sz1)*dcos(x+y)+dreal(sz2)*dcos(x-y)
```

```
     *                  -dimag(sz1)*dsin(x+y)-dimag(sz2)*dsin(x-y)
 155          continue
c
c----------------------------------------------------------------
c    The routine 'epal' is called to accelerate the convergence
c    of the sum of the series considered above
c----------------------------------------------------------------
c
              call epal(f, n+1, epsum, flag1)
              if (flag1) then
                 write(9,5000) t1, t2
                 go to 135
              else
                  parsum(jj+1)=epsum
              endif
 145        continue
            call epal(parsum, n+1, totsum, flag2)
            if (flag2) then
               write(9,5000) t1, t2
               go to 135
            else
               aproxf=0.5d0*totsum*dexp(a1*t1+a2*t2)/(capt*capt)
               exactf=dexp(-t1)*dexp(-2.d0*t2)
               abserr=dabs(exactf-aproxf)
```

```
                 if (exactf .ne. 0.d0) then
                    relerr=abserr/dabs(exactf)
                 else
                    relerr=abserr
                 endif
                 write(9,6000) t1, t2, aproxf, exactf, abserr, relerr
              endif
 135    continue
 125    continue
         capt=capt+0.1d0
         if (capt .le. 0.7d0) go to 101
c        a2=a2+0.25d0
c        if (a2 .le. 1.5d0) go to 101
c
c***************Format statements*************
c
 1000   format(2x,'Transform: F(s1,s2)=1/((s1+1)*(s2+2))')
 1001   format(2x,'Inverse function: f(t1,t2)=exp(-t1)*exp(-2*t2)',/)
 2000   format(2x,'Parameters: a1= ',f6.3,2x,'a2= ',f6.3)
 2001   format(2x,'           capt= ',f6.3,' terms (n)= ',i3,/)
 3000   format(1x,'   t1    t2      aprox val     exact val',
     *         ' abs-error rel-error',/)
 5000   format(1x,2f5.2,1x,'epal failure')
 6000   format(1x,2(1x,f5.2),2(2x,f10.5),2(2x,e9.3))
```

```
c
c ***************Format ends*******************
c
      stop
      end
ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c      End of main program
ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c      The complex function 'zfn' is a user provided
c      function whose inverse to be evaluated using the
c      above program.
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c       ** Parameters**
c
c     z1 & z2:  complex parameters (arguments of the user provided
c              coplex function 'zfn')
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
```

```
c
        double complex function zfn(z1, z2)
c
        implicit double precision (a-h, p-r, t-y)
        implicit double complex s, z, o
c
        double complex z1, z2, one
c
c
        one=dcmplx(1.d0, 0.d0)
        zfn=one/((z1+one)*(z2+2.d0*one))
        return
        end
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c   End of function zfn(.,.)
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c      The subroutine 'epal' is used to accelerate the convergence
c      of series approximation of the inverse function. 'epal'
c      is abbreviated form for the well-known epal algorithm.
```

```
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c
c     ** Input parameters**
c
c     n1    :   number of terms of series used for espsilon alg.
c     f1    :   array of 'n1' elements
c
c         ** Output parameters **
c
c     flag  :   to signal the failure of the epsilon algorithm
c     epsum1:   sum of the series obtained by using epsilon alg.
c
*******************************************************************
      subroutine epal(f1, n1, epsum1, flag)
c
      implicit double precision (a-h, p-r, t-y)
      implicit double complex s, z, o
c
      real*8 denom, sum, epsum1
      real*8 eps(0:2,0:100), f1(100)
c
      integer n1, nn1, i1, j1, k1, k2, k3
```

```
c
      logical flag
c
c
      do 205 i1=1,n1
         eps(0,i1)=0.d0
 205  continue
c
      sum=0.d0
      do 215 k1=1,n1
        sum=sum+f1(k1)
        eps(1,k1)=sum
 215  continue
c
      nn1=n1-1
 9    do 225 j1=1,nn1
         denom=eps(1,j1+1)-eps(1,j1)
         flag=(dabs(denom) .le. 1.d-29)
         if (.not. flag) then
            eps(2,j1)=eps(0,j1+1)+1.d0/denom
         else
            go to 19
         endif
 225  continue
```

```
c
      nn2=nn1+1
      do 235 k2=1,nn2
         eps(0,k2)=eps(1,k2)
 235  continue
      do 245 k3=1,nn1
         eps(1,k3)=eps(2,k3)
 245  continue
      nn1=nn1-1
      if (nn1 .ge. 1) go to 9
      epsum1=eps(1,1)
c
 19   return
      end
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
c
c     End of subroutine 'epal'
c
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
```