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ABSTRACT

The first volume of this proceedings contains an introduction and the plenary, research forum, working group, and discussion group papers. Papers include: (1) "Resources as a Verb: Recontextualizing Resources in and for School Mathematics" (Jill Adler); (2) "Markets and Standards: The Politics of Education in a Conservative Age" (Michael Apple); (3) "Analyzing the Mathematical Learning of the Classroom Community: The Case of Statistical Data Analysis" (Paul Cobb); (4) "The Production of Artifacts as Goal for School Mathematics?" (Cyril Julie); (5) "A Moment in the Zoom of a Lens: Towards a Discursive Psychology of Mathematics Teaching and Learning" (Stephen Lerman); (6) "Diversity and Change in Mathematics Teacher Education" (Chris Breen); (7) "Pilot In-Service Mathematics Teacher Project" (Barbara Jaworski); (8) "Talking Math: Proposal for School Change" (Lena Licon Khisty); (9) "The CAPE Project--Constructing Activities for Peer-Education" (Konrad Krainer); (10) "Reflecting on Mathematics Learning In and Out-of-School from a Cultural Psychology Perspective" (Guida de Abreu); (11) "Reflections on Mathematics Learning and Teaching: Implications of Cultural Perspectives" (Philip C. Clarkson); (12) "A Semiotic Analysis of Students' Own Cultural Mathematics" (Norma Presmeg); (13) "Linking Students' Own Cultural Mathematics and Academic Mathematics" (Marta Civil); (14) "Developing Chains of Signifiers: A New Tool for Teachers?" (Kathryn C. Irwin); (15) "Learning through Problem Solving" (Hanlie Murray, Alwyn Olivier, and Piet Human); (16) "Learning through Problem Solving" (Anne Reynolds); (17) "Supporting Students' Construction of Increasingly Sophisticated Ways of Reasoning through Problem Solving" (Koeno Gravemejier, Kay McClain, and Michelle Stephan); (18) "Supporting Students' Reasoning through Problem Solving--Implications for Classroom Practice" (Susie Groves); (19) "Building the Meaning of Statistical Association through Data Analysis Activities" (Carmen Batanero, Juan D. Godino, and Antonio Estepa); (20) "Building the Meaning of Statistical Association through Data Analysis

Activities" (Michael Glencross); (21) "Graphing as a Computer-Mediated Tool" (Janet Ainley, Elena Nardi, and Dave Pratt); (22) "Symbol-Use, Fusion, and Logical Necessity: On the Significance of Children's Graphing" (Ricardo Nemirovsky); (23) "Advanced Mathematical Thinking" (David Reid); (24) "Algebra: Epistemology, Cognition and New Technologies" (Jean-Philippe Drouhard, Alan Bell, and Sonia Ursini); (25) "Classroom Research" (Donald Cudmore and Lyn English); (26) "Cultural Aspects in the Learning of Mathematics" (Nomma Presmeg, Marta Civil, Judit Moschkovich, and Phil Clarkson); (27) "Geometry" (Maria Alessandra Mariotti and Ana Mesquita); (28) "Research on the Psychology of Mathematics Teacher Development" (Andrea Peter-Koop and Vania Santos-Wagner); (29) "Research in the Social Aspects of Mathematics Education" (Jo Boaler and Paola Valero); (30) "The Teaching and Learning of Stochastics" (John Truran, Kath Truran, and Carmen Batanero); (31) "Understanding of Multiplicative Concepts" (Tom Cooper and Tad Watanabe); (32) "Embodiment, Enactivism and Mathematics Education" (Rafael Nunes, Lawrie Edwards, Joao Filipe Matos, and Stephen Campbell); (33) "Learning and Teaching Number Theory" (Stephen Campbell and Rina Zazkis); (34) "Under-Represented Countries in PME: National Mathematics Education Research Communities and Priorities" (Paola Valero, Bernadette Denys, Renuka Vithal, and Pedro Gomez); (35) "Frameworks for Research on Open-ended Mathematical Tasks" (Peter Sullivan and Shukkan Susan Leung); and (36) "Exploring Different Ways of Working with Videotape in Research and Inservice Work" (Susan Pirie and Chris Breen). (ASK)

Proceedings of the 22nd Conference of the International Group for the **Psychology of Mathematics Education**

Volume 1

PME 22

12-17 July 1998

Stellenbosch, South Africa

Editors: Alwyn Olivier and Karen Newstead

University of Stellenbosch
Stellenbosch, South Africa
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PREFACE

The theme for the 1998 PME conference is *Diversity and Change in Mathematics Education*. The Program Committee hopes that the Plenary Addresses and Plenary Panel Discussion, as well as many personal presentations will create an atmosphere of reflection, examination and discussion on this significant issue.

The papers in the four volumes of these Proceedings are grouped according to the type of presentation, i.e. Plenary addresses, Plenary Panel Discussion, Research Forum, Working Groups, Discussion Groups, Research Reports, Short Oral Communications and Posters. Within each group, papers are sequenced alphabetically by the name of the first author, with the name(s) of the presenter(s) underlined if not all the authors are presenting.

We have included two cross-references to help readers to easily identify papers of interest:

- by research domain, according to the first author (page xxix of Volume 1)
- by author in the address list of presenting authors (page 285 of Volume 1).

We would like to extend our thanks to the Program Committee and to the reviewers for their respective roles in working with the papers in these Proceedings. We would also like to express our sincere thanks to Kate Bennie for her help in preparing these Proceedings.

Alwyn Olivier
Karen Newstead
Stellenbosch, June 1998

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INTRODUCTION

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and aims of PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976.

Its past presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gérard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK) and Carolyn Kieran (Canada).

The major goals of the Group are:

- to promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- to promote and stimulate interdisciplinary research in the aforesaid area with the co-operation of psychologists, mathematicians and mathematics educators; and
- to further a deeper understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

PME membership

Membership is open to people involved in active research consistent with the Group's aims, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees (US$30 or the equivalent in local currency) for the current calendar year (January to December). For participants of the PME22 conference, the membership fee is included in the pre-registration fee. Others are requested to write to either their Regional Contact, or directly to the Executive Secretary.

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THE REVIEW PROCESS

Research Reports

The Program Committee received 180 Research Report proposals. Each paper was sent to three PME members knowledgeable in the research domain specified by the author(s) for blind review.

The Program Committee automatically accepted all papers with two (55) or three (45) acceptances from the three reviewers. The Committee then read all the reviews of papers with one acceptance and two rejections from the reviewers, and where it considered it necessary, a member of the Committee formally reviewed these papers. The Committee also formally reviewed papers with one acceptance and one rejection where the third review was not received, unclear or undecided. As a result of this further review process another 19 papers were accepted, making a total of 119 accepted Research Reports.

All 20 Research Report proposals that had received three rejections from the reviewers were automatically rejected by the Program Committee and not reconsidered. The Committee reconsidered those proposals with one acceptance and two rejections from the reviewers and invited 33 authors to resubmit their rejected Research Report as a Short Oral Communication.

Short Orals and Posters

Each of the 83 Short Oral proposals was reviewed by two members of the Program Committee, and after discussion 65 Short Oral Communications were accepted. Another 9 authors were invited to resubmit their rejected Short Oral as a Poster.

Of the 34 Posters submitted, 32 were accepted by the Program Committee for presentation at the conference.

Research Forum

The Program Committee received 12 proposals for the three themes of the Research Forum. These papers were reviewed by six people knowledgeable in the respective fields. The Committee selected two presentations for each theme on the recommendations of the reviewers and the co-ordinators.

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The PME22 Program Committee thanks the following 192 PME members for their help in reviewing Research Report and Research Forum proposals:

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PME International:

No.	Year	Place	ERIC number
2	1978	Osnabrück, Germany	ED 226 945
3	1979	Warwick, United Kingdom	ED 226 956
4	1980	Berkeley, USA	ED 250 186
5	1981	Grenoble, France	ED 225 809
6	1982	Antwerpen, Belgium	ED 226 943
7	1983	Shoresh, Israel	ED 241 295
8	1984	Sydney, Australia	ED 306 127
9	1985	Noordwijkerhout, Netherlands	ED 411130 (Vol. 1), ED 411121 (Vol. 2)
10	1986	London, United Kingdom	ED 287 715
11	1987	Montreal, Canada	ED 383 532
12	1988	Veszprem, Hungary	ED 411128 (Vol. 1), ED 411129 (Vol. 2)
13	1989	Paris, France	ED 411140 (Vol. 1), ED 411141 (Vol. 2), ED 411142 (Vol. 3)
14	1990	Oaxtepex, Mexico	ED 411137 (Vol. 1), ED 411138 (Vol. 2), ED 411139 (Vol. 3)
15	1991	Assisi, Italy	ED 413 162
16	1992	Durham, USA	ED 383 538
17	1993	Tsukuba, Japan	ED 383 536
18	1994	Lisbon, Portugal	ED-383 537
19	1995	Recife, Brazil	ED 411134 (Vol. 1), ED 411135 (Vol. 2), ED 411136 (Vol. 3)
20	1996	Valencia, Spain	being processed
21	1997	Lahti, Finland	being processed

PME North American Chapter:

No.	Year	Place	ERIC number
2	1980	California (with PME4)	ED 250 186
3	1981	Minnesota	ED 223 449
4	1982	Georgia	ED 226 957
5	1983	Montreal	ED 289 688
6	1984	Wisconsin	ED 253 432
7	1985	Ohio	ED 411 127
8	1986	Michigan	ED 301 443
9	1987	Montreal (with PME11)	ED 383 532
10	1988	Illinois	ED 411 126
11	1989	New Jersey	ED 411 132
12	1990	Oaxtepex (with PME14)	ED 411137 (Vol. 1), ED 411138 (Vol. 2), ED 411139 (Vol. 3)
13	1991	Virginia	ED 352 274
14	1992	New Hampshire (with PME16)	ED 383 538
15	1993	California	ED 372 917
16	1994	Louisiana	ED 383 533
17	1995	Ohio	ED 398 534
18	1996	Panama City	ED 400 178
19	1997	Illinois	being processed

INDEX OF PRESENTATIONS BY RESEARCH DOMAIN

All the Research Reports, Short Oral Communications and Poster presentations are indexed below by research domain, mostly as indicated by the authors on their proposal form. To make reference easy, the papers are described by the first author and page-number.

Rational numbers

Socio-cultural studies

Teacher education

Theories of learning

Philosophy of research

PLENARY ADDRESSES

Jill Adler
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Stephen Lerman

RESOURCES AS A VERB
RECONCEPTUALISING RESOURCES IN AND FOR SCHOOL MATHEMATICS[1]

Jill Adler
Department of Mathematics, University of the Witwatersrand

"More resources" for school mathematics has become a clarion call for all stakeholders in education. However, the universal appeal and understanding attached to the common lament "we lack resources" or "we need more resources" hides the complexity of resources in and for school mathematics practice. That educational practice is a function of available resources, and vice versa, needs no explanation. Yet, we know only too well that 'more' resources does not necessarily or simply equate with 'better' practice. There are wealthy schools that do not offer quality education to their pupils, and there are impoverished schools that succeed against all odds. And still, it remains prevalent, and all too easy, particularly in economically impoverished areas of South African education, and in contexts of curriculum innovation and change, for stakeholders to either blame or explain their educational difficulties on this notion of 'lack of resources'.

In their study of mathematics, science and technology curriculum innovations across 13 Organisation for Economic Co-operation and Development (OECD) countries and 23 projects, Black & Atkin (1996) argue that the critical resources for implementing curriculum innovation and change are "mainly human", requiring enough people who are "willing" and "capable", and that there will always be "inadequate resources to support educational change". Their notion of "resources" extends beyond material objects. Like Clarke et al (1996) they posited the need for "support of materials" and "time released from other work", for planning, action and reflection. Despite the crucial importance of this issue, however, they were "struck" by how little discussion of resources there was across all 23 case reports (Black & Atkin,1996, p.193).

'Resources' in and for school mathematics practice thus needs to be interrogated so that in the use of the term, we enable both our action and our reflection on action. Hence I use the notion 'resources as a verb'. A resourced teacher, for example, then becomes a teacher-*acting*-with-resources and not simply a teacher surrounded (or not) by resources. In mathematics teacher education and research, and particularly in relation to changing pedagogical practices, we need to shift our attention off *resources as objects in themselves* that somehow always enable and enhance mathematics classroom practice, and onto *resources-in-practice-in-context*. In this way our attention will be off simplistic calls for "more", and perhaps more constructively onto how, in our complex contexts and practices, we use the resources we have, how this changes over time, how we integrate new resources into our practices, and with what consequences.

In order to interrogate 'resources' in and for school mathematics practice in this paper, and particularly in relation to changing pedagogical practices, I will start with a discussion of school mathematics practice and of its related resources. I will argue that school mathematics is a *hybrid* practice - a mixture of everyday and academic mathematics, and of learner and teacher-centred strategies. I will then use the concept of *transparency*, and its dual functions of *visibility and invisibility* to interrogate resources for teachers in use in school mathematics practice. Finally, I will argue that conceptualising the term 'resources' as a verb, as practice-in-context, allows for a more dynamic pedagogic practice both in the mathematics classroom and in mathematics teacher education.

Resources[2] and equity, access and change

As mathematics educators and researchers we are concerned with equity and change in relation to access to mathematical knowledge. The starting point of this paper is that resources, in all their complexity, reveal and are revealed by issues of equity, access and change. In South Africa equity, access and change are constantly "in your face", and all are thrown into sharp relief by a general shortage of material and human resources in the country.

Differential distribution of material and human resources in school education is highly visible across South African schools. I will not rehearse here apartheid's legacy of gross disparities across class and race. The relative wealth of schools in historically white middle-class suburbs in contrast with impoverished schools in black townships, in rural areas and in the increasing spread of informal settlements is well known. The recent Schools Register of Needs (Bot, 1997) reveals that a staggering 17% of all schools in South Africa lack basic physical infrastructure. There is serious overcrowding in some of these schools, with classes of up to 100 pupils, and in 23% of all schools there is no running water nor any toilet facilities in or close by the school. In short, not only is there little to draw on for learning and teaching in such schools, but conditions actively detract from possibilities for focussed attention on learning and teaching. What is important, and not surprising, is that the maldistribution of human and material resources across schools in South Africa is mirrored in a maldistribution of matriculation pass rates, and a critically low pass rate in school mathematics[3].

Of course, socio-economic disparities and related maldistribution of resources are global and local phenomena - across and within countries, rich or poor. Resource inequity is not unique to South Africa. The importance of this in education, as Bernstein (1996) has argued, is that the "... maldistribution of resources, certainly outside the school and often within it, affects access to and acquisition of school knowledge" (p.8).

As already noted, resources are also a significant issue in *change* in school mathematics, and across contexts [4]. Any attempts to change practices will bring with them or entail new and different resources and/or new uses for existing resources. This explains why even in educational contexts that are relatively well-resourced, difficulties with change in educational practices are attributed to "lack of resources". A large scale research project on the implementation of the National Mathematics Curriculum in the United Kingdom, for example, reported that lack of availability of resources was given by teachers as a reason for their difficulties, that at 'all key stages teachers felt that they lacked suitable activities in probability, and that they had inadequate teaching materials in handling data'. (Johnson & Millett, 1996, p.62).

At this juncture in South Africa's history, issues of equity are coupled with issues of access and change. There is a crisis in human resource development, particularly with respect to mathematics, science and technology. This is a crisis of *access* to scientific practice in its various forms. Part of enabling and broadening access are attempts to change structures and cultures across former oppressors and former oppressed, all deeply infused with apartheid ideology and practice. Not least are educational institutions, schools, their cultures and critically, their curricula. The demand for radical change is captured the new 'outcomes-based' curriculum in South Africa, popularly known as Curriculum 2005. I point to it here to highlight the political meanings and messages embedded in the slogan "Curriculum 2005". These relate to the need for something new and different from apartheid education. As one of the provincial Ministers of Education was recently quoted as saying: "The only benefit of the discredited system we inherited is the opportunities it necessitates for radical change" (Metcalfe, in The Weekly Mail and Guardian, April, 1998, p. 19). South Africa is thus a good context through which to interrogate 'resources' in use in school mathematics practice. By interrogating resources in school mathematics practice I hope to contribute to action *in* and reflection *on* the diversity and change that constitute current mathematics education practice.

School Mathematics Practice[5] - hybridised content and pedagogy

My discussion of "school mathematics practice" is restricted to the selection of curriculum content (what counts as mathematics) and pedagogical strategies (the relationship between teaching and learning, the ways in which mathematics is mediated).

The ubiquitous challenge for mathematics education is explanation and action that addresses widespread poor and socially maldistributed performance as well as the limitations of procedural knowledge in school[6]. A major debate in the field with respect to the selection of content into the school mathematics curriculum is whether the route to the acquisition of mathematical knowledge that is not restricted to procedural knowledge lies in contextualised or situated activity (mathematics in everday, real life), or in mathematical investigations that mirror the activity of mathematicians (See Dowling (1998) and (Boaler) 1997 for in-depth interrogations of aspects of this debate).

As I have argued elsewhere (Adler, 1998b), mathematical activity in school is by necessity neither everyday activity nor the activity of the mathematician. Solving a mathematical problem in school is simply not continuous with solving mathematical problems in other real world contexts[7]. School mathematic practice contains a selection from applicable and contextualised mathematics on the one hand, and from (academic) mathematics per se - a hybridisation[8].

Resources in and for school mathematics (and this is discussed further in the next section) can be and are drawn from both practices. They are delocated from everyday and mathematical contexts and relocated in the school mathematics context. Because of these recontextualisation[9] processes, their use in and for school mathematics is complicated, and sometimes contradictory. For teachers and teaching, hybridisation produces the important challenge of whether and how to be explicit about mathematical purposes in relation to a resource-based task, and thus about where meanings need to be located to facilitate sense-making, access and success in school mathematics practice. For example, is a population growth graph in the mathematics classroom a resource for learning about 'real reality' (the phenomenon of population growth) or about mathematical modelling (through, say, representing population growth data as a straight line graph)? Christiansen (1997) highlights the barriers that have to be overcome in a modelling course that draws on 'real' situations - that is, in classrooms with "de- and re-located extra-mathematical content" (p. 20). This observation and interpretation is similar to arguments in language education, that with communicative language practices in bi- or multilingual settings, explicitness about subject discourse (in relation to informal, everyday discourse about the subject) is needed to enable access to subject knowledge (JET, 1998).

Explicit or more directed mediational moves by the teacher run counter to current advocacy of a 'constructivist' (less directed, more facilitative) orientation to pedagogy. Underlying assumptions here are about naturally developing learners who will make mathematical meaning on their own or with co-learners if appropriate open tasks and related resources are placed in their hands, with teacher as non-directional facilitator. Thus inter-related with the challenge of hybridised content is the challenge of selection across growing and competing orientations to pedagogical practice in school mathematics and their assumptions of how we come to know mathematics. Under the wider rubric of 'mathematics for all' has come the argument that mathematical rationality has developed in specific contexts and as such it is exclusionary. Ethnomathematics as pedagogy comes to the fore[10] with its assumptions about of learners as cultural subjects and access and meaning thus located in mathematics 'frozen' into cultural artifacts (Gerdes, 1988). Critical mathematics education (Skovsmose, 1994) and realistic mathematics education (De Lange, 1996) assume, respectively, positioned and contextualised learners, with mathematical meaning lying in some form of action in and application to, situations and problems in a power differentiated and mathematically formatted real and everyday world.

What these orientations share is an approach to mathematical knowledge that goes beyond procedures, and, moreover, a commitment to some degree of learner-centred practice. New pedagogical approaches in and for school mathematics are, or attempt to be, respectful of learners, their histories, their meanings and their participation in learning activity. Nevertheless, debates abound, produced by the dichotomy posited between learner and teacher-centred pedagogy, and include the tensions in the dichotomies posited between personal constructions and enculturation (Jaworski, 1994), between participation and acquisition (Sfard, 1998), and between individual creativity and determining social structure (Confrey, 1994; 1995a; 1995b). In the context of the learner/teacher-centred debate, Cuban's (1993) study of American pedagogy over 100 years, supported by Black and Atkins overview of 23 educational innovations across 13 countries, is a convincing and somewhat sobering case for the resilience of teacher-centred practices, particularly in secondary school contexts, and the more limited emergence of a hybridisation - along a continuum - of learner-centred and teacher-centred pedagogical strategies[11].

The issue for this paper is that in a hybrid pedagogy, learner-centred strategies entail handing resources over to the learner. Here, for example, the teacher does not monopolise the resource, using it in a highly directed way to demonstrate an action or task. Rather, learners are provided the means to enact the task themselves, bringing to it their own meanings and interpretations from which to construct their mathematical knowledge. The difficulty lies in the fact that the resources, intended means to mathematics, are not self-explanatory objects, with mathematics shining clearly through them. Mathematical meaning comes in their mediated use and through what I will describe as their relative *transparency*. Hybridisation and transparency are connected analytic tools that enable us to interrogate resources in use in context.

Resources in hybridised school mathematical practice

Having described the significance of resources for equity and change, and school mathematics practice as a hybrid, it is useful to describe the range of resources in this practice as a prelude to the discussion of the use of resources through the analytic lens of the notion of transparency.

Popular approaches to educational resources are focussed on particular material and human resources which can be described as *basic resources*. They are necessary for the maintenance of schooling (though we know there are schools that succeed despite lacking some of these basic resources), and they are determined by the relative distribution of wealth of the country and its schools. *Basic material* resources include the physical infrastructure in the school, the buildings, water, electricity, desks and chairs, paper and pens. *Basic human* resources refer to teacher-pupil ratios (or class size), teacher qualifications, and the knowledge(s) teachers have to draw on in class, though the content of the qualification and optimal class size are both contested issues (Sedibe, 1998, pp. 38,72).

With an understanding of school mathematics, including pedagogy, as a hybrid practice, resources for school mathematics extend beyond basic material and human resources to include a range of other human and material resources, as well as cultural and social resources. The description that follows constitutes a first attempt at categorisation of resources in school mathematics. Of course, a categorisation is always a simplification, and thus can be limiting in its attempts to delineate what goes where. However, what follows has been useful for my own analysis and the development of this paper.

Technologies in school mathematics range from the common and widely available chalkboard to sophisticated computers. Their histories, and the meanings and uses built into them are located in workplace and everyday activity, and not specifically school mathematics activity. *School mathematics materials* are also wide-ranging and include, for example, textbooks, geoboards. These are 'made for school maths'. They thus have built into them both mathematical and instructional intentions and possibilities. *Mathematical objects* arise in the context of the discipline and the academy. They are obviously extensive, and range from the most complex theorem to a simple number line, a magic square, a representation of a triangle, the Cartesian plane, and (perhaps more debatably) conventional procedures[12]. *Everyday objects* include money (most popularly used to make operations of various kinds 'relevant'), stories, calculators, rulers. The determining context of everyday objects that are drawn on in school mathematics has no direct relation to the mathematics classroom, but is constituted by everyday cultural practices like buying and selling, measuring, communicating.

Language as a resource for mathematics teachers (what they can draw on) is at least three-dimensional. It is a cultural resource in that it includes the main language(s) learners bring to class (and their relation to the language of instruction). It is also a social resource as it includes learners' verbalisations during class, as well as communication (talk with and between learners). As Forman argues, "... students need to view themselves and each other as intellectual resources instead of relying solely upon the authority of the teacher and the text ... " (in Steffe et al,1996, p.117, 121). The determining contexts of language(s) brought to class are the home, street and (prior experience in) school. Finally, *time* can also be viewed as a cultural resource, used differently in, for example, urban and rural contexts. Yet, across contexts, time functions formatively in school through time-tables, length of periods, and possibilities for homework. It structures school mathematics practice to produce pacing, sequencing, and time-bound tasks. It also structures teachers' work, and hence their experience of 'lack of time' when attempts at change in school practice disregard teachers' time[13]

Many of these resources bring to and provoke in learners in the mathematics classroom, significations and meanings from practices in other contexts, particularly everday practices. What lies between the resource and school mathematics practice is their use in-practice-in-context - their transparency.

Transparency of resources - situated and relational

For Lave and Wenger, access to a practice entails access to its resources, its artifacts and its social relations (p.91).

> To become a full member of a community of practice requires access to a wide range of ongoing activity, old-timers, and other members of the community; and to information, resources, and opportunities for participation. (Lave and Wenger, 1991, p.101)

Lave and Wenger argue that often, social scientists who concern themselves with learning treat technology as a given and are not analytic about its interrelations with other aspects of a community of practice. Access for Lave and Wenger is thus pivoted on the concept of *transparency*, with its dual functions of *visibility and invisibility* (p.103). If there is to be access to a practice, then the resources in the practice need to be transparent. They need to be visible, seen so that they can be used and so extend the practice, and also invisible, so that they enabled smooth entry into the practice.

I have argued elsewhere that Lave and Wenger's notion of learning as legitimate peripheral participation does not transfer smoothly into mathematical learning in school (Adler, 1998b). Nevertheless, their concept of transparency is illuminating of classroom practices, and particularly in relation to resources and their use. Resources in school mathematics practice need to be seen to be used (visible) and seen through to illuminate mathematics (invisible). Transparency is not an inherent feature of the resource, but rather a function of its use in practice, in context. As resources are harnessed to support and enable learning in a hybrid practice like school mathematics, their transparency becomes more complex. As a result they either enable or block access to mathematical knowledge. Contrary to common sense views, 'more' resources in a hybrid practice make more, rather than less demands on the teacher.

- **Language as a transparent resource**

My interest in resources has its roots in a research project on teachers' knowledge of their practices in multilingual secondary mathematics classrooms (Adler, 1996), and in the shift in orientation to language and learning in bilingual settings away from a deficit model towards seeing the languages pupils bring to class as a resource (Baker, 1993). In this view the language(s) learners bring to class are not viewed as a problem, something to be silenced in school and replaced with the language the learner 'lacks' (what is often referred to as the 'subtractive' model of bilingual education). Rather, they are viewed as a resource - to be drawn on to facilitate meaning-making and access to new knowledge and/or a new language.

In the research project, English-speaking teachers whose classrooms had rapidly deracialised spoke at length of the importance of being explicit about mathematical language in class. This was an access and equity issue since for some pupils, English, the language of instruction, was not their main language. They were thus disadvantaged. Helen, for example, considered talk, and particularly mathematical talk between herself and her pupils, and between pupils themselves as a resource in her mathematics classroom practice. Talk was something to be drawn on for teaching and learning in what she hoped was a more learner-centred practice. She asserted that because her class was now multilingual, she had become more explicit about terms and ways of talking mathematically. She claimed that this act of making mathematical language and talk highly visible in class in fact benefited all pupils, not only those whose main language was not English. It is interesting to note that Helen's practice fits findings of research into bilingual and multilingual education more generally. As mentioned earier, trends identified from a recent survey into multilingual classrooms include the importance of teachers being explicit about school and subject discourses (JET, 1998)

However, as Helen became more self-conscious of her practices, she began to question whether being explicit about mathematical language was always and everywhere a "good thing". She experienced what I called the 'dilemma of transparency' (Adler, 1996; forthcoming). Videotapes of her teaching reflected how, in some moments of practice, explicit focus on mathematical language in fact seemed to obscure mathematical meaning. Instead of mathematical talk being a transparent resource with its dual functions of visibility and invisibility (visible in that it extends the practice, and invisible in that it enables smooth entry into the practice) explicit mathematical language teaching became opaque. The talk itself became too visible, the object of attention rather than also a means to mathematics.

Language related dilemmas, like the dilemma of transparency, arise in contexts where language practices like code-switching (i.e. drawing on learners' main language) and mathematical talk are viewed as resources in school mathematics practice. They highlight that shifts in practice through harnessing new or additional resources, or using resources in a different way, entail consequences, both intended and unintended.

- **Other resources**

If we extend this analysis to wider classroom practices, then we need to understand that resources - be they the widely available chalk board, a textbook, the computer, Dienes blocks, money or talk - need to be both visible and invisible. Learners need to be aware of them and at the same time they need to illuminate the mathematics. Whenever a resource is drawn on in class, it becomes visible, the object of attention. If there is novelty in the resource (e.g. a graphics calculator), time will be needed for learners to become acquainted with the resource and how it is operated. But if the resource is to enhance and enable mathematical learning, then at some point it will need to become invisible - no longer the object of attention itself, but the means to mathematics.

Meira (1995) drew on the notion of transparency in his analysis of tool use in terms of culturally mediated mathematical activity. His focus was an interpretation of classroom episodes where two male primary pupils were working with a purposefully designed gear apparatus that could illuminate mathematical relationships. His analysis of the way the boys used this tool led to the argument that the "instructional quality of physical devices ... relate[d] to the very process of using them". Their making sense of the physical device - a 'made for school maths' resource - was a specific process in a specific context. How they used the resource was not a function of how it was made - of the intended mathematics and pedagogy built into it - but rather a function of the meanings the boys brought to it, the teacher's construction of the task , his mediation of the boys' activity and the classroom culture. As Meira argues, this relational, "cultural view" of tool use is an important shift away from a narrow "epistemic view" of tools where mathematical principles and relations are treated as if they are obviously and clearly intrinsic to the tool, easily perceived on the one hand, and independent of learner meanings, classroom processes and context on the other.

Research into and development of technologies and school mathematics materials that can support mathematics learning in school have brought similar insights. For example, Love & Pimm (1996) have argued that texts, whatever form they might come to take in the mathematics class, will always have to be 'read', and this will be a function of the situation (context) and relationships (practice-in-context) within which the text is being used. Szendrai (1996) has argued that structured mathematics materials are no panacea, leading automatically to some intended mathematical understanding.

In Lave and Wenger's terms these resources need to be seen through (be invisible) if they are to illuminate mathematics. As they are embraced by teachers they take on specific and situated meanings in the practices and context of the mathematics classroom. They become visible and need to be rendered invisible. This is can be particularly complex if the resource has been drawn from an everyday context, and the pedagogical strategies being used embrace learner-centredness.

Money is an example of a popular school mathematics resource with an everyday determining context. When money is used in school as a familiar context that could enhance meaning of various aspects of number, then we need to understand that not only is the meaning of money in a school activity very different from its meaning in real life, but that such meanings are significantly shaped by social class (Walkerdine, 1988, 1990). While everyday practices like buying and selling might well provide a familiar context and hence a system of meaning for mathematics in school, they bring to the classroom, meanings related to the purchasing power of money in real life, and as such they could obfuscate, blocking access to those mathematical meanings they are meant to support. This is why drawing on resources from contexts and practices outside of school mathematics creates significant challenges for teachers. As Muller and Taylor (1995) argue, context crossings can be dangerous and alienating in school, and moreso for some learners than others.

In Lave and Wenger's terms, "supportive artifacts need to be transparent - a good balance between the two interconnecting requirements of visibility and invisibility" (p. 103). Transparency is not a property of the resource, but a function of how the resource is used and understood within the practice in context. Most of the resources teachers draw on in hybridised school mathematics practice bring the challenge of transparency, that is, establishing the balance between visibility and invisibility. In the discussion above I have referred to and exemplified language, everyday objects and school mathematics materials including texts. In the remainder of the paper I will draw on examples from a research project in teacher education to further my argument.

Resources in the South African context

A research team at the University of the Witwatersrand is currently investigating whether and how a formalised INSET programme for mathematics, science and English language teachers shapes their classroom practices. Learner centred strategies and 'resources' are both foci across courses in the programme. A base-line study of the classroom practices of some mathematics teachers in the programme was completed in 1996 with a follow up in 1997 (Adler et al, 1997;1998), and a third data collection phase is planned for 1998. Given a ranging resource-base across schools and classroom contexts in the research project - these ranged from very poor rural schools to urban township schools - and an underlying assumption of curriculum as contextualised social process (Cornbleth, 1990), one focus in the research is on resources, guided by (a) a wider conceptualisation that includes material, human, cultural, mathematical and social resources and (b) the following questions: What resources are available and how these are used over time? What resources do teachers create and/or use anew?

There are numerous examples from the project that reinforce the discussion so far. In 1996, and moreso in 1997, the primary mathematics teachers in particular brought in and used a range of additional material resources in attempts to elaborate their practice. Unfortunately, in almost all cases, the resources (and these ranged from a 'home-made' tangram-like puzzle, to a 3 X 3 magic square, to cuisenaire rods and constructed worksheets) did not shift between being object and means. Instead of becoming transparent resources, they were often opaque[14]. I would like to focus here on two examples from the project, each interesting in different ways.

- **The chalkboard**

For the secondary mathematics teachers in the research project, the chalkboard remained the predominant resource during the lessons observed in 1997, but in contrast to base-line data in 1996, it was now being used in new ways. Unlike what was observed in 1996, the teachers did not spend most of the lesson explaining from the board. Instead, their embracing of learner-centred practice entailed pupils working on exercises in small groups (similar to the textbook ones observed in 1996). They were then invited to share their solutions with the rest of the class by writing these up on the board and explaining them. Mpho[15], for example, was also quite deliberate in whom she encouraged up to the board, selecting groups whose answers were different from each other. When learners wrote up their solutions, they did this silently. Mpho assumed control of the lesson and moved up to the board to work with the whole class on the different solutions. Her focus at that point was on identifying the correct solution and then identifying and correcting the error in the incorrect solution.

Mpho, and the other secondary teachers in the project expanded their pedagogical practice by using the chalkboard in a new way. The chalkboard was used more as a shared resource, as a device for making public diverse pupil responses and for working publicly with learners' errors. The chalkboard made visible (it could be seen through to) greater participatory practice. At the same time, the learners' publicly displayed responses did not include verbal descriptions of process - the 'how' and 'why' in each solution. When Mpho reclaimed centre stage, she did this in part - in relation to the point of error. What remains publicly visible on the chalkboard is a correct solution that was not described nor discussed, and an incorrect solution where only the point of error has been corrected. It is interesting to think about the gains and losses in this practice. In 1996, Mpho was the sole user of the chalkboard. She demonstrated and explained the processes behind 'model' solutions to exercises. I am not beginning to suggest that such modelling is unproblematically taken up by all learners. We know it is not. Rather that we think, with teachers, about the process and consequences of resource use in their classrooms, intended and unintended - who benefits and from what.

I have brought in this example of chalkboard as resource because, besides textbooks and notebooks, the chalkboard is probably the most simple, available and widely used material resource in school mathematics practice. What I have tried to illustrate and concretise here is that even in contexts of seriously limited resources - Mpho teaches upper secondary classes in a very poor rural school - teachers interpret and use what they have in attempts to improve their practice. In using her chalkboard in new ways Mpho rendered the chalkboard transparent with respect to greater participation in her class and so expanded her pedagogical practice. She could do more with the use of the resource. Herein lies some of the challenges produced by hybridised pedagogy with more learner-centred strategies.

The implication here from which mathematics teacher education could proceed is to work with teachers on 'teachers working with resources' for access to mathematics. It is not that the chalkboard is good or bad (as in the decrying of "chalk and talk"), but how it is used, for what and for whose benefit. Nor is it simply that teacher is herself good or bad, but how she optimises her resources.

- **Time as a resource**

A great deal has been written on 'time' and teachers' work. The calls for 'more resources' in the context of curriculum innovations have included time in this call and I referred to this at the beginning of the paper. This is not only a function of more being expected of teachers (in working with new materials, for example), without any change to the structuring of their time on a daily basis. It is also a function of both the preparation and the time in class required for more learner-centred practice. Here diversity in the class needs to be taken into account in terms of content and pedagogy. This is more time demanding in relation to the pacing, selection and mediation of tasks.

What emerged in the research project during data collection in 1997, a somewhat different focus, was the significance of time and how it appeared to be working in the schools. What follows is anecdotal and speculative at this stage rather than systematic - we plan to follow this up more carefully in this year's data collection. Through an examination of pupils classwork books we noticed, for example, that in some schools pupils did no written work for extended periods of time. In these same schools, pupils continued to arrive well over an hour after the official start of the day, and many left at various points in the day. We also became aware that in some cases there were no clearly visible timetables. Absenteeism was high and because continuity could not be assumed, teaching tended to fragment into self contained half hour pieces. And teachers talked about how they never had enough time because pupils arrived late, left early, missed work and so on. In contrast, where 'time' was visible to the 'outsider' in the school, for example, timetables existed and were displayed, bells rang, gates (symbolic in places where there were none) closed at particular times, homework was expected and done), the school appeared to function well, with an appropriate focus on teaching and learning. In the conceptual framework if this paper, in the latter case time had become invisible in the daily practices in the school, structured and structuring, but there as a transparent resource - a means to teaching and learning. In both innovation and change in school mathematic practice, and the turning around of ineffective schools, attention to time as a transparent resource might well be enabling.

Conclusions and implications

In this paper I have foregrounded 'resources', spurred by my own research and practice and by the observation that despite widespread acknowledgement of the importance of resources in and for education, that is where discussion stopped. In effect, 'more resources' has become a clarion call, but it is not a solution. I am not beginning to suggest that impoverished schools like those described at the beginning of the paper do not need resources. Of course they do. Rather that resources in-practice-in-context need to be talked about, and concepts like hybridisation and transparency could enable us to work with their complexity. I have taken examples of chalkboard use, language and time - three universally obtainable resources - the most common of resources to all situations - and argued that through a clear understanding of the concept of 'resources as a verb' and the visibility/invisibility dynamic of resources in their use in context, teachers can elaborate their practice through more transparent use of resources in the classroom and so enable access to and change in school mathematics. Effective use of these kinds of available resources might go some way to explain why some poorly resourced schools manage to succeed. What is also implied is that school mathematics practice in contexts where, for example, there is a wealth of sophisticated technological resources will also be enabled by a similar interrogation of those resources. As we work for equity and change in school mathematics practice, *resources in practice in context* needs to be firmly on our agenda.

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NOTES

1. My gratitude to Taffy Adler and Lynne Slonimsky for helping me shape and sharpen my ideas, and to Margot Berger for comments and criticism.

2. By 'resources' I mean those objects and actions in our material, social and cultural world that we draw on and use to sustain, transmit and change our ranging practices.

3. In 1997, 45% of all matriculation candidates wrote mathematics, 44% passed with fewer than 10% at the higher grade. The nett result of all the matric candidates in 1997 is that only 20% passed mathematics (Shindler, 1998).

4. I have found two additional references to the importance of resources in situations of change. Carré (1993, p.194) discusses how locating resources is a major issue for starting teachers; Gray and Wilcox (1996, p.100) discuss how turning around ineffective schools involves deploying resources in different ways.

5. By 'practice' I mean activity and social relations. In this paper I focus more on activity and less on social relations.

6. Procedural, teacher-centred activity still dominates South African mathematics classrooms, particularly at the secondary level. The resources in this dominant practice, are largely prescribed textbooks, and one or more public display boards (a chalkboard, whiteboard, overhead projector). Pedagogical strategies are typically explanations and demonstrations by the teacher, followed by repetition and practice of procedures by learners. These practices often appear in exaggerated form in contexts of limited resources, as well as in multilingual contexts where learners are

having to simultaneously learn the language of instruction while they learn mathematics.

7. I understand context as both structural and socio-cultural.

8. In her wider longitudinal study of different mathematical practices in two schools in the UK, Boaler (1997) commented on a study that found attempts at an integrated approach to both principles and procedures in mathematics - a hybrid of a kind - were not feasible. Procedural knowledge still dominated and limited ways of knowing and doing mathematics. This raises an interesting point - beyond the scope of this paper - about the continuum and relative worth of varied hybrid forms.

9. My use of 'recontextualision' is inspired by Bernstein (1996) but attempts no more than to capture movement across contexts and with this, changing meanings.

10. This is not to deny epistemological debates on ethnomathematics - just to foreground the pedagogical.

11. As Black & Atkin explain, "Teachers ... have developed routines for helping students. The routines may look unambitious ... but they serve complex purposes, and meet definable expectations. In all these studies teachers used these routines to fashion ... new forms of activity, like group work ..." (1996, p. 130).

12. I mean here, procedures that have been transformed via the practice from processes to objects (reification - Sfard (1994)). They have become objects to be acquired in the practice.

13. See Hargreaves (1994) for an extended discussion of teaching and time.

14. The research report describes these uses in detail and is able in the context of the wider report to unpack the relational aspects of these practices which inevitably are a function of the teacher and her history and knowledge-base, the context of the school, the nature of the resource being used, the pedagogical strategies adopted and the mathematical learning intended.

15. This is a pseudonym.

MARKETS AND STANDARDS
THE POLITICS OF EDUCATION IN A CONSERVATIVE AGE

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Introduction

We have entered a period of reaction in education. In many nations, our educational institutions are seen as failures. High drop-out rates, a decline in "functional literacy," a loss of standards and discipline, the failure to teach "real knowledge" and economically useful skills, poor scores on standardized tests, and more – all of these are charges leveled at schools. And all of these, we are told, have led to declining economic productivity, unemployment, poverty, a loss of international competitiveness, and so on. Return to a "common culture," make schools more efficient, more responsive to the private sector. Do this and our problems will be solved.

The threat to egalitarian ideals that these attacks represent is not usually made explicitly, since they are often couched in the discourse of "improving" competitiveness, jobs, standards, and quality in an educational system that is seen as in total crisis. Education is a site of struggle and compromise. It serves as a proxy as well for larger battles over what our institutions should do, who they should serve, and who should make these decisions. And, yet, by itself it is one of the major arenas in which resources, power, and ideology specific to policy, finance, curriculum, pedagogy, and evaluation in education are worked through. Thus, education is both cause and effect, determining and determined. Because of this, no one paper could hope to give a complete picture of this complexity. What I hope to do instead is to provide an outline of some of the major tensions surrounding education as it moves in conservative directions.

A key word here is directions. The plural is crucial to my arguments, since there are multiple and at times contradictory tendencies within the rightist turn.

The rightward turn – what I have elsewhere called the conservative restoration (Apple, 1993; 1996) – has been the result of the successful struggle by the right to form a broad-based alliance. This new alliance has been so successful in part because it has been able to win the battle over common-sense. That is, it has creatively stitched together different social tendencies and commitments and has organized them under its own general leadership in issues dealing with social welfare, culture, the economy, and as we shall see in this chapter, education. Its aim in educational and social policy is what might best be described as "conservative modernization" (Dale 1989b).

There are four major elements within this alliance. Each has its own relatively autonomous history and dynamics; but each has also been sutured into the more general conservative movement. These elements include neo-liberals, neo-conservatives, authoritarian populists, and a particular fraction of the upwardly mobile new middle class. I shall pay particular attention to the first two of these groups here since they – and especially neo-liberals – are currently in leadership in this alliance to "reform" education. However, in no way do I want to dismiss the power of these latter two groups.

Neo-liberals

Neo-liberals are the most powerful element within the conservative restoration. They are guided by a vision of the weak state. Thus, what is private is necessarily good and what is public is necessarily bad. Public institutions such as schools are "black holes" into which money is poured – and then seemingly disappears – but which do not provide anywhere near adequate results. For neo-liberals, there is one form of rationality that is more powerful than any other – economic rationality. Efficiency and an "ethic" of cost-benefit analysis are the dominant norms. All people are to act in ways that maximize their own personal benefits. Indeed, behind this position is an empirical claim that this is how all rational actors act. Yet, rather than being a neutral description of the world of social motivation, this is actually a construction of the world around the valuative characteristics of an efficiently acquisitive class type (Apple 1996; Honderich 1990).

Underpinning this position is a vision of students as human capital. The world is intensely competitive economically, and students – as future workers – must be given the requisite skills and dispositions to compete efficiently and effectively. Further, any money spent on schools that is not directly related to these economic goals is suspect. In fact, as "black holes," schools and other public services as they are currently organized and controlled waste economic resources that should go into private enterprise. Thus, not only are public schools failing our children as future workers, but like nearly all public institutions they are sucking the financial life out of this society. Partly this is the result of "producer capture." Schools are built for teachers and state bureaucrats, not "consumers." They respond to the demands of professionals and other selfish state workers, not the consumers who rely on them.

The idea of the "consumer" is crucial here. For neo-liberals, the world in essence is a vast supermarket. "Consumer choice" is the guarantor of democracy. In effect, education is seen as simply one more product like bread, cars, and television (see Apple 1990). By turning it over to the market through voucher and choice plans, it will be largely self-regulating. Thus, democracy is turned into consumption practices. In these plans, the ideal of the citizen is that of the purchaser. The ideological effects of this are momentous. Rather than democracy being a political concept, it is transformed into a wholly economic concept. The message of such policies is that of what might best be called "arithmetical particularism", in which the unattached individual – as a consumer – is deraced, declassed, and degendered (See Ball 1994 and Apple 1996).

The metaphors of the consumer and the supermarket are actually quite apposite here. For just as in real life, there are individuals who indeed can go into supermarkets and choose among a vast array of similar or diverse products. And there are those who can only engage in what can best be called "postmodern" consumption. They stand outside the supermarket and can only consume the image.

The entire project of neo-liberalism is connected to a larger process of exporting the blame from the decisions of dominant groups onto the state and onto poor people (Apple 1995). Yet, with their emphasis on the consumer rather than the producer, neo-liberal policies need also to be seen as part of a more extensive attack on government employees. In education in particular, they constitute an offensive against teacher unions who are seen to be much too powerful and much too costly. While perhaps not conscious, this needs to be interpreted as part of a longer history of attacks on women's labor, since the vast majority of teachers in so many nations are women (Apple, 1988; Acker, 1995).

There are varied policy initiatives that have emerged from the neo-liberal segments of the new hegemonic alliance. Most have centered around either creating closer linkages between education and the economy or placing schools themselves into the market. The former is represented by widespread proposals for "school to work" and "education for employment" programs, and by vigorous cost-cutting attacks on the "bloated state." The latter is no less widespread and is becoming increasingly powerful. It is represented by national proposals for voucher and choice programs (Chubb and Moe 1990). Behind this is a plan to subject schools to the discipline of market competition (see Wells 1993, Smith and Meier 1995, and Henig 1994).

However, there is increasing empirical evidence that the development of "quasi-markets" in education has led to the exacerbation of existing social divisions surrounding class and race. In his own review of the international evidence, Whitty argues that while advocates of choice assume that competition will enhance the efficiency and responsiveness of schools, as well as give disadvantaged children opportunities that they currently do not have, this may be a false hope (Whitty 1997, p.58). These hopes are not now being realized and are unlikely to be realized in the future "in the context of broader policies that do nothing to challenge deeper social and cultural inequalities." As he goes on to say, "Atomized decision-making in a highly stratified society may appear to give everyone equal opportunities but transforming responsibility for decision-making from the public to the private sphere can actually reduce the scope for collective action to improve the quality of education for all" (Whitty 1997, p.58).

There is a second variant of neo-liberalism. This one is willing to spend more state and/or private money on schools, if and only if schools meet the needs expressed by capital. Thus, resources are made available for "reforms" and policies that further connect the education system to the project of making our economy more competitive. Two examples can provide a glimpse of this position. In a number of states in the US, legislation has been passed that directs schools and universities to make closer links between education and the business community. In the state of Wisconsin, for instance, all teacher education programs must include identifiable experiences on "education for employment" for all of its future teachers; and all teaching in the public elementary, middle, and secondary schools of the state must include elements of education for employment in its formal curricula.

The second example is seemingly less consequential, but in reality it is a powerful statement of the reintegration of educational policy and practice into the ideological agenda of neo-liberalism. I am referring here to Channel One, a for-profit television network that is now broadcast into schools (many of which are financially hard-pressed given the fiscal crisis) enrolling over 40% of all middle and secondary school students in the nation. In this "reform", schools are offered a "free" satellite dish, 2 VCRs, and television monitors for each of their classrooms by a private media corporation. They are also offered a free news broadcast for these students. In return for the equipment and the news, all participating schools must sign a 3-5 year contract guaranteeing that their students will watch Channel One every day (Apple 1993).

This sounds relatively benign. However, not only is the technology "hard-wired" so that only Channel One can be received, but broadcast along with the news are mandatory advertisements for major fast food, athletic wear, and other corporations that students – by contract – must also watch. Students, in essence, are sold as a captive audience to corporations. Since, by law, these students must be in schools, the US is one of the first nations in the world to consciously allow its youth to be sold as commodities to those many corporations willing to pay the high price of advertising on Channel One to get a guaranteed (captive) audience. Thus, under a number of variants of neo-liberalism not only are schools transformed into market commodities, but so too now are our children (Apple 1993; see also Molnar 1996).

As I noted, the attractiveness of conservative restorational politics in education rests in large part on major shifts in our common-sense – about what democracy is, about whether we see ourselves as possessive individuals ("consumers"), and ultimately about how we see the market working. Underlying neo-liberal policies in education and their social policies in general is a faith in the essential fairness and justice of markets. Markets ultimately will distribute resources efficiently and fairly according to effort. They ultimately will create jobs for all who want them. They are the best possible mechanism to ensure a better future for all citizens (consumers).

Because of this, we of course must ask what the economy that reigns supreme in neo-liberal positions actually looks like. Yet, far from the positive picture painted by neo-liberals in which technologically advanced jobs will replace the drudgery and under- and unemployment so many people now experience if we were to only set the market loose on our schools and children, the reality is something else again. As I demonstrate in a much more complete analysis in **Cultural Politics and Education** (Apple 1996, pp.68-90), markets are as powerfully destructive as they are productive in people's lives.

Let us take as a case in point the paid labor market to which neo-liberals want us to attach so much of the education system. Even with the growth in proportion in high-tech related jobs, the kinds of work that are and will be increasingly available to a large portion of the US population will not be highly skilled, technically elegant positions. Just the opposite will be the case. The paid labor market will increasingly be dominated by low-paying, repetitive work in the retail, trade, and service sector. This is made strikingly clear by one fact. There will be more cashier jobs created by the year 2005 than jobs for computer scientists, systems analysts, physical therapists, operations analysts, and radiologic technicians combined. Further, 8 of the top 10 individual occupations that will account for the most job growth in the next 10 years include the following: retail salespersons, cashiers, office clerks, truck drivers, waitresses/waiters, nursing aides/orderlies, food preparation workers, and janitors. It is obvious that the majority of these positions do not require high levels of education. Many of them are low-paid, non-unionized and part-time, with low or no benefits. And many are dramatically linked to, and often exacerbate, the existing race, gender, and class divisions of labor (Apple

1996). This is the emerging economy we face, not the overly romantic picture painted by neo-liberals who urge us to trust the market.

Neo-liberals argue that by making the market the ultimate arbiter of social worthiness, this will eliminate politics and its accompanying irrationality from our educational and social decisions. Efficiency and cost-benefit analysis will be the engines of social and educational transformation. Yet among the ultimate effects of such "economizing" and "depoliticizing" strategies is actually to make it ever harder to interrupt the growing inequalities in resources and power that so deeply characterize so many societies.

This very process of depoliticization makes it very difficult for the needs of those with less economic, political, and cultural power to be accurately heard and acted upon in ways that deal with the true depth of the problem. This is because of what happens when "needs discourses" get retranslated into both market talk and "privately" driven policies.

For our purposes here, we can talk about two major kinds of needs discourses. There are first oppositional forms of needs talk. They arise when needs are politicized from below and are part of the crystallization of new oppositional identities on the part of subordinated social groups. What was once seen as largely a "private" matter is now placed into the larger political arena. Sexual harassment, race and sex segregation in paid labor, and affirmative action policies in educational and economic institutions provide examples of "private" issues that have now spilled over and can longer be confined to the "domestic" sphere (Fraser 1989, p.172).

A second kind of needs discourse is what might be called reprivatization discourses. They emerge as a response to the newly emergent oppositional forms and try to press these forms back into the "private" or the "domestic" arena. They are often aimed at dismantling or cutting back social services, deregulating "private" enterprise, or stopping what are seen as "runaway needs." Thus, reprivatizers may attempt to keep issues such as, say, domestic battery from spilling over into overt political discourse and will seek to define it as purely a family matter. Or they will argue that the closing of a factory is not a political question, but instead is an unimpeachable prerogative of private ownership or an unassailable imperative of an impersonal market mechanism" (Fraser 1989, p.172). In each of these cases, the task is to contest both the possible breakout of runaway needs and to depoliticize the issues.

In educational policy in the United States, there are a number of clear examples of these processes. In the state of California, for instance, a recent binding referendum that prohibited the use of affirmative action policies in state government, in university admission policies, and so on was passed overwhelmingly as "reprivatizers" spent an exceptional amount of money on an advertising campaign that labelled such policies as "out of control" and as improper government intervention into decisions involving "individual merit." Voucher plans in education – where contentious issues surrounding whose knowledge should be taught, who should control school policy and practice, and how schools should be financed are left to the market to decide – offer another prime example of such attempts at "depoliticizing" educational needs. Both show the emerging power of reprivatizing discourses.

A distinction that is useful here in understanding what is happening in these cases is that between "value" and "sense" legitimation (Dale 1989a). Each signifies a different strategy by which powerful groups or states legitimate their authority. In the first (value) strategy, legitimation is accomplished by actually giving people what may have been promised. Thus, the social democratic state may provide social services for the population in return for continued support. That the state will do this is often the result of oppositional discourses gaining more power in the social arena and having more power to redefine the border between public and private.

In the second (sense) strategy, rather than providing people with policies that meet the needs they have expressed, states and/or dominant groups attempt to change the very meaning of the sense of social need into something that is very different. Thus, if less powerful people call for "more democracy" and for a more responsive state, the task is not to give "value" that meets this demand, especially when it may lead to runaway needs. Rather, the task is to change what actually counts as democracy. In the case of neo-liberal policies, democracy is now redefined as guaranteeing choice in an unfettered market. In essence, the state withdraws. The extent of acceptance of such transformations of needs and needs discourses shows the success of the reprivatizers in redefining the borders between public and private again and demonstrates how a people's common-sense can be shifted in conservative directions during a time of economic and ideological crisis.

Neo-Conservatism

While neo-liberals largely are in leadership in the conservative alliance, the second major element within the new alliance is neo-conservatism. Unlike the neo-liberal emphasis on the weak state, neo-conservatives are usually guided by a vision of the strong state. This is especially true surrounding issues of knowledge, values, and the body. Whereas neo-liberalism may be seen as being based in what Raymond Williams would call an "emergent" ideological position, neo-conservatism is grounded in "residual" forms (Williams 1977). It is largely, though not totally, based in a romantic appraisal of the past, a past in which "real knowledge" and morality reigned supreme, where people "knew their place," and where stable communities guided by a natural order protected us from the ravages of society (See Apple 1996 and Hunter 1988).

Among the policies being proposed under this ideological position are national curricula, national testing, a "return" to higher standards, a revivification of the "western tradition," and patriotism. Yet, underlying some of the neo-conservative thrust in education and in social policy in general is not only a call for "return." Behind it as well – and this is essential – is a fear of the "other." This is expressed in its support for a standardized national curriculum, its attacks on bilingualism and multiculturalism, and its insistent call for raising standards (see, e.g., Hirsch 1996).

Behind much of this is a clear sense of loss – a loss of faith, of imagined communities, of a nearly pastoral vision of like-minded people who shared norms and values and in which the "western tradition" reigned supreme. It is more than a little similar to Mary Douglas's discussion of purity and danger, in which what was imagined to exist is sacred and "pollution" is feared above all else (Douglas 1966). We/they binary oppositions dominate this discourse and the culture of the other is to be feared.

This sense of cultural pollution can be seen in the increasingly virulent attacks on multiculturalism – which is itself a very broad category that combines multiple political and cultural positions (see McCarthy and Crichlow 1994) – on the offering of schooling or any other social benefits to the children of "illegal" immigrants and even in some cases to the children of legal immigrants, in the conservative English-only movement in the US, and in the equally conservative attempts to reorient curricula and textbooks toward a particular construction of the western tradition.

In this regard, neo-conservatives lament the "decline" of the traditional curriculum and of the history, literature, and values it is said to have represented. Behind this set of historical assumptions about "tradition," about the existence of a social consensus over what should count as legitimate knowledge (Apple 1990), and about cultural superiority. Yet, it is crucial to remember that the "traditional" curriculum whose decline is lamented so fervently by neo-conservative critics "ignored most of the groups that compose the American population whether they were from Africa, Europe, Asia, Central and South America, or from indigenous North American peoples" (Levine 1996, p.20). Its primary and often exclusive focus was often only on quite a narrow spectrum of those people who came from a small number of northern and western European nations, in spite of the fact that the cultures and histories represented in the United States were "forged out of a much larger and more diverse complex of peoples and societies" (Levine 1996, p.20). The mores and cultures of this narrow spectrum were seen as archetypes of "tradition" for everyone. They were not simply taught, but taught as superior to every other set of mores and culture (Levine 1996, p. 20).

As Lawrence Levine reminds us, a selective and faulty sense of history fuels the nostalgic yearnings of neo-conservatives. The canon and the curriculum have never been static. They have always been in a constant process of revision, "with irate defenders insisting, as they still do, that change would bring with it instant decline" (Levine 1996, p.15. See also Apple 1990 and Kliebard 1995). Indeed, even the inclusion of such "classics" as Shakespeare within the curriculum of schools in the United States came about only after prolonged and intense battles, ones that were the equal of the divisive debates over whose knowledge should be taught today. Thus, Levine notes that when neo-conservative cultural critics ask for a "return" to a "common culture" and "tradition," they are oversimplifying to the point of distortion. What is happening in terms of the expansion and alteration of official knowledge in schools and universities today "is by no means out of the ordinary; certainly it is not a radical departure from the patterns that have marked the history of [education] – constant and often controversial expansion and alteration of curricula and canons and incessant struggle over the nature of that expansion and alteration" (Levine 1996, p.15).

Of course, such conservative positions have been forced into a kind of compromise in order to maintain their cultural and ideological leadership as a movement to "reform" educational policy and practice. A prime example is the emerging discourse over the history curriculum – in particular the construction of the United States as a "nation of immigrants." In this hegemonic discourse, everyone in the history of the nation was an immigrant, from the first Native American population who supposedly trekked across the Bering Strait and ultimately populated North, Central, and South America, to the later waves of populations who came from Mexico, Ireland, Germany, Scandinavia, Italy, Russia, Poland, and elsewhere, to finally the recent populations from Asia, Latin America, Africa, and other regions. While it is true that the United States is constituted by people from all over the world – and that is one of the things that makes it so culturally rich and vital – such a perspective constitutes an erasure of historical memory. For some groups were colonized. Others came in chains and were subjected to state sanctioned slavery and apartheid for hundreds of years. And others suffered what can only be called bodily, linguistic, and cultural destruction (Apple 1996).

This said, however, it does point to the fact that while the neo-conservative goals of national curricula and national testing are pressed for, they are strongly mediated by the necessity of compromise. Because of this, even the strongest supporters of neo-conservative educational programs and policies have had to also support the creation of curricula that at least partly recognize "the contributions of the other." This is partly due to the fact that there is an absence of an overt and strong national department of education and a tradition of state and local control of schooling in the US. The "solution" has been to have national standards developed "voluntarily" in each subject area (see Ravitch 1995). Indeed, the example I gave above about history is one of the results of such voluntary standards.

Since it is the national professional organizations in these subject areas – such as the National Council of Teachers of Mathematics – that are developing such national standards, the standards themselves are compromises and thus are often more flexible than those wished for by neo-conservatives. This very process does act to provide a check on conservative policies over knowledge. However, this should not lead to an overly romantic picture of the overall tendencies emerging in educational policy. Since leadership in school "reform" is increasingly dominated by conservative discourses surrounding "standards", "excellence", "accountability", and so on, and since the more flexible parts of the standards have proven to be too expensive to actually implement, standards talk ultimately functions to give more rhetorical weight to the neo-conservative movement to enhance central control over "official knowledge" (Apple 1993) and to "raise the bar" for achievement. The social implications of this in terms of creating even more differential school results are increasingly worrisome (Apple 1992).

Yet it is not only in such things as the control over legitimate knowledge where neo-conservative impulses are seen. The idea of a strong state is also visible in the growth of the regulatory state as it concerns teachers. There has been a steadily growing change from "licensed autonomy" to "regulated autonomy" as teachers' work is more highly standardized, rationalized, and "policed" (Dale 1989a). Under conditions of licensed autonomy, once teachers are given the appropriate professional certification they are basically free – within limits – to act in their classrooms according to their judgement. Such a regime is based on trust in "professional discretion." Under the growing conditions of regulated autonomy, teachers' actions are now subject to much greater scrutiny in terms of process and outcomes. Indeed, there are states in the US that have specified not only the content that teachers are to teach, but also have regulated the only appropriate methods of teaching. Not following these specified "appropriate" methods puts the teacher at risk of administrative sanctions. Such a regime of control is based not on trust, but on a deep suspicion of the motives and competence of teachers. For neo-conservatives it is the equivalent of the notion of "producer capture" that is so powerful among neo-liberals. For the former, however, it is not the market that will solve this problem, but a strong and interventionist state that will see to it that only "legitimate" content and methods are taught. And this will be policed by state-wide and national tests of both students and teachers.

As I have demonstrated elsewhere, such policies lead to the "deskilling" of teachers, the "intensification" of their work, and the loss of autonomy and respect (see Apple, 1988; Apple 1995). This is not surprising, since behind much of this conservative impulse is a clear distrust of teachers and an attack both on teachers' claims to competence and especially on teachers unions.

The mistrust of teachers, the concern over a supposed loss of cultural control, and the sense of dangerous "pollution" are among the many cultural and social fears that drive neo-conservative policies. However, as I noted earlier, underpinning these positions as well is often an ethnocentric, and even racialized, understanding of the world. Perhaps this can be best illuminated through the example of Herrnstein and Murray's volume, **The Bell Curve** (Herrnstein and Murray 1994). In a book that sold hundreds of thousands of copies, the authors argue for a genetic determinism based on race (and to some extent gender). For them, it is romantic to assume that educational and social policies can ultimately lead to more equal results, since differences in intelligence and achievement are basically genetically driven. The wisest thing policy makers can do would be to accept this and plan for a society that recognizes these biological differences and does not provide "false hopes" to the poor and the less intelligent, most of whom will be black.

The consequences of such positions are not only found in educational policies, but in the intersection of such policies with broader social and economic policies, where they have been quite influential. Here too we can find claims that what the poor lack is not money, but both an "appropriate" biological inheritance and a decided lack of values regarding discipline, hard work, and morality (Klatch 1987). Prime examples here include programs such as "Learnfare" and "Workfare" where parents lose a portion of their welfare benefits if their children miss a significant number of school days or where no benefits are paid if a person does not accept low paid work, no matter how demeaning or even if childcare or healthcare are not provided by the state. Such policies reinstall earlier "workhouse" policies that were so popular – and so utterly damaging – in the United States, Britain, and elsewhere (Apple 1996).

Conclusion

Because of the complexity of educational politics, I have devoted most of this paper to an analysis of the conservative social movements that are having a powerful impact on debates over policy and practice in education and in the larger social arena. I have suggested that the conservative restoration is guided by a tense coalition of forces, some of whose aims partly contradict others.

The very nature of this coalition is crucial. It is more than a little possible that the conservative modernization that is implied in this alliance can overcome its own internal contradictions and can succeed in radically transforming educational policy and practice. Thus, while neo-liberals call for a weak state and neo-conservatives demand a strong state, these very evident contradictory impulses can come together in creative ways. The emerging focus on centralized standards, content, and tighter control paradoxically can be the first and most essential step on the path to marketization through voucher and choice plans.

Once state-wide and/or national curricula and tests are put in place, comparative school by school data will be available and will be published in a manner similar to the "league tables" on school achievement published in England. Only when there is standardized content and assessment can the market be set free, since the "consumer" can then have "objective" data on which schools are "succeeding" and which schools are not. Market rationality, based on "consumer choice", will insure that the supposedly good schools will gain students and the bad schools will disappear.

When the poor "choose" to keep their children in underfunded and decaying schools in the inner cities or in rural areas (given the decline and expense of urban mass transportation, poor information, the absence of time, and their decaying economic conditions, to name but a few of the realities), <u>they</u> (the poor) will be blamed individually and collectively for making bad "consumer choices." Reprivatizing discourses and arithmetical particularism will justify the structural inequalities that will be (re)produced here. In this way, as odd as it may seem, neo-liberal and neo-conservative policies that are seemingly contradictory may mutually reinforce each other in the long run (Apple 1996).

Yet, while I have argued that the overall leadership in educational policy is exercised by this alliance, I do not want to give the impression that the elements under the hegemonic umbrella of this coalition are uncontested or are always victorious. This is simply not the case. As a number of people have demonstrated, at the local level there are scores of counter-hegemonic programs and possibilities. Many schools in many nations have shown remarkable resiliency in the face of the concerted ideological attacks and pressures from conservative restorational groups. And many teachers, community activists, and others have created and defended educational programs that are both pedagogically and politically emancipatory (see, especially, Apple and Beane 1995 and Smith 1993).

Thus, in the face of all of these structural, financial, and political dilemmas, the fact that so many groups of people have not been integrated under the alliance's hegemonic umbrella and have created scores of local examples of the very possibility of difference, shows us in the most eloquent and lived ways that educational policies and practices do not go in any one unidimensional direction. Even more importantly, these multiple examples demonstrate that the success of conservative policies is never guaranteed. And this is absolutely crucial in a time when it is easy to lose sight of what is necessary for an education worthy of its name. Our task is to help ensure that this does not get lost in the search for profit and control.

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ANALYZING THE MATHEMATICAL LEARNING OF THE CLASSROOM COMMUNITY: THE CASE OF STATISTICAL DATA ANALYSIS

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Abstract. *In recent years, we have seen an increasing emphasis on the socially and culturally situated nature of mathematical activity. In this paper, I focus on a notion that is central to this general orientation, that of participation in communal practices. In developing this notion, I ground the discussion in my own and my colleagues' work in classrooms. My immediate goal is to clarify how we analyze students' mathematical reasoning as acts of participation in the mathematical practices established by the classroom community. In approaching this issue, I present episodes from a recently completed classroom teaching experiment that focused on statistics. Against the background of this analysis, I then broaden my focus in the final part of the paper by developing the themes of change, diversity, and equity.*

Orientation: Developmental Research and the Classroom Microculture

The type of research that I and my colleagues conduct involves classroom teaching experiments of up to a year in duration. In the course of these experiments, we develop sequences of instructional activities and analyze students' mathematical learning as it occurs in the social situation of the classroom. Research of this type falls under the general heading *of developmental research* in that it involves both instructional *development* and classroom-based *research*. Gravemeijer (1994) has written extensively about the first of these aspects, instructional development, and clarifies that the designer initially conducts an anticipatory thought experiment in order to formulate conjectures about both 1) possible trajectories for students' learning and 2) the means that might be used to support and organize that learning. These tentative conjectures are then tested and modified during the teaching experiment on the basis of an ongoing analysis of classroom events. It is here that the second major aspect of developmental research, classroom-based analyses, comes to the fore.

The interpretive framework that I and my colleagues currently use when conducting these analyses differentiates three broad features of the classroom microculture (Cobb & Yackel, 1996). The first of these, classroom *social norms*, provides a means of documenting the participation structure that the teacher and students establish in the course of their ongoing interactions (Erickson, 1986). Examples of social norms that typically become explicit topics of discussion in the classrooms in which we work include explaining solutions, attempting to make sense of explanations given by others, indicating understanding or non-understanding, asking clarifying questioning, and articulating alternatives when differences in interpretations have become apparent. These norms, it should be noted, are not specific to mathematics but apply to any subject matter area. For example, one might hope that students would explain and justify their reasoning in science or history classes as well as in mathematics. The second aspect of the interpretive framework focuses on normative aspects of classroom actions and interactions that are specific to mathematics. Examples of these so-called *sociomathematical norms* include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation. As we have noted elsewhere (Yackel & Cobb, 1996), the analysis of sociomathematical norms has helped us to understand the process by which the teachers with whom we have collaborated fostered their students' development of what might be called a mathematical disposition.

Our motivation for teasing out a third aspect of the classroom microculture, *classroom mathematical practices*, stems directly from our concerns as instructional designers. Recall that the designer develops conjectures about the possible trajectory of students' mathematical learning. These conjectures cannot be about the anticipated mathematical learning of each and every student in a class given that there are significant qualitative differences in their mathematical reasoning at any point in time. It is, however, feasible to view a hypothetical learning trajectory as consisting of conjectures about the collective mathematical development of the classroom community. This in turn indicates the need for a theoretical construct that allows us to talk explicitly about collective mathematical development. I illustrate how the notion of a classroom mathematical practice can serve this function by presenting episodes from a recently-completed teaching experiment that focused on statistical data analysis.

Background to the Teaching Experiment

The teaching experiment was carried out in an American seventh-grade classroom with 30 twelve-year-old students and involved 34 lessons conducted over a ten-week period. A member of the project staff served as the teacher for the first 21 classroom sessions, and two members of the research team shared the teaching responsibilities for the remaining 13 sessions. The overarching mathematical idea that served to orient our instruction design effort was that of distribution. We therefore hoped the students might come to view data sets as entities that are distributed within a space of possible values (Konold et al. 1996; Wilensky, 1997). Notions such as mean, mode, median, skewness, spread-outness, and relative frequency would then emerge as characteristics of distributions. Further, in this approach, various statistical representations or inscriptions would emerge as different ways of organizing and structuring distributions. In general, this focus on distribution allowed us to frame our instructional intent as that of supporting the gradual emergence of a single, multi-faceted mathematical notion rather than a collection of, at best, loosely-related concepts and inscriptions.

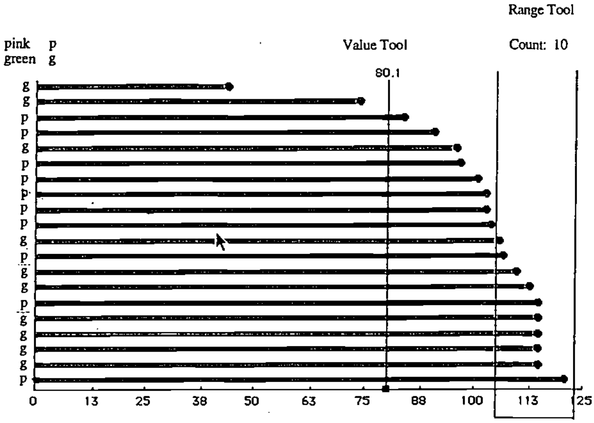
In refining our instructional goals, we also distinguished between additive and multiplicative reasoning about data (cf. Harel & Confrey, 1994). Briefly, the hallmark of additive reasoning about data is that students partition one or more data sets in ways appropriate to the question or issue at hand and then reason about the number of data points in the various parts of the data sets in part-whole terms. This can be contrasted with multiplicative reasoning about data wherein students reason about the parts of a data set as proportions of the whole data set. Our goal for the learning of the classroom community was that reasoning about the distribution of data in multiplicative terms would become an established mathematical practice that was beyond justification.

With regard to the starting points of the conjectured learning trajectory, a pilot study indicated that students frequently do not view data as measures of particular features of situations that are judged to be relevant with regard to the issues at hand. Instead, data analysis for them involves "doing something with the numbers," frequently by using methods derived from their prior instructional experiences with statistics in school (McGatha, Cobb, & McClain, 1998). An immediate concern at the beginning of the teaching experiment was therefore to ensure that the first mathematical practices established in the classroom actually involved the analysis of data. In the approach that we took, the teacher talked through the data creation process with the students in some detail. She then introduced the data the students were to analyze as resulting from this process. We conjectured that as a consequence of participating in such discussions, the data would have a history for the students so that the data would be grounded in the situation and would reflect particular purposes and interests.

Beyond this general instructional strategy, we developed two computer minitools for the students to use as integral aspects of the instructional sequence. These minitools offered students several ways of structuring data and were designed to fit with their reasoning at particular points in the envisioned learning trajectory while simultaneously serving as a means of supporting the reorganization of that reasoning. The students used these minitools in 27 of the 34 classroom sessions. Typically, they worked at computers in pairs to conduct their analyses and then the teacher organized a whole class discussion in which a computer projection system was used. I describe these two minitools below when presenting an analysis of two of the classroom mathematical practices that emerged during the experiment.

Emergence of the First Mathematical Practice

The first computer minitool was introduced during the fifth classroom session. This minitool was designed to provide students with a means of ordering, partitioning, and otherwise organizing sets of up to 40 data points in a relatively immediate way. When data was entered into the minitool, each individual data point was inscribed as a horizontal bar. The students could select the color of each bar to be either pink and green, thus enabling them to enter and compare two data sets. For example, Figure 1 shows data generated to compare how long two different brands of batteries last. Each bar shows a single case, the length of time that one of the batteries lasted. The students could sort the data by size and by color. In addition,



they could hide either data set, and could also use what they called the value tool to find the value of any data point by dragging a vertical red bar along the axis. Further, they could find the number of data points in any interval by using what they called the range tool.

During the first whole class discussion in which the minitool was used, many of the students did not initially appear to be analyzing data but instead described differences in two sets of numbers inscribed as colored bars. However, the teacher was able to initiate a shift in the discourse during this session such that the students began to speak about the bars as attributes of individual cases that had been measured. This shift continued during the second discussion conducted with the minitool when the students explained how they had analyzed the data shown in Figure 1. The green bars showed the data for a brand of battery called Always Ready and the pink bars the data for a brand called Tough Cell. The first student who gave an explanation directed the teacher to use the range tool to bound the ten highest values (see Figure 1).

Casey: And I was saying, see like there's seven green that last longer.

Teacher: OK, the greens are the Always Ready, so let's make sure we keep up with which set is which, OK.

Casey: OK, the Always Ready are more consistent with the seven right there, and then seven of the Tough ones are like further back, I just saying 'cause like seven out of ten of the greens were the longest, and like.....

Ken: Good point.

Janice: I understand.

Teacher: You understand? OK Janice, I'm not sure I do so could you say it for me?

Janice: She's saying that out of ten of the batteries that lasted the longest, seven of them are green, and that's the most number, so the Always Ready batteries are better because more of those batteries lasted longer.

Although Casey spoke of "the greens ," her comment that they lasted longer suggests that each bar signified how long one of the batteries lasted. Janice certainly understood Casey's explanation in these terms and in revoicing it, both stated an explicit conclusion ("the Always Ready batteries are better") and justified it by summarizing the results of Casey's analysis ("because more of those batteries lasted longer"). In doing so, she contributed to the gradual emergence of an initial practice of data analysis.

As the episode continued, the teacher asked Casey to explain why she had focused on the ten batteries that lasted the longest.

Casey: Alright, because there's ten of the Always Ready and there's ten of the Tough Cell, there's 20, and half of 20 is ten.

Teacher: And why would it be helpful for us to know about the top ten, why did you choose that, why did you choose ten instead of twelve?

Casey: Because I was trying to go with the half.

Significantly, Casey's justification for the way she organized the data and thus for the statistic she used did not make reference to the question at hand, that of comparing the two brands. It is also noteworthy that none of the students asked her for such a justification before the teacher intervened.

The next student to explain his reasoning, Brad, directed the teacher to place the value tool on 80.

Brad: See, there's still green ones [Always Ready] behind 80, but all of the Tough Cell is above 80. I would rather have a consistent battery that I know will get me over 80 hours than one that you just try to guess.

Teacher: Why were you picking 80?

Brad: Because most of the Tough Cell batteries are all over 80.

Possibly as a consequence of the questions the teacher had asked Casey, Brad justified the statistic he had used without prompting. Further, in doing so, he interpreted a feature of the inscription ("there's still green one's behind 80") as indicating a difference in the two brands of batteries that he considered significant. In this respect, his explanation involved a significant advance when compared with those that Casey and Janice had given.

Later in the discussion, Jennifer compared Casey's and Brad's analyses directly.

Jennifer: Even though seven of the ten longest lasting batteries are Always Ready ones, the two lowest are also Always Ready and if you were using those batteries for something important then you might end up with one of those bad batteries.

Significantly, Jennifer justified her preference for the statistic that Brad used by focusing on the pragmatic consequences of the two analyses. The obligation of justifying particular ways of organizing the data with respect to the practical issue at hand gradually became taken-as-shared during the remainder of the session. For example, towards the end of the discussion, one of the students observed:

Barry: The other thing is that I think you also need to know something that or whatever you're using them [the batteries] for.

Teacher: You bet.

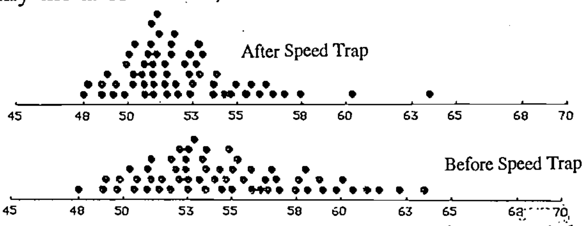
Barry: Like, if you're using them for something real important and you're only going to have like one or two batteries, then I think you need to go with the most constant thing. But if you're going like, "Oh well, I just have a lot of batteries here to use," then you need to have most of the highest.

In making this comment, Barry explicitly referred to the situations in which the qualities of the two brands being assessed by the two statistics would be relevant. It is also worth noting that, during the latter part of the discussion, four students volunteered that they had changed their judgments as a consequence of others' arguments. Their comments gave every indication that they experienced the discussion as an investigation in the course of which they developed insights into the issue at hand, that of the relative merits of the two brands of batteries. In this respect, the discussion had the spirit of a genuine data analysis.

The characteristics of data sets that emerged as significant in this discussion and in subsequent classroom sessions included the range and maximum and minimum values, the number of data points above or below a certain value or within a specified interval, and the median and its relation to the mean. However, the arguments that the students developed as they reasoned with the first minitool were generally additive rather than multiplicative in nature. In the first sample episode, for example, Casey, Janice, and the teacher jointly developed an argument that focused on how many of the ten batteries that lasted the longest were of each brand. In doing so, they compared two data sets that they had structured in part-whole terms. This argument can be contrasted with an alternative argument that focuses on the proportion of each data set that is among the ten highest values. An argument of this type would involve comparing two data sets that have been structured multiplicatively. The absence of such arguments even when the students compared data sets with unequal numbers of data points indicates that data sets were constituted in public classroom discourse as collections of data points rather than as distributions. The mathematical practice that emerged as the students used the first minitool can therefore be described as that of *exploring qualitative characteristics of collections of data points.*

Emergence of the Second Mathematical Practice

The students first used the second of the two computer minitools during the twenty-second session of the teaching experiment. This tool was designed to allow them to analyze one or two data sets with a total of up to 400 data points (see the figure below). The tool provide students with a variety of options for structuring data sets. The first, called "Create Your Own Groups", involved dragging vertical bars along the axis in order to partition the data set into groups of points. The remaining four options were: 1) partitioning the data into groups of a specified size (e.g., ten data points in each group), 2) partitioning the data into groups with a specified interval width, 3) partitioning the data into two equal groups, and 4) partitioning the data into four equal groups. The students could also hide the data, thereby leaving only the axes and the, vertical bars visible.



The practice of data analysis that emerged as the students used this minitool can be illustrated by focusing on episodes from two whole class discussions. In each of these discussions, two members of the project staff shared the teaching responsibilities. The first of these discussions focused on the question of whether the introduction of a police speed trap on a road with a 50 miles per hour speed limit had slowed down the traffic speed and thus reduced accidents. The data the students analyzed is shown above. To begin the discussion, one of the teachers asked Janice to read the report she had written of her analysis.

Janice: If you look at the graphs and look at them like hills, then for the before group the speeds are spread out and more than 55, and if you look at the after graph, then more people are bunched up close to the speed limit which means that the majority of the people slowed down close to the speed limit.

This was the first occasion in public classroom discourse where a student had described a data set in global, qualitative terms by alluding to its shape. One of the teachers legitimized Janice's interpretation and indicated that it was particularly valued by drawing the "hills" on the projected data. Both teachers then capitalized on Janice's contribution in the remainder of the discussion by treating other students' analyses as attempts to describe qualitative differences in the data sets quantitative terms. For example, Karen explained that she had organized the data sets by using a fixed interval width of five.

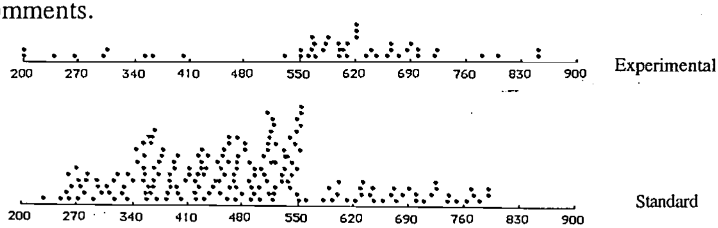
Karen: Like, on the first one [before the speed trap was introduced], most people are from 50 to 60, that's where most people were on the graph.

One of the teachers checked whether other students agreed with her interpretation. Karen then continued:

Karen: And then on the top one [after the speed trap was introduced], most people were between 50 and 55, because, um, lots of people slowed down...so like more people were between 50 and 55.

The same teacher then recast Karen's analysis as a way of characterizing the global shift of which Janice had spoken. As a consequence of this revoicing, it gradually became taken-as-shared that the intent of an analysis was to investigate global trends or patterns in data that were significant with respect to the issue under investigation.

In the second illustrative episode, which occurred a week later, the students had compared two treatment protocols for AIDS patients by analyzing the T cell counts of people who had received one of the two protocols. Their task was to assess whether a new experimental protocol in which 46 people had enrolled was more successful in raising T cell counts than a standard protocol in which 186 people had enrolled (see first figure below). The subsequent discussion focused on the reports that the students had written of their analyses. The inscription from the first report showed global differences in the way the two sets of data were distributed (see second figure below). The students judged this report to be adequate and made a number of comments.



OLD PROGRAM	NEW PROGRAM
200 - 525	200 - 525
130	9
525 - 850	525 - 850
56	37

Janice: I think it's an adequate way of showing the information because you can see where the ranges were and where the majority of the numbers were.

David: What do you mean by majority of the numbers?

Teacher 1: David doesn't know what you mean by the majority of the numbers.

Janice: Where the most of the numbers were.

Teacher 1: Sharon, can you help?

Sharon: What she's talking about, I think what she's saying, like when you say where the majority of the numbers were, where the point is, like you see where it goes up.

Teacher 1: I do see where it goes up (indicates the "hill" on the lower diagram).

Sharon: Yeah, right in there, that's where the majority of it is.

Teacher 1: OK, David.

David: The highest range of the numbers?

Sharon: Yes.

Teacher 1: The highest range?

Several Ss: No.

Teacher 1: Valarie

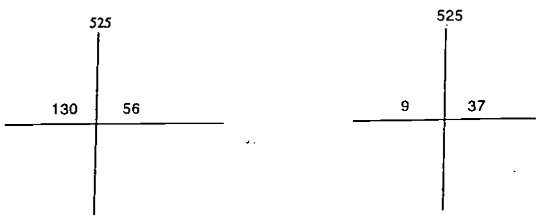
Valarie: Out of however many people were tested, that's where most of those people fitted in, in between that range.

Teacher 1: You mean this range here (points to lower and upper bounds of one of the "hills")?

Valarie: Yes.

It is evident from this exchange that when the students spoke about "the majority" or "most of the people", they were talking about data organized multiplicatively as qualitative proportions (Thompson, personal communication). Janice had previously introduced the term "the majority" during the discussion of the speed trap data when she had described hills in the data. A concern with global patterns in the way that data are distributed in fact assumes that the data are structured multiplicatively. In describing hills, Janice had reasoned about qualitative relative frequencies. However, this notion of the majority of the data did not become an explicit topic of conversation until the students analyzed data sets with unequal numbers of data points.

During the remainder of the discussion, the teachers attempted to guide the gradual refinement of the taken-as-shared notion of qualitative proportionality. For example, recall the third report described previously. One of the teachers clarified with the students during the subsequent exchange that the writers of the report had chosen the statistic of the number of patients with T cell counts above 525 because the majority of the data points in the old treatment were below this value and the majority in the new treatment were above it. One of the students then suggested drawing graphs to show the results of the analysis. These graphs are shown below. One of the teachers



then made the following argument that reflected an additive interpretation of the graphs.

Teacher 2: Could you just argue that this shows really convincingly that the old treatment was better, right, because there were 56 scores above 525, 56 people with T cell counts above 525, and here (points to the right graph) there's only 37 above, so the old one just had to be better, there's more people, I mean there's 19 more people in there so that's the better one surely.

The initial arguments the students made when rejecting this claim involved reasoning in terms of qualitative proportions. However, Ken made the following proposal:

Ken: I've got a suggestion. I don't know how to do it (inaudible). Is there a way to make 130 and 56 compare to the 9 and 37, I don't know how.

Teacher 2: I'll tell you, how many of you have studied percentages?

In the ensuing exchange, several students calculated the percentages of data points above the T cell count of 525 in each distribution. As the discussion continued, it appeared to be taken-as-shared that the results of these calculations provided a way of describing global differences in the two distributions in quantitative terms.

Interviews conducted with the students shortly after the teaching experiment was completed document that most could readily interpret graphs of unequal data sets organized into equal interval widths, an analogue of histograms, and into four equal groups, an analogue of box plots, in terms of global characteristics of distributions. The classroom mathematical practice that had emerged as they developed these competencies can be described as that of *exploring qualitative characteristics of distributions.* Participation in this practice involved reasoning about data multiplicatively while using the computer minitool to identify global patterns and to describe them in quantitative terms. Konold et al. (1996) argue that a focus on the rate of occurrence of some set of data values within a range of values is at the heart of what they term a statistical perspective. As participation in the second practice involved a concern for the proportion of data within various ranges of values, the students appeared to be well on the way towards developing this statistical perspective.

Change

My overall purpose in presenting the sample episodes has been to illustrate a theoretical approach that involves analyzing the mathematical learning of the classroom community. It is important to stress that the account I have given does not focus on the mathematical development of any particular student. Instead, I have been concerned with changes in public mathematical activity and discourse. It should also be apparent from the sample episodes that the use of tools and symbols is integral to both mathematical practices and the reasoning of the students who participate in them (cf. Dorfler, 1993; Kaput, 1991). In this regard, the theoretical viewpoint that I have illustrated is consistent with the basic Vygotskian insight that the tools students use profoundly influence both the process of mathematical development and its products, increasingly sophisticated ways of reasoning.

A second aspect of classroom mathematical practices that complements the emphasis on tool use is that of argumentation. I can best clarify this point by following Krummheuer (1995) and Yackel (1997) in using Toulmin's (1969) scheme of conclusion, data, warrant, and backing. In this scheme, Toulmin refers to the support one might give for a conclusion as data. In the case of the analysis of the battery data, for example, a student might merely point to the two data sets and state the conclusion that one of the brands of batteries is superior. In doing so, the student treats the conclusion as a self-evident consequence of the data. If questioned, the student would be obliged to give a warrant that explains why the data support the conclusion. For example, Casey justified her preference for the Always Ready batteries by explaining that she had focused on the ten batteries that lasted the longest and noted that seven of them were Always Ready batteries. In giving this warrant, Casey explained how she had structured and interpreted the data sets. In Toulmin's scheme, the warrant can be questioned and it is then necessary to give a backing that indicates why the warrant should be accepted as having authority. Casey was in fact challenged by the teacher, who asked her why she had chosen to focus on the ten batteries that lasted the longest. The backing that Casey gave, namely that ten was half of the data set of 20 points, was delegitimized as the episode progressed. Instead, it became taken-as-shared that a particular way of structuring data had to be justified by explaining why it was relevant to the question or issue at hand. The standards of argumentation inherent in these warrants and backings were relatively stable and also capture the structure of classroom discourse when the students used the second minitool and participated in the second mathematical practice. These standards are in fact quite general and apply to data analysis more broadly, indicating that the students were being inducted into what might be termed an authentic data analysis point of view.

It should be clear from the illustrative analysis that an approach of this type takes what are traditionally called issues of mathematical content seriously. For example, the contrast between the two mathematical practices is characterized, at least in part, by the distinction between additive and multiplicative reasoning about data. However, this approach also calls into question the metaphor of mathematics as *content*. The content metaphor entails the notion that mathematics is placed in the container of the curriculum, which then serves as the primary vehicle for making it accessible to students. In contrast, the approach I have illustrated characterizes what is traditionally called mathematical content in emergent terms. For example the mathematical idea of distribution was seen to emerge as the collective practices of the classroom community evolved. This theoretical orientation clearly involves a significant paradigm shift in how we think about both mathematics and the means by which we might support students' induction into its practices. However, this approach does have the merit of being compatible with the view of mathematics as a socially and culturally situated activity (cf: Bauersfeld, 1992; John-Steiner, 1995). I therefore suggest that it is an approach that is worth pursuing.

Diversity

Thus far, in focusing on collective practices I have emphasized the taken-as-shared ways of reasoning, arguing, and using tools that are established by a classroom community. It is therefore important to acknowledge that the students' participated in the two mathematical practices I have discussed in a variety of different ways. In order to account for this diversity in their reasoning, I and my colleagues find it essential to coordinate the strong social perspective we take on communal practices with a psychological perspective that brings the qualitative differences in their ways of participating to the fore. Further, in viewpoint that has emerged from my our work in classrooms, the relationship between the two perspectives is taken to be reflexive. This is an extremely strong relationship and does not merely mean that individual students' reasoning and the practices in which they participate are interdependent. Instead, it implies that one literally does not exist without the other (Mehan & Wood, 1975). Thus, when adopting a psychological perspective, one analyzes individual students' reasoning as they participate in the practices of the classroom community. Conversely, when adopting a social perspective, one focuses on communal practices that are continually generated by and do not exist apart from the activities of the participating individuals. The coordination at issue is therefore not that between individual students and the classroom community viewed as separate, sharply defined entities. Instead, the coordination is between two alternative ways of looking at and making sense of what is going on in classrooms. What, from one perspective, are seen as the norms and practices of a single classroom community is, from the other perspective, seen as the reasoning of a collection of individuals who mutually adapt to each others actions. Whitson (1997) emphasizes this point when he proposes that we think of ourselves as viewing human processes in the classroom, with the realization that these processes can be described in either social or psychological terms.

Elsewhere, I have described how this process of coordinating perspectives has grown out of and yet remains deeply rooted in our attempts to support students' mathematical development in classrooms (Cobb, in press). For my present purposes, it suffices to note that the theoretical viewpoint I have outlined has two major ethical implications. The first is that all students must have a way to participate in the mathematical practices of the classroom community. In a very real sense, students who cannot participate in these practices are no longer members of the classroom community from a mathematical point of view. This situation is highly detrimental given that to learn is to participate in and contribute to the evolution of communal practices. One of our primary concerns when conducting a teaching experiment is therefore to ensure that all students are "in the game." To this end, we adjust the classroom participation structure, classroom discourse, and instructional activities on the basis of ongoing observations of individual students' activity. In doing so, we are coordinating psychological and social perspectives and contend that an approach of this type is necessary if not sufficient when addressing concerns of equity at the microlevel of classroom action and interaction.

The second ethical implication is closely related to the first and concerns the view one takes of students whose ways of participating in particular classroom practices are less sophisticated than those of other students. In the theoretical orientation I have presented, the mathematical interpretations that students make are not viewed as individual cognitive characteristics, but as characteristics of their ways of participating in communal mathematical practices. In other words, the differences in the students' reasoning are seen to be socially situated and to reflect the history of their prior participation in particular practices. As a consequence, I and my colleagues do not take a cognitive deficit view of the students who made less sophisticated interpretations. Instead, our reflections on the teaching experiment have focused on the evolving mathematical practices that constituted the immediate social situation of the students' mathematical development, and on the nature of their participation in those practices. In framing the issue in this way, we have treated academic success and failure in the classroom as neither a property of individual students nor of the instruction they receive. Instead, we have cast it as a relation between individual students and the practices that they and the teacher co-construct in the course of their ongoing interactions. In the last analysis, the ethical dimension of this perspective on success and failure in school is perhaps the most important reason for adopting a viewpoint that brings the diversity of students' reasoning to the fore while simultaneously seeing that diversity is socially situated.

Equity

In this paper, I have focused on the issues of change and diversity as they relate to the concerns of instructional design at the classroom level. It is therefore important to acknowledge that the teaching experiment we conducted did not take place in a social vacuum. Instead, the classroom in which we worked was itself located within the sociopolitical setting of one particular school and community, and ultimately, within the activity system that constitutes schooling in the United States. The work of a number of participants in this conference makes us aware that schooling involves a number of taken-for-granted policies and practices that foster inequity due to race, gender, class, and economics status (Apple, 1995, Zevenbergen, 1996). In addition, issues of equity come to the fore even when we restrict our focus to the immediate concerns of instructional design. In the case of the seventh grade teaching experiment, a question that we had to address was that of why statistics should be taught in school. Our answer to this question focuses on the implications that increasing use of computers in society has for the discourse of public policy and thus for democratic participation and power (Cobb, G., 1997). It is already apparent that debates about public policy issues tend to involve reasoning with data. In this discourse, policy decisions are justified by presenting arguments based on the analysis of data. Inability to participate in this discourse results in de facto disenfranchisement that spawns alienation from and cynicism about the political process. Cast in these terms, statistical literacy that involves reasoning with data in relatively sophisticated ways bears directly on both equity and participatory democracy. The guiding image that emerges for instructional design is then that of students as increasingly substantial participants in the discourse of public policy. The important competencies for this participation are those of developing and critiquing data-based arguments.

This rationale for the importance of statistics in students' mathematics education led us to make an important design decision when planning the statistics teaching experiment. As I have noted, in most of the instructional activities, the students did not generate data themselves but instead analyzed data sets created by others. However, we were also aware that data does not speak for itself but is instead the product of a sequence of interpretive decisions and judgments (Latour, 1987; Roth, 1997). We therefore anticipated that the students would not initially be able to "look through" data to the situation from which they were generated. It was for this reason that we developed the approach of talking through the data creation process so that the data might have a history for the students. It is apparent from the sample episodes I have presented that this approach worked reasonably well. As the teaching experiment progressed, the students in fact assumed increasing responsibility for asking questions that related to the data creation process. Further, although most of the classroom discussions focused on analyses that the students had conducted, in the last few classroom sessions they developed arguments on the basis of graphs created by others. In the course of this transition, the students were developing the very competencies that make increasingly substantial participation in public policy discourse possible.

In terms of the broader literature on equity, the approach we have taken to statistics instruction is broadly compatible with Delpit's (1988) admonition that students should be explicitly taught what she calls the culture of power. It is also makes contact with Banks and Banks' (1995) equity pedagogy that aims to help students from diverse cultural backgrounds develop the ways of knowing needed to participate effectively within and maintain a just, democratic society. Thus, although we look for further inspiration from scholars whose work focuses on global, structural characteristics of schooling and society, we also contend that a concern for equity is critical when considering issues traditionally addressed by mathematics educators. In particular, it is essential that we scrutinize the overall goals we have for students' mathematics education and examine whether they can be justified in terms of participation in a democratic society. I will be more satisfied if our work in the area of statistics can serve as a paradigm case in this respect.

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THE PRODUCTION OF ARTEFACTS AS GOAL FOR SCHOOL MATHEMATICS?

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INTRODUCTION

There appears to be at least two expectations when an address such as this one is made. The first one is that there should be a reflection on work, within the area of interest of the community who frequent the conference, that the presenter has been involved with for a substantial number of years. If the presenter does not work within some demarcated area of the frequenting community then the expectation is that she/he should at least have done fairly substantive work related to the conference topic in order to broaden the community's knowledge base of the conference topic so that they can make decisions regarding the incorporation of such work in their further scientific endeavours. Neither of these two conditions are satisfied.

In this paper nothing substantially novel is proposed or relayed. The non-novel declaration arises from an observation that too frequently it is found within the mathematics education research literature that ideas, notions and "theories" that are being held up as novel have been propagated earlier. This assertion is not to be taken as the discarding of Wilder's (1978) fourth law of the simultaneous emergence of novel mathematical ideas. Rather, Wilder's (1981:164) advice to "not worry too much about being first..." is taken seriously. This article thus is more suggestive than proclamationist; more opinionated than conversionalist and, hopefully, more conversation-instigationist than rejection-suppresionist.

Notwithstanding the afore-mentioned declarations and disclaimers there are antecedent work that served as inspiration for what is being presented here. In particular, inspiration is taken from a long-standing interest in the notion of a practice and mathematics as a practice. (Julie, 1989). Broadly, this interest is not on the *how* of a practice but on the *what* in a practice. This what is qualified in the sense of "what are the things that are being made in the practice of the mathematical sciences?" In asking a somewhat similar question to tailors in West Africa, Lave (1989: 312) got the answer that apprentice tailors are learning to make "hats, children's underwear, short trousers, long trousers, Vai shirts, sports shirts, Muslim prayer gowns, women's dresses, and Higher Heights suits..." Lave further investigated the mentioning of this list of garments and concluded "that the list of garments in fact encoded complex, intertwined forms of order integral to the process of becoming a master tailor." (Lave, 1989: 312). Rather than concentrating on the insights that resulted from Lave's further investigation, we restrict ourselves to the things that are produced in a practice and the essence of this paper is to suggest that the production of artefacts from the practice of the mathematical sciences should receive more prominence in school mathematical education.

THREE CONCERNS ABOUT DEVELOPMENTS IN SCHOOL MATHEMATICAL EDUCATION

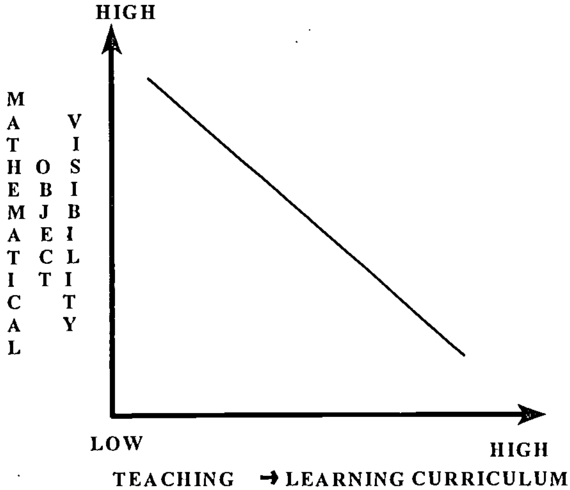
In recent years there has been a shift towards describing school mathematics curricula in terms of desirable behaviours learners should acquire. This shift is fairly natural given the advances made in understanding learners' mathematical behaviour; the insights gained from and theories developed about mathematical practices; an anxiety about desirable school mathematical education for at least the first part of the next millennium; the availability of technologies which can do much of the mathematics normally taught in schools; socio-political considerations and so forth. The different proposals for school mathematics curricula are strongly influenced by these advances. This is typified by statements such as

> One goal was to enable students to build a strong conceptual basis for effective problem solving, to be achieved by offering them [the students] a consistently more thoughtful curriculum

which Maher, Davis and Alston (1991: 220) assert about a school mathematics program they helped to establish. They continue by describing an incomplete set of desirable behaviours that students should exhibit in mathematics classrooms. These descriptions of desirable behaviours are similar to those found in reform documents for mathematical education in schools. This is to be expected since there are genuine attempts to move from, in Lave's (1989: 318) terminology, a "teaching curriculum" to a "learning curriculum." A feature of these documents is a strong tendency towards Freudenthal's prediction of

> the disappearance of mathematics as individual subject matter in education sometime around the year 2000, because it would by then be merged with integrated thematic strands. (Streefland, 1993: 5)

What is being describe in documents are processes such as generalisation, abstraction, conjecture, pattern recognition, and so forth. The explicit mentioning of mathematical objects is suppressed and processes are highlighted. This non-naming of explicit mathematical objects is also found in some learning materials where, for example, activity names are mentioned rather than the mathematical objects.[1] The movement from a teaching to a learning curriculum thus brings about a diminishing visibility of mathematical objects as indicated in the graphic illustration below.



This decrease in visibility of mathematical objects seems odd if, as social practice theory suggests, mathematical practice is just another practice. And as is further evident from the Lave quote above on the explicit mentioning of the making of an array of garments, practices are characterised by the products that they produce.

A second reason for proposing the highlighting mathematical artefacts in school mathematical education is more parochial. Within South Africa a mathematical sciences school curriculum based on an outcomes-based education philosophy is in the process of being introduced. A major motivating factor for the adoption of this theoretical and ideological educational 'paradigm' for schooling in South Africa is that schools should produce graduates who can more easily fit into some undefined future world of work[2]. This economic imperative is accompanied by a desire to develop and introduce a school mathematics program based on insights gained from research into and recent proclaimed theories regarding the learning of mathematics. The conclusion drawn from these insights is that there must be a shift from teaching to learning and a notion that "we should start from where the learners are." The adoption of a constructivist perspective is clearly evident but is reserved for learners. However, this very perspective is ignored regarding teachers making sense of their teaching in that teachers are expected to act and behave in a manner which the curriculum planners and their collaborators want (and in many instances instruct) them to act and behave. And as Davis (1996: 11) concludes in his insightful analysis of instructions given to teachers for the implementation of constructivist-inspired pedagogies

> The authority of the teacher no longer resides in a display of knowledge of mathematics but in the policing of pedagogic space so that the students, who are conceived of as autodidacts, can generate mathematical knowledge themselves under favourable conditions.

It is not an uncommon occurrence that when teachers are presented with activities which purportedly allow learners to construct their own knowledge that the question "but where is the mathematics?" is being raised in various guises. Without pathologising teachers from disadvantaged, marginalised and developing communities (which is my primary sphere of operation), it is so that their frame of reference is fairly firmly located within a discourse of mathematics impregnated by the existing products of mathematics--this is what they are used to, this is the environment within which they operate and this is where they are. Making the products of the mathematical sciences visible will contribute to the alleviation of the anxiety and conflict mathematics teachers are experiencing between, what one teacher called, "the real mathematics and the reform mathematics." A mistake that is being made is to assume that a clear specification of the products of the mathematical sciences will lead to a now-despised expository form of teaching. I contend that in suppressing the explicit expression of mathematical products, which the South African curriculum is veering towards, ignores the paradox which Cohen (1990: 343) frames as "If...it is implausible to expect students to understand math simply by being told, why is it any less implausible to expect teachers to learn a new math simply by being told?"

A final reason for advancing the introduction of the production of artefacts into school mathematical education is a socio-political one. It overlaps with the insight that learners from disadvantaged and working class communities receive a kind of mathematical education which parallels the treatment these social groupings receive in order to keep them in a position of subjugation. This is one of the conclusions Da Silva (1988: 78) reaches in a study of Brazilian working class schools and suggests that

> ...pedagogy should be concerned with providing working class children with the knowledge that has been socially produced, but whose appropriation has been the privilege of the dominant classes, constituting in fact one of the bases of their dominance [and that] exclusion from knowledge which is collective property...is central to the reproduction of relations dominance subordinancy.

In expressing the view that

> Our classrooms are the primary source of mathematical experiences...for our students, the experiential base from which they abstract their sense of "what the mathematical sciences are

all about",

Schoenfeld (1990: 13) is pointing to the importance of schools as primary sites for access to a practice such as the mathematical sciences. This is more so for students from disadvantaged backgrounds. As is well rehearsed, students from advantaged backgrounds have recourse to a wealth of resources from which to "abstract their sense of 'what the mathematical sciences are all about'. " Students from disadvantaged backgrounds do not have these resources readily and easily available. They are, in fact, highly dependent upon schools for the provision of an environment within which they can experience what the mathematical sciences are all about. The production of artefacts from the mathematical sciences is a significant component of this "all about." In denying learners from disadvantaged and marginalised backgrounds opportunities for engagement in the production of artefacts of the mathematical sciences there is the real danger that they might be betrayed and end up being in a situation of "happy idiocy" (Aronowitz, S and Giroux, H, 1988: 154).

The three reasons advanced above are central to my proposal that the production of mathematical artefacts should be a substantive part of school mathematical education.

WHAT IS AN ARTEFACT?

> **ar·te·fact**, *U.S.* **ar·ti·fact** (árti-fakt) n. **1.** An object produced or shaped by human workmanship; especially a simple tool, weapon, or ornament of archeological or historical interest. **2.** *Biology*. A structure or substance not normally present, but produced by some external agency or action; especially a structure seen in a microscopical specimen after fixation that is not present in the living tissue.[Latin *arte*, "by skill", ablative of *ars*, ART + *factum*, something made from past participle of *facere*, to make]
>
> Reader's Digest Universal Dictionary

In the following elaboration of an artefact the first sense of artefact above is adopted as point of departure. An artefact is any object constructed to serve a particular purpose. It is situated within a specific community of practitioners. This community plays a valuable role in terms of the validation of the artefact as an authentic object worthy for inclusion in the set of artefacts of the practice; performing the necessary quality control; of recruitment and induction of the next generation into the practice and the preservation of the repertoire of artefacts for later generations. The popular notion of an artefact is that it must be something concrete and that it had to be produced generations before its concrete form is dug up, analysed and assigned a particular place in the social life of a "primitive" society. This notion is now being discarded. The incorporation of, for example, current machines and instruments and other non-concrete productions of scientists as artefacts signifies a shedding of the archeological and historical-attachment notion of an artefact. In this regard Suchman and Trigg (1993:145) refer to "...representations of the world as texts, diagrams, formulas, models..." as artefacts and Latour (1986) includes facts as one of the things scientists produce.

A further issue coming into play is the difference between artefacts and tools. Polanyi (1964) provides a mechanism to distinguish between tool and artefact (where artefact is taken as object.) For him tool and artefact are separated by an internal and external (to the user) dimension with tool in the internal and artefact in the external domain. He states:

> While we rely on a tool or a probe, these are not handled as external objects...they remain on our side...forming part of ourselves, the operating persons. We pour ourselves into them and assimilate them as parts of our existence. We accept them existentially by dwelling in them. (Polanyi, 1964: 59)

Tools are thus essentially extensions of ourselves. By way of explaining Polanyi's existential acceptance we can consider the factorization of quadratic trinomials. For a particular purpose factorization may be of importance. In such a consideration the technique itself, its adequacy, its effectiveness and elegance for dealing with various kinds of quadratic trinomials, say, and issues surrounding its computerizability are the focus of study. Considering and dealing with factorization in this sense is taking factorization as an artefact, an object. For some other purpose factorization is a tool. This happens when the user utilizes the technique to realize a goal not necessarily within the domain of factorization. For example, in dealing with a phenomenon whose mathematical model is a quadratic form, factorization can be used as a transparent access to interrogate the mathematical model.

As an aside it is so that artefacts can be used for purposes they were not explicitly designed for. Skemp (1979: 21) in his discussion about mathematical concepts warns that we "classify cars as vehicles, time-savers, and perhaps status symbols, and use them in accordance with these functions. Fewer also see them as potential lethal objects."

Tool and artefact are intrinsically linked and an artefact created for one purpose can serve as a tool for another.

THE CENTRALITY of ARTEFACTS in a PRACTICE

A careful consideration of social practice theory (Lave and Wenger: 1991) and the social study of science (Latour: 1987, Woolgar: 1988) subtly draws attention to the centrality of the production of artefacts in a practice. Social practice theory draws attention to the production of artefacts deemed to have mainly low discursive saturation[3] and social study of science to those with mainly high discursive saturation. Notwithstanding this distinction, what stands out is that within any practice the central activity is the production of artefacts. It is around the production of artefacts that other issues such as the fashioning of identity, induction into the practice, the acquiring of the practice-specific visible and invisible behaviours and the necessary production skills are actualised.

As an illustration of the primacy of the production of artefacts is in the training and education of doctoral students in, say, mathematics education can be taken. In this case it is so that the candidate must produce a text, the artefact, in a particular area of interest. During the production of this text the candidate gets various forms of support (and non-support) from his/her supervisor or promoter. Also, dependent upon the candidate's permanency (e.g. being a lecturer, research assistant, etc) or not (a doctoral candidate not working in an institution with research as one of its major tasks) in the academic environment, her or his level of acceptance as a legitimate peripheral participant on her or his way to become a full participant in the area of practice is graduated. The candidate involves her-/himself in various tasks, in the same way as an "old-timer" in the practice would do, on the way to produce the artefact. For the execution of some of these tasks, the candidate would have had forms of formal education and training. For others, the candidate arranges for and makes requests for contributions to expand her/his knowledgeable skill in order to advance the production of her/his artefact. These arrangements and requests are made to her supervisor/promoter, to other knowledgeable "old-timers" in the practice or to other candidates whom she/he knows are capable of providing her/him with guidance. In still other cases the candidate just enskills her-/himself so that she/he can proceed with the artefact production activity. Along the way the candidate might produce and make available for the scrutiny of the community-of-interest fairly self-contained units (articles for publication, conference presentations) of the artefact. This wider circulation of work-in-progress serves further the enskilling and on-the-way-to-become-competent in producing a community-of-interest-approved artefact. All along the way the candidate has a fair idea of what the final product should look like. She/he has seen quite a number of them and in most instances her/his supervisor/promoter would have provided her with some exemplary pieces of varying degrees of quality. There might also be cases (especially in university settings) where she/he her-/himself would have been either a supervisor/promoter or would have been an adjudicator/examiner of an artefact's suitability for inclusion in the collection of artefacts of the practice.

In addition to the above knowledgeable skill acquisition and intermediate production and "testing" activities, there are tacit knowledge which are "picked up" along the way by the candidate. Dilemma-resolution, problem-solving, and creative thinking skills are applied all along the way and is further advanced and enhanced through mere application. Large chunks of time are spent on reading, interrogating, evaluating and communicating about existing artefacts (theses and dissertations, articles) in the area of interest. Ideas related to presentation format, argument presentation, and so forth are gathered from these existing pieces. In some instances these ideas are imitated in the own-produced artefact. Once the candidate has a completed artefact to her/his and supervisor/promoter's satisfaction it is submitted to other members of the practice for evaluation. The number of members involved in this evaluation varies but it is generally accepted that the majority of members should not have been directly involved in the production process. These experts will give judgement on the artefact's suitability for inclusion and they also indicate the degree of the quality of the artefact. Thus they will make pronouncements such as "This dissertation should be published" or "Chapter X and Chapter Y offer some innovative perspective and their further development can lead to an publishable article" or "A pass with distinction (cum laude) is recommended." Not all the submitted artefacts are found acceptable upon first submission. The assessor experts will reject, partially reject and offer suggestions for repair where rescue is possible.

The candidate will finally get her/his approval as a certified member of the practice upon graduation. This public ceremony in a sense bestows on the candidate the right of practice. The fact, as is tradition in some institutions, that the candidate kneels before the chief custodian of the academic practices of the institution, who in her/his turn "caps" the candidate is the public bestowal and acceptance of full membership as practitioner, protector and recruiter for the sustaining of the practice.

This rather extended exposition is to indicate the centrality and primacy of the production of artefacts in a practice. It is thus suggested that the production of mathematical artefacts should also be central to school mathematical education.

SOME ARTEFACTS of the MATHEMATICAL SCIENCES

Before embarking on a delineation of artefacts some comments on what is understood by school mathematics in the context of this paper is necessary. These comments are necessary since there is too easy a slippage to view school mathematics as pure mathematics. Wittmann (1995: 359) draws a distinction between "specialized mathematics as found in university departments of mathematics" and

> mathematical work in the broadest sense [which] includes mathematics developed and used in science, engineering, economics, computer science, statistics, industry, commerce, craft, art, daily life, and so forth according to the customs and requirements specific to these contexts

The specialised mathematics Wittmann is referring to is what is normally termed pure mathematics and in recent times strong critiques have been developed against "mathematics used in...craft, art, daily life..." (See for example, Vithal and Skovsmose (1997) for a critique of ethnomathematics as an exemplar of "mathematics developed and used in craft, art.") We have come to use the term 'Mathematical Sciences' which include pure mathematics (specialized mathematics), applied mathematics (mathematics used and developed in and science, engineering, economics, industry, commerce, craft and art), statistics and the non-hardware aspects of computer science (computer science). For the identification of artefacts this notion of the mathematical sciences is used. Obviously such an identification will be flavoured by one's personal lens and the one I use is that of "the artefacts produced and used by the producers and users of the Mathematical Sciences." These artefacts are readily available, identifiable and open for scrutiny. The communities of practitioners are easily identified and their ways of working are more and more being made public. In searching through these practices seven artefacts were identified. These artefacts contain some fairly well-known ones and some that are not so well-known.

Algorithms:

These are generally step-by-step procedures and has moved beyond the earlier arithmetic notions such as Hogben's (1969: 243) "...rules for calculation." or Crump's (1990: 4) "...more or less standard procedures for applying [the numerical vocabulary of one's own language] in culturally defined contexts." That the construction of algorithms should receive explicit attention is especially important in this age where mathematical algorithms are embedded in technology and thus invisible. That they can be implemented faultily has been strikingly brought to the fore with the well known millennium bug. This is but one algorithm that was wrongly implemented. Dershowitz and Reingold (1997: xviii - xix) draw attention to others.

By asserting that "Mathematicians...create algorithms, they do not merely memorize algorithms and recall them as needed.", Davis and Maher (1990: 77) draw attention to the production of algorithms as an artefact in the Mathematical Sciences. Within mathematics education research there is evidence that learners can create algorithms. What is disconcerting is that this is not made explicit to learners and little is seen on how the learner-constructed algorithms are taken through processes of revision and refinement to give learners a way-in to what, with respect to the construction of calendrical algorithms, Derhowitz and Reingold (1997: xvii) calls "...the number of revisions, and, ...polishing what [algorithms for calendrical calculations] we have."

Models:

Within the Mathematical Sciences symbolic, compact, idealised mathematical models of physical, mathematical, economic and other social phenomena are produced and used. They are normally dealt with under the rubric of the 'applications of and modelling in mathematics.' Some strides have been made in securing a place for the applications of and the modelling in mathematics beyond the mere sugar-coating of pure mathematics as end-of-section exercises. But a similar phenomenon is rearing its head where contextual problems have become mere sugar-coatings for start-of-section exercises. In this case the starting point is pure mathematics and then a search is made for appropriate "real" and "not-so-real" contexts which act as concrete carriers of conceptual ideas. In many instances the contexts are not returned to in order to assess the appropriateness of the learner-devised models. Rather than following through the complete traditional modelling cycle, the cycle is stopped at the formulation of a mathematical formula since the end goal is the pure mathematical ideal.

Included in the construction of mathematical models is the construction of critiques of models. This issue is of particular import since there are more and more installations of mathematics to organise the affairs of society. And as Davis and Hersh (1986: xv) warns "*The social and physical world are being mathematized at an increasing rate...We'd better watch it, because too much of it may not be good for us.*" (Italics in original). Based on this development, arguments are emerging that all citizens should have the skill of critique of mathematical models. I would not go that far since the mathematical models regulating societal affairs are complex. Coming to grips with this complexity might require years of study and specialization which for practical reasons is beyond realization for all learners. This does not mean that "expertocracy" is advocated. Rather, all citizens should know that mathematizations are constantly being installed to regulate and justify societal affairs. They should be aware of the practices leading to these installations. But more so, they should have knowledge of whom they can consult to analyse these models for them so that they can take appropriate action regarding the desirability or not of the mathematical installations. As a case in point the Financial Programming Model the International Monetary Fund uses to determine the amount of financial aid it would give to a country can be considered. This model makes, amongst other things, no provision for the "increase [in] the supply of goods so that people [can] have something to spend their money on." Hanlon (1996: 28). If, as is the case in most developing countries, compulsory education will only be up to grade 9, then the likelihood is slim that the majority of future citizens will have the mathematical sophistication and expertise to disect a mathematical model in such detail to reveal its warranted and unwarranted intentions. To reiterate, however, provision should be made within school mathematical education to build awarenesses of the mathematical installations in societal affairs and the avenues available to the citizenry to raise questions about their feasibility.

Equivalent Representational Forms:

Within the Mathematical Sciences functionality of objects is an important issue. Producing different representational forms of a mathematical artefact to serve particular purposes best are constructions built within the Mathematical Sciences.

Notation:

That preferred notational forms are produced in the Mathematical Sciences is self-evident.

Systematic Accounts:

Systematic accounts are extended pieces of work dealing with particular areas and/or themes in the Mathematical Sciences. They are generally organised around some central ideas the producer identifies as the major project of her/his exposition. Products of this nature vary in their degree of sophistication dependent upon the audience the producer has in mind. In his discussion on assessment de Lange (1993: 11) expresses the desire that at school level learners should be afforded the opportunity to produce systematic accounts. His motivation for this suggestion is that "the real professional mathematician is never judged by a test he has to pass but by the papers he writes. It remains unclear why we cannot use this tool at a lower level."[4]

Proofs:

In his motivation for a thinking curriculum which fosters creativity, Davis (1984: 21) views the making of "original proofs of geometric theorems" as a central goal. The school curriculum has now developed to such an extent that "geometric theorems" can be replaced by a more general "mathematical theorems."

Extra-mathematical Accounts:

These are products where objects of the Mathematical Sciences are used to give expression to issues outside the "formal" of the Mathematical Sciences. In the consideration of the issues, the constructs of the Mathematical Sciences are used as "objects to think and represent with" and not as "the objects." The tools and machinery used in the construction of these artefacts are not exclusively internal to mathematics. For the consideration of such issues, the tools and machinery of other disciplines and/or fields of study are used.

Philosophy, history, sociology, anthropology and political science are some of the more popular disciplines providing tools for the construction of artefacts characterised as extra-mathematical accounts. Some of the more known available artefacts in this category are: *Alice in Wonderland; Flatland; Escher, Gödel and Bach* and some articles appearing in the *Humanistic Mathematics Network Journal.*

CONSTRUCTIONS, PRODUCTIONS and ARTEFACTS

The central issue in this paper is that in school mathematical education there should be an explicitness about the making of things that are being made in the mathematical sciences. This "making of things" is not something new and has been propagated by various schools of mathematics education research and development. Two well-known schools are the Freudenthal Institute and the Epistemology &Learning Group: Learning & Common Sense Section of the Massachussets Institute of Technology.

The Epistemology &Learning Group: Learning & Common Sense Section has been propagating the idea of "to *do* mathematics rather than merely learning *about* it." (Papert 1972: 249) for more than twenty five years. The "to do mathematics" was later developed into the theory of constructionism which lays strong emphasis on the construction of a thing that is visible, concrete and manipulatable. The thing must be something which is personally meaningful to and hence personally owned by the learner. As Resnick (1996: 2) states "constructionism adds [to constructivism] the idea that people construct new knowledge with particular effectiveness when they are engaged in constructing personally meaningful products." The products that are constructed emanate from the interest world of the constructer although projects may be suggested by someone else as Harel (1991: 358) instructed her nine-year olds to "design a program that teaches a younger child about fractions." However, there appears to be lurking in the project on constructionism something akin to "they must be taught something somewhere else, then we come in and let them do the sort of things we get excited about." In Harel's project the learners were taught fractions in their regular classes and their project design was done in their computer lessons. Resnick (1990: 5) refers to "In many cases, the students have previously "learned" [the] concepts in the classroom." [5] In proposing that consideration be given to the production of artefacts from the mathematical sciences, it is not only about concepts but the larger structures of which concepts are but parts. Artefact production is more in line with Harel's project but with a quest to combine the "teaching something somewhere else" with the mathematical artefact production so that there is a seamless movement between becoming knowledgeable and skilful with tools, artefact production and understanding.

The term, "free productions" (Streefland, 1990), was coined by developmental researchers at the Freudenthal Institute in the mid-eighties. Productions, in the

Freudenthaller's sense, differ from constructions in that "...productions [result] from reflection on construction." (Streefland, 1990: 33). These "free productions" are also constructed more or less at the completion of a substantial part of mathematics and are thus revisitational in character with the intention, as Freudenthal has asserted, to allow "children to think about their own learning process" (Goffree, 1993:41). Notwithstanding Freudenthal's prediction that school mathematical education will disappear in its current form around the year 2000, the Freudenthallers are more insistent than the Papertian school that the productions must be mathematical artefacts. What can, however, be said of the Freudenthalian school is that "certain perceived elements of the universe of everyday experience are selected and brought together in such a way that the non-mathematical is (to be) backgrounded and mathematics is (to be) prioritised..." (Davis, 1994: 9). Learners are not informed of this prioritisation but they accept it because of all the indicators at the particular time and place signifying school mathematics. One sometimes wonder if an activity such as "Into how many groups of 7 can 81 be divided.?" will not elicit the same kind of pupil constructions as

> One pot of coffee holds seven cups of coffee, every parent gets one cup. How many pots of coffee are to be brewed for the 81 parents?[6]
>
> (Gravemeijer, 1990: 15)

Barring issues of the above nature and the way the environment within which learners find themselves structure the way they perceive tasks (Säljö and Wyndham, 1993) the notion of the production of artefacts has a lot in common with the "construction and production" notions of the Freudenthallers.

CONCLUSION

One thrust of this paper is that a variety of artefacts are produced in the mathematical sciences. Some, such as algorithms and proofs, have been highlighted and privileged for decades. Even the current reform movements tend to concentrate on a privileged set of artefacts with a celebration of the diversity of methods learners construct. This proposal calls for an expansion of the diversity of artefacts learners are allowed to construct in school mathematical education. It remains a question whether such diversification would address the three concerns expressed earlier. The economic notion of diversification, however, stresses that investments must be spread in order to minimise losses. In some strange way school mathematics is used to maximise loss through its being used as a device for social stratification. The broadening of the universe of allowable artefacts to be constructed in school mathematical education might just point in a direction to seek answers to minimise the losses in this sense. Whether it is a fruitful direction the conversation-instigationist hope expressed above will teach us.

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NOTES

1. As an example, a unit named "Wet and Dry Numbers" (Encyclopaedia Britannica Educational Corporation, 1994) deals with integers and recently a teacher asked "What are wet and dry numbers?" because she thought these were a set of numbers she was not aware of. The issue is not whether or not context-related titles should be used to name mathematical units. It is about the non-mathematical signification that such labelling carries.

2. The notion that school mathematics or any other school subject can contribute towards economic development is questionable. For an argument in this regard see Jansen (1997).

3. Dowling (1994) introduced the construct discursive saturation. His demarcation criterion is a linguistic one. Thus artefacts that can be transmitted, interrogated and mastery demonstrated only in linguistic form are deemed of high discursive saturation and those that are "distributed" in a form other than the linguistic are

deemed of low discursive saturation. The use of the adjectives "high" and "low" are unfortunate since it attaches a status value to artefacts in different practices.

4. The paper from which this quotation is drawn is a draft version of a chapter for a book (T A Romberg (ed), *Reform in School Mathematics and Authentic Assessment*. SUNY Press, Albany NY). No comment is made about the use of the male gender pronoun since it is accepted that in the final form this male gendering would have been attended to. However, it is noted that in our moments of free constructions, there should be vigilance about the perpetuation of male dominance construction.

5. Papert would argue that this need not be the case and that it is possible to provide learning activities for "constructions" of which learners had no prior "taught" experiences. In fact, in demonstrating his "*thingness principle*: object before operation" with the Logo primitive RANDOM, he demonstrates that "RANDOM can be presented...for its connections with mathematical ideas." (Papert, 1996: 99). The issue at stake in this paper is: Whether opportunities are provided to students to construct RANDOM, an algorithm, and in this production process also make "connections with mathematical ideas." Papert admits that "RANDOM is a mathematical object" but this object (artefact) is not constructed by the learner. I see here a move away from the earlier Papert which draws a parallel between the doing of mathematics in schools and the "creative work in language and plastic arts." (Papert 1972: 250). Surely in this "creative work in language and plastic arts." what learners are producing are the artefacts of that practice. Similarly RANDOM is the artefact of the practice. Papert would obviously argue that his school mathematics education, which is firmly rooted in the availability of computers and other technologies, is very different because in a strict sense the constructionist project is not about school mathematics as it is normally perceived. Rather it is about an "alternative mathematics education (ame)...as *different* as possible from SME (School Mathematics Education) in identifiable, theoretically possible ways" where use come before understanding and "new media open the door to new content." (Papert, 1996: 95 - 99).

6. This task is given as one which give learners access to develop the division algorithm. Although there are various ways in which this task can be solved, it is probably the learners' experience that "in this class we get our mathematics problems this way" and thus they produce these mathematics-like answers. There is evidence that if division problems stated in the form as given by the prior wonderment question, that learners (and fairly well-qualified adults) use a similar anchoring strategy as given by Gravemeijer to deal with the task. (See Julie (1990)).

A MOMENT IN THE ZOOM OF A LENS: TOWARDS A DISCURSIVE PSYCHOLOGY OF MATHEMATICS TEACHING AND LEARNING

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Abstract *Psychology has undergone considerable changes in the last decade or so, in line with developments in cultural studies, feminist research, postmodernism etc. In mathematics education research one can see the evidence of such developments, albeit as yet in a limited way. In publications over the last few years I have argued the case for a cultural psychology and for a move away from constructivism. In this paper I attempt to outline a research programme for mathematics education which is firmly based in cultural, discursive psychology. I discuss theoretical issues, refer to some of the important literature, and present some research in aspects of the programme. (1)*

In the Beginning was the Word

Mathematics education as a theoretical field has its roots in mathematics classrooms, from nursery age to those in University. It draws on a range of theoretical resources in developing its own body of theory to account for what goes on in those settings in relation to teaching and learning. The concern of workers in the field is to find a language with which to describe the process of the acquisition of mathematics, and through which to draw inferences for what teachers might do to bring about that acquisition by as many students as possible. That language has to account for a great deal, much of it tacit in teaching, such as: a notion of development as being towards something; the goals of education in general and of mathematics education in particular; the relationship between teaching and learning; and the particularities of the nature of mathematics. That language has to be informed by empirical studies, and has therefore to incorporate a process for its own continuing elaboration. At the same time it needs to take account of relations of power, of voice and of silence of any theory, of any account of teaching and learning, of any set of goals for education, and for any notion, usually implicit, of development.

Traditional psychology, for all that its field of study is human behaviour, has offered little that can help to improve society.

> modern psychology has been incapable of making serious contributions to Third World (*sic*) development... it is important to point out that mainstream psychology has also failed to make significant contributions to national development and the lives of the poorest sectors in Western societies. (Harré, 1995, p. 54)

> In the process of individualizing its view of students, it (mathematics education) has lost any serious sense of the social structures and the race,

gender and class relations that form these individuals. Furthermore, it is then unable to situate areas such as mathematics education in a wider, social context that includes larger programs for democratic education and a more democratic society. (Apple, 1995, p. 331)

I believe they are right to say that psychology cannot provide such a language, at least psychology as understood to be the study of learning as the individual's cognitive reorganisation, albeit *caused by* social, physical or textual factors (von Glasersfeld, 1994, p. 6) through equilibration. I want to suggest, though, that psychology can be seen as a moment in socio-cultural studies, as a particular focusing of a lens, as a gaze which is as much aware of what is not being looked at, as of what is. This is an adaptation of Rogoff's planes of analysis, into a dynamic metaphor in which one might envisage a researcher choosing what to focus on in research through zooming in and out in a classroom, as with a video or still camera, and selecting a place to stop. Rather than seeing social factors as *causative* of learning, they can be seen as *constitutive* (Smith, 1993). As such, I will argue that psychology from this perspective can respond to Apple's critique. A discursive, cultural psychology locates its interpretation of the individual at the intersection of overlapping language games in which the person has developed and thus is necessarily rooted in the study of cultures and histories. Draw back in the zoom, and the researcher looks at education in a particular society, at whole schools, or whole classrooms; zoom back in and one focuses on some children, or some interactions. The point is that research must find a way to take account of the other elements which come into focus throughout the zoom, wherever one chooses to stop.

In this paper I address the issue of what is the terrain of such a psychology, that is, what counts as an appropriate language, for mathematics education. I will describe the move in psychology over the last decade or so (Cole, 1996; Harré, 1995) to one which is fully cultural and focused on the way in which consciousness is constituted through discourse. That move can be seen as part of the reaction in sociology and philosophy to the nineteenth century challenge by Durkheim and Marx to the image of the individual as the source of sense-making and as the autonomous builder of her/his own subjectivity. It is also a response to Wittgenstein's later work on language, and to the anti-essentialism of poststructuralism. I will attempt to engage with the elements of a theory for mathematics teaching and learning which I mentioned above: what might be understood as teleology/development; the connections between teaching and learning; the process of acquisition; the particularities of mathematics; the inevitable coercion and denial of voice of any theory; and the role of empirical study. I am choosing to undertake an impossibly large agenda for one talk as I wish to present as comprehensive a map as possible. It is more of a recipe for a life's work, and in that it is a recounting of developments in my own learning and research and a setting out of a programme upon which I am engaged with colleagues and with some of my students, it is self-consciously broad and over-ambitious. I will, however, endeavour to sketch the main outlines of this programme and give some illustrations of relevant research, my own or that of others, where appropriate. These illustrations will inevitably be briefly described, due to limitations on time and space.

Vygotsky, whose work has become better known in the mathematics education community in recent years (e.g. Bartolini Bussi, 1991; Lerman, 1992; Boero et al, 1995) is a major figure in the development of cultural and discursive psychology (Cole, 1996, p. 107; Harré & Gillett, 1994, p. viii). Feminist research, in particular, has invited us, researchers and writers, to be forthcoming about our biographies and to own up to where we are located in our work. I happily confess that I became fascinated and excited by Vygotsky's ideas when I first came across them some eight years ago and immediately found a strong resonance with the way in which I perceive myself to be culturally and socially situated (Lerman, forthcoming, a).

The theoretical resources to which I referred above come from outside mathematics education, and I will endeavour to signal from where I draw inspiration. I will also attempt to do justice to those ideas in spite of the limitations of time and space, an essential task for researchers. Whilst recontextualisation is inevitable (Bernstein, 1996) it is incumbent on researchers to take theories seriously. Elsewhere (Lerman, 1996) I have argued that the consequences of superficial readings can result in incompatible theories being conflated.

Development and Teaching/Learning

The question of what is consciousness and how it develops was the subject of Vygotsky's (1924/1979) first major public paper. In his subsequent writing he argued for development to be seen, from the first moments, as brought about by communication:

> instruction and development do not meet for the first time at school age; rather, they are in fact connected with each other from the very first day of a child's life. (Vygotsky, 1956, in Wertsch, 1985, p. 71)

Bruner's studies of babies illustrate vividly Vygotsky's descriptions of voluntary attention as learned, and of meaning as initially intersubjective, mediated by the adult (Harré (personal communication)):

> Bruner was interested in the very early moment of the development of intentionality and causality. Here's an attractive object. And an infant, a very small infant, reaches for it. Causal relation. Object is there...infant reaches out like that. What happens? Mother gives it to him. At that point the relationship between the reaching and the grasping changes...the mother has introduced into this game, baby wants, baby intends. Baby just acted causally...didn't get, so mother says baby wants. Next step, so she's completing the action, she is giving him the interpretation of his action. Next step, baby reaches forward, and compresses the air in his little chest and goes, 'Ah.' So, mother gives it to him. We've now turned the intentionality into a babble, the first primitive verbal utterance. Causality turning into intentionality. And the mediating role is the caretaker, who has done the defining, and done the supplementing. (2)

This account of how the baby becomes conscious exemplifies Leont'ev's (1981, p. 57) description of how the mental plane is constituted in the process of internalisation. In this sense, learning is inseparable from 'teaching'. That teaching may not be deliberate and intentional, as it is in school or in parental instruction. In everyday situations of the child's life s/he learns how to be, in gendered, ethnic, class and other historical, socio-cultural identities. Learning how to be, or to become, is motivated by desire, goals and needs, to be accepted, to emulate a desired person, or to join a group. Lave and Wenger's (1991) account of learning in workplace situations presents that theory of learning as becoming, and Lave (1996) and Winbourne and Watson (1998) discuss the notion in relation to the classroom.

Concomitant with this view of learning and development is an interpretation of concepts and knowledge which is neither of the two choices rejected by Piaget, empiricism and innatism (1970/1972), nor the individualistic, constructivist world which he proposed in their place. It is one where meanings historically precede the individual, which the individual internalises, and through which the individual perceives the world.

> How do I know that this colour is red? - It would be an answer to say: 'I have learnt English'. (Wittgenstein, 1958, 381)
>
> I did not get my picture of the world by satisfying myself of its correctness; nor do I have it because I am satisfied of its correctness. No: it is the inherited background against which I distinguish between true and false. (Wittgenstein, 1969, section 94)

Those meanings are not static and singular. As they are experienced and internalised from a range of discursive situations they may well carry the meanings, integrally imbued with affect, of those situations (Evans, 1993).

Vygotsky emphasised the presentation of scientific concepts to students and opposed the idea that they need to rediscover the development of mankind for themselves (1988). This formulation is taken to be very close to a transmission style of teaching by some. However, Vygotsky was opposed to merely telling learners. He was centrally concerned with the mediation of cultural tools and of metacognitive tools.

> scientific concepts... just start their development, rather than finish it, at a moment when the child learns the new term or word-meaning denoting the new concept (Vygotsky, 1988, p. 159).

Two aspects of a discursive psychological approach to development will be reported on here, first that of Vygotsky's argument for teaching from the general to the particular and, second, cultural mediation in learning, through Lave's work on learning as social practice.

i Vygotsky drew on Marx's notion of ascent from the abstract to the concrete in his theory of the acquisition of scientific concepts, and one development of this perspective has been towards the teaching of general principles to students, with particular questions being seen as instances in which the general principles need to be identified and applied (Galperin, 1969; Talyzina, 1981). This runs contrary to the usual tendency to work inductively from a range of everyday examples to general principles. We have been working on utilising this approach in mathematics classrooms. In a study of students in three inner city schools in the UK, across all levels of achievement according to national tests, principles for calculations of rates of processes were taught (Day, forthcoming). Students were shown a generalised model for the conception of the values involved and the relations between them, in the form of a visual structure for combining given data, such as time and rate, and using it to calculate the quantity. The teaching was orientated towards success in using the model for a range of problems of increasing complexity, both set by the teacher and invented by the students. That success was measured by a dynamic assessment procedure based on the amount and type of assistance they required. Analysis of quantitative data and of videos is currently taking place. Interim results show a level of achievement and change in attitude, across the ability range, which has surprised the classroom teachers and ourselves, and we have certainly found that the results support the argument for a 'theoretical learning approach' (Karpov & Haywood, 1998).

ii The metaphor of students as passive recipients of a body of knowledge is terribly limited: so too is the metaphor of students as all-powerful constructors of their own knowledge, and indeed of their own identities. Lave's (e.g. 1996) focus on the shaping of identity in social practice, extended by an analysis which takes account of the differences between schooling and the practices which she has studied (Lerman, forthcoming, c), emphasises the centrality of the social relationships constituted and negotiated during classroom learning. Lave talks of learning as "an aspect of participation in socially situated practices" (*ibid.*, p. 150). Provided we do not expect those practices to be those of the mathematics teacher, or of the mathematician but, instead, of the practices of the classroom culture, her description of learning can be very fruitful, as is shown by Winbourne (1997), for example. In that study he described the demonstration of creativity and expertise in the use of graphical calculators amongst a class of 13 year-old girls, and the subsequent display of mastery and learning through participation in mathematical activities, which he has since described in terms of local communities of practice (Winbourne & Watson, 1998).

To summarise, Vygotsky outlined a method for accounting for development which is rooted in an historical, sociocultural notion of mind. This method brings together teaching and learning. In terms of a *telos*, or direction for development, Wertsch (in Cole & Wertsch, 1996) argues that Vygotsky offered, although not explicitly, a somewhat confused account of a *telos* of abstract rationality, an enlightenment principle, and one of a 'harmony of imagination', a kind of mythical thinking. The former is evidenced in his and Luria's studies in Uzbekistan (Luria, 1976), and the latter in Vygotsky's *The Psychology of Art*. These two *teloi* co-exist in dialectic with each other, much like thinking and speech. Lave offers an interpretation which is, inevitably, more recent and partial, and hence more appropriate to discursive psychology than Vygotsky's, although it has clearly grown from his ideas, that of the desire of the individual to 'become'. Again I mean here such desires as: to please parents; to emulate a sibling; to be a member of a desired group; to fulfil goals, etc.

The Process of Acquisition - The ZPD

For psychologists of education, as distinct from sociologists, cultural theorists and others who also study the situations in which meanings are manifest, the concern is with the process of acquisition of meanings. Vygotsky introduced the zone of proximal development in a lecture given in March 1933 (Van de Veer & Valsiner, 1991, p. 329), although he pointed out that the idea was not originally his own. He died only fifteen months later and clearly had not been able fully to elaborate his thoughts on the zpd. Along with Newman & Holzman (1993) I take it to be the explanatory framework for learning as a whole, both in intentional settings, such as schooling, and in informal settings; in other words all socio-cultural milieus. It recognises the fundamental asymmetry of the teacher-student(s) relationship, an asymmetry often denied or underplayed by more individualistic approaches. It provides the framework, in the form of a symbolic space (Meira & Lerman, forthcoming); for the realisation of Vygotsky's central principle of development:

> In our conception, the true direction of the development of thinking is not from the individual to the socialised, but from the social to the individual. (Vygotsky, 1988, p. 32)

Such a definition opens a space for a unit of analysis of consciousness that incorporates affect and cognition:

> When we approach the problem of the interrelation between thought and language and other aspects of mind, the first question that arises is that of intellect and affect. Their separation as subjects of study is a major weakness of traditional psychology... (Vygotsky, 1988, p. 10)

The interpretation of Vygotsky's zpd upon which we draw (Meira & Lerman, forthcoming) is closer to that of Davydov (1988, 1990) and is less 'internalist' than Vygotsky's own version might appear. He wrote that the instruction process creates the zpd, but he also wrote that instruction is "fruitless if it occurs outside, below, or above the zpd" (1987, p. 213). He probably intended to emphasise jointly the roles of the learning activity and the learning potential of the child. The zpd is often described as a kind of force field which the child carries around, whose dimensions must be determined by the teacher so that activities offered are within the child's range. According to Davydov and followers, on the contrary the zpd is created in the learning activity, which is a product of the task, the texts, the previous networks of experiences of the participants, the power relationships in the classroom, etc. They speak of the ideas offered by one student potentially pulling other students into their zpd (Lerman, 1994a). The zpd is the classroom's, not the child's. In another sense the zpd is the researcher's: it is the tool for analysis of the learning interactions in the classroom (and elsewhere).

Teachers use a variety of strategies to try to create a zpd, including reminders of past lessons, events and language (Edwards & Mercer, 1987), references to outside school objects or meanings, etc. Much can be learned by the researcher, focusing on these strategies of the teacher. Similarly, pairs of students can create their own zone of proximal development if they are motivated, taught how to share ways of working, have an appropriate personal relationship, and/or other factors. Students can be, and very often are, pulled into their zpd by imposition. For reasons of desire to become like another person, or to please another person, to be accepted into a group, or achieve other such goals people will copy/emulate another, and subsequently that behaviour may become part of that person. However, it is certainly not the case that learning always takes place. In both teacher-student interactions and student-student interactions the participants may not engage together in the activity. They may act separately or one and/or the other may not act at all. In Meira & Lerman (forthcoming) we give some instances of a zone of proximal development being created, in which a nursery teacher almost 'grasps' a child's attention and orientates it towards what she wants the child to learn. In one scenario the interaction is initiated by the child's questioning gesture and as observers we were unable to ascertain whether the child was pointing at the objects on which the teacher chose to focus. This did not matter, though; the child responded to the teacher and became involved in the activity. We also offer instances where, despite the teacher's best efforts, a zone of proximal development is not created in the activity, the teacher missing the child's experience immediately preceding her intervention and offering something not relevant. In another study (Lerman, forthcoming, b) I argued that as much may be learned from incidents where a zone of proximal development is not created as when it is. Two thirteen-year-old students were engaged, ostensibly together, on a task to simplify a ratio expression, *ab:ab*. A close analysis revealed that they chose different methods and although they spoke, one might say, at each other about their methods, they did not pay attention to each other and so did not move forward. As we have described it elsewhere, students may not catch each other's ideas (Vile & Lerman, 1996) and hence not create a zpd. Creating a zpd is more about mutual orientation of goals and desires than about the intended content of the interaction. In that study I was looking at the interactions between the subjects of the video, trying to identify when they were communicating their ideas and reasoning to each other. Acts of communication, as objects of study, are the signs of sociogenesis, the social origins of psychogenesis and internalisation. I was looking for clues from all the elements of the data set, videos, transcripts, and interviews, to help in drawing inferences about the nature of that communication. For instance in that study I indicated that the behaviour of the teacher and the students with regard to their mathematical activity is framed by a discourse of ability. This constituted one of the students as more able and therefore more powerful in the interactions between them. Some time after the interaction used in the study the teacher looked at and read the video and the interactions of the students as confirming her evaluation in terms of ability.

This is the relevant extract from the transcript.

1. M: What? Equals ab? [pause, D looks on M's page] Equals ab?
2. D: Yeah.
3. M: No, it equals 1.
4. D: Wait a second...
5. M: 'Cause 1, [punching calculator buttons] 12 times tw... no. One, look, look, look. One times 2, divide 1 times 2...it shouldn't equal 4. [M appears to be substituting the values 1 and 2 for a and b]
6. D: [laughs]
7. M: Um, yeah, it's, 'cause I'm doing [punching buttons] 1 times 2, divide 1 times 2, equals 1.
8. D: So that's cancelled. The two b's are cancelled out.
9. M: Equals 1.
10. D: Right? The two b's are cancelled out.
11. M: Hey, where'd my pen go? No come on, look, look, look, look. You've got to do BODMAS. Watch, watch, watch, watch. [punching buttons] One times 2, divide 1...come on, 1 times 2. That's stuffed up. [with emphasis] One.

The outcome was that the 'more able' student M ignored the ideas of the 'less able' D and firmly tried to impress his method on D. D seemed to lose heart quickly and did not press his method. In order to give an account which is adequately framed in a theory, and which also offers a description of the objects about which one, as researcher, is making statements, one has to delimit one's text. A Piagetian model would argue that power, in the form of authority, inhibits equilibration. Vygotsky's zpd, working from a sociogenetic perspective, assumes imbalances in social relations as part of being human and communicating, and therefore these were identified as an element of my account.

Following an account of teaching and learning and development from the perspective of discursive psychology, I have argued that the zpd offers a sociogenetic mechanism for interpreting learning particularly suited to microgenetic studies. In mathematics education research activity theory has been used in this way to take account of goals and needs as they change over time, by Crawford & Deer (1993), Bartolini Bussi (1996) and Lerman (1997).

Mathematical Meaning-Making

In mathematics education we are confronted with powerful models of the process of mathematics learning, based on Piaget's constructivism. According to Steffe and Wiegel (1996) a model of mathematics learning consists, at least, of a meaning for operations and for representation. The former "are part of a system of operations that is goal directed" (p. 486) and the authors draw distinctions between Piaget's notion of actions and activity and the Soviet notion in activity theory. The latter are of greatest concern to me here.

> Many accounts of knowledge representation are misleading because they are based on the assumption that concepts are things - mental objects - "out there" to be represented... we regard mathematical concepts as mental acts or operations, and it is these operations that are represented. (*ibid.*, p. 487)

Piaget's familiar ontological choices, between reality imposing itself on a person empirically or platonically, and the person constructing her/his own internal individual world, are evident here. There is another option, that mathematical concepts are social acts and tools. Consciousness is constituted in historical, socio-cultural settings and cultural tools are internalised in the strong sense that the mental plane is formed in that process. Thus, cultural tools (analogous with Marx's thesis concerning physical tools) both transform the person and the world for that person, and these cultural tools precede the individual. Words and symbols are mediators of thought, "It is the world of words which creates the world of things" (Lacan, 1966, p. 155) and objects, including concepts, have meanings only within relations of signification (Walkerdine, 1988).

From the perspective of teaching and learning mathematics the research programme would therefore be to study empirically the semiotic mediation of those objects. The language of semiotic mediation, whereby the person and the world for that person are transformed by the acquisition or appropriation of cultural tools, is a theoretical resource that engages with the fact of signification as well as the specificity of relations of signification. It rejects the notion of decontextualised, abstract concepts. It offers a medium through which one can account for cultural specificity, such as the mathematical meanings of Aboriginal students (Klein 1997) and of multiple subjectivities as a result of the overlapping social practices of gender, ethnicity, class, family relations etc., in which people develop. The richness of the social and cultural implications of a discursive approach to psychology and the range of theoretical resources (e.g. sociolinguistics, cultural theory, semiotics, postmodernism) on which it draws is enough to support its adoption.

In Vile (1996) and Vile and Lerman (1996) students' work in linear equations and co-ordinates were examined as case studies for the elaboration of a developmental semiotics, taking together the science of signs and the functioning of the process of mediation in learning, which is interpreted as making meaning which allows appropriate use in relevant contexts. In the study of co-ordinates (Vile, 1996) students were asked a series of questions in clinical interviews concerning the distance between two points, in two, three and four dimensions. The intention was to examine the meanings which students gave when working in the different dimensions and to try to identify which meanings corresponded with successful transfer to four dimensions.

> Students who retained a concrete, measuring meaning for the distance between two points were unable to cope with the distance between 4 dimensional co-ordinates... Those students who did make the appropriate symbolic meaning but only as a process... were unable to make recourse to a more iconic

representation if they were unsure… Students who were able to make the more generalised sign-sign foregrounded meaning early in their development are able to transfer to the more generalised dimension with relative ease. (p.176-177)

This study pointed to the kinds of meanings that can mediate the generalisations needed in mathematical thinking, and hints at the classroom activities that might encourage their appropriation. The transformation in thinking and acting, when students learn mathematics using such software as *Cabri Géomètre* or technological tools such as the graphics calculator, is accessible with the notion of semiotic mediation, which also opens the possibility for analyses of social transformations of the classroom (Winbourne, 1997).

In another study (Finlow-Bates, 1997) we examined undergraduate students' mathematics proof activities through an analysis of proof as a process of social negotiation of meaning, rather than an 'understanding' or its lack.

Wertsch (1991) has argued that focusing on mediation offers a unity for analysis which neither the individual nor mathematical knowledge can offer, with their implied separation of subject and object. Drawing on meaning as the mediation of cultural tools enables the study of other aspects of the positioning brought about in learning, through the social and political associations of concepts, or knowledge as power. For example, recent sociological studies (Cooper & Dunne, 1998) offer insights into how contexts mediate differently for students of different social groups. In the first stage of their research on mathematics assessment items set in realistic (everyday) settings and esoteric ('pure' mathematical) settings, they found that working class children "failed to demonstrate competences they have" (p. 115), through mis-readings of the realistic settings. Cole (1996) has argued that a focus on the mediation of cultural tools does not take account of action on the world, in the sense of tool use that Marx described. I have suggested above that in the zone of proximal development one can study the mediation of cultural tools but that activity theory is more fruitful for longer-term studies, taking account of goals and needs. There is a dialectical unity in these two methodologies in that, whilst both are rooted in the cultural psychology of Vygotsky, mediation is a generalising principle, looking for similarities, whilst activity theory is a specialising one.

I gave this section the title 'Mathematical Meaning-Making', rather than mathematical understanding, with the intention of writing this paper without using the term. Indeed I have scarcely used it, apart from in this paragraph. The term is part of the 'regime of truth' which locates power in the hands of teachers who can say when a child understands or doesn't, independently of what s/he produces, verbally or in writing (Watson, 1995). Its entirely internal nature makes it a rather useless notion (Lerman, 1994b) whilst its association with closure places it in a positivist paradigm. Much of Wittgenstein's later work can be seen as a deconstruction of attempts to find essences behind social meanings. His well-known argument (e.g. 1974, p. 64) that to understand a concept is to know its use is to locate meanings in grammar and in rule-following.

Voice

Confrey (1995) argues that constructivism offers a space for individuality of interpretation, or voice, that Vygotsky's emphasis on scientific concepts replacing spontaneous concepts appears to deny. This aspect of his theory has often been interpreted, wrongly in my view, to recall the possibility of learning through transmission. First, spontaneous concepts do not disappear under scientific ones, which might be seen to lead to a uniformity which denies the possibility of individual voice. In general they coexist with spontaneous concepts, through a splitting of subjectivities, the child having learned in which situations the differing meanings are appropriate. As a rather simplistic example, a child might know to use "My half is bigger than your half" in the playground but not in the mathematics classroom. This offers a discursive interpretation of intuitions in mathematics (Fischbein, 1987). Second, as I discuss below, the notion of the zpd requires from the teacher, (desired) peers and texts, the particular experiences of individuals.

In my view the method which Vygotsky's work offers is also often misunderstood, in large part because of the time and forms in which it was used. Vygotsky died in 1934, at the age of 38. The theoretical discourses available at that time, and especially the particular circumstances of the Soviet revolution, limited the perspective for the theorising and therefore for the choice of research programmes of Vygotsky and his colleagues. This is inevitable and is actually an application of Vygotsky's own theory that concepts are related to their time and place. Thus Luria's work in Uzbekistan (1976) presents a strong image of the valuing of a particular interpretation of advanced societies as against primitive ones, and of progress. But Vygotsky's method is, through his argument for the priority of the intersubjective, to enable the study of consciousness as the internalisation of sociocultural meanings, the appropriation of cultural tools, and the transformation that this effects for the individual and for her/his world. The origins of individual meanings being located in socio-cultural tools roots 'voice' in its proper framework. It is not the individualism of private world views which has dominated the debate around subjectivity and voice in recent decades. In cultural, discursive psychology individuality is the uniqueness of each person's collection of multiple subjectivities, through the many overlapping and separate identities of gender, ethnicity, class, size, age, etc., to say nothing of the 'unknowable' elements of the unconscious.

Discourses which dominate in the classroom, and everywhere else for that matter, distribute powerlessness and powerfulness through positioning subjects (Evans, 1993). Walkerdine's (1989, p. 143) report of a classroom incident in which the emergence of a sexist discourse bestows power on five year-old boys, over their experienced teacher, dramatically illustrates the significance of a focus on discourse not on individuals. In some research on children's interpretations of bigger and smaller (Redmond, 1992) we found some similar evidence of meanings being located in practices.

> These two were happy to compare two objects put in front of them and tell me why they had chosen the one they had. However when I allocated the

> multilinks to them (the girl had 8 the boy had 5) to make a tower . . . and I asked them who had the taller one, the girl answered correctly but the boy insisted that he did. Up to this point the boy had been putting the objects together and comparing them. He would not do so on this occasion and when I asked him how we could find out whose tower was the taller he became very angry. I asked him why he thought that his tower was taller and he just replied "Because IT IS." He would go no further than this and seemed to be almost on the verge of tears. (p. 24)

Many teachers struggle to find ways to enable individual expression in the classroom, including expressing mathematical ideas, confronting the paradox of teachers giving emancipation to students from their authoritative position. But this can fruitfully be seen as a dialectic, whereby all participants in an activity manifest powerfulness and powerlessness at different times, including the teacher. When those articulations are given expression, and not denied as in some interpretations of critical pedagogy (Lerman, 1998), shifts in relations between participants, and crucially between participants and learning, can occur (Ellsworth, 1989; Walcott, 1994). Learning is predicated on one person learning from another, more knowledgeable, or desired, person.

In the classroom Davydov's learning activity structure of a lesson encourages, and actually requires 'voice', the expression of individual life experience and perspective. When a teacher offers an activity in a classroom, say to share 2 oranges between 3 children, the different answers offered by the children arise from their previous experiences, what has been called the zone of actual development, and potentially pull the others, including perhaps the teacher, into their zones of proximal development. Similarly, powerful technologies can offer possibilities for novel ideas by children which create zones of proximal development for other participants and change the social relations in the classroom.

The account of a discursive psychology for mathematics education which I have attempted to develop in this paper incorporates action, goals, affect, power and its lack, based on sociocultural origins. A psychology focused on the individual making her/his own sense of the world does not engage with social and cultural life: other theoretical discourses, such as approaches to sociology which merely describe, are not adequate for mathematics education either. I go along with Harré (1997) when he writes, referring to discursive psychology: "Psychology is the study of the skills necessary to live as a human being with others" (p. 189). It should be clear that such a definition, particularly when related to education, is open to contestation concerning what is valued as development and what constitutes cultural capital. A cultural, discursive psychology places that contestation at the heart of what constitutes consciousness, meaning-making and, in this paper, mathematics teaching and learning.

Theoretical and Empirical Fields

Theories need to account for their on-going development in relation to their empirical work. Brown and Dowling (1998) propose that "the research process itself is properly conceived of as the construction of the theoretical and empirical as increasingly coherent and systematically organised and related conceptual spaces" (p. 11). Since Kuhn (1970) researchers have been forced to recognise that they create the objects of their research, they are not entities existing independently of the research discourse or the researcher. This is not to prioritise theory but to recognise the dialectic between the two fields, the empirical and the theoretical, and it distinguishes between mathematics education as a set of practices and mathematics education as a field of knowledge (Patricio Herbst, personal communication). I began this paper by pointing out that there is an overlap, since all mathematics education has its roots in the classroom whether its aim is to say something about practice or about how one might think and speak about mathematics education. In the main, though, this paper has been about the latter. My intention has been to map out the field, from the point of view of a discursive psychology. I have also tried to indicate its implications for mathematics education as a set of practices, through the examples of research and other classroom illustrations.

Steffe & Wiegel (1996) challenge researchers to provide an account of the self-reflexivity of their theories, although why this should be a sign of a good theory is not spelt out, except as a counter to naiveté. They argue that, according to radical constructivism, theories of learning can be seen as making what they call second order models of students' understandings, which are understood as first order models constructed by students to order their experiences. This symmetry is very appealing.

In that the objects of research, the products of research, the theories drawn upon, the methodologies used, etc. are all cultural products, texts, the theoretical programme outlined here is, in its entirety, reflexive. Language precedes phenomena, which precede experience - this the sense I take from the sentence which heads this paper. To refer again to Kuhn, however, researchers are forced to admit their allegiances to their theories. In one direction, empirical research leads to elaboration of theory, as our work on, for example, the zpd, on developmental semiotics and on teaching general principles demonstrates. In the other direction, theory, as outlined in this paper, provides the resource for interpretation, and for methodology and its justification. Rarely does one's theory as a whole change (although see Lerman, 1989, in comparison with this paper!).

The metaphor of the zoom lens is part of my theory: to sustain the metaphor a little further, it has framed my writing here and thus offered me both possibilities and limitations. It is a rhetorical tool for expressing the need to take into account all of the social and cultural life of the classroom, but it cannot quite capture the histories of the participants, or the classroom, and perhaps it is too linear. However, if it is the zoom lens of a video camera it can capture development and change. How we read the tapes remains the challenge for research.

Notes

(1) My thanks to Peter Winbourne and Ros Sutherland for comments on an earlier draft.

(2) This quote comes from the transcription by Anne Watson of a tape recording of a seminar with Rom Harré held in Oxford University on December 6th 1997.

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PLENARY PANEL DISCUSSION

Theme: *Diversity and change in mathematics teacher education*

Co-ordinator: Chris Breen

Panelists: Barbara Jaworski
Lena Licón Khisty
Konrad Krainer

Plenary Panel: Diversity and Change in Mathematics Teacher Education

Chair:	Chris Breen, University of Cape Town, South Africa
Panel Members:	Barbara Jaworski, University of Oxford, England Lena Licón Khisty, University of Illinois at Chicago, USA Konrad Krainer, University of Klagenfurt/IFF, Austria

Introduction

This year's panel will take an innovative form in that panel members as well as the PME conference audience will be asked to take part in the simulation outlined below where panel members have been set the task of sending in project proposals in response to an advertisement for an inservice mathematics education project. The project is set in a fictitious Cape Town site against the background of some realities which ensure that the issue of diversity and change cannot be ignored.

One of the real dilemmas of any inservice school intervention is the problem of asserting that any particular change has come about as a direct result of the intervention. As a result of this the main focus of the panel will be on the research and evaluation of the intervention - funding agencies are becoming increasingly insistent that their money can be seen to produce results.

On the next pages, you will find the advertisement and an information sheet which contains additional details of the context in which the project is to take place. This is followed by the project proposals that have been received by the three panellists. The panel session will take the form of a mass interview where the short-listed applicants (the panel members) will be interviewed by the chair of the panel as well as the PME audience to explore the strengths and weaknesses of each intervention and evaluation proposal.

Pilot In-Service Mathematics Teacher Project

The FINDUS Corporation in collaboration with the South Western Education Department for Enlightenment(SWEDE) invites applications from international academic mathematics educators to lead a consortium of mathematics educators in a pilot project for an inservice programme for secondary school mathematics teachers in the South Western Cape.

SWEDE finds itself in a position where a severe shortage in central government funding has resulted in its being forced to retrench 3000 temporary teachers in 1998 with the probability that there will be more cuts in 1999. This has obviously had a demoralising effect on the teaching cadre. Central government has also insisted that secondary school class sizes should increase to an average of 40 - 45 by the year 2000 The Government is also in the process of introducing the revolutionary Curriculum 2005 programme in 1998 which is underpinned by a new outcomes-based approach.

In view of this serious background, SWEDE and FINDUS have agreed to prioritise a major initiative to ensure that teachers in the South Western Cape meet these challenges of change and diversity and maintain the province's record for achieving the most outstanding results in the end of year final school-leaving national examinations. Since it is noticeable that those schools previously disadvantaged by apartheid have recorded the poorest final results in 1997, we have decided that this project should focus on these teachers in township schools. With this in mind, it has been decided that the one year pilot project will initially focus on a cluster of 5 High Schools in the Khagulitsha township. A videotape of one of the schools in the cluster and its surrounds will be made available to applicants on request so that they can gain a sense of the schools they will be dealing with.

Applicants should submit a plan for how they propose to work with teachers in the pilot project phase. Since it is obviously crucial for the pilot project to deliver results it will be essential for applicants to indicate clearly how they intend to research the pilot project so that they can prove the success of their intervention. If appropriate, applicants can include in their proposal the use of the services of staff from a local mathematics teaching inservice project which is familiar with the schools and the local situation, as well as researchers from the local university staff. Funding for the stage two expansion of the project into the whole region, will depend on the proven success of the pilot project.

Since the research aspect of the pilot project will be of the utmost importance, it has been arranged for shortlisted applicants to appear before an interview panel of about 400 international mathematics educators during the PME conference to be held in Stellenbosch between 14 - 18 July 1998 where applicants will be required to justify their choice of intervention and method of researching its success.

Information Sheet

The pilot phase of the project will be conducted over the 1999 school year, which starts in January and consists of 4 terms each of about 10 weeks. There are Easter and September breaks which are each of about 10 days with a 4 week holiday in June/July and a further 5 weeks at the end of the year.

The five pilot schools are all high schools which cover standards 6 - 10 (US Grades 8-12). Each school has a register of about 1200 pupils at the school (grades 8 - 12) with 48 teachers, meaning that basic class size is in the region of 50 pupils. The official language of instruction in the schools is English but this will be the second or third language for the overwhelming majority of the teachers and pupils. The majority home language will be Xhosa.

Mathematics is a compulsory subject in Standards 6 and 7 and then optional after that and pupils can take the subject at higher or standard grade. The classes in 6 and 7 are usually very large and taught in the pilot schools by teachers with impoverished mathematics backgrounds. The basic day's classes are the same for the whole range, except that there are about 8 subjects in 6 and 7 and only 6 after standard 7. A Standard 10's Mathematics Higher Grade teacher will take that class for their mathematics the whole year. The day is divided into 8 periods on average, and in a week of 40 periods on average each teacher will teach about 35 periods. Each class ends up with 6 or 7 periods of maths a week so this means each teacher is likely to take 5 classes (in the project schools some of the teachers take all five of the Standard 7 classes).

Each of the project schools normally has six mathematics teachers, half of whom will have university mathematics qualifications. Teachers qualify in one of two ways. One way is to do a degree at university and then a 1 year postgraduate diploma with two subject methods. The mathematics requirement is supposedly at least a second year level of maths to teach maths, but the reality has always been that those with any university maths (or sometimes even a good school leaving result) will end up teaching maths. The other route is to study at one of the teacher training colleges where the qualification is a 4 year teachers diploma, where mathematics content and methodology classes of varying standards have been taken (in some cases these have an intended equivalence to second year university courses). Most of the older teachers in the pilot schools have College backgrounds and are suspicious of university trained graduates.

The State has no money to run inservice courses at present. In general teachers do any inservice work voluntarily through Non Governmental Organisations (NGO's) who work with those teachers who want to work with them. Otherwise there are formal qualification routes through either a Further Diploma in Education (FDE) which is school-focused or through higher degrees which are generally research-focused.

Pilot In-Service Mathematics Teacher Project .

A proposal from Barbara Jaworski – University of Oxford

Rationale

Curriculum 2005 and the proposed intervention

Curriculum 2005 poses a radical, exciting and demanding challenge to education in South Africa. Any intervention into teacher education has to take Curriculum 2005 very seriously. The price of failure is high. The document offers a *theoretical* perspective which needs to be translated into an agenda for action. The proposed intervention seeks to develop mathematics teaching within C2005 and is designed to turn theoretical principles into practice. Like C2005, the Curriculum for the Intervention (CI) will be opportunity based, will integrate knowledge and training, and will fit into a wider programme of lifelong learning for its learners who are classroom teachers of mathematics. In particular, CI seeks to develop the principles:

- Learners should be actively involved in, and at the centre of, the learning process.
- Learners should work co-operatively and collaboratively in groups.
- Learners should become analytic and creative thinkers, well equipped to take decisions and exercise judgement.
- Learners should understand and play a role in directing the processes of learning in which they are engaged. (C2005 pp 9, 12, 27)

CI will emphasise the importance of *consistency* at all levels, relating the learning of teachers to the learning of their students in the classroom. It will be people-centred, designed to empower the teachers who are learning within it by promoting critical thinking, rational thought and deeper understandings.

As with C2005, CI will be based on outcomes. These will involve teaching development leading to students' successful learning of mathematics. Also, CI will "foster learning which encompasses a culture of human rights, multi-lingualism, and multiculturalism, and a sensitivity to the values of reconciliation and nation building" (C2005 p.1). Outcomes to be achieved are:

O1 Teachers will gain enhanced understanding and skills in mathematics and its learning and teaching.

O2 Teachers will learn to work supportively and collaboratively, valuing mutual respect and dealing with issues in social justice..

O3 Teachers will become reflective practitioners and gain experience of research methods and processes and their contribution to developing teaching and students' learning..

O4 Teaching and learning of mathematics will develop according to the expectations of Curriculum 2005. Students will gain fluency and conceptual understanding of mathematics and will be successful at all levels of national testing or examining.

O5 Schools and the wider community will understand and learn from the practices and processes employed in the project.

Theoretical perspectives underpinning the intervention

The intervention draws on research, theories and practices elaborated in the professional and academic literature. Only a brief treatment can be included.

1. Mathematics (Learning and Teaching)

Unavoidably central to all considerations, is that the learners in this programme are teachers of *mathematics*. They are largely responsible for the delivery of the learning area 'Numeracy and Mathematics' in C2005. Thus, the intervention takes account of the nature of mathematics relative to involvement, skills, creativity, collaborative activity, critical thinking and social justice. See for example, Polya (1945), Davis & Hersch (1980), Mason, Burton & Stacey (1982), Mellin-Olson (1986), Bishop (1988), Ernest (1991).

2. (Mathematics) Learning and Teaching

Learning and teaching in the project will draw on research and development in the *learning* and *teaching* of mathematics. This will include work based on, in and relative to *learning* theories, addressing particularly children's strategies and errors, constructivist and socioculturalist approaches, classroom negotiations and argumentation (e.g. Hart, 1981; Steffe et al, 1988; Cobb, Yackel & Wood, 1992; Bauersfeld, 1995). It will draw on research into mathematics *teaching*, focusing particularly on innovative teaching approaches which emphasise guiding principles in the rationale above. See, for example, Simon and Schifter (1991), Jaworski (1994), Fennema (1996), Wood (1997).

3. (Mathematics) Teacher Education

The intervention will draw strongly on research and development projects which have been influential in promoting innovative practices in line with the guiding principles above. Examples include the *Low Attainers in Mathematics* Project (e.g. Ahmed et al, 1985) and the *Mathematics Advisory Teacher* Project (e.g. Nolder & Tytherleigh, 1990) from the UK; the *PFL* Mathematics Project from Austria (Krainer, 1993); the *Teachers Raising Achievement in Mathematics* Project from New Zealand (Britt et al, 1993); the *MCPT Model* Project from South Africa (Goldstein, forthcoming). Central to the proposed intervention will be (1) teacher research and reflective practices in the development of teaching (e.g. Zack, Mousley & Breen, 1997; Jaworski, 1998); (2) relationships between teachers' thinking and beliefs and their classroom practices (Thompson, 1984; Lerman, 1986); (3) the impact of innovatory teacher-education programmes on the wider community (e.g. Wood, Cobb, Yackel & Dillon, 1993); (4) issues highlighted in Cooney's review of teacher education literature in the U.S. (Cooney, 1994);. The proposal also draws on the proposer's experiences in teaching development work in Pakistan where under-resourcing, lack of a professional development culture and teachers' difficulties with mathematical content reflect needs in common with South Africa (e.g. Jaworski, 1997). Findings from the TIMSS Video study concerning teaching development in Japan will feed into collaborative materials production (e.g. Stigler & Hiebert, 1997).

Key principles and issues for the proposed intervention

There is considerable research which shows that teachers develop understandings of teaching and learning mathematics through doing mathematics themselves alongside colleagues, and relating this directly and indirectly to classroom practices (Murray et al, forthcoming; Markovitz &Even, forthcoming; Simon, 1995). However, previous research in South Africa has shown that small scale interventions outside the classroom have ultimately little impact on teaching and learning (Murray et al). For an effect on teaching and learning, teachers need the opportunity to learn within the field, to have in-built opportunities for collegial collaboration and support, and to have input of various forms from experienced practitioners (Goldstein et al, forthcoming). Research elsewhere has shown the value for teaching development of teachers' questioning, teachers' critical reflection on their classroom work and of teachers conducting research into their own teaching (Jaworski 1994, 1998). Thus, consistently with recommendations in C2005, the proposed intervention takes very seriously: the field-based nature of the intervention; support in diverse ways from colleagues and 'experts'; critical questioning, reflective practice, and classroom research. In order to engage in collaborative activity, reflect on and evaluate the outcomes, and produce evidence of development, time needs to be built into day to day operation of the teacher participants. One consequence of these requirements is that creating the opportunity for effective outcomes is expensive in time and people provision.

The Proposed Intervention

This will be a one year research-based development programme spent in collaboration between teachers and teacher-educators in SWC. The main features of the programme will be paired teachers researching their classroom teaching; collaborative school groups including teachers and educators; whole project sharing and addressing of practices and issues. *All participants in the programme will be researchers.* Inputs to the programme will be made by experts at all levels including teachers, educators, and policy-makers. Links with wider school and community will be fostered.

Participants in the project will be project leaders (PL - 2); the research support team (RST) consisting of university researchers (UR- 3) and members of the SWC Mathematics Teachers' In-service Project (MTIP - 5); mathematics teachers in the five schools plus recruited retrenched teachers of mathematics (MT & RT - approx 30 of each), research assistants (2) and administrative support staff (2).

In advance of the programme, PL will meet with Head Teachers (HT) of the 5 schools along with SWEDE personnel to initiate the project, and gain agreement and support for its requirements. HTs' understanding of and support for the project will be treated as an important component of development. PL will work with HTs and members of SWEDE on recruitment of RTs, needs and responsibilities of schools with respect to the project, and on needs for the development of teaching in SWC more widely.

Every MT is to pair with one RT to teach jointly and collaboratively a set of classes within the school equal to one teacher's workload. Certain classes, designated as 'research-focused', will provide data related to teaching and learning. Teachers will prepare jointly for these classes and research jointly the processes in which they engage and learning outcomes they observe. Some joint teaching will be encouraged for gaining trust and familiarity and establishing coherent and consistent approaches for the students. Separate teaching will allow time for individual thinking, planning, materials preparation, reading, etc. It will be up to the two teachers how they organise teaching of their non-research-focused classes.

As the success of the project will depend largely on the way these teachers work together, recruitment of RTs must be extremely sensitively done with knowledge of the existing teachers' particular situations and needs. All responsibilities within a pair should be joint, and RTs should have equal roles and status with established teachers within the school during the year they are there. PL will work with HTs to develop collegial school communities.

One member of MTIP will join each school to conduct research and provide support on a day to day basis. Research as a whole will be co-ordinated and supported across schools by UR. Emphasis will be on collegiality, shared aims, complementary roles and establishing ways of working. PL will co-ordinate the project as a whole. The project starts 4 weeks in advance of the school year (40 weeks) and includes two weeks in long vacation time (46 weeks in total).

In weeks 1 and 2 (of 46) PL meet with the RST to prepare the programme. Planning must translate aims into lines of action. PL and RST will provide input to the project in terms of expertise and leadership. They have the job of establishing working relationships, trust, and a recognition that *all* are co-dependent learners - encouraging teachers to overcome initial reluctance and fears, to become energised and excited by the project and its potential, and to see themselves as researchers and participants in the teaching-learning process.

In weeks 3 and 4, all teachers will join PL and RST for workshops on doing and thinking mathematics, exploring preconceptions, establishing ways of working, and for setting up aims and objectives for the year including classroom work. Emphasis will be placed on mutual support, collaborative work, and mathematical and educational thinking. The meaning of research and the processes it involves will be introduced in order to dispel myths and fears. The mathematical education and achievement of students will be at the forefront of all thinking and activity. Aspirations of parents and educators regarding students' examination achievement will be central considerations. A shared interpretation of C2005 will be sought. A mathematics syllabus review will be undertaken (with attention to past examination papers) to identify key curriculum areas in which teachers will focus during the year. Needs for content enhancement for particular teachers will be acknowledged, and those who have expertise in the identified areas will be noted. These will form 'expert'-groups, in or across schools, to develop workshops and materials for their colleagues. A major aim of the project will be to develop teaching support materials for use by other teachers either in the project or beyond it. The principle teaching focus in the programme will ~~be~~ on developing students' conceptual understandings of mathematics alongside an emphasis on fluency and mental skills to promote general numeracy and examination success. In school-weeks, schools will have been asked to timetable all mathematics lessons in the first four days, leaving Fridays available for project teachers to work on a variety of tasks. The details of this and problems arising (for example, if some teachers teach another subject besides mathematics) will be thoroughly negotiated with HTs. At the end of each 4 weeks, there will be a Saturday workshop for the entire project. In each of the longer breaks, there will be one "INSET" week where all project staff share experiences, discuss issues and reflect on development. Workshops and seminars on Fridays and Saturdays will include:

- Developing schemes of work related to the curriculum, C2005;
- Sharing planned activities and their classroom outcomes;
- Teaching each other agreed areas of mathematical content;
- Developing curriculum support materials for use by other teachers.
- Reading and discussing texts relevant to ongoing work;

Materials development will proceed alongside teaching. Schemes of work and lesson plans will be shared, discussed with, and tested out by, other teachers. Written versions will be given to teams in the other schools, to implement and critique relative to mathematical learning of students.

Expertise in classroom approaches and research methods is likely to be more developed in the RST than it is in most of the participating teachers. Sensitive levels of support, scaffolding and input by the RST should be built into day to day work in school and beyond. These will involve (a) the MTIP member in a school working alongside teachers in the classroom, either supporting the teacher in (jointly planned) teaching, or in demonstrating possible approaches; (b) school-wide, or project-wide, seminars on research, led by the RST, on a Friday or Saturday, drawing on teachers' own experiences. RST will provide readings to support and enhance teachers' knowledge and experience.

Meetings, formal and informal, will be held to share the work and thinking in the project with other members of a school. Curriculum integration (C2005) will be addressed overtly, enabling non-project teachers to gain awareness of the value of a professional team approach to teaching. This may occur by the mathematics team in the school involving teachers from other curricular areas in discussions on theme or task planning to highlight links between mathematics and other subjects. Planning might result in, for example, a scientific, geographical or historical project or field trip where students use mathematics in the field to address issues and questions in the associated area.

Understandings of parents and the wider community will be sought through open meetings held in late afternoon or Saturday sessions to include joint activities, sharing of research findings, and airing of aspirations. Mathematics in home and community will be a major feature of such meetings.

For roles and responsibilities of participants of the project, see Appendix 1
For accrediation related to the project, see Appendix 4.

The Evaluation of the Project

Ongoing evaluation of the project against the outcomes O1 to O5 (listed at the end of page 1) will be constructive and formative. All activities will be fully documented. All participants in the project will contribute to collection of data and its evaluation at different levels. Teacher pairs will provide records of their thinking, planning, teaching and evaluation in focused classes. MTIP researchers will meet with and observe teachers, and keep records of the developing work and thinking, issues arising and their resolution, questions and problems. All participants will keep journals, extracts from which will be shared with participants at all levels on different occasions. Evaluation of the impact of the project on the school will proceed through interviews with HTs, other teachers and students.

A project data-base will be established where agreed forms of data will be entered throughout the project. The chief sources of data will be (1) bi-weekly summaries (in agreed format) from teacher pairs, MTIP members, UR and PL, which capture significant elements of activity and learning, thinking, questions, issues and problems; (2) data from interviews with teachers headteachers, students, and parents; (3) Qualitative accounts from participants' journals, and from discussions in project meetings; (4) Video recording of teachers in action in the classroom; (5) Selected test and examination papers and students' results. Further details are included in Appendix 2.

Teachers will reflect critically, singly and jointly, on their own teaching and research, on a regular basis, keeping written records. Bi-weekly written summaries (in agreed format) of this reflection, developments noted, issues raised and questions asked will be provided and entered into the project data-base. These will be read and analysed by the UR, and results recorded and fed back to participants. Thus an ongoing record of teacher and teaching development will emerge. This will be formative in encouraging reflection to take place, and providing feedback for further reflection. IMPT members will liaise with teachers in this process, ensuring that records are produced as agreed.

IMPT members of the RST will write their own personal reflections on their own and their school teachers' development and produce bi-weekly summaries to be treated as above. Along with UR, they will devise interview schedules for their school in which they interview, with agreed regularity, project teachers, other teachers, students and selected parents. These interviews will be transcribed (by administrative support staff) and analysed by members of the RST and research assistants. Transcriptions and analyses will be entered into the data-base. UR will be responsible for co-ordinating this process.

PL and UR will keep an overview of the entire project, visiting schools and classrooms and talking with all participants. They will observe or participate in a selection of interviews. Records will be kept of all conversations, and individuals will write their own journals. Bi-weekly summaries will be produced and treated as above. PL will be responsible for the overall analysis of data and its use in measuring actual outcomes against expected outcomes. PL will read all summaries and associated analyses. They will read analyses of interviews and check findings against randomly chosen transcripts

All electronic recordings will be transcribed. Analysis will be conducted alongside data collection and will be closely linked to the desired outcomes O1 to O5. Research staff will feed emerging patterns back to researchers in the field. Termly meetings to report outcomes and progress will be held between PL and a steering committee agreed with SWEDE.

Analysis will be led by PL and UR who will initially read all documents or transcripts, and decide on appropriate forms of coding. These will be tried out with research assistants, looking critically at outcomes and refining the analytical process. Once stable and reliable procedures are in place, research assistants will continue data analysis, in continuous consultation with UR. PL will regularly scrutinise the analytic process, refining it wherever need is suggested. A randomly chosen portion of the data will be reserved for checking against emergent findings and patterns, for validatory purposes. Procedures and outcomes of the analytic process will be fed back to and discussed with teachers periodically, and responses sought for participant validation. Reports will be produced by PL at the end of each term and discussed with the Steering Committee. A final report will be produced at the end of the year.

Results of the Project

In addition to a final report offering an assessment of the achievements of the project relative to desired outcomes O1 to O5, the following tangible outcomes will result from the project:

- 5 schools with highly effective mathematics teachers relative to Curriculum 2005, and other teachers in the process of gaining awareness of collaborative working, team building and interpretation of C2005 in their own subject areas. These schools can become models for future development in other schools.
- 6 HTs with enhanced understanding of the interpretation of C2005 within their schools, able to work with other HTs for future development.
- 30 teachers (originally the RTs) with considerable experience of the developmental process, who can be used in development in further schools.
- Materials for use by teachers in all schools as part of further development.

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TALKING MATH: PROPOSAL FOR SCHOOL CHANGE

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This proposal focuses on issues of making mathematics accessible to students in South African classrooms. Access is defined by two dimensions. First, that mathematics instruction is linguistically comprehensible. Second, that mathematics instruction develops subject literacy which is the masterful control of the discourse of mathematics; this also emphasizes teaching for meaning. Overall, the issues pertain to making mathematics instruction linguistically and culturally sensitive for students--and teachers--whose home language is not the language of instruction. The proposal describes a plan for achieving reform based on the foregoing.

Need

It has been a long held myth that mathematics is a subject that is language free. However, current reform documents in mathematics, such as the NCTM *Standards* (1989) and the South African *Curriculum 2005,* have emphasized communication--talking, reading, listening, and writing--as an important factor in mathematics learning. There clearly are great cognitive benefits for students when they orally or in writing explain their strategies and when they listen to others' explanations (e.g., Khisty, 1996). Also reading mathematical problems provides another context for developing language arts skills. In essence, the mathematics context is becoming very language-rich in terms of students' participation.

However, mathematics has never been language free; learning mathematics has always been dependent on language since talk forms the heart of interaction between persons as they come to have shared meanings or understandings whether formally or informally. Language is simply that singular human characteristic by which common understandings are accomplished. Also, we know that mathematics has a cultural life of its own in that there are certain ways of thinking, knowing, and doing that pertain to mathematics alone. It has its own symbols, syntax, terminology, and ways of speaking; it even has a way of taking ordinary words and giving them special meanings, as for example, "side", right as in "right angle", and "quarter" (e.g., Pimm, 1987). As in basic human interactions, common understandings are negotiated, overtly or covertly, and established in order to carry out further activity. For children--or adults--to acquire new concepts, be able to use them, and have command of them, they must be able to comprehend what someone else who already has this knowledge is talking about. This is not to suggest some didactic process of transmitting knowledge. Rather, it suggests that when human beings--teachers, parents, older siblings, or peers, and the learner--interact, in this case regarding mathematics, the parties must be able to comprehend each other. When there is confusion about meanings or what is actually spoken or written, the learner can be put at a disadvantage--and this is even more so in multilingual contexts (e.g., Khisty, 1993).

Consequently, in reformed mathematics learning contexts that emphasize rich and varied use of language (i.e., reading, writing, speaking, and complex meaning-making), being considerate and conscious of language becomes a critical pedagogical tool for teachers. Moreover, South African classrooms are clearly multilingual contexts. While English is the medium of instruction, it is not the home language, or the language of most familial, social, or religious activities, of a significant portion of students. It even may be the second language for many teachers; but students and teachers may not share the same home language. In essence, there may be at minimum three cultural languages operating; and add on top of this the language of mathematics (i.e., its register) with its specialized forms for expressing unique and complex concepts, and mathematics becomes a linguistically complex learning environment. The need to create linguistically and culturally sensitive learning environments then becomes paramount. Given this and given that teachers, like others, take talk for granted, there is a need to develop teachers' strength 1) in better recognizing the role and nature of talk in learning mathematics; 2) in utilizing strategies that provide linguistic support to students who are second-language learners (i.e., who come from homes where a language other than that used in instruction is dominant); and 3) in instructional problem-solving whereby teachers can develop solutions for their own contexts so that students' home and community, including language, remain as learning capital.

This proposal addresses this need. It also intends to positively impact educational reform based on the assumption that improvement in students' learning outcomes is a direct result of modifying teachers' behaviors, particularly those related to communication. This is not to suggest that teachers be defined as having deficits and in need of remediation. Instead, it suggests that teachers are the primary agents of educational change, and as such, any activity related to developing them must be grounded in processes of empowerment. This means that changing teachers' behaviors must utilize a process by which they become independent thinkers with strengths including proficiency in a language other than English. The following section provides the theoretical underpinnings for the proposed project.

Theoretical Background

This proposal is informed by primarily two areas of inquiry: first, the work of sociocultural theorists such as Vygotsky (1978) and Bakhtin (1981); second, the work in bilingual education which includes second language acquisition theory. Relevant aspects of these works will be presented here as a rationale for the activities proposed by this project.

Sociocultural theory suggests that the nature of learning is not an individual accomplishment but a social activity (Vygotsky, 1978), and in classrooms, it involves interactions between students and between student and teacher. Also, in classroom contexts, it is the teacher who is the more experienced or "enabling other" (Vygotsky, 1978), and as such, is the person who plays the critical role not only in engineering learning contexts, but also in creating environments that foster students' appropriation of desired modes of thinking and in enculturating the novice into mathematics. In mathematics, the social activity inherently involves interactions as part of doing mathematics as mathematicians do it. Consequently, teacher and students interact and construct an "ecology of social cognitive relations" (Erickson, 1996). In light of this, for theorists such as Bakhtin (1981), understanding the nature of talk or discourse is fundamental to understanding learning. Freire (1973) and Gee (1989), extend the issue of talk to issues of empowerment. To them, the power of the *word* is more than talk, conventions, or linguistic forms; the power of the *word* is literacy or the masterful control of discourses associated with social institutions such as schools (Gee, 1989). Therefore, the issue of language in mathematics is not simply a matter of teaching and learning new vocabulary. The issue is providing students access to mathematics, teaching for meaning, and the role of talk in these processes. From this perspective, the teacher's own mode of speaking mathematically, using questions, and giving explanations are part of the content of mathematics and can affect students being either remedial or more advanced in their learning. What teachers need to better understand, therefore, is the nature of their interactions and talk in creating the ecology of the classroom and the role of their *words* in students' learning mathematics and becoming empowered.

The South African classrooms present issues of teaching and learning in two languages and creating learning contexts that are linguistically and culturally sensitive. Therefore, it is only reasonable that teachers be well grounded in the literature of bilingual education and what it provides in terms of effective practice for multilingual classrooms. This body of knowledge is very rich and some concepts are too complex to completely present here; therefore, the reader is referred to the note at the end of this paper for additional sources. This proposal uses as part of its rationale three ideas related to this area: 1) the role of the home language in learning; 2) the nature of providing linguistic support for students' learning; and 3) the nature of comprehensible input.

One of the key concepts that emerges from this literature is that there is a significant difference between conversational and academic proficiencies in a second language and that strength in one area does not mean strength in the other (Cummins, 1981). It is easier for students to develop proficiency in conversational skills in a second language because the context of conversing with someone else naturally provides a good deal of linguistic support or clues such as gestures, realia, or visuals. A student can easily comprehend the phrase, "get the ball", when he/she is on the playground, a ball rolls past, and someone points at it as the words are spoken. However, academic language is characterized by its abstractness and lack of visual cues; it is mostly text from which meanings must be inferred and it emphasizes one's own production of language as in writing. Consequently, proficiency in academic language takes considerably longer to develop and must be taught with greater consideration and attention. A second language learner cannot simply "pick it up" as one can conversational proficiency. Therefore, academic content may best be learned in the students' home language--the language understood best. If students acquire understanding of mathematical concepts in the home language, this knowledge is transferred to the second language. In fact, learning content in the home language may actually help develop the second language--but if nothing else, it ensures that students keep up academically (Cummins, 1981). This idea is so counter-intuitive that teachers require a great deal of time, discussion, and practical evidence to understand and apply it. It is also important to keep in mind that the home language is part of one's self-definition, and its maintenance or loss affects how valued and valid one feels. Consequently, teachers need opportunity to consider the characteristics of academic language and how to include the home language in the classroom as a learning resource.

The research in bilingual education also provides concepts, principles, and applications of second language acquisition theory. While the home language is important for concept development and for affirming identity in the classroom, there are times and circumstances when the second language is used for instruction. The second language is best learned through context, and for academic proficiency, the content provides a good context (Cummins, 1981). This suggests that as much attention is given to teaching the second language form of mathematics as is given to teaching concepts and skills. Effective teachers of mathematics with bilingual students have been found to accomplish this by noting possibly troublesome words or phrases and teaching them in context using various strategies (Khisty, 1993). Some teachers develop vocabulary as part of the introduction to new concepts and others use word problems and student written explanations as a context to teach the both the cultural and mathematics languages (Khisty, 1993; Khisty, 1995).

But one of the most important ideas from this area is that instruction must be linguistically comprehensible. We can consider this as providing students with access to the interactions or talk that are critical to learning. One way to accomplish this is to ensure that instruction does not force a second-language learner to learn only by listening. Listening is the weakest skill in a second language, and consequently, it critically affects comprehension. Instruction should be context-embedded which means that what one says is accompanied by realia, visuals, and gestures which serve to aid comprehension (Cummins, 1981). Again, in studies of effective teachers of mathematics (e.g. Khisty, 1995), teachers were found to frequently write what they said as they said it or they did whatever was necessary to contextualize learning and provide support so that students could both see and hear what was said. They also consciously used their talk as a teaching tool, for example, by using their voice to draw attention to words and concepts and by revoicing positively and unobtrusively what students said in order to model how the language of mathematics is used (e.g., Khisty, 1993; Khisty, 1995). Consistent with second language acquisition principles (Krashen, 1985), teachers also modified their speech to have clear enunciation, clear word boundaries, fewer idioms, and a slightly slower rate.

Access in this project in part refers to providing instruction which is comprehensible via its talk. It can also be thought of as providing students with opportunities to acquire control of the discourse related to the community of practice of mathematics. It is a mistake to assume that teachers only need to acquire skills related to modifying speech or contextualizing instruction. A teacher's talk can be the primary source for acquiring the more formal aspects of the second language in that the context of academic work is where students will hear how the language is suppose to sound (Wong Fillmore, 1985). Likewise, the teachers' talk is also the source for hearing and acquiring the thinking of mathematics as reflected in the discourse embedded in the practice of mathematics (Khisty, 1995; Khisty & Viego, in press). The acquisition of mathematical discourse is strongly related to mathematics learning, and is developed by being immersed in mathematical practice. Consequently, teachers need to consider how their talk reflects the kind of mathematics being proposed by current reforms.

Project Objectives and Design

The primary need addressed in this proposal is how to provide students who come from homes where the language there is not the same as the language used for instruction with access to mathematics, and how to help teachers accomplish this. Given the foregoing, it is proposed that a group of twenty teachers begin a long term inquiry into creating linguistically and culturally sensitive learning environments based on utilizing current research from bilingual education (theory and practice) and integrating it with current research on reforming mathematics teaching. Specifically, for an academic year and a half, teachers will engage in activities which emphasize 1) collaborative analysis and problem solving regarding their current discourse and communication strategies; 2) developing appropriate to their own classrooms instructional strategies that integrate and apply communication constructs; and 3) testing, analyzing, and revising communication strategies in classrooms.

To meet the need and objectives noted above, the following activities will be conducted.

I. Gathering Baseline Data

Although there is a growing body of literature on the role of discourse in learning (e.g., Hicks, 1996) that can inform project participants, it is still appropriate to gather qualitative data on the current nature of talk in actual classrooms. This data will assist to identify more specifically the language, discourse, and communication issues that exist. It also serves to make more concrete to teachers what these issues are and allows them to reflect and think more deeply about constructs that are too easily overlooked or taken for granted. It further will provide the context for teachers' construction of knowledge about discourse. Project participants will be asked to videotape themselves as they teach mathematics over a specified period of time. They will be taught to do discourse analysis and will practice these skills using their own and others' data tapes. Teachers will also develop skills for analyzing interactions such as participant structures as a means for understanding how their talk affects students' participation in class. The questions guiding ongoing assessment of this phase include: How comprehensible is teachers' current talk? What is the nature of the content of teachers' talk; for example, how are questions used or how are concepts explained? How does the teacher use talk to connect with students' background and experiences? Given that English may be the second language for teachers also, what are the strengths and weaknesses of teachers' talk? Lastly, to what extent does each teacher currently demonstrate a conscious consideration of his/her own talk?

II. Developing New Conceptual Understandings

A major task, which will take up most of the project time, will be to change teachers' conceptions of the role of language and talk in mathematics. This includes reconsidering their own discourse (e.g., whether questions promote critical thinking or how their talk fosters meaning-making). Teachers will engage in long-range activities whereby they simultaneously a) develop a working understanding of current literature on bilingual education and second language acquisition theory and practice, communication in mathematics, and other work related to discourse; b) engage in self-designed short-term and long-term inquiries to test applications of this understanding; and c) collaboratively develop demonstration lessons that reflect the application and integration of new knowledge, which also can be used to assist other teachers to improve instruction. During this process, data (particularly interviews, observation fieldnotes, and artifacts) will be gathered to answer the following questions. How do teachers perceptions and understandings of the role of language in learning mathematics change over time? How do they make conceptual and concrete connections between the findings from bilingual education research and mathematics education research? What issues arise as teachers consider and use their talk as a teaching tool? What support do teachers need as they reform their talk? How well do development activities especially teacher educators' own discourse model the target concepts and practice?

III. Indicators of Success

As we work with teachers of mathematics to have them rethink their talk and to problem-solve strategies for using their talk to better foster learning especially in multilingual contexts, what would we look for to indicate success in our efforts? First, are teachers conscious of their talk and do they concretely consider what and how they communicate as an important component in learning mathematics? Second, can teachers explain why and how talk should be considered, modified, and/or used? Third, do teachers plan and implement concrete ways to validate in classrooms students' home language even if they are not proficient in the students' home language. Fourth, do teachers modify their talk to be more comprehensible? Fifth, do teachers miss fewer opportunities during instruction to develop both the second language and the language of mathematics? Sixth, do teachers consider and implement ways to teach for meaning; do teachers' talk more often address meanings or reflect meaning-making? Sixth, do students speak more and better mathematically, does their talk reflect mathematical practice, and can they appropriately express mathematical meanings; in other words, can we trace student discourse back to the teacher's discourse? Seventh, does the teacher educator/researcher's actions model the target understandings and skills, and in essence, can we trace the teacher's discourse back to the researcher's discourse? Eighth, integrating basically three knowledge bases (mathematics education, bilingual education, and pedagogical reform), even though they at times seem to overlap, raises questions about complexity in thinking; what specific questions have emerged based on observing teachers' changes? What has been learned from the project's activities that could help us better understand how to facilitate complex and complicated integrations and the change process among teachers?

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The CAPE project - Constructing Activities for Peer-Education

Short Proposal
for a Pilot In-Service Mathematics Teacher Project funded by
the FINDUS Corporation in collaboration with the South Western Education
Department for Enlightenment (SWEDE) in South Africa

submitted by the
Center for Interdisciplinary Research and Development of Austrian Universities (IFF), Sterneckstraße 15, A-9020 Klagenfurt; *http://www.uni-klu.ac.at/iff/schule*
Project managers and consultants: Konrad Krainer (Project director), Marlies Krainz-Dürr, Christa Piber, Peter Posch and Franz Rauch

in co-operation with the
Association for Research and Development in Mathematics Education (README)
in South Western Cape, Cape Town

1 Initial Remarks on the current status and the intentions of the commissioning bodies and the CAPE project

1.1 Brief outline of the situation of schools in the (South) Western Cape

(South) Western Cape schools are confronted by numerous challenges and problems which clearly have a major impact on the culture of learning and teaching, such as the cutting back of 3 000 temporary teachers, the increase of class sizes to an average of 40 - 45 pupils, the introduction of a totally new curriculum starting in 1998 (grades 1 and 7, then grades 2 and 8 in 1999, etc.) in which traditional subjects disappear completely, poor results for 1997 in the final school-leaving national examinations and in the international TIMS-study (Mathematics and Science literacy), significant differences between schools (in particular schools previously disadvantaged by apartheid), further disparities in teachers' education, problems with the language of instruction and in places a lack of motivation in both teachers and pupils.

1.2 Basic intentions of the commissioning bodies

The goal and the kind of intervention and procedure intended by the FINDUS Corporation and the South Western Education Department for Enlightenment (SWEDE) can briefly be summarized as follows:

Goal and kind of intervention: Improving the quality of mathematics teaching in five High Schools in the Khagulitsha township (poor final results in 1997) through a one year in-service education pilot project in 1999 (involving all mathematics teachers at these schools). If the project is successful the plan will be to extend it to all schools in the South Western Cape.

Procedure: International consortiums of mathematics educators are invited to submit proposals for this intervention, including a plan for the evaluation of the project. FINDUS and SWEDE will decide which proposal is to be realized. The two organisations are in contact with the five schools and have obtained their willingness to participate in this pilot project (agreement on general conditions, e.g. teachers' additional work load and time needed by the project).

1.3 Basic intention of the CAPE project

The key issue of the CAPE project is the active involvement of teachers and pupils (one class from each of the participating teachers) in a joint effort to improve learning and teaching in their mathematics classrooms. Therefore it is crucial that the development work initiated and supported by the project and its evaluation reflect the concrete needs of pupils and teachers. Both pupils and teachers are seen as active members of CAPE. Based on the assumption that learning and teaching are not isolated activities that only take place within classrooms the project intends also to take into consideration the situation of the whole school and the intentions of the school authorities and relevant organisations (FINDUS, SWEDE, ...), e.g. to regard the project as a possible starting point for systematic efforts to promote the quality of the school system in the whole region. Therefore the evaluation of the project not only focuses on the further development of teachers (individually and as a team) but also investigates the implications of the project for the pupils, for their parents, for the culture of the whole school and for the school authorities and relevant organisations.

The project is based on the theoretical assumption that complex practical situations and problems (e.g. the improvement of learning and teaching) cannot be resolved outside the practice through general propositions to be transferred (e.g. through in-service education) as "ready knowledge" to practitioners who then would only have to apply this knowledge in practice. In contrast to this model of "technical rationality", the project is grounded on a model of "reflective rationality" which assumes that complex practical situations and problems need particular solutions that only can be developed in the specific context of their appearance (see e.g. Schön 1991, Posch 1996). Therefore action research (defined as the systematic reflection of practitioners on action, see e.g. Elliott 1991 or Altrichter/Posch/Somekh 1993) is the central feature of development and research activities within CAPE. The design of this project is based on experiences with in-service education programs and school development projects in Austria working on a "teacher as researcher"-basis (see e.g. Krainer 1994 or Brunner et al. 1997) and on other international experiences (see e.g. Cooney/Krainer 1996, Crawford/Adler 1996 or Jaworski 1998).

In order to give teachers and pupils enough freedom of scope to define their own way of improving learning and teaching the most important part of the evaluation is based on self-evaluation carried out by pupils, teachers and the facilitators. This will be accompanied by additional elements of evaluation, e.g. by pre and post questionnaires and by a pre and post test using items of the TIMS-study with regard to mathematics literacy in order to generate some further data about the effects of the project.

In conclusion, the CAPE project aims at building on existing strengths in the South Western Cape school system and at taking professional communication and peer-education among teachers and facilitators as a chance for improving learning and teaching and for increasing theoretical and practical knowledge about learning and teaching mathematics. Everyone involved in the project is seen both as a learner and as someone who facilitates the learning of other people.

2 Core activities within the CAPE project

2.1 Groups involved in the CAPE project

In the project the following groups of people will play major roles:

- Pupils: It is proposed that each participating teacher chooses one class that he/she will teach in the school-year 1999 (be it a class which will follow the new curriculum or not). The pupils are motivated to take co-responsibility for the whole project and to support their teacher's efforts to improve and evaluate his/her teaching. Assuming that each class has about 50 pupils and that 30 teachers participate in the project, a total of about 1 500 pupils will be involved.
- Teachers: Each of the (assumed) six mathematics teachers in each of the five schools is expected to carry out in one of his/her classes some teaching innovations and to investigate systematically his/her teaching through an ongoing action research process. There will be extensive communication within the group of mathematics teachers at each school (being "critical friends" to each other) and periodic meetings of all 30 mathematics teachers participating in the project.
- Facilitators: Five South African teacher educators (from a local mathematics teaching project and the local university staff) are responsible for continuously supporting the teachers' action research process (e.g. observing teaching if invited by a teacher), by leading in-service workshops (modules on topics proposed by the participants or concerning important topics relevant for all participants), by carrying out different elements of evaluation and by reflecting their own intervention process.
- Project managers: The five facilitators will be supported both at facilitation and evaluation level by two Austrian researchers (with experience in mathematics education, quality development and project management) who are also responsible for coordinating the whole project (including the evaluation), for writing the final project report and for presenting it officially to FINDUS and SWEDE, in all cases in co-operation with the facilitators. (If the pilot project is successful the plan will be to give the local facilitators more responsibility for coordinating the project.)
- Schools: Each group of mathematics teachers at the five schools is supported by a facilitator. There will be a few meetings of the group or its mathematics head teacher with the principal and there will also be two opportunities for feedback to the whole teaching staff at the school in order both to inform the other teachers about the ongoing process and eventually to motivate them to start similar processes.
- Parents: Parents will receive updates on the progress of the project periodically. Apart from their right to know about what is happening in their children's classroom, there are three reasons in particular: firstly, to get them involved in thinking about the quality of learning and teaching; secondly, (hopefully) to enhance their anticipation

that education is important and that they can contribute to the progress of their children; thirdly, for them to realize that their child's school (teacher) is undertaking a specific effort to improve the quality of teaching. Parents will also be invited to write down some observations with regard to their child (concerning the project) each week to promote their involvement in the project and their thinking about education.

- FINDUS/SWEDE and the local school authority: These organisations will periodically receive information on the progress of the project. In a final presentation representatives of these organisations should get authentic evidence of the progress of the pupils and teachers. This final presentation should include a discussion about the challenges and needs of pupils, teachers, parents, principals, facilitators and schools and about activities aiming at supporting those challenges and needs.

2.2 Core activities within the CAPE project

Figure 1 (see next page) sketches the core activities within the project. In the following section a short description of these activities and the basic elements of the evaluation of the project (see 2.3) is outlined. The project consists of three phases: the *pre phase* (October - December 1998), *the main phase* (January - December 1999 = school year 1999) and the *post phase* (January - March 2000). The *total amount of teachers' additional work load with regard to the project* is about 30 days (table):

Days (add. work load)	Pre phase	Main phase	Post phase	Total
Teachers' self-reflection		5 days (= periodic meetings) + 5 (individual work on the case reports)	2 (individual work on the case reports)	12
Teachers' reflection supported by facilitators		4 (periodic meetings)	2 (periodic meetings)	6
Teacher in-service modules	6 (two three day workshops for all 30 teachers)	6 (teachers attend several shorter workshops specific to his/her needs)		12
Total (days)	6	20	4	30

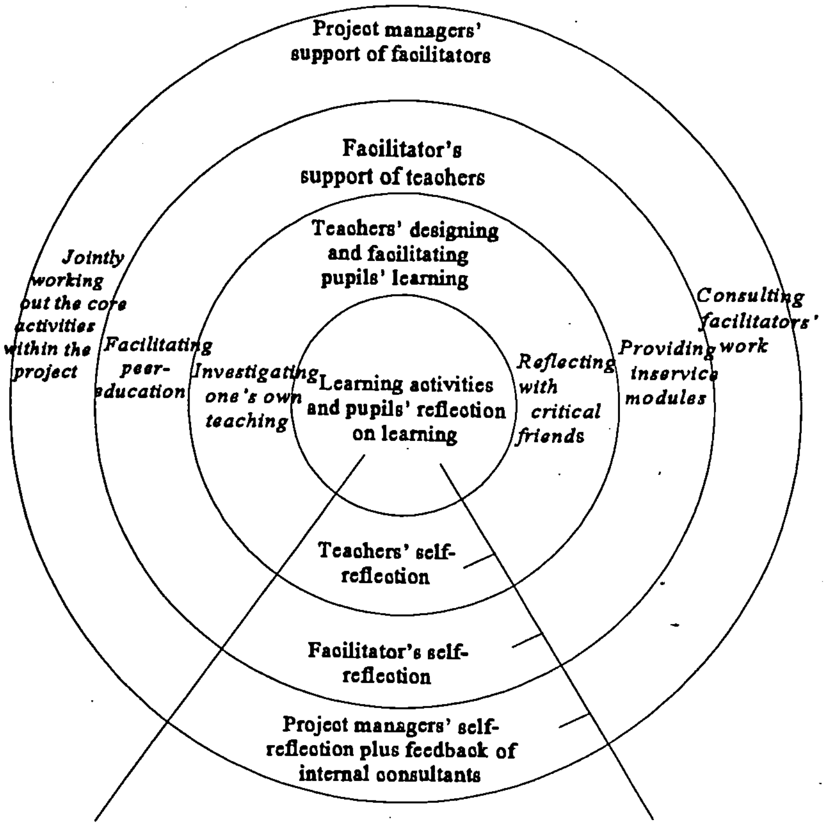
Teachers' designing and facilitating pupils' learning

Pre phase: Teachers formulate their visions for the further development of their mathematics teaching, develop ideas for first steps (teaching innovations, e.g. another approach to trigonometry) that could be taken in the next school year and define realistic criteria for the success of the project.

Main phase: Pupils are informed about the intentions and the aims of the project and the steps the teacher wants to take. The teacher explains the main topics to be covered in this school year (e.g. on a poster which remains in the classroom). Time is reserved for the pupils reflecting on these intentions and topics and to become co-producers of the project (negotiation of "rules" etc.): they reflect on their possible contribution to improve learning and teaching and work out criteria for their success within the project. As a starting point to discuss their current abilities in mathematics a subset of the TIMSS-mathematics literacy test is used. Each class works out a joint slogan for improving learning and teaching in this school year (e.g. "Math is more than rules") and designs some steps. Pupils are invited to write their experiences down at the end of each mathematics lesson ("math diary") and to inform the parents about the project on a periodic basis. Throughout the whole school year teaching innovations are carried out and evaluated by pupils and teachers.

Figure 1

Core activities within the CAPE project



Presentation of the results of the project (by pupils, teachers, facilitators, ...) to "relevant environments" (FINDUS, SWEDE, school authorities, principals, teachers, parents, ...)

Additional elements of evaluation (e.g. TIMSS, questionnaires on beliefs, ...) used within the project

Teachers' self-reflection

Pre phase: Teachers reflect on their learning processes within the in-service courses provided by the facilitators and exchange experiences with other teachers with regard to their goals and intended steps concerning the project.

Main phase: Teachers reflect on the in-service courses provided by the facilitators and on their support for teachers' action research processes. More importantly, teachers reflect on their teaching innovations, both together with their colleagues (one hour per week on average = in total five days) and individually, in particular through continuously working on their case report (also one hour per week on average).

Post phase: Teachers reflect on what had been achieved within the project (e.g. using the criteria for the success of the project defined at the beginning of the process) and finalize their case report.

Facilitator's support of teachers and self-reflection

Pre phase: Teachers are informed about the intentions and the aims of the project. Time is reserved for reflecting on these intentions and for them to become co-producers of the project (negotiation of "rules" etc.). The teachers are introduced by two three day workshops into the "methods of action research" and into "aspects of quality of learning and teaching mathematics", e.g. reflecting on the importance of clear goals (understood by the pupils), of challenging tasks, of freedom of scope for pupils to define their own ways and activities in the learning process, of a mixture of teaching methods, of mathematics as an activity (and not as a "ready knowledge" to be transferred to pupils, ...), including enough freedom of scope for teachers to define their own plans for teaching innovations in their mathematics classrooms (based on their current status of teaching and on their strenghts). The facilitators work out criteria for the success of their facilitation activities within the project and reflect on their interventions.

Main phase: The teachers are supported in designing the communication and learning process in the mathematics classrooms. Periodic meetings and professional exchange of experiences within the group of mathematics teachers at each school are facilitated, in particular with regard to analysing teachers' and pupils' efforts to improve mathematics learning and teaching. The teachers can choose among different in-service education courses provided by the facilitators, e.g. dealing with specific mathematical topics, with problems concerning the teaching language or other issues important for the teachers.

Post phase: Facilitators provide support for teachers in interpreting their data, in finalizing the case reports and in preparing their results for a final presentation.

2.3 Basic elements of the evaluation of the CAPE project

The evaluation of the project is based on three basic elements:

a) Self-evaluation

The most important part of the evaluation of the project is self-evaluation. Self-evaluation is an integral part of the facilitation and in-service education process and is continuously practised by pupils, teachers, facilitators and project managers. Each group defines its own criteria for their success within the project and assesses them at the end

of the project. A basic method of gathering data is keeping a "math diary" (pupils) or a "research diary" (teachers and facilitators). Further elements of self-evaluation:
- The pupils write a story about their learning process at the end of the school year, in particular relating to the question of how their view of mathematics has changed.
- The teachers investigate their own teaching (e.g. making or organizing classroom observations and conducting or organizing interviews with pupils) and write a case report (starting with a mathematical curriculum vitae) in which they document their development process (including reflections on the development of the group of mathematics teachers at their school).
- Each mathematics head teacher writes a short report at the end of the school year in which he/she reflects on the development of his/her group of mathematics teachers.
- Each principle writes a short report at the end of the school year in which he/she reflects on possible implications of the project for the whole school.
- Each group of teachers conducts interviews with principals and other colleagues at their school before and after the project in order to gain an insight into their view of mathematics (teaching).
- The parents are invited to write a short feedback to the teachers about their observations with regard to their own children (based on their weekly notes).
- The facilitators write a case study reflecting on their facilitation process (including also reflections on the development of the group of facilitators).
- The project managers write a case study reflecting on their management and facilitation process.

b) Cross-case and meta analyses

- Each facilitator writes a cross-case-analysis of the case reports of one group of mathematics teachers which will be discussed with the teachers.
- The project managers write a cross-case-analysis of facilitators' case studies and a meta-analysis of facilitators' cross-case-analyses which both will be discussed with the facilitators.

c) Additional elements of evaluation

In addition to self-evaluation some further elements of evaluation are carried out by the facilitators. In all cases the idea and the results of the evaluation will be discussed with the teachers and can also be used for the self-evaluation process (e.g. to generate hypotheses about the progress within the project):
- Field notes of facilitators, in particular with regard to teachers' language (as used in classroom or while reflecting about the learning and teaching of mathematics) and teachers' understanding of their role in the classroom. Other issues might be developed within the project, e.g. proposed by the teachers.
- Analyses of pupils' results with regard to a mathematics literacy test (selected items from the TIMS-study) at the beginning of the school year in order to get an insight into pupils' competence in mathematics (literacy) and on which to base (at least partially) the planning of the teaching innovations. Comparison of the test results with those of a post test using a similar sample of items of the TIMS-study.

- If possible and meaningful: Comparison of national test results of the participating classes at the end of this school year with those in former school years.
- Pre and post questionnaires for pupils, teachers and parents in order to get a general overview of possible changes in beliefs, attitudes and experiences. (The data will show the absolute and relative changes in comparison with the pre questionnaire. With regard to the comparison between the five schools it may be better to publish only the relative changes.)

The final report about the CAPE project will include summaries of all data (case reports and studies, cross-case and meta analyses, results and interpretations of the additional elements of evaluation, ...). The progress of the project will be discussed on different levels:

- pupils (as individuals and as a class; knowledge, beliefs, ...)
- teachers (as individuals and as a group; knowledge, beliefs, ...)
- schools (importance and "visibility" of mathematics, possible influences on other subjects or social and organisational improvements)
- school system (changed view about education and mathematics by pupils, teachers, parents, school authorities, public, ...)
- (theoretical and practical) knowledge about the improvement of learning and teaching of mathematics, of teacher education and facilitation processes, of the management of such projects (including a reflection on supporting and impeding factors).

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RESEARCH FORUM

Theme 1: ***Mathematics in and out of school***

Co-ordinator: Alan Bishop

Presentation 1:

Reflecting on mathematics learning in and out-of-school from a cultural psychology perspective

Guida de Abreu

Reaction: Philip C. Clarkson

Presentation 2:

A semiotic analysis of students' own cultural mathematics

Norma Presmeg

Reactions: Marta Civil
Kathryn C. Irwin

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REFLECTING ON MATHEMATICS LEARNING IN AND OUT-OF-SCHOOL FROM A CULTURAL PSYCHOLOGY PERSPECTIVE

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In this paper three analytical scenarios are presented as a basis for reflecting on the nature of mathematics learning in and out-of-school: (1) uses of mathematics in a traditional sugar-cane farming practice; (2) encounters with new mathematics in the process of social change or innovation of farming technology and management; (3) learning of mathematics by a new generation, the children who have been exposed to the coexisting traditional and innovative practices. These scenarios were unique in terms of the salient features of the sociocultural context analysed and they influenced my reflections on situated learning and cultural psychology.

The background

The relationship between the learning and the uses of mathematics in school and out-of school has been discussed in the last decades from several distinct perspectives. Investigators in Brazil have offered an outstanding contribution to this debate. The peculiarities of Brazilian society have enabled them to ground their theoretical perspectives firmly in the realities of people's lives. They were faced with a country where an active working class force has emerged independent to a great extent of formal schooling. For researchers this posed a main problem: how could the same population experience failure in school and be quite skilled in their jobs? Two disciplines that have approached the problem from a sociocultural stance are ethnomathematics education (D'Ambrosio, 1985) and developmental psychology (Nunes, Schliemann & Carraher, 1993). At first glance they appear to be quite distinctive, ethnomathematics educators who are concerned with historical and anthropological analysis of the mathematics of different sociocultural groups and psychologists who study the psychological processes involved in learning and using mathematics in specific sociocultural contexts. However, in fact, they appear to be concerned with the same phenomena at different levels of analysis, the former at a sociogenetic and the latter at an ontogenetic level. Reflecting on the Brazilian experience one can see that both ethnomathematics educators and psychologists agree in one quite critical issue. That is, both demand the legitimacy of the forms of knowledge associated with out-of-school practices. However, they diverge in the forms of legitimacy they emphasize. Nunes et al. (1993) stresses the cognitive consequences, that is, how specific cultural tools mediate mathematics cognition. D'Ambrosio (1985) calls attention for the relationship between social-political order and individual learning: how the value social groups attribute to certain forms of mathematics could have consequences for the learner.

It seems the integration of the complementary views on mathematics learning derived from Nunes et al and D'Ambrosio will be useful. In fact, that is the way my own empirical work has evolved: from a perspective that considers the mediating role of cultural tools to one that also accounts for the mediating role of social valorisation. Much of this development was initially data driven, or grounded in ethnographic type descriptions. However, the current developments in the field of cultural psychology seems to provide a sound basis for a new research agenda. In the next section I will briefly introduce this "new field". Then I will use my own studies: (a) to illustrate the need to take valorisation into account in psychological studies of mathematics learning in and out-of-school; (b) to speculate on the relationships between valorisation and the emergence of diversity (a phenomenon central to human learning and in cultural psychology).

A cultural psychology perspective on mathematical learning

Over the last three decades psychologists studying learning in sociocultural contexts, both in and out-of-school, have been moving from a cross-cultural experimental psychology, passing through a Vygotskian-based sociocultural psychology, to the most recent calls for a cultural psychology (Bruner, 1990, 1996; Cole, 1990; Cole, 1995; Cole, 1996; Lucariello, 1995; Price-Williams, 1980; Shweder, 1990).

Nancy Much (1995) claims that the impetus for "cultural psychology" is related to the publication of the book: "Cultural psychology: essays on comparative human development" by Stigler, Shweder and Herdt (1990). The idea of cultural psychology, however, is not recent. Cole (1990) reminds us that it was present at the birth of psychology as a science. Wundt divided psychology into two halves (Cole, 1990; Miller, 1962). One half, that of experimental psychology, was focused on simpler mental functions (sensation, memory, perception), and can be studied in laboratories. The other half, named by Wundt as "Volkerpsychologie" and translated by Toulmin (1980, cited in Cole, 1990) as "cultural psychology", consisted of the higher mental functions involved in human thinking. For Wundt the sociocultural character of these higher functions, such as influences of "linguistic habits, moral ideas, and ideological convictions" (Miller, 1962, p. 38) meant that they could not be studied by the traditional experimental methods. Instead he seems to have suggested that they "can be explored only by the nonexperimental methods of anthropology, sociology and social psychology" (Miller, 1962, p. 38-39).

It is now acknowledged that a network of several approaches towards a cultural psychology have been emerging , not only in the last three decades, but during this century (Cole & Engestrom, 1995). For instance, these include, Bartlett's approach to the study of human memory, published in his classic book "Remembering" (Bartlett, 1932), Vygotsk's conception of human learning as socioculturally mediated (Vygotsky, 1978), and Herbert Mead's account of mind as emerging out of interactive and communicative processes in particular societies, also in a classic book "Mind,Self and Society" (Mead, 1934). The common thread in these approaches is their attempt to understand the interplay between sociocultural contexts and human psychological functioning. It is beyond the scope of this paper to engage in a deep analysis of the differences between approaches to cultural psychology. Instead I will try to focus on how the general orientation of the perspective can offer a useful framework for understanding mathematical learning in and out-of-school.

Diversity in human psychological functioning

A first reason why a cultural psychology perspective could be useful to explain the uses and learning of mathematics in and out-of-school is the recognition that there is **diversity** in human psychological functioning. Diversity has been used as a key criterion to establish the uniqueness of the perspective and of the interests of those that follow it. Stigler, Shweder and Herdt (1990) remark that:

> *The basic idea of a cultural psychology implies that an "intrinsic psychic unity" of humankind should not be presupposed or assumed. It suggests that the processes decisive for psychological functioning (including those processes promoting within-group or within-family variation and the replication of diversity) may be local to the systems of representation and the social organisation in which they are embedded and upon which they depend* (p. viii).

Wertsch (1991) suggests that the renewed interest in cultural psychology is linked to the possibility of following an alternative research agenda that focus on the aspects of mental functioning that are socioculturally specific. Quoting his own words he states that:

> *Many psychologists have concerned themselves with the universals of mental functioning, and this emphasis on mental processes which are assumed to be universal, has dominated research in contemporary western psychology. In contrast, my focus emphasises what is socioculturally specific. In this sense it is in accord with "cultural psychology" (....) In recent years, a variety of factors (Cole, in press) have inspired renewed interest in the issues this discipline addresses. At a general level, this renewed interest is grounded in the assumption that "cultural traditions and social practices regulate, express, transform, and permute the human psyche, resulting less in psychic unity for humankind than in ethnic divergences in mind, self, and emotion (Shweder, 1990, p.1)" (Wertsch, 1991, pp.6-7).*

Although I agree with Wertsch that recognition of diversity involves a focus on what is socioculturally specific it seems to me it needs more than that. It will require a different account of the interplay between person and socioculturally specific environments (Shweder, 1990; 1995). For me a true account of diversity will need to cover the phenomena of individual variation. For instance, it will need to provide insights on why the same given-context, e.g. a school maths lesson, is re-constructed in different ways even by children apparently belonging to homogeneous home communities. The reason for stressing this point is my reading of the way diversity has been approached in the Vygotskian-based approaches. Taking as an example the studies on mathematical cognition, diversity was first recognised in variations between cultures (e.g. Liberian Kpelle versus US Americans, in Gay and Cole, 1967); then it was extended to understand variations in mathematical thinking within one culture (e.g. out-of-school mathematics versus school mathematics in Brazil, Nunes et al. 1993). At an empirical level analysis of diversity stops at this point: variations between individuals were overlooked. But, a theory of human development and learning that aims to be relevant to practice cannot stop here. Indeed, this is a problem teachers are faced with: they can try to explain differences between children that have different social and cultural backgrounds with reference to their membership in particular groups, but they are stuck when they try to provide reasonable explanations as to why children from a similar background have different performances.

Having decided that the main research agenda for cultural psychologists is to understand human diversity some new questions need to be asked: What constitutes diversity? How does it emerge? How is it maintained? These are quite complex issues. Indeed, diversity can be defined according to different criteria and this might have been one of the reasons for the neglect of valorisation as a mediator in human learning. A brief review of Wertsch (1991) analysis of major positions on heterogeneity could be quite illuminating at this point. Wertsch reserves the term "diversity" to refer to the variety of mediational means that can be used by human beings and following Tulviste (1986) uses "heterogeneity" to refer to a variety of forms of thinking. Different versions or positions on heterogeneity "can be distinguished from others on the basis of what is being ranked" (p.97). Table 1 summarises Wertsch's outline of three major positions organised around the genesis of the forms of thinking and its power and efficacy.

It is quite interesting to note how changes in positions of what constitutes diversity require different views on the way people come to master and use the mediational tools available in their cultures. Thus, from a position that sees *heterogeneity as genetic hierarchy* the appropriation is always based on a unidirectional developmental trend both at sociogenetic as well as ontogenetic levels. However, the other two positions require a different account of appropriation and uses of knowledge. What leads cultural groups and individuals to make selective use of apparently equivalent mediational tools? Wertsch introduced the notion of "privileging" as an attempt to account for these types of phenomena. For him "Privileging refers to the fact that one mediational means, such as a social language, is viewed as being more appropriate or efficacious than others in a particular sociocultural setting" (p.124). The dynamics of these phenomena remain to be explained and cultural psychologists suggest this needs a review of the way the person has been theorised in sociocultural theories of human learning.

Table 1: Wertsch's outline of three major positions on heterogeneity

Genesis and power or efficacy	Acquisition of mediational tools
Heterogeneity as genetic hierarchy A form of representation acquired "'later' is viewed as more powerful (and often, at least implicitly, as better)" (p. 97)	"The tools are acquired in a certain order and are therefore inherently organised along a continuum from lower to higher, or from less powerful to more powerful." (p.100)
Heterogeneity despite genetic hierarchy "Although some forms of functioning emerge later than others, they are not inherently better" (p.97)	"Different tools are acquired at different developmental stages, but they have no inherent ranking with regard to power or efficacy." (p. 102)
Nongenetic heterogeneity "There is no inherent ranking, either in terms of genesis or in terms of power, of the various forms of representation and action in human mental functioning. (p.97-98)	The variation on the acquisition of the tools is not tied to developmental stages. (p. 103)

Person, society and culture

The second reason why a cultural psychology perspective could be useful to the understanding of mathematics learning is its "triple focus upon the person, the social structure or social system, and the cultural symbol system" (Much, 1995, p. 103). Much refers to these as three interrelated and mutually constitutive systems and defines each in the following way:

> The first system - "*is a person, with a distinctive biological make-up and unique history of experience.*"
>
> The second system - "*is a 'society', more precisely, the local social structures (for example, the family and other institutions) of a society or culture.*"
>
> The third system - "*is culture in its symbolic sense, culture as a representational system, the collective symbol systems and institutionalised meanings for interpretation and organisation of experience and action in local social contexts*" (1995, p. 100)

Referring to the person, instead of cognition, involves an attempt to bring the concept of self, human agency or identity into the psychology of human learning and development (Lucariello, 1995; Shweder, 1990; 1995). Thus, the way the person is conceptualised in this approach depends, in my opinion, on three complementary foci of analysis:

- How personal knowledge and meanings are constructed from the systems available in particular cultures;
- How personal identities are constructed from roles and valorisations of practices and institutions in specific social structures;
- How the construction of personal knowledge interacts with the construction of personal identities.

Hardly any studies on mathematics cognition in sociocultural contexts have attempted to address issues related to how the person constructs identities. For instance: Why does the person get engaged in a particular practice? How does the person feel about a specific type of participation in a particular practice? What motivates an individual to continue participating in a specific practice? What are the consequences for the individual construction of social identities of successful and unsuccessful participation in practices, such as school mathematics?

With few exceptions (e.g. Walkerdine, 1988; Grossen, 1997) the main tendency in the studies of mathematical learning in sociocultural contexts has been to focus either on relationships between cognition and culture or between cognition and social interactions (Abreu, in press). In addition, they centre on a particular type of relationship between the first system and third system, that is, the personal construction of cultural systems or tools for mathematical representation. So, for now there is a weak conceptualisation of the constitutive role of the second system (society). This explains the lack of analysis of valorisation and related construction of social identities in mediating human learning. Consequently, it is also unclear how these might contribute to emergence of diversity.

In the next section I move to an analysis of my own empirical research, and I hope it will help to clarify the development of my own thinking, so that I come to find in a cultural psychology perspective a useful theoretical and methodological framework.

Studying uses and learning of mathematics in a sugar-cane farming community in the Northeast of Brazil

The three scenarios presented below were constructed on the basis of two distinct phases of empirical research carried out in a community of sugar-cane farmers in the Northeast of Brazil. The first and second scenarios are based on phase one. This involved an investigation into the use and understanding of mathematics by sugar-cane farmers (Abreu, 1988, 1991). Both theoretically and methodologically, this study was informed by the so-called everyday cognition approach (Rogoff & Lave, 1984), and by a Vygotskian view of mathematics learning as the internalisation of sociocultural tools.

The third scenario is based on a second phase of investigations conducted in the same community, but focused on the way school-children experience the relationship between home and school in terms of the mathematical knowledge and its associated values (Abreu, 1993, 1995). It was based on a framework which shared with that of the culture and cognition theorists a focus on children's mathematical activity-in-social cultural context, but which goes beyond that of these theorists by including questions relating to valorisation.

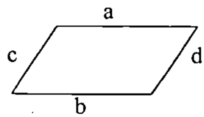
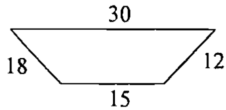
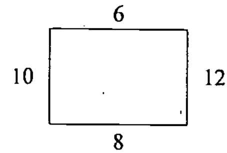
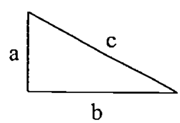
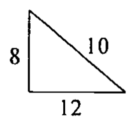
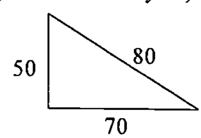
First scenario: the use of mathematics in traditional farming practices

Focusing the analysis on farmers' traditional practices enabled me to document the existence of mathematical tools specific to sugar-cane farming. The ethnographic approach provided the basis for a description of specific mathematics used by the farmers which differed from school mathematics. For instance, they had specific length and area measures, formulae to calculate areas and also a variety "oral" strategies to solve sums involving both additive and multiplicative structures. In addition, findings from the interviews about the strategies farmers used to solve mathematical problems were quite revealing about the way their experiences with the use of specific tools mediated their cognition. For instance, in problems related to the amount of fertiliser applied by area they were sensitive both to the units of measurement and to the numerical relations. When solving problems that involved uses of mathematics farmers chose as mediators specific forms of representation closely linked to their practices. The image we got was that situatedness of cognition relates to the use of particular tools, in specific contexts of practice. These tools enabled the user to function efficiently and perform meaningful cognitive operations.

When I look back at my data it seems obvious that my research was a good example of an approach which was looking for diversity between groups within the wider Brazilian culture, but was still marked by a homogeneous bias in the "within group" type of analysis. By focusing on the similarities of use of tools among farmers I overlooked observations that showed unique appropriation of the tools, and did not explore the mechanisms behind this (Wertsch, 1995). Nevertheless, this latter type of analysis could shed some light on the emergence of diversity between individuals who participate in similar cultural practices.

Looking retrospectively at the data I can see two distinct patterns of re-construction of cultural knowledge by the farmers. Table 2 provides some examples based on their

Table2: Conventional versus unique appropriation of cultural tools

Conventional formulae used in farming	Example following the convention	Example of unique appropriation
Quadrilaterals  The area of a quadrilateral plot of land is found by summing the opposite sides, halving each sum, and then multiplying the two resultant numbers. Area = [(a+b)/2] x [(c+d)/2]	**Unschooled farmer, 46 y-old**  Farmer: here is 30 braças, here is 12, here 15 and here 18. I put these two together, 30 and 15, add them up gives 45; now here 18 and 12 (…) gives 30. Then here is 22 and a half. And here 30, half is 15. [he then multiplies 22.5 by 15]. it is 3 contas and three hundred and seventy five cubes. Area = [(12+18)/2] x [(30+15)/2]	**Schooled farmer, 29 y-old**  Farmer: (1) performs the following calculations: 12 x 8 = 96 10 x 6 = <u>60</u> 156 : 2 = 78 Area = $\frac{(b \times d)+(a \times c)}{2}$ (2) then demonstrates the conventional
Triangles  The area of a triangular plot of land is found by multiplying the average of two opposite sides by one half of the length of the remaining side. Area = [(a+b)/2] x (c/2)	**Unschooled farmer, 64 y-old**  Farmer: Here is 12 divide is 6, here 10 by 8, is 18, is 9. 9 times 6, 9 times 5, 45 and 9 times 1, 9, is 54. Area = (12/2) x [(10+8)/2]	**Unschooled farmer, 50 y-old** (participate in adult literacy programme for 1.5 year)  Farmer: 70 by 25 (…) Here is 50 we have to divide. *(I: you did not average the 70 with the 80?)* You cannot do that. It's wrong. (*He then explains that the largest side can be seen as equivalent to* the *diagonal of a qualdrilateral*).

procedures for calculating the areas of quadrilateral and triangular plots of land. In the first pattern, which was prevalent, farmers followed the convention. It is as if personal knowledge copies the conventional cultural knowledge, a truly Vygotskian account of a re-construction of the social at a psychological level. The second pattern seems to give some indication of a more complex process. There is an indication of uniqueness: personal knowledge is grounded in cultural knowledge, but it is not a copy. If this is the case, a crucial issue to understanding the emergence of diversity is to try to get some insight into what motivates certain individuals to produce these new forms of knowledge: Cognitive understanding? Valorisation of knowledge?

Second scenario: the impact of changes imposed by macro-social structures on farmers' mathematics

The second scenario illustrates the farmers' struggle to cope with changes imposed by the macro-social structures and which involved varying degrees of exposure to new mathematical tools. At the time of my field work the farming community was experiencing one of these external demands. Changes in the Brazilian economy at a macro-level led the government to impose new criteria for the payment of sugar-cane: since 1984 the system of payment for sugar-cane according to quality had been imposed by law. In the past the criterion had only been a function of the weight of sugar-cane produced, independent of quality. This was the only system the farmers had experienced in their whole lives, and mathematically was quite simple.

Following the old system the farmer could calculate how much he would receive from the sugar-mill factory by just multiplying the amount (tons of sugar-cane) he had produced by the price per ton. In the new system he needed to deal with the variables that define the quality of the sugar-cane. In the end the price could be found by multiplying the index of quality, times the amount produced, times a fixed price per ton. The new system posed some difficulties for the farmers, such as: (a) The index of quality was calculated by highly sophisticated computing and laboratory technology located in sugar-mill factories. (b) The index of quality did not apply to the total quantity of sugar-cane a farmer produced, but the tests were carried out on each specific delivery. The consequence was that the price per ton was variable. For the farmer to find out an average price he needed to go through some complex calculations (not straightforward, because the information was presented in sophisticated forms, with values for the different factors, and the need to read complex tables).

This scenario could be analysed from a Vygotskian-based perspective focusing on the limits on cognition imposed by not mastering the new mediating tools. This would emphasise a particular type of relationship between the third system (cultural tools) and the first system (personal knowledge). Again, such orientation, as already exemplified in the first scenario, seemed to be predominant in my thinking at that stage, and can be noted in the guidelines for "questions" to be covered when interviewing the farmer (Abreu, 1988):

1. *Investigate the farmer's opinion and basic understanding of the new system of payment for sugar-cane:*
   1.1. *What do you think about the payment for sugar-cane by amount of "sacarose" in it?*
   1.2. *Do you know what "agio" means?*
   1.3. *And, what is "desagio"?*
   1.4. *Which system do you think is better for you and why?*
2. *How did you calculate your income before the new system? And now?*
3. *In case you have an "agio" of x% (5% and 10%) what will it represent in gain?*
4. *In case you have a "desagio" of x% (5% and 10%) what will it represent in loss?*
5. *If you apply 2 tons of fertilizer in an area of 5 hectares, and this results in an "agio" of 5%, was it a good financial investment?*

Apart from the first question which left some room for the farmer to articulate his experience in an overall perspective, the other questions limited the focus to mathematical understandings. However, research, as a human enterprise, is not a unilateral process shaped by the researcher (Grossen , 1997). My mathematical focus did not prevent farmers from re-interpreting the questions and articulating answers revealing different facets of their experiences with the new system. Table 3 summarizes the impact of the new technology on farmers' lives considering the three systems of cultural psychology.

It stands out from the summary in Table 3 that, for a person, changes in macro-social structures can have different types of impact, which can be linked both to mathematical knowledge itself, and to how knowledge constitutes identities. The first impact of the change was that it required a type of mathematical knowledge that most of farmers did not have. Traditional mathematics (Abreu, 1988) enabled the farmers to grasp some understanding of the new system, such as when comparing whether they were gaining or losing money, but it was limited when they had to read and interpret tables combining different variables, and when required to understand concepts such as percentages, decimals, positive and negative numbers.

The second type of impact was on the farmers' identities. The changes made them experience loss of control, it brought uncertainty, and threatened their standing. They were not sure where they stood, whether they could contract services, borrow money from the bank, and perhaps more than that survive in business.

The third type of impact was that exposure to technological innovation and modern institutions (e.g. schools; banks) over time has raised farmers' awareness that some forms of knowledge are more powerful than others, and also that some are more accepted than others. For instance, a contract in a bank could either be signed or stamped by a finger print. The farmer who signed might be functionally as illiterate as the one who stamps his finger print, neither of them could read the contract. However, the first method enabled the person to feel part of the literate society, and

Table 3: An example of the impact of new technology in a farming community

		Old technology	New technology
System Three Cultural tools		Minimum of oral mathematics with multiplicative structures.	Minimum of written mathematics, but better with the use of modern technology, such as calculators.
System Two Social structures		Knowledge "owned" by the farming community and able to be transmitted in practice. Relatively low status.	Knowledge "owned" by outside groups and driven by forces located in macro-economic structures imposed on the farming community (e.g. competition of the product in the international market). Relatively high status.
System One Person	Knowledge	Knowledge appropriated by most of the farmers and an understanding of the traditional mathematics obvious.	Access to knowledge restricted to those that mastered written arithmetic to the minimum level of coping with percentages and decimals.
	Identity	Clear positioning of the local farmers in the wider farming community, and other institutions determined by their average production, which was used as a "code" for decision making and communication (e.g. with employees and banks).	Uncertainty generated by a loss of control. Not having clear indicators about final income brought difficulty to all the levels of administration and threatened the established identities.
	Interaction: Knowledge X Identity	Traditional knowledge enabled full participation in farming practices, but restricted entry into literate society.	Widened the gap between the type of knowledge required to participate in traditional farming and the one required by a modern technological society.

the second assigned the person to the category of "illiterate". The ability to sign was then highly valued by the group, while use of the finger print a reason to be ashamed. The same applied to the whole of traditional farming mathematics when compared to school mathematics. I referred to this phenomenon as a group's valorisation of knowledge. This seemed to reflect the status of the practices in the wider social structure.

This second scenario shows some of the shortcomings of the explanations about the learning of mathematics in and out-of-school which were based on studies conducted in traditional practices (Duveen, 1996). In modern societies it is more likely that both adults and children will frequently experience exposure to change and co-existing practices. In these circumstances their type of experiences might be more close to those of the farmers when new technologies were introduced. This will involve a reference both to the cultural representation of the tool and to the social value of the groups that own the tool. It follows that if mathematics of specific practices is experienced both in terms of knowing and valuing, why should one expect that the teaching and learning of the new generations should be mainly based on a competency or expert-novice models as advocated in Vygotskian-based approaches?

Third scenario: the engagement of the new generation in the traditional and new practices

While in the first two scenarios the actors were an adult generation, full participants in a practice, in the third scenario there was a shift to young actors, school-children growing up in a farming community. This made it possible to turn participation in the practices and understanding of valorisation into objects of empirical analysis. This study explored how they experienced the relationship between their home and school mathematics. It followed a multi-method approach combining the use of ethnographic and clinical interview techniques. The data were collected in two primary schools by interviewing the children and their classroom teachers, through classroom observation, videotapes, and notes from school documents and pupils' files. A sample of twenty school-children, aged between 8 and 16 and enrolled in third to sixth grade classes, was selected as case studies, balancing successful and unsuccessful pupils and also boys and girls (for a detailed description see Abreu, 1993, 1995).

The results were quite complex. First they provided the empirical confirmation of the existence of diversity among children growing up in what seemed to be a quite homogeneous sociocultural background and this can be seen in the summary provided in Table 4. The first column shows the child's performance in school mathematics. The second column describes the child's participation in home mathematics practices: 'less-engaged' means that when the child engaged in home practices she or he was not usually in charge of the mathematics and 'more-engaged' means that the child was in charge of the mathematics. The third column shows the kind of experience the child had in the farming: 'none' means that neither the child nor the family worked directly in farming; 'family' means that the head of the family had a job linked to farming; 'child' means that the child had personal experience of farming either as a part-time job or helping a parent.

Table 4: Patterns of diversity among children growing up in the same farming community and studying in the same schools

School	Home	Farming experience
Successful (9)	Less-Engaged (7)	none (3)
		family (4)
		child (0)
	More-Engaged (2)	none (0)
		family (1)
		child (1)
Unsuccessful (11)	Less-Engaged (2)	none (0)
		family (1)
		child (1)
	More-Engaged (9)	none (1)
		family (3)
		child (5)

Diversity was apparent in various dimensions, such as, performance in school mathematics, engagement in home mathematics, and levels of exposure to sugar-cane farming. Although we can see a variety of patterns, it seems that, in this particular community the success in school was associated with less-engagement with the home maths, while lack of success with more-engagement. The dynamics that create this phenomenon are still open for future investigation.

Secondly, the findings showed that children representations of mathematics involved valorisation. Although these findings were more clear in the texts of the children's interviews (Abreu, 1993), the summary on Table 5 gives some indication of these valorisations. No recognition of the use of mathematics in farming was taken as indication of "low value".

Table 5: Patterns of identification of mathematics in farming compared with children's competence and recognition of mathematics in a non-farming out-of-school practice.

Child recognised that mathematics was used in the following situations:		Competence in farming mathematics	
		YES (13)	NO (7)
Farming	YES	4	2
	NO	9	5
Non-farming (market stalls)	Yes	11	5
	NO	2	2

While the majority of children could see that people in popular markets or shops use mathematics, the tendency was to deny the use of mathematics in sugar-cane farming. The justification for the denial was the social status of the work (Abreu, in press). Children had developed an awareness of the valorisation of the different tools and how they situated the person socially. The findings showed a clear relationship between the degree of participation and knowledge of specific tools. However, there is not such a direct relationship between participation and valorisation. The different patterns created by the combination of the child's competence and recognition of uses of mathematics in another familiar out-of-school practice illustrated this complexity. To overcome this complexity an alternative is to subdivide the concept of valorisation into: (a) the understanding of social markers (such as, social rules about where the use of the tool can be expected, the status of groups that use the tools, etc.); (b) the positioning each individual assumes towards it (Lloyd and Duveen, 1990). In case study analysis some particular positionings become quite clear (Abreu, 1993). Nevertheless, it is necessary to refine methodologies to separate these two aspects, so that particular positionings do not become confounded with the understanding of the status of knowledge in the social structure. This might also help to obtain a more clear picture of the mediating role of valorisations on learning.

So, now it seems clear from this scenario that the way learners experience their participation in school and out-of-school practices apart from exposure to specific tools, involves exposure to social valorisations. And, that the interplay of cultural tool (system 3) and social structures (system 2) supports not only the personal re-construction of knowledge, but also of social identities.

The next step: extending the approach to children's learning in other contexts where they are exposed to different cultures at home and at school

Within the new cultural psychology agenda and on the basis of the empirical work presented above one key issue that needs further investigation is the emergence of diversity. There was some indication on Table 4 that some patterns can support inclusion on the practices valued by society (e.g. school success) while some can be associated with exclusion (e.g. school failure). However, it is unclear the type of dynamics and developmental processes that can explain such phenomena.

These issues are central in an ongoing research project (Abreu & Cline, 1997), which attempts to expand the insights from the Brazilian research to an understanding of children's learning in multiethnic primary schools in England. Some of the important new methodological dimensions in this study are:

1. The inclusion of a sample of children (British South Asian) where the differences between home and school mathematics will be linked to parents' experiences of a different school system (e.g. going to school in their home country), rather than parents' experiences with a distinctive non-school mathematics belonging to a non-literate culture as was the case in the Brazilian study;

2. The inclusion of an English monolingual sample in which, in theory, the parents have been through the same school system, so that the differences between the home and school mathematics, might be more linked to valorisations and positioning, rather than to the tools.

3. The inclusion of children in different stages of their primary schooling (years 2, 4 and 6, aged between 7 to 11) to explore developmental processes that might reflect changes over time with additional schooling.

In following this line of research, which attempts to get a wholistic understanding of how children experience the relationships between their home and school mathematics, we aim to offer concrete suggestions to improve their education.

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Reflections on mathematics learning and teaching:
Implications of cultural perspectives

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This paper begins with two examples which seem to the author to connect with Abreu (1998) concerns. The discussion then moves to taking a perspective through teachers' eyes of pupils' learning, and considering the interplay of constructs that Abreu develops. Are there concerns for teachers if what pupils actually assessed as being worthwhile for them when doing mathematics, were factored into their teaching role? The paper finishes by reflecting on whether we know what teachers actually value when they are in the act of teaching.

Connections

In the late 1960s when Australia finally realized that Papua New Guinea would become an independent nation, beginning steps were taken to establish a comprehensive primary education system and a minimalist secondary system for Papua New Guinea nationals. At this time the two original tertiary institutions were also founded. Because of this new concern for education in the country a number of researchers became interested in the notion of what should be the composition of the Papua New Guinea curriculum, including the mathematics curriculum. Some began to report on practices that might, they thought, be important for education. But it did not always turn out that way. It was noted for example that the area of a garden was measured in some villages in a way that was more akin to finding the perimeter of the garden, viewed from a western perspective. This type of example, it was readily decided, clearly had no place in the school mathematics curriculum. Even though the size of a garden had to be important, this method did not fit into the proper cognitively sequenced set of experiences that was necessary if a 'real' understanding of 'mathematics' was to be achieved. It would only confuse the pupils about the 'real' concept of area. It was many more years before the same type of education literature started to take seriously the notion that the area of a garden in the eyes of these villagers was a rather a minor quality of the garden; the richness of the soil, how steep was the garden, how many large trunks of felled trees inconvenienced planting, the distance from the sleeping huts, were considerations of far more importance. Only now is the question of what really should be in the Papua New Guinea mathematics curriculum being taken seriously (see Kaleva, 1998).

With such an interplay of cultures, what should the teacher do?

In Melbourne in the late 1990s many immigrant children study mathematics the Australian way. In some classrooms, up to ten different cultural groups may be represented. This multicultural context is often acknowledged; indeed many primary (elementary) school teachers take positive steps to incorporate the diverse cultural backgrounds of their students into the learning environments that they seek to create. But the exception is when it comes to teaching mathematics. For this part of the curriculum it is often assumed that culture has little or no impact. There is little acknowledgment that, for example, the common algorithm for an arithmetic operation may be country dependent. More importantly, one culture may value more highly different aspect of mathematics; for example rote recall or memorization compared to longer, more involved generalized, and at first confusing strategies to solve problems.

What does the teacher do in such classrooms?

Teachers and teaching

Abreu's paper draws together important notions in a new juxtaposition, which gives interesting insights into learning contexts. From this interplay of constructs she reevaluates some of her own work and suggests directions for new research. I found the paper to be very thought provoking. Instead of analyzing it in a sequential manner, I wish to make contact with Abreu's notions and analyses from another vantage point.

Given that there is an interplay of the constructs that Abreu suggests, what does the teacher actually do?

If you as the teacher are working with a group of pupils and you know they come from different cultural backgrounds, you may not be surprised to find that there is a variety of ways that the pupils are framing up the lesson. You may anticipate that differences will occur 'because of the cultural backgrounds'. As an aside, it is of interest to note that when pupils from different background seem to be constructing ideas in a similar manner, you may not be at all surprised, particularly if this is akin to the way you also conceptualize that part of mathematics. I think you might simply say to yourself "Well, that is 'the' way to handle mathematical ideas", and think nothing more of it. Interesting that the similarities are of little surprise to so many teachers, but the differences do catch our attention. May be if we were really aware of the many differences that could occur between individuals, we might be really amazed that there are any similarities at all. However, clearly teachers do attribute some of the differences to cultural background.

However, teachers are also well aware that there are many differences occurring between individuals, even when the pupils come from what appear to be homogeneous home communities. The teachers could argue that these differences may be cognitively based, or may come from the affective domain, depending on the situation in their classroom.

One of the notions that Abreu raises for us is the notion that within the pupils' personal identities, valorization also has an important role to play. It seems that this is something that we as researchers, and teachers in the classroom, may have overlooked for too long. Some individual pupils may choose a particular strategy to solve a problem because they see that it is more efficient or more cognitively meaningful to them. It may be simply habitual. But it may also be that a pupil uses a particular strategy chosen on rather different grounds.

One example of this can be found in some research Lloyd Dawe and I have been conducting (Dawe & Clarkson, 1997). We have been interviewing primary aged bilingual children concerning the languages they use when solving particular mathematical word problems. Sometimes they found the word problem easier to solve after translation because they better understood the meaning of the words when they were translated into Vietnamese. At other times they would said they considered computing the algorithm simpler when they used the method learnt in the Arabic school that they attend after day school? Well sometimes one of these, among a number of other reasons, were the replies to our probing. But we have also been told "Well mum showed me this way", or "This one looks like one that I did for homework and big brother said this was a quicker way to do it", each time processing the solution in the language of the home. These were explanations that we thought might be given from time to time, but perhaps not so often as they were. They were explanations that few of these pupils' teachers expected. But these explanations seem to us to be derived from situations which the students remember and assessed as being worthwhile, and they were willing to repeat the process, at least to some degree, in the rather different context of speaking to a strange researching person from a university who was visiting their school.

Dawe and I still have not explored this issue with teachers in much depth. I think that few teachers actually believe that pupils really choose to do this bit of mathematics this way or that, since there is only one correct way to do it anyway; the way I just taught you. This traditional approach also has the underlying belief that doing mathematics is actually a sequence of doing lots of bits, that some how come together to form an appropriate whole. It does not allow for a broader, holistic, planned approach to a solution. However, there are some teachers, say in problem solving situations, or may be if they really are taking constructivism notions seriously, who do assume or even encourage pupils to make choices. I wonder what assumptions these teachers make when reflecting on why pupils make the choices they do make.

I suspect a teacher's comments, made after listening to pupils outline their solution strategies to the class, will emphasize that there are indeed a number of strategies that can be used to 'get the correct answer'. Given the common attitude of many teachers that there is only 'one right way to do a mathematics problem', this emphasis should not be scoffed at. However if any other comment is made I suspect it might note the quickness of the strategy, or may be fewer number of steps or moves needed for a solution. Such comments clearly emphasize efficiency. Or the teacher may applaud the way a pupil has drawn on mathematical concepts that the teacher did not expect pupils to use. Such notions are good and should be affirmed. We recognize these as being firmly embedded in what we know as mathematics.

But how do teachers respond if the pupil says, "My dad showed me this way"? What if the pupil says "I just like to do it this way"? Is this part of mathematics? How should the teacher respond so they recognize publicly that 'what' the children assess as worthwhile is also appropriate in doing mathematics? When we find appropriate ways of valuing such answers, we will to some extent be recognizing that we are teaching children through the vehicle of mathematics.

One of the issues that Abreu perhaps does not speculate on is the influence of 'the' teacher, and what role this plays in the matrix of constructs she develops to view the learning context. It is an issue that may be worthwhile exploring. If teachers are not fully aware that their pupils will be bringing their whole personality to bear on the doing of mathematics, are they aware that they too bring all their personality to bear when they are teaching mathematics, including what they value as worthwhile? Hence there is an assumption behind the above discussion: It is assuming that teachers actually do recognize what they value in the process of teaching mathematics. It would seem to me that this is different to what teachers actually believe. What they value is their beliefs in action, and hence on display for their pupils (see Bishop & Clarkson, forthcoming).

Finally then, what of Papua New Guinea and school mathematics there? At least in Melbourne and Sydney where Dawe and I have been working, most teachers come from a western background and have a shared notion of what is western mathematics. I have speculated that even in this situation, as does Abreu, the processes are most difficult to tease out. In Papua New Guinea where it is common for a teacher to come from quite a different culture to that of the school in which s/he may be teaching (there are over 700 different language groups in Papua New Guinea), the recognition of the mathematics in the community that would be useful to incorporate into the school curriculum is difficult and time consuming (Kari, 1998). Going further and recognizing the interplay of constructs that Abreu has outlined may be even more daunting, and the patterns may well turn out to be quite different to that finally seen in western countries.

For example, there is some evidence to suggest that Papua New Guinea tertiary students see their lecturers and the mathematics they study in different ways compared to their western counterparts (Clarkson, 1984; Hanrahan, 1997). It will be exciting to see what contributions there will be to this debate from the very rich educational environment in Papua New Guinea.

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A SEMIOTIC ANALYSIS OF STUDENTS' OWN CULTURAL MATHEMATICS

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Significance of the research program.

Concerns about equity in mathematics education have been coming to the fore in many countries in the world (Keitel et al., 1989). Mathematics in particular was the subject, more than any other, that was considered to be value- and culture-free: hence the view of many educators was that mathematics education had no need to take the growing diversity of student populations into account. That this position is untenable and contributes to the differential effectiveness of educational systems has been demonstrated convincingly (Bishop, 1988; Powell and Frankenstein, 1997). In countries as diverse as Brazil (D'Ambrosio, 1985), USA (Wilson & Mosquera, 1991), and Mozambique (Gerdes, in press), educators have called for the recognition not only that mathematics is a cultural product, but that the *ethnicity* (unique sociocultural history) of students can be used in a powerful way in the learning of school mathematics. Connections are advocated between mathematical content and the home cultures of learners, as well as between different branches of mathematics, various disciplines in which mathematics is used, historical roots of mathematical content, and connections with the real world and the world of work (Civil, 1995; Powell & Frankenstein, 1997). In various parts of the world, the need for such connections in mathematics education is being acknowledged and explored, e.g., in the "Realistic Mathematics" project which is ongoing in The Netherlands (Treffers, 1993).

In the USA there is a growing awareness amongst mathematics educators that "the American educational system is differentially effective for students depending on their social class, race, ethnicity, language background, gender, and other demographic characteristics" (Secada, 1992, p. 623). Situated in a broader base of multicultural education literature (Banks & Banks, 1995), recent writings show that acceptance and understanding of cultural diversity is not an option, but a necessity, for those who teach in multicultural schools (Ogbu, 1995). Ogbu's writing, in particular, suggests that the "collective identities" of minority students are complex phenomena. Without understanding of the issues, there is no guarantee that even a well-designed multicultural program will be successful in eliminating inequities (Ogbu, 1995). How then, can mathematics teacher educators prepare prospective and practicing teachers to cope with the challenge of cultural diversity in their classrooms? This paper documents an ongoing research program which investigates some of these issues. There are three components in the research, as follows.

1. Investigation of ways that students may use their cultural identities and practices in constructing mathematical ideas that belong uniquely to them, in a graduate course called "Ethnomathematics".

2. Investigation of ways that a teacher may facilitate students' construction of such uniquely personal cultural mathematical ideas, in a high school classroom.

3. Development of a grounded theoretical framework in which to situate the two previous components, using semiosic chaining in analyses based on Lacan's inversion of Saussure's model (Walkerdine, 1988; Whitson, 1997).

The first and second components involve the pioneering activity of students constructing mathematics curriculum which is based on their own cultures and shared with the class. The third component emerged from a need to situate this activity in a theoretical model which allows interpretation of the processes involved in linking mathematics in and out of school. This model resonates with an emergent perspective (Cobb, Jaworski, & Presmeg, 1996), and also with a developmental research paradigm (Gravemeijer, 1994). The first component, involving graduate students, has been ongoing since this mathematics education course was first taught in 1993. The high school research project was conducted during the 1995-96 school year. The semiotic framework is currently being used to analyze student projects from the work of the past five years. Only a few of the rich examples of student work can be suggested here, along with the methodology of the investigations.

The research described in this paper resonates with and complements that of Abreu, Bishop, and Pompeu (1997), since both programs address issues concerning home and school mathematical practices, the implemented mathematics curriculum, and investigation of ways in which the experienced mathematics curriculum may or may not resonate with home mathematical cultural practices of learners. Abreu analyzed the sociocultural organization of mathematical practices linked to school and home culture, in terms of the structure of the practice, the patterns of social interaction within practices, and prior knowledge of learners. The research described here complements Abreu's research, using a different theoretical framework. The purpose of a semiotic framework for analysis of students' projects is to investigate ways in which signs facilitate the identification by learners themselves, and their teachers, of patterns and structures in cultural practices, so that these patterns may become the basis for mathematical constructs in classrooms, thus bridging these two forms of knowledge which some scholars have described as incommensurable (Dowling, 1995).

The three components of the research program are described further in the following sections.

I. Principles for a mathematics course which takes culture into account.

I shall suggest some reasons for choosing an approach based on ethnomathematics, defined as *mathematics of cultural practices* (Presmeg, 1994), for a teacher education course which addresses cultural diversity (rather than another theoretical framework). There are other approaches to diversity and equity, such as critical race theory (Ladson-Billings & Tate, 1995; Tate, 1997) which have proved useful in addressing minority achievement in mathematics. However, for curriculum development in mathematics, because of the potential for construction of mathematical ideas which are uniquely students' own (Presmeg, 1996), ethnomathematics provides a broader and more viable framework which resonates with the "intellectual property" aspect of critical race theory, in which ownership is important. Thus for purposes of cultural course development in mathematics I have used a theoretical framework grounded in ethnomathematics. I agree with Vithal and Skovsmose (1997) that definitions of ethnomathematics are problematic (see Presmeg, in press, for further discussion of this issue). But in view of the difficulty in defining their own perspective of critical mathematics education, at this time ethnomathematics still appears to be viable as a basis for cultural curriculum development in mathematics.

Literature on the use of cultural practices in learning school mathematics (Bishop, 1988, Ascher, 1991; Civil, 1995; Nieto, 1996), along with the foregoing considerations, provided a basis for the Ethnomathematics course. The following were the principles used.

1. Each student is considered as having a unique sociocultural history; each student has ethnicity, and so does each teacher.

2. This ethnicity is a mathematical resource; mathematics may be developed from associated cultural practices.

3. Students and teachers can use their ethnicity in developing mathematical activities for sharing in mathematics classrooms.

4. Since the sharing of elements of one's cultural or ethnic practices may be a sensitive issue, those who belong to a social group of a certain culture should be involved in making decisions about who should share the mathematics of its practices, and which practices should be shared.

Students in the course (which is offered regularly) are practicing or pre-service mathematics teachers, and the principles imply that the ethnicity of learners is a resource for mathematics teachers at all levels (investigated in part II). This approach entails not only learners' cultural *backgrounds,* but their cultural *foregrounds* too (Vithal & Skovsmose, 1997), since their lived experiences and hopes for the future are taken into account. In spite of cultural conflicts (Presmeg, 1988), when the position is taken that *all* students, and the teacher, have ethnicity, the way is opened to view that ethnicity as a resource and an asset rather than as a liability in a mathematics classroom.

The research involved in this component was an investigation of ways in which graduate students could take ownership of a mathematics education curriculum (that is, make it their own, engage in *developing* curriculum) sufficiently to construct mathematical ideas from cultural practices which belonged to them in some way, and to integrate this mathematics in the curriculum by sharing both the mathematical ideas and the underlying cultural elements. An open and exploratory qualitative methodology was appropriate; data consisted of students' weekly journal entries, fieldnotes following student presentations, and students' final project reports. Analysis of the data yielded multiple ways in which students took ownership as described in the foregoing. Categories of ways in which students chose topics for their personal projects include the following.

1. <u>Looking to the past.</u> An example in this category is Debra Stocking's project, which arose out of of her family tree, which was traced back to the "Domesday Book" of William the Conqueror about the year 1083. She chose to analyze *The Mathematics in Stonehenge*, in the area of England which was the ancient family seat of her ancestors.

2. <u>Looking to the future.</u> Linda Burke chose to examine her potential future investment in the US stock market, rather than a topic from her Caribbean ancestry. Her project was *A Personal Evaluation of Stock Investment and the Mathematics Involved in the Selection of Outback Steakhouse.*

3. Looking to the present. Most of the student choices fell into this category; for example, there were many projects in each of the following subcategories:

* current American sports, e.g., *baseball* (Michael Knight), *mountain biking* (Vivian Knowles), *racquetball* (Kim Crook).

* games around the world, e.g., *Mah Jongg* - China (Lilly Sun), *dominoes* - Cuba (Isabel Tamargo and Raquel Casas), *the Dreidel game* - Israel (Amy Solomon), *contract bridge* (Curtis Ruder).

* navigation, cars, boats and planes, e.g., *traffic flow* - USA (Michael Capps), *flight navigation* - USA (Flora Joiner), *construction of boats* - Haiti (Jean Louis), *traditional navigation* - North Pacific (Denise Gardiner, from the Peace Corps).

4. Looking to national and religious practices and emblems. I have included emblems such as national flags in a category of their own (rather than including them in the previous one) since these analyses frequently involved past and present, the history of a country along with its national and religious practices and underlying philosophies, e.g., *South Korean flag* (In-Gee Lee), *Jamaican flag* (Genevieve Burke), *Gematria* - Israel (Bonnie Jeroslow), *I Ching* - China (Lilly Sun).

5. Looking to arts and crafts. This category and the following one also involve past and present practices, e.g. *Lithuanian folk textile ornamentation* (Trina del Valle), *American patchwork quilting patterns* (Ann Zamanillo, Leonda Bussell, Kim Scales) *Islamic art* (Shabana Ahmad), *Cherokee Indian artwork* (Leatisha Brown), *cuckoo clocks* - Germany (Karen Chandler).

6. Looking to music and dance. Examples include *Flamenco music and dance* (Sandra Davis), *a Jamaican folk song* (Clova Jobson), *construction and playing of the tabla* - India (Parmjit Singh), *Gospel music* - USA (Vanetta Grier).

These examples give a small taste of the rich diversity of cultural practices chosen by students in the more than 170 projects collected to date. (I am putting together an edited collection of some of these.) The significance of this research stems from the excitement generated in students as they took ownership of their personal projects and the mathematical elements of these, shared in class, and learned about mathematical elements in cultural practices authentic to their peers. The implications of this ownership factor were investigated in a high school in the second component of the program.

II The high school research project.

The purpose of the project (conducted with research assistants Stephen Sproule and Bineeta Chatterjee) was to work with students and teachers to develop viable ways of using the authentic cultural experiences of students as a resource for the learning of mathematics. Although it was envisaged that this development might include the introduction of ethnic instructional materials (as described in Zaslavsky, 1996, for example) for the learning of mathematical content in mathematics classrooms, the purpose was not to use such materials in "cookbook" fashion (i.e., as recipes), but as examples for students, of mathematics of cultural practices, i.e., of ethnomathematics. Thus an important part of the project was the development of a bank of personally meaningful student activities which had the potential for classmates to construct mathematical content at various levels. Qualitative research involving interviews with a small number of learners was suitable for the purpose of investigating authentic student activities. These interviews were video- or audio-recorded, took place on a regular basis, and were semi-structured to facilitate comparison but allow the flexibility to pursue unanticipated avenues.

This year-long project started in Fall, 1995, with seven students in a multicultural high school. The research also included observations in their Algebra II classroom, interviews with their teacher, and the teaching of one lesson which was video-recorded. Consonant with the goal of investigating cultural practices authentic to these diverse students, and their beliefs about mathematics and whether it was involved in their practices, we asked the students to talk about their Histories, Hobbies, Hopes (career aspirations), and Homelands, and mathematical ideas which they could discern in these "four Hs". The results may be summarized as follows (see Presmeg, 1996, & in press, for greater detail).

1. Students in this project entertained beliefs about the nature of mathematics which prevented them from "seeing" mathematical ideas in their own practices.

2. Nevertheless, they described rich and diverse cultural practices which the researchers considered to have potential for the construction of mathematics.

3. It was concluded that use of cultural practices in traditional classrooms requires renegotiation of both the social norms (involving patterns of social interaction) and the sociomathematical norms (involving what is taken to constitute mathematics) (Cobb et al., 1997).

Based on the evidence of the interview data and classroom observations, a strong case can be made that traditional mathematics teaching does not facilitate a view of the nature of mathematics which encourages students to see potential for mathematics in their lives outside the mathematics classroom. All but one of the students interviewed, Big Al in the final interview, perceived mathematics as a "bunch" of numbers, equations, and formulas, used to solve (school) word problems. These beliefs about the nature of mathematics persisted even after involvement in the project and the one ethnomathematics lesson. It was not possible to do more than this one lesson because the teacher felt pressured to complete the syllabus, and in fact expressed impatience that the activities had taken longer than intended. It may be concluded that introduction of ethnic and home activities into mathematics classrooms needs to be accompanied by a recognition of the value of such activities, and such recognition may involve a change of belief about the nature of mathematics, on the part of teachers as well as students. Pompeu (1992) wrote of the necessity of changing teacher attitudes in a project which attempted to integrate ethnomathematics in the school curriculum in Brazil.

In contrast to this somewhat negative conclusion about beliefs concerning the nature of mathematics, a very positive aspect of the research was the richness of the mathematical aspects of the cultural and home activities described by the high school students, even if these students themselves did not always identify these aspects as mathematical. Examples of such potentially mathematical activities were racing around barrels on horseback in Central Florida, international coin collecting, American football, basketball, volleyball and cheerleading, carpentry and housepainting.

The project, which seemed merely to scratch the surface of a much larger undertaking, does however suggests that ethnomathematics, taken broadly to encompass mathematical elements in everyday activities of students, has an important role to play in making mathematics more meaningful in the lives of students. The idea is that a cultural activity which is authentic to at least one student would be described in class by that student; then small group work and whole class sharing would be used to identify and symbolize patterns as a basis for developing mathematical concepts from the activity. For instance, one of the project students, Keri, grew up on a farm in central Florida. She described rich patterns involved in the practice of barrel racing on horseback (e.g., "hairpin", "candle", "cloverleaf"). These patterns are important for students in Keri's class, because they are part of Keri's culture; but they may not have meaning for students in other classes who do not know Keri. Mathematical principles of symmetry, distance and time, and sharing of culture, are involved simultaneously. (Incidentally, Keri herself, when asked if these activities were mathematical, replied, "Naw! Well, maybe; if you count the barrels, one, two, three, four, five, six.")

Many questions remain. For instance, should it be a case of "out-of-school mathematics on Fridays", while "academic mathematics" is taught on the other days? Contrary to this view, academic mathematics may be viewed as a form of ethnomathematics too (since distinct cultural practices are involved). This may be linked by semiosic means to cultural practices outside school, for instance in symbol systems which become the *counterpart* - acted upon as objects in their own right (Pimm, 1995) - of patterns in cultural practices (see part III). Thus there is a place in the curriculum for "mathematics for its own sake," the advantage of this broad characterization being that cultural links are not ignored. In the groupwork following our one "ethnomathematics" lesson, mathematical constructions of the high school students were of varying degrees of mathematical sophistication, and certainly not all were algebraic - and it was an Algebra II class! Many of the student constructions had potential for geometric rather than algebraic ideas. Would this matter? The questions are legion. However, the project has shown enough promise for me to believe that there is a possible didactic interface, mediated by semiosis, between cultural practices and academic mathematics, and that pursuing such an interface is of value in the search for equity and meaning. Current trends have teachers playing a greater role in the development of curricula than was the case in some countries previously (Posner, 1992). What is suggested here goes a step further, in that teachers would be facilitating the development of curriculum by students.

Dowling (1995) wrote powerfully about the incommensurability of the domains of everyday knowledge and academic knowledge, which he characterized as belonging in what he called the public domain and the esoteric domain respectively. I agree with the basic tenets of his characterization, but wish to suggest that semiosis, and in particular, the chaining of signifiers, provides a missing link between the domains, which has the potential to "open up the availability of academic discourse to all" (Dowling, 1995, p. 223), while retaining the personal ownership factor which was crucial in the first two components of this research program.

III Semiotic analysis.

This theoretical component of the program arose from the previous two components in the need for a model to use in analyzing processes used in constructing links between cultural practices and academic mathematics. In investigating factors in the discrepancy between the facility of the graduate students in making these links, and the almost total lack of perception of links by the high school students, a theoretical framework was needed in which to examine these processes in more detail. The analysis of student projects is ongoing and will be described in the presentation. There is room only for a brief sketch of the model here.

Using Lacan's inversion of Saussure's diadic model of semiosis (activity of signs; see Whitson, 1997, p. 99), it is possible to analyze chaining of signifiers which involve mathematical symbolism, leading in two or more steps from an initial signified which is situated in a cultural practice, through various symbolic signifiers, to a mathematical structure which is isomorphic to the structure of the original practice and retains some of its properties (Presmeg, 1997). As Walkerdine (1988) pointed out, resonating with Dowling's (1995) characterization, very different discourse patterns and power relationships exist in cultural practices and in mathematics classrooms: the subjectivities of those positioned in these practices are involved. The significance of a semiotic framework for analyzing the links lies in its capacity to retain, not gloss over, these differences, while unfolding structural isomorphisms which allow for the construction of mathematics. In Dowling's (1995) view, "a sign may quite easily be carried between activities, but its signification will of necessity be transformed, because it will participate, relationally, in distinctive systems" (p. 214). The chaining component in semiosis brings about several successive transformations, while retaining essential structures which are isomorphic in relationally different domains. The knowledge constructed rests squarely in the domain of academic mathematics but retains some meanings from Dowling's (1995) "public domain" in a sliding under effect, which was also illustrated in a different context by Cobb et al. (1997).

I have analyzed Marcia Ascher's characterization of the Warlpiri kinship system as a dihedral group of order 8, in terms of a semiotic model, in a different paper (Presmeg, 1997). Here I shall include only the diagram from that analysis as a first example of semiosic chaining which links cultural practice and academic mathematics.

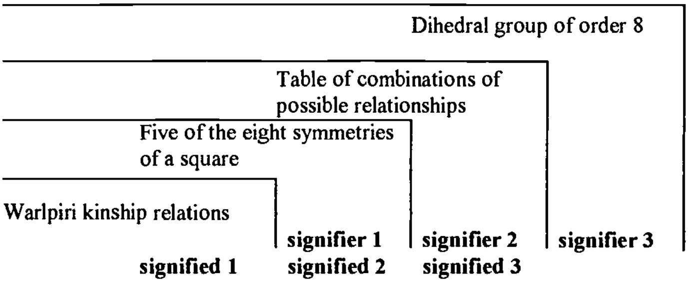

Figure 1: Chaining of signifiers in a progression of generalizations from the Warlpiri kinship system to a dihedral group of order 8.

There are increasingly abstract systems of symbolism in this example, in the sense that there is progressive distancing from the cultural practice in which the recognition of structure originated. In the use of the sides and symmetries of a physical square, say, made of cardboard (signifier) to illustrate the structure of the Warlpiri system (signified), the symbolism may not yet be of a level of generality to satisfy some definitions of academic mathematics. In the next link of the chain, the concrete square gives way to more abstract symbolism in a table. Finally, a generalized structure called "a dihedral group of order 8" becomes the signifier for this specific table, which is now no longer the signifier, but the signified, in an academic mathematical structure. Each sign in turn is subsumed under a new signifier, which belongs to a different set of cultural practices, with different goals, norms, expectations, discourse patterns, and power structures. However, because the whole sign, including signifier and signified, slides under and is incorporated in the new signifier, some of the implicit meanings and associations of the old sign with its sociocultural norms, are still present in the new sign with its own set of different sociocultural norms (Cobb et al., 1997). In this way, some of the associated personal meanings may be retained in this transformation.

What follows are two further sketches of semiosic chaining from student projects which linked personal cultural practices and academic mathematics.

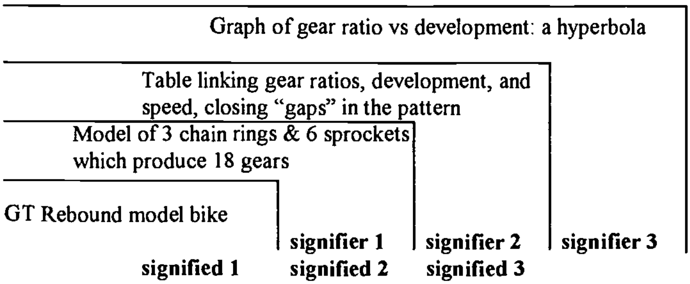

Figure 2: Chaining of signifiers in "Mountain Bike Mathematics", by Vivian Knowles.

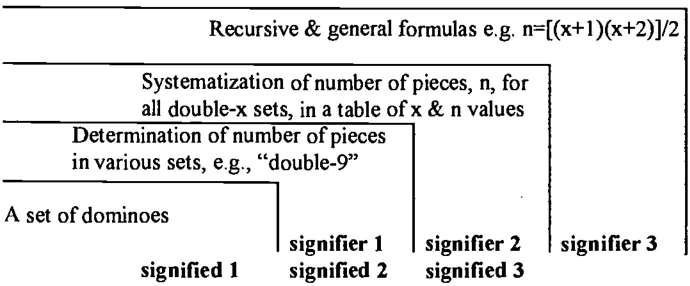

Figure 3: Chaining of signifiers in "Dominoes: A Tradition in Cuban Families" by Raquel Casas and Isabel Tamargo

In figure 3, the chain of signifiers could continue. The values of n in Raquel and Isabel's table constitute the triangular numbers, which could be signified by diagrams of various kinds. The mathematization could then be extended to other figurate numbers (e.g., square, oblong, and patch numbers), with their associated formulas and diagrams, and even to mathematical relationships between figurate numbers, all squarely in the domain of academic mathematics.

In this way, semiosic processes may be used to illustrate connections as symbol systems are constructed in a bridge between cultures. Thus, symbolism provides possible connections between mathematical ideas "frozen" in practices (Gerdes, in press), and academic mathematics. Different symbolism would facilitate the construction of different mathematical structures and concepts (Pimm, 1995; Presmeg, 1997). The role of abstraction and formalization is clear from the analysis, resonating with Noss's (1997) implied answer to his question, "Are we justified in talking about a (unique) mathematical idea represented in various ways?" He continued, "Or should we better acknowledge that there are no mathematical ideas without representation, and that a change in the mode of representation necessarily entails a change (however subtle) in the idea itself?" (p. 290).

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LINKING STUDENTS' OWN CULTURAL MATHEMATICS AND ACADEMIC MATHEMATICS

A reaction paper to Presmeg

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Presmeg's paper contributes to a fast growing research body in the general area of culture and mathematics. In particular it presents a research program that aims at connecting students' own cultural mathematics with school[1] mathematics. It is an ambitious research program that has implications for teacher education, curriculum development, and learning theory. I will address each of these areas in this reaction. I will do so by mostly raising issues that perhaps can be discussed during the research forum.

Teacher Education

Presmeg's approach to the course described in her paper centers around the idea of students' ownership of mathematics. By working on projects grounded on their backgrounds or foregrounds, these teachers and teachers-to-be take ownership of the mathematics embedded in their projects. When I first heard about Presmeg's work in this area at PME in Lisbon (Presmeg, 1994), I was drawn to it as it fit very nicely with my own research interests. Our goal is the development of curriculum units grounded on the students' knowledge and experiences in everyday life. The teachers we work with receive ethnographic training and then visit the households of some of their students to conduct extensive interviews to learn about their "funds of knowledge" (experiences, knowledge, resources that exist in any household). They then develop curriculum units (in collaboration with University based researchers as well as with their students and often some parents) that build on these funds of knowledge. Yet, we have not worked on building on the teachers' own funds of knowledge, which is what Presmeg's work does in my opinion. Presmeg's approach allows teachers to experience the process of exploring / researching their own mathematics, perhaps as a first step towards the involvement of their own students in this activity.

To me, the main reason why we would want experiences such as Presmeg's in our teacher education programs, is because I think that in looking for mathematics "out there," we are forced to address our beliefs and values about what mathematics is, and in a teacher education context, about its teaching and learning. Presmeg addresses this point with respect to the High School project, but I would predict that the teachers' projects (or their discussion with them) offer a window to their beliefs about what counts as mathematics. On the one

[1] In this paper I will use school mathematics and academic mathematics somewhat interchangeably to refer to a vision of mathematics in school that attempts to emulate "mathematicians' mathematics." I am aware that these two types of mathematics can be very different (Civil, 1995).

hand, students (such as the high school ones in Presmeg's paper, or the ten-year-olds in my work (Civil, 1994)) may have a hard time seeing mathematics outside the domain of school mathematics. On the other hand, in my work with teachers, and I wonder if it is also the case in Presmeg's work, I sometimes feel a tendency to see mathematics everywhere. In the NCTM standards (1989), one reads "to some extent, everybody is a mathematician and does mathematics consciously. To buy at the market, to measure a strip of wallpaper, or to decorate a ceramic pot with a regular pattern is doing mathematics" (p.6). Is it? The recent reform in mathematics education in trying to move away from a very narrow image of what counts as mathematics, seems to be interpreted as implying that "mathematics is everywhere." I question this omnipresence of mathematics.

Engaging teachers in actually looking for mathematics outside their direct experience with school mathematics provides us with situations (e.g., Presmeg's students projects) in which we can discuss "where is the mathematics?" Unfortunately, in the paper there is not much elaboration on what the mathematical content of these projects was. What I would like to know is not only what mathematics we, as mathematics educators, may be able to read into these projects, but what mathematics the students themselves saw in the projects and how they described it.

In our research project, a key challenge is the uncovering of mathematics in everyday life situations. By this I mean going beyond a description of more or less superficial uses of mathematics in these everyday situations. As we analyze the household interviews, or the more focused occupational interviews that we have recently started (e.g., an interview with a student's father who works as a car mechanic), or projects along the lines of those by Presmeg's students, I see different aspects in the analysis. On one hand, I need to see the mathematics in them. In my case, I am aware that my own training in "formal" or "academic" mathematics limits my analysis. Miller's (1992) paradox rings true in our work: "How can anyone who is schooled in conventional Western mathematics "see" any form of mathematics other than that which resembles the conventional mathematics with which she is familiar?" (p. 11) Not knowing about the practices themselves is perhaps the main problem (this became evident to me in a recent discussion on carpet and tile laying or in the work of the car mechanic). This certainly raises implications for the classroom: to which extent can a teacher (or anyone) develop mathematically rich experiences that are grounded on practices with which he/she is not familiar? An example of this was the mountain bike example in Presmeg's paper. My lack of knowledge about this type of bikes made the example hard to follow. Once these bikes were described to me, I still had some difficulties because I needed to visualize the gears but I could follow the mathematics behind it. On another hand, our work is with teachers and they need to see the mathematics too. This is why I raised earlier the question about what mathematics the teachers themselves saw in their projects. Of course this is not to say that "not seeing the mathematics" presents a problem. In fact, for us, it provides an opportunity for collaborative work: we discuss and analyze the mathematical content in the different situations, which in turns allows for another approach to professional development, one in which we each bring our different areas of expertise. Finally, a third aspect in the analysis has to do with connecting this "out-of-school" mathematics with the required curriculum, or more generally with school mathematics. Presmeg undertakes this in her semiotic analysis to which I will return later.

Curriculum Development

Although there seemed to be some curricular implications in the teachers' projects, these are not developed in the paper. Thus, I will focus on the High School project as an example of curriculum development. A piece that seems to be missing in Presmeg's paper is the role of the high school teacher. Who was this teacher? Was he/she one of the teachers who had participated in the Ethnomathematics course? Linking the two parts of this research program seems like a desirable undertaking (i.e., first the teachers themselves explore their own cultural mathematics; then, in their classrooms, they engage their students in doing the same thing). The high school project, as presented in the paper, seems to be conducted by "outsiders" (Presmeg and the research assistants). If the goal is to gather evidence that students' cultural and home activities are mathematically rich, then that is fine. From a research point of view, I think that analyzing and disseminating examples of mathematics in activities such as the practice of barrel racing is indeed important as it allows us to expand our visions of what counts as mathematics and of where it can be found. We are doing something similar in our own research project when we "mathematize" the practice of a car mechanic or a seamstress. In involving the teachers we work with in this mathematization, we hope to create a community of inquiry among ourselves that will allow us to then transfer this process to the classroom with the students. However, in this part of our work, we are leaving the parents out: we are the ones imposing our mathematical analysis on their practices. This is certainly a point that I hope gets to be discussed in this forum.

From a curriculum development point of view, I would like to suggest that a next step in Presmeg's line of work in the high school project could be the involvement of the teacher and the students in this research / curriculum construction endeavor. Presmeg does seem to suggest this when she writes "what is suggested here goes a step further, in that teachers would be facilitating the development of curriculum by students" (p. 10). This would probably address some of the problems encountered such as the level of acceptance of the project on the part of the teacher, or the issues around what mathematics is reflected in the different students' projects and how does it fit the required curriculum. This latter point is by no means a trivial one. Presmeg's work was with seven students. How would this translate with the whole classroom? Would we have thirty or so projects? (Actually for high school teachers, it would mean many more than that since they teach several groups.) Students developing their own projects in school subjects is not new from a logistic point of view. However, what Presmeg has in mind, I think, is much more ambitious: it involves a restructuring of the instructional approach. These projects would serve, in part, as a basis for the exploration of mathematics for everybody in the class. One key aspect of that section of the paper is the importance of developing a community of learners in the classroom in order for this proposed approach to succeed. Presmeg's reference to renegotiation of social norms and sociomathematical norms and later on, her remark that "these patterns are important for students in Keri's class, because they are part of Keri's culture" (p. 9), seem to indicate the need for an atmosphere in the classroom that may be different from that in traditional high school mathematics classes.

Presmeg's observations on students' beliefs about the nature of mathematics are no surprise. In our work, we recently had middle school students in a largely Hispanic working-class school fill out a questionnaire that addressed, among others, their perceptions as to whether and how their parents used mathematics in their everyday life. In the past we have also given questionnaires to elementary school students to see how they think they used mathematics in their everyday life. With few exceptions, most of the answers in both cases reflected an arithmetic meaning of mathematics (e.g., counting, adding, multiplying). Mathematics is equated to school mathematics, in fact to a narrow view of school mathematics. It is well documented that what we (as mathematics educators) may perceive as mathematical processes in everyday practices often are not perceived as such by the people involved in these practices and are in fact given less value (e.g., Abreu, 1995; Bishop, 1991; Nunes & Bryant, 1996; Spradbery, 1976).

Thus, developing a classroom learning community and "changing" students' (and teachers') beliefs about what counts as mathematics (and the value accorded to different kinds of mathematics) seem to be two key conditions towards the success of the research program presented in Presmeg's paper. In my earlier work in fifth grade classrooms (ten-year-olds), most of our effort was spent on exactly this--developing a learning community and challenging students' ideas about what mathematics is (Civil, 1994, 1995). Yet, I was not sure about what we had accomplished from a "learning point of view." What had these students learned? How could I tell? It was not till recently when my reading of van Oers (1996) and Forman (1996) brought some light to that work, and thus to our current research. I think that these readings may be relevant to Presmeg's work too. For example, Forman (1996) refers to looking at classroom discourse for indices of learning, and states that "the establishment of a community of practice with a common communication system, norms, values would also be evidence of learning" (p. 128). Presmeg and I seem to share a similar concern, that is opening the patterns of participation, or as Dowling (1995, in Presmeg's paper, p. 10) says to "open up the availability of academic discourse to all" (p. 223). In my own research, as long as the discussions were grounded on students' experiences, we had wide participation, but as the discussion turned more "mathematical" (e.g., an exploration on the angles of [illegible]r polygons), the traditional patterns of participation came back (Civil, 1995). In Presmeg's work, are these students' projects seen as starting points for mathematization (Noss & Hoyles, 1992), examples of protomathematics (Chevallard, 1990), from which mathematics may be derived through careful reflection? (on whose part?; see also Dowling, 1998, on the myth of emancipation). Actually, what seems to be an important characteristic in Presmeg's conception of these projects (whether with teachers or with the high school students) is the retention of the "personal ownership factor" as we move onto the academic mathematics. Presmeg presents the idea of semiosic chaining as a way to preserve that ownership while linking everyday and academic mathematics. Will it? Or is it more like a "tool" for the researcher? I address this in the next section.

Learning Theory

Presmeg refers to the semiosic chaining as a theoretical model to analyze "processes used in constructing links between cultural practices and academic practices" (p. 10). My referring to this section of my reaction as "Learning Theory" is perhaps a bit strong. However, if students (whether teachers as in the first part of the project, or high schoolers (or other grade level) as in the second part) engage actively in the development of semiosic chaining for their projects, then I can see the potential for meaningful (for the students) learning while allowing for development of academic mathematical knowledge. It is in this sense that I am conceiving of this semiosic chaining as contributing to the body of research on theories of mathematical learning, in particular from a sociocultural perspective (Forman, 1996; van Oers, 1996).

I read Presmeg's work on semiosic chaining from two points of view--as a research tool and as a learning (in school) tool. I will focus my remarks on each of these aspects. From a research point of view, I find the idea intellectually inviting. I can see myself using it to analyze, in a way, the occupational interviews we have conducted (e.g., car mechanic, seamstress, construction worker), or the curriculum units we have developed (e.g., games, garden theme). By the way, although I might be able to develop a semiosic chaining for the mathematics in the practice of a car mechanic, I will still not know how to fix a car (Dowling, 1998).

As I read through the three examples presented in the paper, I see a notion of hierarchy. I find it hard to accept that on one hand we can have an isomorphism and a preservation of some of the original properties, in particular the "personal ownership factor" (p.10), yet on the other hand we are moving across different discourse patterns and power structures, as Presmeg very accurately points out. I question, thus, the retention of personal meanings in this process. In a sense when I look at the first level (or step) in the semiosic chaining, I see an example of what Abreu (1998) in her paper for this research forum describes as knowledge "owned" by the people involved in the practice. But, when I look at the last level (and actually at levels prior to that), I see an example of knowledge "owned" by people outside the practice. As Abreu remarks, the former has a lower status than the latter. I am not saying that Presmeg is imposing a hierarchy, I am wondering, though, if it may not be perceived that way. What makes something a signified and something else its signifier? Who decides this?

Nonetheless as a student of mathematics, I think I would have appreciated an example such as the chaining of signifiers for the Walpiri kinship system in my study of abstract algebra. Pedagogically, I think the three examples provide a level of concretization of abstract ideas (dihedral group; hyperbola; formulas) that would be welcome in the study of mathematics, even though the chaining is done by someone else--the researcher in this case. This brings me to the second point of view mentioned earlier, the use of the semiosic chaining as a learning tool. I gather from the paper that the examples presented are the result of Presmeg's analysis, not that of the students. However, if I understand correctly Presmeg's intentions, her idea would be to involve students in the development of semiosic chainings for their own projects. This would provide a way for students to develop their own mathematics and preserve the personal ownership. What will the teacher's role be as students develop a variety of semiosic chainings? As Presmeg notes, "the mathematical constructions of the high school students were of varying degrees of mathematical sophistication" (p.10). How can we "move" them towards the academic mathematics yet preserving their personal ownership of the projects? Note that I say "how," not whether. I work primarily with working-class minority students who traditionally are left behind in the educational system. My goal is to help them succeed in this system (even though I may disagree with what takes place in it). This means an academically challenging program that does prepare them for the more formal and abstract aspects of mathematics.

van Oers' (1996) considers young children's semiotic ability (in, for example, the context of their play activity) as a possible prerequisite for their later work in mathematics. Reading his work and Presmeg's paper reminded me of something we tried a few years ago with a class of ten-year-olds. The anthropologist in the project was interested in how students may mathematize a children's story. The students were familiar with the story and participated in the development of a mathematical symbolization (that relied heavily on their recently acquired knowledge of Logo). Could this little experiment have been an example of negotiation of meanings of symbols (van Oers, 1996), or an example of integration of the everyday and the mathematics register (Forman, 1996)? I will confess that I did not view any of this at the time. In fact, I am not sure I would see it now. Maybe because all we did was a transformation from everyday language into "mathematical" language (or pseudo). We did not move beyond the story, we did not look for mathematical ideas in the story, which seems to be what Presmeg's analysis does. Letting students develop their own symbolization (even within some pre-agreed parameters) is not unproblematic. I will just mention two brief examples from our recent work around a garden theme. Students' "symbolization" of perimeter with human beings seems to have been the most powerful image for a student (not part of the group that developed it) when it came to his concept of perimeter in a task-based interview. From his approach, perimeter of a shape was equated to the number of sides because he was visualizing one person for each side. In another case, students were asked to come up with their own graphical representation of the growth of a plant that they were measuring daily. One student chose to graph how much the plant had grown everyday. When I mentioned to the teacher the connection between what this student had done and calculus ideas (in terms of looking at first differences and the relationship with the first derivative), she was surprised and told me that she had been puzzled by this student's approach but had not thought there was much to it.

In the reality of the classroom how are teachers and students going to tackle the diversity in mathematizations? How much of this mathematization can we reasonably expect to be able to do in the classroom? Also, as Sierpinska (1995) points out, "to see mathematics in a situation, one must already know some mathematics and be 'mathematically tuned'" (p. 4). I hope Presmeg will continue this research on semiosic chaining with a focus on students' (teachers and school age) development of their own chains. How does participating in this activity affect students' learning of mathematics? Also, I would be interested in students' affective reaction towards this activity. Let's keep in mind Pimm's (1995) observation that "bell ringing and square dancing can be found fascinating in their own right and may leave permutations and transformation groups out in the cold!" (p. 26). Thus, I wonder, for example about domino players' reaction to the chaining presented in figure 3 in Presmeg's paper. Did the students who developed this project focus on the number of pieces in a domino set (which is more a counting problem than a "domino" problem)? How does the proposed chaining relate to the game of domino?

I will conclude my remarks by reiterating some questions that I hope will get discussed during the forum. How can we ensure that students' projects grounded on their own cultural practices and interests do indeed become opportunities for learning academic mathematics and not mere superficial examples of "mathematics is everywhere"? Who is in charge of their mathematization (e.g., in this case, the semiosic chaining)? For what purposes?

In closing, I would like to thank Norma Presmeg for a very stimulating paper. Her approach has certainly helped me theorize some aspects of my research and opened up new directions. I am looking forward to the discussions in the research forum.

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DEVELOPING CHAINS OF SIGNIFIERS: A NEW TOOL FOR TEACHERS?

A response to Norma Presmeg's paper

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In reading Norma Presmeg's paper, I found myself in sympathy both with her graduate students who enjoyed analysing the mathematics in their cultural practices and with the high school students who could not see the relevance of this activity. New to me is her model of chains of signifiers between cultural activities and mathematics. Behind the development of this model is her concern to make mathematics accessible to students from diverse cultures in the hope that this will help all to achieve equitably. I will comment on both of her programs while raising questions for discussion about her semiotic model. Overriding questions would include:

How does the theory advance our understanding of the use of situated mathematical activities in class?

Can it lead to more equitable outcomes in mathematics education?

Ethnomathematics for adults. In her initial section she describes a graduate class in which students explored the mathematical ideas in one of their cultural practices. I understand that some of these studies are being written up for a book, which I look forward to reading. Like Norma, I have taught graduate classes for teachers in which they analysed the mathematics in their everyday activities. In my class the teachers then went on to analyse the mathematics used by children in their leisure activities. In contrast to Norma's class, my students were mostly elementary school teachers with little mathematics above that which was taught in school. These adults identified the mathematics in their daily lives, from the morning alarm clock, judging the time it took to drive to school, the geometry of parking, and on through the day. Some teachers found that they used other mathematical strands, such as the geometry - largely estimated - used by a man who was building a house for his mother some distance away, and needed to take trailer loads of hardboard down with him, a situation in which it was punishing to carry either too much or too little. In analysing the mathematics that they had used, these teachers were surprised to find how basic it was. Almost all of it involved either reading numbers, estimating or doing simple calculations.

In the recent first issue of the NCTM's *Mathematics Education Dialogues* a similar point is made by Dudley in his article "Is mathematics necessary?" (Dudley, 1998). He answers his own question with a resounding "no". Rarely do people in most occupations use any mathematics above the level reported by my students. There are other reasons for doing mathematics.

On the other hand, these same teachers were fascinated by the ethnomathematics studies that they read. In reading about the mathematics of the Oksampin in New Guinea, the Kpelle in Liberia, the spatial concepts of Aboriginals or the understanding of ratio of Brazilian workers (Gay & Cole, 1967; Harris, 1991; Saxe, 1991; Schliemann & William, 1993) they not only learned about other cultures but gained new insight into the mathematics that the writers of these articles had identified. After reading about these studies they were ready to look more closely at the mathematics that could be pulled out or the activities that they observed. One teacher explored synchronized swimming, identifying algebraic patterns in it. Another described the geometric and strategic activities of 4-year-old children who were trying to get a large cable reel out of a sandpit. A third went on to do a masters thesis on the functional relationships used by Maori women in cooking for large groups (McMurchy-Pilkington, 1995). The success of such a class raises relevant questions, like those below:

Does the culture of a university make exploring similarities between culture and mathematics a legitimate activity? If so, do these teachers use, or benefit from seeing, a chain of signifiers between cultural practices and mathematics? Does this analysis help maintain the integrity of the cultural practice and the academic mathematics?

I agree with Norma that finding the mathematics in cultural practices is a fascinating activity, and helpful for adults in seeing that they can apply the mathematics that they already know to it. I do not think that it gives their culture any additional legitimacy. For the teachers whom I work with their culture already has that, as the culture of academic mathematics has its legitimacy. It would be interesting to know if there is a perhaps unconscious chaining that goes on when identifying mathematics in other activities.

Do university students have enough understanding of mathematics to enable them to see the mathematics in their cultural actives, while others may not have enough understanding?

Piaget's analysis of reactions to conflict includes a gamma reaction, which he admits does not involve perturbation, but reflects a larger understanding, in this case of a mathematical concept, which can absorb new, related ideas because of the depth and flexibility of the mathematical understanding (Piaget 1987/1975). Perhaps these adults who can see recursive formulae in sets of dominoes or synchronized swimming have this flexible understanding. Do they use a linear relationship of chaining signifiers or a less ordered procedure that involves unmeditated insights?

Ethnomathematics as a guide for teaching. If teachers are enthusiastic about seeing links between cultural practice and school mathematics they may want to use this experience in teaching. This link is crucial to Norma's paper.

Should teachers use children's cultural concepts to teach mathematics?

For several years my course ended with deep conversations about whether or not teachers had the right to take children's activities and use them as the basis for teaching mathematics. The general opinion was that they did not. These leisure or cultural activities were interesting to adults who understood the mathematics, but the activities belonged in a different culture, that of childhood play or leisure. Their play was the children's intellectual property. Not only did the children have ownership of these activities, which the teachers did not have the right to appropriate, but if a teacher tried to do use such activities in mathematics class it was thought very likely that they would make mistakes. One man who had done a study of the mathematics of surfing made it quite clear that if a teacher were not a surfer he would get it wrong, thereby antagonising the surfers in his class and bewildering the non-surfers.

What is the value to these teachers of seeing the links between their cultural practices and mathematics?

Would seeing the relationship between these activities and mathematics have made the children involved better swimmers or domino players? None of us thought so. But perhaps seeing these relationships would help children understand mathematics better. Perhaps chaining signifiers is one way among many of doing this, thus connecting a concept to a learner's world. Many teachers speak of this as providing hooks for students to hang ideas on. If a value of a semiotic analysis is to maintain the integrity of both the cultural activity and the mathematical interpretation, does this imply that this integrity is lost in other methods of grounding mathematical ideas? Perhaps it is, especially if the teacher's hooks appear to appropriate a cultural activity. Alternatively, these hooks may act as memory aids which are only loosely tied to culture. They may involve an illustration, a word, or a story which could be seen as part of a chain of signifiers. In this way children might develop a rich personal mathematics. Perhaps an awareness of chaining would increase this. This is a question to be researched.

If it proves difficult to link cultural practice and classroom mathematics, then the answer to improving the achievement of culturally different students may lie elsewhere. There is a suggestion from our Kura Kaupapa, or Maori immersion schools, that part of the answer lies in the self confidence of the learners. If their culture and language are affirmed in many ways at school, everyday, the students' confidence to try 'hard mathematics' increases their success. In these schools the Maori community has taken ownership of the mathematics curriculum to the extent that they have written their own curriculum document which incorporates cultural concepts. It is in line with the curriculum of English medium schools but is not a direct translation.

If the main purpose of chaining to illustrate a bridge between culture and mathematics, my preference would be for a deeper embedding, as described in Ballenger's description of a science lesson arising out Haitian beliefs about cleanliness.

Ethnomathematics for students. Can students see the link between cultural practices and mathematics? The teachers whom I worked with would not have been at all surprised that high school students did not see the value of making a linkage between academic mathematics and the students' cultural practices. There are questions for discussion related to why this is.

Do school students know enough mathematics to enable them to make these links?

I suggest above that adults can see the mathematics in cultural practices because of their flexible mathematical concepts. If this is so, children who have less competence will have difficulty seeing the mathematics in more than the simple examples commonly presented in their classrooms. These may be the well-worn examples related to money, figuring out how many combinations of shorts and tee shirts you can make, or drawing graphs of baby-sitting rates. These have become classroom activities that have some relationship to cultural activities but differ from the way a child would operate in their culture where, for example, they are limited in choice of clothing combinations by what tee shirts are clean or personal favorites among their clothes. Students rarely think up such examples before they have been given similar ones as part of their classroom mathematics programs. When they do invent such problems, these problems sound like stereotyped imitations of other classroom problems, and show little relation to their culture. It has to be the teacher who sees these relationships and finds examples for the mathematics that they are teaching. One approach to getting students to develop links themselves might be to ask them to think up some problems that much younger children could do. In this case they could use the mathematics that they do have command of.

Is the culture of school mathematics clearly defined by children in a way that excludes situated mathematics?

Both of our studies suggest that the culture of schooling is clearly defined by children. Norma makes this point in her reference to Dowling (1995). School children by their actions are strong advocates of this separation between cultural practices and school mathematics. One example from my research of the separation of everyday and school mathematics came from interviewing children of different ages about where they saw decimals, or "numbers with a dot in them' outside school. Eight-year-olds, who had not received instruction in decimals, described a rich variety of places out of school where they saw these numbers. Yet children of 10 and 12, who were receiving instruction in decimals, thought of very few settings beyond those examples used in school. For these children, decimals were something that you saw in maths books or in money, the main context that schools use as an example of everyday use. Once an idea was taken up by the school, in-school and out-of-school practices were clearly separated. My paper in these proceedings (Irwin, 1998) discusses an attempt to reunite the practices, but I was the one who appropriated and engineered the use he children's out-of-school mathematics. The students did the problems to satisfy me, and perhaps learned some mathematics along the way. I do not think they believed that school mathematics and everyday mathematics could be joined.

In New Zealand our school children often describe school mathematics as "doing your pluses", and adults think of mathematics first as being about knowing your tables. Although the students readily do as told, and investigate such things as the symmetries in rafter patterns, they know that rafter patterns were not developed using school mathematics. Teachers may go out of their way to show the relevance of a topic that they are teaching, but in return they are very likely to be greeted with a question like "What do we need to know for the test, Miss?" In saying this, these students are acknowledging the point put by Riley, (1998) emphasising the importance of passing mathematics tests in order to get into higher education. They and their parents know about mathematics as a social filter without giving it that name. They know that passing tests is what matters, not the relationship of mathematics to their culture.

Would chaining between cultural practices and mathematics classrooms enable each to keep its different discourse and power relationships while demonstrating similarities?

Norma suggests that a semiotic analysis permits the power relationships of cultural practices and mathematics to stay separate while demonstrating underlying commonalties. If the teachers draw the connection there may be a risk that the power that they hold as teachers overrides their attempt to keep power with the student who is the expert in the social practice. Perhaps students' separation of mathematics and cultural practices is a defense mechanism that permits them to keep the power of their cultural practices.

What are the similarities and differences with the Dutch Realistic Mathematics program and this chaining of signifiers?

I am not sufficiently familiar with realistic mathematics to address this question, but as that program appears to be particularly successful this needs to be discussed. It does appear to be a program that starts with a context and draws out the mathematics from this context. It may be relevant to note that they warn about the difficulty of using a context that is too close to children's experience, as this may prevent students from concentrating on the mathematics in the exercise. Reasons for the success of realistic mathematics may lie in its systemic application, and the fact that the examples that it uses are well within the grasp of the learners. I don't know how well the program is handled by culturally diverse students.

Like all good papers, Norma Presmeg's leads to more questions. The ideas in it are worthy of considerable discussion, some aspects of which are raised above. These questions could well lead to further research that would be of benefit to the Mathematics Education community.

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RESEARCH FORUM

Theme 2: ***Learning through problem solving***

Co-ordinator: Anne Teppo

Presentation 1:

Learning through problem solving

Hanlie Murray, Alwyn Olivier & Piet Human

Reaction: Anne Reynolds

Presentation 2:

Supporting students' construction of increasingly sophisticated ways of reasoning through problem solving

Koeno Gravemeijer, Kay McClain & Michelle Stephan

Reaction: Susie Groves

LEARNING THROUGH PROBLEM SOLVING

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Introduction

By 1988, our group had completed several studies on young students' understanding of particular concepts before, during and after instruction. These experiences led us to conduct two small scale and several informal teaching experiments based on the idea that the teacher should pose problems to students for which they do not yet have a routine solution method available, and that learning would take place while the students were grappling with the problems. The outcomes of these experiments helped us to develop the following *tentative* model for learning and teaching mathematics:

> *Learning occurs when students grapple with problems for which they have no routine methods. Problems therefore come before the teaching of the solution method. The teacher should not interfere with the students while they are trying to solve the problem, but students are encouraged to compare their methods with each other, discuss the problem, etc.*

In the years since 1988, we have come to realise that this naïve description actually represents an enormously complex *series* of learning situations. *Some* of the issues that we were confronted with to research further and resolve as well as we could, were:

The role of the teacher. To what extent should the teacher be part of the problem solving process? Keeping in mind that the problem solving process involves mathematical as well as social processes, do different processes demand different types of support and intervention?

The classroom culture. Although the classroom culture includes the didactical contract between teacher and students (their mutual expectations and obligations), it also includes the ways in which the learning situations are physically set up and the rules under which they operate. What did we have to learn about this?

Interaction patterns among students. These interaction patterns depend on the role which the teacher has assumed, the classroom culture and the way in which the teacher sets up learning situations, reflecting her beliefs about how mathematics is learnt. To what extent do different interaction patterns influence learning, and might there be different kinds of interaction patterns for different learning situations (i.e. different kinds of tasks)?

The kind of problem posed. Mathematical tasks or activities come in a variety of guises: investigations, projects, traditional story sums, real-life problems, abstract problems, puzzles, etc. Were all of these suitable for learning through solving problems, or were some more suitable than others?

The mathematical structure of the problem. Would it be a good idea if Grade 1 students worked at addition-type problems for a long time, so as to establish strong understandings of the operation and its solution methods? Would it matter if Grade 1 students never met a proportional sharing problem?

Sustained learning. It might be possible to achieve single, successful learning episodes, or even the satisfactory development of a group of students over a period of weeks, but would such a programme maintain students' mathematical development over a number of years?

The type of response elicited from the student. Should the teacher accept verbal answers and explanations, or should she insist on written explanations, and if so, for what purpose and in what format?

Teacher awareness, understanding and co-operation. We knew from experience that many teachers of the lower elementary grades were to some extent aware that young children invented their own methods and, provided they received *sufficient* information and some continued support, achieved significant success. We were concerned about large-scale implementations which necessarily implied brief training sessions with large groups of teachers.

Informing the larger community. Would we be able to communicate well enough with the larger community (parents and other members of the public, pre-school, elementary and high school teachers and lecturers at local colleges and universities, and remedial teachers and educational psychologists) so that these different groupings could understand and identify with our basic principles and appreciate the quality of the mathematics that students were doing?

Theoretical base

Our present theoretical base can be described as follows:

Contrary to an empiricist view of teaching as the transmission of knowledge and learning as the absorption of knowledge, research indicates that students construct their own mathematical knowledge irrespective of how they are taught. Cobb, Yackel and Wood (1992) state: "... we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of traditional direct instruction," and "The central issue is not whether students are constructing, but the nature or quality of those constructions" (p. 28).

A problem-centred learning approach to mathematics teaching (e.g. Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Olivier, Murray & Human, 1990) is based on the acceptance that students construct their own knowledge and therefore attempts to establish individual and social procedures to monitor and improve the nature and quality of those constructions. We hold the view that the construction of mathematical knowledge is firstly an *individual* and secondly a *social* activity, described as follows by Ernest (1991):

a) "The basis of mathematical knowledge is linguistic knowledge, conventions and rules, and language is a social construction.
b) Interpersonal social processes are required to turn an *individual's subjective* mathematical knowledge, after publication, into *accepted objective* mathematical knowledge.
c) Objectivity itself will be understood to be social." (p. 42, our italics.)

Social interaction serves at least the following purposes in problem-centred classrooms:

- Social interaction creates opportunities for students to *talk* about their thinking, and this talk encourages reflection. "From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics ... To verbalise what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted." Also, "... leading students to discuss *their* view of a problem and their own tentative approaches, raises their self-confidence and provides opportunities for them to reflect and to devise new and perhaps more viable conceptual strategies" (Von Glasersfeld, 1991, p. xviii, xix).
- Through classroom social interaction, the teacher and students construct a consensual domain (Richards, 1991) of taken-to-be-shared mathematical knowledge that both makes possible communication about mathematics and serves to constrain individual students' mathematical activity. In the course of their individual construction of knowledge, students actively participate in the classroom community's negotiation and institutionalisation of mathematical knowledge (Cobb et al, 1991).

Whereas a traditional, transmission-type approach necessarily leads to subjective knowledge which is largely reconstructed objective knowledge, our version of a problem-centred learning approach reflects the belief that subjective knowledge (even if only in young children) should be experienced by the students as personal *constructions* and not re-constructed objective knowledge. (When we aim at children creating their own knowledge, as opposed to reconstructing existing objective knowledge, we do not imply that children are actually creating knowledge that does not already exist as objective knowledge; we do state that the children in this approach assume that they are creating *their* knowledge as new.)

We therefore regard problem-solving as the *vehicle* for learning.

It is necessary to distinguish sharply between learning *to* solve problems and learning *through* solving problems. Davis (1992) describes the process of learning through solving problems as follows: "Instead of starting with 'mathematical' ideas, and then 'applying' them, we would start with *problems or tasks*, and as a result of working on these problems the children would be left with a residue of 'mathematics' – we would a[illegible] mathematics is what you have left over *after* you have worked on problems. V[illegible]he notion of 'applying' mathematics, because of the suggestion that you *start* with mathematics and then look around for ways to use it." (p. 237). Also: "According to Dewey (1929), these relationships and understandings are what is left after the problem has been resolved" (Hiebert et al, 1996, p. 15).

However, no matter how well-designed a problem or sequence of problems is, the amount and quality of the learning which takes place depend on the classroom culture and on students' and teachers' expectations. "Tasks are inherently neither problematic nor routine. Whether they become problematic depends on how teachers and students treat them." (Ibid, p. 16). Neither do we imply that learning *to* solve problems is not important in its own right, nor that *routine* problems should never be posed. This is discussed again later.

Implementation

In 1988, one of the local Departments of Education made available to us eight schools of their choice as experimental sites, and identified a control school for each of the experimental schools. We conducted a two-day workshop for the Grade 1, 2 and 3 teachers of the experimental schools at the end of 1988, and these teachers started implementing a problem-centred approach at the beginning of the school year in 1989. A small team of three researchers and three subject advisors from the Department of Education supported the process by brief visits to classrooms and short follow-up meetings.

All the students involved in the project wrote two sets of tests in the course of a school year for evaluation purposes. It was already clear after the first test in August (six months after inception) that there was a marked improvement in both the understanding of word problems and in computational skills (Malan, 1989).

The Department of Education accordingly requested us to provide in-service training for the teachers of another sixteen schools. In the years that followed, all the elementary schools of the Department from Grade 1 to Grade 6 eventually became involved. The approach also spread to four other Departments of Education. By 1993, the lower elementary grades of more than a thousand schools from five Departments of Education were involved. This very wide implementation was against our advice, because we support an organic model of growth with sufficient teacher support.

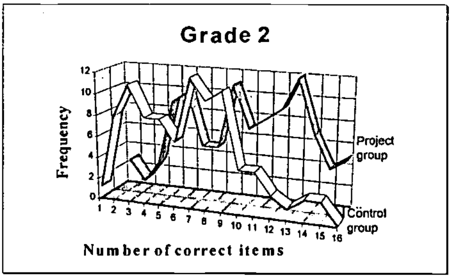
The problem-centred approach was also introduced to the elementary grade teachers of more than 50 special schools (schools for mentally and physically handicapped students). In a small number of these schools teachers have adopted and are using the approach with success.

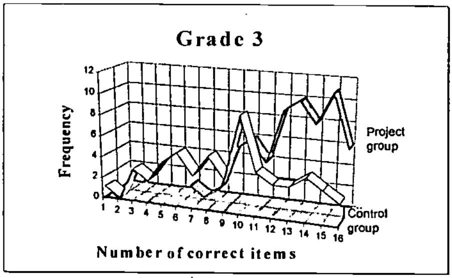
In the majority of schools involved, the implementation has petered out in the sixth and seventh grades, partly because in-service training received by the upper elementary teachers was very brief, and partly because the documentation they received was not comparable to the extensive and detailed teachers' guide, compiled by subject advisors, teachers of the first eight experimental schools and researchers, which the lower elementary teachers received.

It is important to make clear that the subject advisors and teachers made immense contributions to the development of the approach: We really *did* approach the original group of teachers with our naïve idea of how learning through problem solving works, but because we were not prescriptive, we could observe how different teachers had made sense of their workshop experiences by initiating different practices in their classrooms. It was our great fortune that we could observe these different practices and their effects on students' learning over time, enabling us to identify useful and potentially problematic practices.

Evaluation

All the Departments of Education involved in the project evaluated the approach independently by comparing second, third and fourth grade students' performance on written tests with the performance of students of the same school in the previous year on similar tests. Two Departments also compared experimental and control schools. The tests included straight calculations and word problems. Students at all grade levels in all the project schools showed marked improvement. When the frequencies of the number of correct answers for a particular test are plotted, the graphs typically show a consistent shift to the right for project students, i.e. more students did well than students in traditional teaching. We give two examples from different schools.





After six years the Cape Education Department commissioned a large-scale independent evaluation which also yielded positive results (Taylor, Glover, Kriel & Meyer, 1995), as did the James and Tumagole (1994) and Newstead (1996, 1997) evaluations.

De Wet's study (1994) found that fourth grade remedial students in a problem-centred approach progressed better than a matched group in a traditional approach (compare Thornton, Langrall & Jones, 1997).

Concerns from the larger community

The fears, accusations and arguments of the "California Math Wars" are very similar to those that arose a few years ago around the problem-centred approach in South Africa, giving rise to heated public debates in the letter columns of newspapers and magazines.

These fears and objections have at least two distinct sources (there are more).

Firstly, *different views* about what mathematics is and what is mathematically valuable; how children learn mathematics; and what our version of a problem-centred approach entails.

Secondly, very real concerns about the implementation of an innovative approach for which teachers *may not have the necessary mathematical and didactical knowledge and skills*, and for which there may be a shortage of appropriate materials.

Here are some examples of specific questions:

- How and when will students learn to perform the basic operations? (Implying that if the students do not learn the standard vertical algorithms, they have not learnt "the operations" for whole number arithmetic.)
- How will students learn their bonds (number facts) and multiplication tables if there is no drilling and memorisation?
- If students cannot do long division, how will they do algebraic division?
- What about the weaker student? Weak students cannot construct their own methods, they need to be shown.
- Communication plays an important part in this approach. What about students who receive instruction in a second language?

These problems will be discussed later.

Research results

The wide "forced" implementation of the approach within a relatively short time led to teachers receiving some quite divergent messages during and after their in-service training. Although this by itself is unfortunate, and caused uncertainty among teachers and parents, it did enable us as researchers to identify the crucial elements of the approach, and to research the reasons for less effective implementations. We have also been able to chart the *development* of young students' number concept and multiplication and division strategies, their *initial conceptions* of fractions and division, and social interaction patterns which lead to improved learning. These results have been described in nine PME papers from 1989 to 1996.

A long-term study is of value in at least three respects: It provides information on the long-term effects of specific interventions on students' learning, it makes possible the identification of unforeseen gains in conceptual development and understanding of the properties of numbers and operations, and it provides information on teacher development (Murray, Olivier & Human, 1995).

We now describe some issues which have emerged over the years. Some of these issues enabled us to clarify and refine our ideas on learning environments and the design of material which sustains development; other issues, notably establishing the link between arithmetic and algebra, are not yet resolved.

The social component of the learning environment

The role of the teacher. In the effort to help teachers make the paradigm shift from believing themselves to be the sources of knowledge and their main responsibility to transmit knowledge, towards accepting that students construct their own knowledge, we initially gave some teachers the impression that teachers should involve themselves as little as possible in the learning process. Later on, when some teacher groups received very sketchy in-service training, this impression was wide-spread and caused concern in the community.

It is necessary to clarify the role of the teacher in a problem-centred approach very thoroughly. We find Piaget's classification of mathematical knowledge as physical, social and logico-mathematical knowledge (Kamii, 1985) to be of great use in this respect. Teachers then realise that they have to provide the necessary information (social knowledge) for students to understand the problem, to communicate with each other and to capture their thoughts on paper in a generally acceptable (intelligible) way. They also have to show students how to use tools like measuring instruments and calculators, and they have to lay down (or negotiate with students) the social norms which govern interaction and general classroom behaviour.

It is only when student activity is *mainly* focused on the construction of logico-mathematical knowledge that the teacher should not interfere, except to monitor the social procedures and social needs.

The classroom culture. We suggest very tentatively that in the lower elementary grades at least, the classroom culture and the quality of students' interactions when solving problems have a greater influence on the students' mathematical constructions than the facilitatory skills of the teacher during discussions (Murray, Olivier & Human, 1993). We have not had the opportunity to research this further, but if the hypothesis holds, it has important implications for the general viability of an inquiry-based problem-solving approach. Many experiments and projects reported on internationally which use similar approaches leave the impression that the teacher involved has well-developed mathematical knowledge and didactical skills. Such reports serve as good examples of delicate and suitable teacher interventions, but most of our teachers, even the supposedly well-trained ones, do not possess similar skills.

Interaction patterns among students. Since this topic has received much attention during the past few years from researchers on a descriptive as well as a theoretical level, we mention only the contentious (but we perceive possibly *very* contentious) idea that we hold on this issue, and that is the need for grouping students into like-ability groups or allowing them to group themselves in such a way for tasks that are mainly focused on the construction of logico-mathematical knowledge (Murray, Olivier & Human, 1993). This must immediately be qualified as follows:

- The streaming or tracking of students into different lower-, middle- and upper-ability *classes* is not implied.
- Rigid groupings that are maintained for *all kinds* of mathematical tasks is not implied.

Using Piaget's classification of different kinds of mathematical knowledge as a guide, we propose that the acquisition of mainly *physical* and *social* knowledge is probably eased by co-operating with more knowledgeable peers or with the teacher (Vygotsky, 1978, p. 86). Logico-mathematical knowledge, however, needs to be constructed where the student's thinking space is not invaded by more advanced ideas, as happens when weaker or slower thinkers are physically near faster thinkers.

We therefore propose that for student interaction, the kind of mathematical *knowledge* that has to be constructed and the kind of *task* (e.g. routine/problematic/measuring/constructing) should guide possible groupings, as well as the physical facilities available. We have found that where students know what is expected of them, *they* are able to choose suitable partners.

The problem or task component of the learning environment

The kind of problem posed. Although in no way denigrating the role that investigations and projects can play in students' mathematical development, we base our learning trajectories on "problems" which present sufficiently clear *structures* for students to respond to. (Such problems need not have only one answer, but frequently attempt to elicit controversy as to the "best" answer.) Such "problems" may have realistic, abstract or imaginary contexts, although we tend to avoid contrived situations. This is later discussed more fully. In the discussion that follows, "problems" therefore mean *problems* and not, for example, projects or investigations.

Such problems include situations where the development of basic skills are directly addressed, not through drill or memorisation, but by turning them into problematic situations (Hiebert, et al, 1996) and at a meta-level, discussing ways of mastering these facts through relationships and patterns. (For example, how many different ways are there to make 7 + 8 easier to calculate?)

The mathematical structures of the problems. The mathematical structures of the problems posed are important for a variety of reasons. Problems cannot be posed simply because they are good or seem interesting – the choice of problems should be based on thorough content analysis and a good understanding of how students develop concepts and misconceptions.

Problems also have different functions, or are used for different purposes, for example:

- Some problems are more suitable than others for initially establishing an inquiry-type classroom culture
- Problems may be used to introduce students to a valid *problem area* (e.g. calculus), so that analysis of the problem and reflection on its structure may in this case be more important than solving the problem
- In general, when students solve problems, they should be provided with opportunities to actualise existing (but not yet explicit) knowledge and intuitions; to make inventions; to make sense and assign meanings; and to interact mathematically.

Our views on suitable problems and on learning sequences were originally based on the thinking of some Dutch researchers, and developed further during our classroom research and observations.

The Van Hiele levels. Pierre-Marie and Dina van Hiele suggest that students should first be immersed in activities involving new concepts which they engage in informally, using what ever insights or skills they have available. "The aim is to acquire a rich collection of intuitive notions in which the essential aspects of concepts and structures are pre-formed. This, then, is laying the basis for concept formation." (Treffers, 1987, p. 248). The teacher gradually introduces more generally acceptable terminology and more rigorous reasoning processes as the students become able to give meaning to these. The Van Hieles describe the next level as having constituted the ground level concepts as abstract entities, *related* to other advanced entities (Van Hiele, 1973).

Progressive schematisation. Although the problem posed may remain similar over a period of time, the students' solution strategies should develop towards more numerically-aware, more advanced strategies (Treffers, 1987, p. 200).

The Van Hiele levels and the idea of progressive schematisation *can and are* interpreted in different ways.

Paul Ernest mentions "accepted objective mathematical knowledge" (Ernest, 1991, p. 42). But to what extent does the "accepted, objective" knowledge influence the learning trajectories of young children through the first Van Hiele level and along the path of progressive schematisation? In other words, what is perceived to be the *desired* objective mathematical knowledge for young children? We contend that this question is in fact one of the crucial points the mathematics education community should be debating: Where do our learning trajectories lead? The endpoint of each trajectory depends on a value judgement, and the learning trajectory for each topic may have a different goal. For example, some well-documented learning trajectories close towards specific algorithms or notations (e.g. Treffers, 1987, pp. 200–209). Why?

The answer to this question lies in our perspectives on what mathematics is, what it is used for (the needs of society) and why children have to study mathematics. For example, a learning trajectory may end with a particular algorithm or conceptualisation. What was the aim of the trajectory – the (objectively substantiated) need for the particular algorithm, or the learning and development of general concepts or theorems (theorems in action) which occurred in the process? This depends on the topic, on the associated algorithms and techniques of the topic, and on the utilitarian value or societal value attached to these algorithms and techniques.

For example, should a learning trajectory for the addition of whole numbers end in the vertical addition algorithm? This is debatable. We believe, for example, that the *[illegible]dge of properties of numbers and operations*, and the *flexible number sense*, shown by Orlando's (Grade 5) solution method for $784 \div 16$, are more to be valued than the application of a (socially acceptable) version of a long division algorithm:

$$\begin{array}{llll} 1600 \times 100 \rightarrow & 1600 \div 2 \rightarrow & 800 - 16 \rightarrow & 784 \\ & 100 & 50 & 49 \end{array}$$

answer: 49.

On the other hand, should a learning trajectory for calculating areas at Grade 6 or 7 level end with students' knowing (and understanding) the formula for the area of a circle? Most decidedly!

Learning trajectories can also start in different ways. It now seems to be generally accepted that the Van Hiele first level experiences need not be *concrete* and need not be *real-world problems* either, but may be any experiences or tasks which make sense to the particular group of students, and which they are able to identify with and problematise. We go further than this.

We believe that, where possible, these ground level experiences should be *authentic* in the sense that they represent situations of which the ideas or concepts we wish to develop are natural parts. We therefore find it useful to reflect on the situations or problems which initially gave rise to the development of the mathematical tool or concept (the genetic principle, e.g. Klein, 1924) and seldom make use of contrived situations, although we admit that in a way almost all school problems *are* contrived.

For example, in our earlier research on young children's understanding of directed numbers and operations with directed numbers, the children *themselves* used patterns, analogies with the natural numbers and logical reasoning to make sense of this novel environment, and showed confusion when they were confronted with contexts like debt to give meaning to directed numbers, even though they understood the idea of debt itself (Murray, 1984; Malan, 1987; Hugo, 1987). Not debt, but mathematics, created the need for directed numbers. Likewise, when our students learn to count and calculate, they do so in contexts which suggest the original situations where the need for counting and calculating arose. There is therefore limited use of *pre-structured* counting materials, and the problem-solving situations cover a deliberate mix of problem structures.

Furthermore, we have ample evidence that if the Van Hiele ground level experiences are aimed at developing understandings of specific cases, or mathematically streamlined situations, *limiting constructions* form very quickly. Studying special or easy cases first does not make the development of concepts and skills easier; it merely hampers understanding.

Limiting constructions: This is a phrase used by D'Ambrosio and Mewborn (1994) to denote the type of misconceptions which arise through limited exposure to a concept or through experiences of a particular (limited) kind. For example, the idea that "multiplication makes bigger" is viable in whole number arithmetic, but severely hampers students when they have to perform operations involving fractions.

There were some teaching practices in our traditional lower elementary classrooms that we knew about, but did not address in our initial in-service training sessions, because we were not yet aware of the severe impact they would have in the long run. Teachers did not pose a wide variety of problem types (for example, they tended to favour division problems of the sharing type and neglected grouping problems). They tended to "block" the four basic operations by starting off the school year with three months' addition, followed by three months' subtraction, then multiplication and then division. When teaching addition, they first dealt with the special cases where no "carrying" of a ten from the units to the tens is necessary, and no "borrowing" of a ten is necessary. Finally, many of their number concept development activities highlighted only the tens-structure of the decimal number system.

What happened was that students developed strong, stable methods for addition based on decimal decomposition of the numbers involved. For example, for 27 + 35, the most popular methods constructed were

$20 + 30 \rightarrow 50 + 7 \rightarrow 57 + 5 \rightarrow 62$

or $20 + 30 = 50;\ 7 + 5 = 12;\ 50 + 12 = 62$

In analogy with this, the following division method might then be constructed to calculate $79 \div 13$:

$70 \div 10 = 7$
$9 \div 3 = 3$
$7 + 3 = 10$

i.e. decimal decomposition, and "combining" tens-parts with tens-parts and units-parts with units-parts.

We therefore realised that a learning trajectory for whole number arithmetic required that problem types be mixed, not blocked, that the special ("easy") cases of adding and subtracting are not presented first, and that number concept activities and problem situations emphasise *multiples and factors, and not only decimal decomposition.* These practices had immediate and long-term positive effects on students' number sense, estimation abilities and especially on their construction of powerful multiplication and division strategies (Murray, Olivier and Human, 1994).

We have also reported on the limiting constructions about common fractions that third graders had built up as a result of teaching as opposed to first graders in the same school (Murray, Olivier & Human, 1996). Similarly, we observe that ninth graders who have studied linear functions and linear graphs for 18 months to the exclusion of all other functions, find even the *idea* of any other type of graph difficult to accept.

The role of time in the development of concepts. It has been possible for us to trace the development of groups of students and in some cases of individual students over periods of time of various lengths, i.e. a single lesson (of about 35 minutes), a set of three lessons, several weeks, several months and for one school, six years. Our data show clearly that many (if not the majority) of students who seem to be mathematically weaker or slower than others can and do construct powerful mathematical concepts and generalisations provided the integrity of their thinking is preserved (i.e. somebody doesn't decide they need help and start demonstrating methods to them), the tasks they are presented with remain challenging and are not made easier, and the inquiry nature of the mathematics classroom is maintained. Our description of the development of division strategies in a third-grade classroom clearly illustrates this (Murray, Olivier & Human, 1992).

Lower elementary grade teachers have found our model of number development (Murray & Olivier, 1989) of much practical use because it sensitised them to the fact that at different points in time students are at different levels of conceptual development and should not be forced to function at levels of abstraction which they have not (yet) reached, but which they will reach, given time.

In fact, our informal observation is that students with a weaker number sense show great ingenuity and understanding of the properties of whole numbers and their operations by their (the students') use of theorems in action to make a calculation easier. For example, Niel (Grade 3) calculates 470 × 7 as follows:

$$\begin{array}{l} 10 \times 470 \rightarrow \underset{10}{4700} \div 2 \rightarrow \underset{5}{2350} + \underset{2}{940} \rightarrow \underset{7}{3290} \end{array}$$

Claire-Anne (Grade 5) calculates 27 × 35:

$$\begin{array}{l} 27 \times 10 = 270 \\ 27 \times 10 = 270 \\ 27 \times 10 = 270 \\ 27 \times 5 = 270 \div 2 = 135 \\ 270 + 270 + 270 + 135 = 945 \end{array}$$

(Both of these students are obviously already at what we describe as level 3 number concept, but it is important to realise that level 3 methods are not invented if students are not confronted with numbers big enough to create the need for these methods. However, the teacher cannot *force* or *demonstrate* such methods; students *produce* them when they are able to do so.)

Anticipatory transformations. The above two examples clearly illustrate the ability of students to transform a given task into equivalent sub-tasks that *they know they can manage.*

The essential nature of any non-counting computational algorithm is that it is a set of rules for transforming a calculation into a set of easier calculations *the answers of which are already known or can easily be obtained.* This process of changing the task to an *equivalent* but easier task involves three distinguishable sub-processes, illustrated here with reference to a procedure to calculate 17 × 28 (Murray, Olivier & Human, 1994):

- Transformation of the numbers to more convenient numbers, e.g.

  $28 = 30 - 2$

  The ability to transform numbers in this way depends on the student's *number concept development.*

- Transformation of the given computational task to a series of easier tasks, e.g.

  $17 \times 28 = 17 \times 30 - 17 \times 2$

  The ability to transform the task to an equivalent task depends on the student's awareness of certain *properties of operations* or theorems-in-action (here the distributive property of multiplication over subtraction).

- Calculation of the parts, e.g.

$$\begin{aligned} 17 \times 28 &= 17 \times 30 - 17 \times 2 \\ &= 510 - 34 \\ &= 476 \end{aligned}$$

When a Grade 3 student solves the problem "Find half of 237" as follows, he has clearly chosen to decompose 237 into numbers that *he anticipates* he can halve:

237	100	100	
	15	15	
	3	3	
	½	½	answer: 118½

The anticipatory transformations may not always be appropriate for a variety of reasons. They may still prove to be too difficult, or upon reflection, uneconomical and tedious. In the following example Marianne (Grade 3) *underestimated* her abilities and adjusted her transformations to a more sophisticated level.

Trying to solve 338 ÷ 13, she starts off by subtracting thirteens, then writes down "this will take too long" and switches to multiplying and doubling:

$$\begin{array}{ccccccccc} 130 \times 10 & \rightarrow & 130 + 130 & \rightarrow & 260 + 52 & \rightarrow & 312 + 26 & \rightarrow & 338 \\ & & 10 & & 20 \quad 4 & & 2 & & 26 \end{array}$$

answer: 26

The type of response elicited from the student. Davis (1992) rightly states: "Mathematics sometimes employs written notations of various sorts, but these symbols are not the mathematics itself, any more than lines drawn on a map are actual rivers and highways" (p. 255). Yet we have found access to appropriate notations to be crucial to young students' mathematical development. Social interaction and effective communication are essential to the approach and access to appropriate notations to capture methods so as to share them with others is a part of this. The ability to capture thought on paper is essential for individual reflection and analysis. In traditional mathematics classrooms at the *elementary* level, students have had no a[illegible]e tools at hand to express (or capture) their thinking in writing. What they had available was a number sentence to be used in a prescribed format, and computational algorithms to be presented in prescribed formats. In such a situation Davis' stricture holds completely.

If, however, we make available notational tools *which students can match* to their thinking processes, they find it an important aid to individual as well as social construction of knowledge (Skemp, 1989, p. 103).

In mathematics, recording has different functions. It eases communication and serves as a thinking aid for the individual. Written explications which aim at proving something or convincing others also *need to measure up to standards* which are not applicable when the individual is simply trying to solve the problem.

For these reasons, the arrow notation, which can be used when the equal to sign would be incorrect, was introduced, and even in the lower grades students are required to record their thinking clearly and logically enough so that others can also follow it. It was found that recording skills take time to develop, and if teachers simply accept verbal explanations in the lower grades, some students experience serious problems from the fourth grade onwards, when they need a written record as a personal thinking aid and then do not know how to express themselves.

Answering the community

Community fears which are based on a lack of knowledge about the approach should of course be addressed by trying to provide the community with the necessary information. During the period when only a few schools were involved, information sessions with parents were very successful. As the number of schools involved increased, and teacher in-service training became sketchy, parents became increasingly badly-informed. A very small group of university lecturers in pure mathematics was (and remains) opposed to the approach. We understand their opposition to be rooted in an unawareness about the substantial international and local results available on the effects of the traditional teaching practices on students' thinking, the changed aims of mathematics education caused by the demands of a radically changed society and workplace, and the need for mathematics and mathematically related skills to be made accessible to the whole community.

Concerns that students who receive instruction through the medium of a second language are at a disadvantage in an approach where communication plays an important part are very valid. Yet the alternative seems to be a classroom where more time is spent on context-free mathematics and mainly teacher-directed explanations and examples. Would this be better?

In the environment of the handicapped with moderate to severe language problems, teachers have told us that it is necessary to stimulate children to listen and to try to communicate while they are doing mathematics. Simply trying to teach them arithmetic divorced from word problems does not equip them to handle everyday problems. Goodstein, himself deaf, strongly supports this view (Goodstein, 1992).

Also, one of the third-grade classrooms that we described (Murray, Olivier & Human, 1993), consisted of 32 students with ten different home languages. The majority of these students had a very poor command of English (the medium of instruction), yet the teacher was able to establish one of the best implementations of the problem-centred approach we have been able to document, and the students' mathematical development was very good.

However, the language issue remains serious and should be researched and debated further.

It is, however, very possible that a problem-centred approach to learning mathematics may not succeed if firstly, the problems posed are not chosen for their mathematical structures and the sequence in which they are posed is not well-planned (we have discussed this, with special reference to the prevention of limiting constructions).

Secondly, activities which are aimed at the development of *routine skills* should not be neglected. It is important, for example, that a flexible and wide-ranging knowledge of the multiplication tables be developed in the elementary grades, not by memorising the tables, which is an extremely inefficient way, but through encouraging students to use relationships and patterns. (Niel and Claire-Anne for example, showed flexible number knowledge.)

Thys (Grade 4) solved $711 \div 9$ mentally, explaining as follows: "720 take away 9 is 711. So the answer is not 80, it is 79."

Schoenfeld (1994) states: "*Some* such skills are important for students, if only because not to be fluent at them means that one's clumsiness at them will get in the way when one needs to see past them" (p. 60). The following comment by Askey (1997) was obtained from the Internet: "Then NCTM tried Agenda for Action and later the Standards. Both of these were built on the idea that if you could solve problems, then you could do mathematics. You can, but at too low a level. All three are needed – problems, technique and structure."

To summarise, we think it unlikely that a problem-solving approach will be effective if:

- Progressive schematization is not encouraged, either through discussion or by the teacher posing the problem in such a way that more exact or more abstract responses are required even, and especially, for supposedly weaker students.
- The necessary important content is not covered.
- Useful mathematical techniques are not developed and sufficiently practised.
- Classes of problems do not achieve coherence (e.g. the function concept, algebra, statistics), so that the associated concepts and relationships cannot be constituted at an abstract level.

In conclusion

The problem-centred approach in the lower elementary grades is based on an over-simplified model called the three pillars:

- well-planned number concept activities, including activities which promote the building of patterns and relationships
- well-planned problems
- effective discussion

Neglect of any of these "pillars" shows in students' behaviour or understanding, even only after a few months.

Hiebert et al (1997) identify five critical features of the very similar approach to the teaching and learning of mathematics they describe, and then go on to say: "The essential features are intertwined and work together to create classrooms for understanding. They define a system of instruction rather than a series of individual components. It makes little sense to introduce a few of the features and ignore the rest; their benefits come from working together as a coherent, integrated system." (Hiebert et al, 1997, p. 172).

Initiating and sustaining mathematical development through posing problems that students have to work on has been found to be a successful way of learning mathematics, *but only if* the problems are well-designed and well-sequenced, and the classroom culture in its full complexity supports learning.

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LEARNING THROUGH PROBLEM SOLVING
A REACTION TO MURRAY, OLIVIER, AND HUMAN

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Overview

In *Learning through problem solving*, Murray, Olivier, and Human (this volume) elaborate their extensive research into a problem-centered learning approach to mathematics teaching. At a time when various mathematics reform efforts are under attack (for example, the current California 'math wars' in the United States) these researchers have documented children's mathematical development and competence in a problem solving setting. This adds significantly to the evidence for the success of an approach to mathematics teaching which values and supports children's "conceptual development." In this paper they have provided insight into their successes and concerns in implementing a problem-centered approach and raised questions for discussion and further research.

The paper provides a clear and compelling theory base grounded in constructivism. Within this framework the researchers develop their approach to a problem-centered learning environment, paying close attention to the role of social interaction. In the Introduction, Murray et al. state what they describe as their *tentative* model for learning and teaching mathematics when this research project began in 1988:

> *Learning occurs when students grapple with problems for which they have no routine methods. Problems therefore come before the teaching of the solution method. The teacher should not interfere with the students while they are trying to solve the problem, but students are encouraged to compare their methods with each other, discuss the problem, etc.*

They characterize this starting point as a "naïve description" of "an enormously complex *series* of learning situations." The paper discusses these complexities and in doing so elaborates some significant findings from their research, as well as highlights questions still in need of investigation. I find it interesting that although the group, in hindsight, acknowledged that they had a "naïve" view of what it would mean to implement such a model, the model itself has proved to be robust. The research they report in the paper provides the basis for continued support of the model.

In the remainder of this paper I discuss some of the implications of Murray et al.'s research as reported here and raise some questions for further consideration. I have organized my response in sections around different themes. This presents a problem in the fact that each section can appear separate and unrelated, a result of the written medium, which is of a linear nature. However, the different ideas that are raised as I read the paper are interconnected and not linear.

Thoughts/questions/implications

Curriculum

Murray et al. argue that problem types "be mixed, not blocked." They give examples from research classrooms in the early stages where teachers tended to provide tasks that dealt with just one type of problem. Students developed "strong, stable methods" within that setting. However, they also found evidence that students developed limiting constructions when problems were blocked into addition, subtraction, multiplication, and division types and when "special cases" like addition with no "carrying" were treated separately. Once problem types were "mixed" there was "immediate and long-term positive effects on students' number sense, estimation abilities and especially on their construction of powerful multiplication and division strategies."

This raises questions about the mathematics curriculum. In most, if not all countries, students routinely progress from grade to grade. There have been some variations, especially in the early years. In some cases, for example, the first three years of schooling are within a "non-graded" setting. However, for the most part, students are thought of as progressing each year from a 'lower' to a 'higher' grade. The mathematics curriculum is "blocked" in the same way. Yet, Murray et al. provide compelling evidence that students need to be engaged with a variety of mathematical ideas *at the same time*. I do not think they are suggesting that young children, as they are engaged in numeric thinking, should be routinely experiencing examples of numbers in the thousands. However, they do suggest that a variety of ideas be incorporated into the tasks that students are experiencing; also that "methods are not invented by students if students are not confronted with numbers big enough to create the need for these methods."

> **How should a school system, school, or educator design a mathematics curriculum if it takes this group's research seriously?**

Learning

Student learning, as Murray et al. describe it, is a complex endeavor. As outlined in the previous section, they advocate providing variety in tasks used to foster students' mathematical thinking. Their problem-centered approach is designed to challenge students to construct meaningful mathematics: "We therefore regard problem-solving as the *vehicle* for learning." Such variety and challenge implies that students would be engaged in constructing relationships among several, if not many, ideas at one time. This approach is supported by research into how the brain develops, particularly during the first decade of a person's life. Brain researcher Marian Diamond states:

> *Where society once viewed the child's brain as static and unchangeable, experts today see it as a highly dynamic organ that feeds on stimulation and experience and responds with the flourishing of branching, intertwined neural forests. ..... But it has a dark side, as well, if the child's mind is understimulated and underused.*
>
> (Diamond & Hopson, 1998, p.1)

Murray et al. use the term "learning trajectory" in discussing the focus or goals of students' problem solving experiences. As I think about the rich interconnections they are describing, I am troubled by this word "trajectory." It is a metaphor (in the sense that Lakoff & Johnson, 1980, use this term) that suggests linearity in a student's development of mathematical ideas and "progressive schematisation." Lakoff (1987) argues that the metaphors we use to describe our thoughts constrain how we think: "Without them our experience would be an undifferentiated mush; but because of their internal structure they constrain our understanding and reasoning" (p.206). I suggest that we would be better served by a metaphor that captures the complex interweaving of mathematical relationships that Murray et al. propose are important in a problem-centered learning setting. Lakoff developed the notion, Idealized Cognitive Models, to describe the development and refining of interconnections between ideas as we construct meaning for our experiences. Maybe the notion could be best expressed as "a web of interconnections," to adapt Capra's (1996) phrase: "The web of life." Diamond and Hopson (1998) use a "forest" metaphor with words like "branching" (as quoted above) to capture the essence of their thinking about how neurons in the brain interconnect. These are just some metaphors that come to mind.

> **What metaphor might best express the idea that learning experiences need to be varied and problem types need to be "mixed" and not "blocked?"**

<u>Procedures, algorithms, addition and multiplication facts, notations</u>

The paper contains an important discussion of "goals," centered around and related to the discussion of "learning trajectories." Murray et al. contend that this is a crucial question for the mathematics education community:

> **Where do your learning trajectories lead? The endpoint of each trajectory depends on a value judgment, and the learning trajectory for each topic may have a different goal.**

As these researchers maintain, procedures, addition and multiplication "facts," and mathematical symbols or notations are necessary to the mathematical development of our students. Their research highlights the importance of students developing quick and accurate procedures in dealing with number and appropriately recording their mathematics in written symbolic form if they are to construct increasingly sophisticated mathematical patterns and relationships. On the other hand they challenge the notion that a particular "learning trajectory for the addition of whole numbers" should "end in the vertical addition algorithm." They provide examples of students' invented procedures that show these students have constructed meaningful number patterns and relationships (as well as efficient procedures).

The group agrees with (and quotes) Davis (1992): "Mathematics sometimes employs written notations of various sorts, but these symbols are not the mathematics itself, any more than lines drawn on a map are actual rivers and highways" (p. 255). However, they bring an interesting 'twist' to that reference: "In traditional mathematics classrooms at the *elementary* level, students have had no appropriate tools at hand to express (or capture) their thinking in writing. What they had available was a number sentence to be used in a prescribed format, and computational algorithms to be presented in prescribed formats. In such a situation Davis' stricture holds completely." At the same time, they do not quite agree with Davis' analysis. They suggest, if I am interpreting their argument in a way they intended, that in a problem-centered classroom the symbols, procedures, and so forth, through being problematized, become meaningful parts of the mathematics.

This raises some questions for me. Should a goal ever be procedural? Are the procedures and symbols an *incidental*, though important, sideline in the construction of mathematical patterns and relationships? To paraphrase the group's question: Should a learning trajectory for the addition of whole numbers end in *an* addition algorithm? Should there be such a thing as a learning trajectory for the addition of whole numbers? Would we be better served in expressing our goals in such terms as "quantitative reasoning" and "multiplicative reasoning" (Harel & Confrey, 1994; Thompson, 1993)?

What should our "goals" for elementary students be?

Problems/tasks

Murray et al. argue: "Problems cannot be posed simply because they are good or seem interesting—the choice of problems should be based on thorough content analysis and a good understanding of how students develop concepts and misconceptions." Situations that encourage the development of "basic skills" are included in their description of what makes a worthwhile problem. While not dismissing problems that emerge from investigations and projects, they propose that problems need to have clear structures that can lead to significant mathematical thinking and discussion. These problems may have "realistic, abstract or imaginary contexts;" however, they place emphasis on the *mathematical* structure and the potential that these problems have to encourage "progressive schematisation." In elaborating these ideas they give an interesting example from their research into children's understanding of directed number. In earlier research they provided a setting using debt for children to explore this idea. They found that, while these children understood the debt situation quite well, it was not helpful to them as they gave meaning to directed number (the children "showed confusion"). Instead, their meaning emerged from numeric patterns and relationships that the children constructed: "Not debt, but mathematics, created the need for directed numbers."

In the other paper in this research forum, Gravemiejer, McClain, and Stephan (this volume) develop their arguments for appropriate instructional activities within the Realistic Mathematics Education tradition. Both papers share several common elements when discussing appropriate mathematics tasks. However, there appear to be important differences, particularly in the discussion of symbolization. Murray et al. encourage the introduction of symbols when students are engaged in mathematical activity which would be enhanced by their use, but such introduction occurs as the particular mathematical ideas emerge: "If, however, we make available notational tools *which students can match* to their thinking processes, they find it an important aid to individual as well as social construction of knowledge." Symbolizing emerges within children's problem solving activity (Reynolds & Wheatley, 1994). Gravemeijer et al. build such symbols into the instructional activities in a deliberate way, using the symbolizing as a vehicle for the development of the particular mathematics. While recognizing, as Murray et al. point out that "in a way almost all school problems *are* contrived" I wonder if these researchers would consider the example used by Gravemeijer et al. as "contrived."

As I read Murray et al.'s elaboration of the kinds of problems they consider further students' mathematics learning other questions arise for me. There has been a call in various reform efforts for mathematics tasks to be more "real world" so that students experience some relevance of mathematics in their lives. This has led to teachers being encouraged to use projects and investigations as a starting place, within which the mathematics can emerge. Coupled with this, an emphasis has been placed on making connections, not just within the various aspects of mathematics, but across the curriculum in other content areas. Yet, Murray et al. suggest that students develop strong mathematics from situations where the tasks are much more structured; in fact, their example from using a debt situation suggests that investigations and projects may in fact be confusing to children as far as their mathematical thinking is concerned. At the same time their research supports attempts to develop a more wholistic view of mathematics; they recommend, for example that problem types be "mixed, not blocked." Their research implies that students need to experience problems where they are encouraged to build connections with various mathematical ideas, rather than isolated problem sets.

What makes a "good" problem, one that leads students toward significant mathematics?

The social setting, collaboration and negotiation of social norms

Murray et al. make a case for collaboration in like-ability groupings, if one wants to promote mathematics learning, or as they describe, in Piaget's terms, "logico-mathematical knowledge." Cobb (1995) supports this also in his analysis of student's interaction in pairs. This is different from Vygotsy's (1978) notion of learning through interaction with a more knowledgeable peer or adult. However, Murray et al. suggest that "*physical* and *social* knowledge" is more appropriately supported using Vygotsy's ideas. I would suggest that the inquiry classroom elaborated in their paper is based on a learning model that does not fit easily with Vygotsy's ideas (Cobb & Bauersfeld, 1996). How is it possible to support the student's autonomous learning in mathematics as detailed here by Murray et al., and at the same time, establish social norms using a Vygotsyian approach?

How do the social norms that form the basis of a problem-centered classroom become established?

Diversity and change

The theme of the conference is "Diversity and change in mathematics education." Murray et al. address how a problem-centered learning setting enhances the mathematical development of diverse learners. One particular concern is when the formal language of instruction is a second language for students. While they suggest that further research is needed, they give a compelling example from a third-grade classroom that indicates that a problem solving approach is successful in addressing these students needs. They also provide evidence for enhanced learning in different special education settings. In discussing the role of time in the development of concepts they state: "our data show clearly that many (if not the majority) of students who seem to be mathematically weaker or slower than others can and do construct powerful mathematical concepts and generalisations provided the integrity of their thinking is preserved (i.e. somebody doesn't decide they need help and start demonstrating methods to them), the tasks they are presented with remain challenging and are not made easier, and the inquiry nature of the mathematics classroom is maintained." There is an unspoken, though perhaps not unrealistic, notion that the more competent students are well served in a problem solving setting.

The authors suggest that the classroom culture, with its emphasis on constructing meaning and negotiating with one's peers, addresses the needs of diverse learners. I suggest it does more than this. It challenges the values of the dominant group, and brings into question the assumptions of a particular culture or social system. In a session hosted by the SIG Philosophical Studies in Education, American Education Research Association Annual Conference, in April, 1988, several papers raised questions about the implications for diversity of a constructivist orientation in education. I would like to refer to one of them here. Boyles (1988) suggests that traditional epistemology holds three claims:

(T1) that there is an external world;

(T2) that we have some descriptions of the world that are correct; and

(T3) that we know *which* of our descriptions are correct.

From an inquiry based standpoint Boyles rejects T3 and suggests "that 'knowing' necessarily reveals itself to be a 'democratic' process that does not end in static knowledge claims separate (or separable) from the community of knowers from which those claims emerge, change, and reformulate" (p.1). Boyles uses democratic here in the sense in which Dewey elaborated it. He argues that acceptance of T3 leads to a 'traditional' approach in education where answers, and not reasoning, is valued. He suggests that, in such a setting, "students need not justify their answers" (p.11). A democratic, inquiry based classroom, on the other hand, encourages teachers and students to ask: "How do you know?" He makes the following strong statement in addressing the implications of his argument: "Dewey's differentiation between individual*ism* (separated, disconnected, egotistical) and individual*ity* (linked to the purposes of the larger society, while still distinguishable and unique among a crowd) justifies a democratic epistemology which does not *value* diversity, it *requires* it" (p.17).

I suggest that Murray et al.'s problem-centered classroom setting has much in common with the democratic classroom described by Boyle. As such there are some important implications for diversity and change in mathematics education. Perhaps this is an underlying reason for the considerable tension in implementing this approach in the local school districts in South Africa. Certainly, opposition to similar approaches in the United States comes in part from groups who see diversity as a threat to mainstream society (Routman, 1996).

Does a problem-centered approach to mathematics learning value and perhaps *require* diversity?

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SUPPORTING STUDENTS' CONSTRUCTION OF INCREASINGLY SOPHISTICATED WAYS OF REASONING THROUGH PROBLEM SOLVING

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During the last several years we have collaborated with Paul Cobb and Erna Yackel on classroom teaching experiments in grades one, three, and seven. Each of the teaching experiments lasted from 10 to 36 weeks and involved daily collaboration with the classroom teacher. The approach that we take toward classroom research emphasizes the teacher's proactive role, the contribution of carefully sequenced instructional tasks, and the importance of discussions in which students explain and justify their thinking. These interrelated aspects of instruction play a critical role in supporting students' development of powerful ways of reasoning and problem solving. A primary focus while working in classrooms is then on the question of how to support the development of both collective mathematical meanings and the understandings of individual students who contribute to their emergence. Answers to these research questions comprise characteristics of the classroom culture, the role of the teacher, and the role of instructional activities. In this paper, we will focus on the latter. The research question then is narrowed down to: How can the design and enactment of instructional activities support the development of sophisticated ways of mathematical reasoning? In addressing this question, we try to build upon an approach to instructional design that has been developed within the tradition of Realistic Mathematics Education (RME). This approach incorporates "guided reinvention" which implies beginning with students' informal mathematical activity while aiming to proactively support the emergence of increasingly sophisticated ways of symbolizing and understanding. In our research, the role that students' models of their informal mathematical problem solving play in supporting their transition to more formal yet personally-meaningful mathematical activity has become a central focus.

The RME approach is rooted in Freudenthal's (1971, 1973) description of mathematics as an activity that involves solving problems, looking for problems, and organizing a subject matter resulting from prior mathematizations or from reality. The heart of this process involves mathematizing activity in problem situations that are experientially-real to students. As a result, discussions of general activity and of reasoning with conventional symbols and tools frequently fold back to referential activity or even to activity in the setting (McClain & Cobb, in press). The development of both collective mathematical practices and of the understandings of individual students who participate in them therefore appears to be a recursive, multi-leveled process (Pirie & Kieren, 1994). This approach stands in contrast to traditional instruction where the goal is a linear progression through mastery of isolated topics.

Our research is carried out within the overarching framework of the developmental research paradigm (Gravemeijer, 1994). This consists of cyclic processes of thought experiments and teaching experiments at a variety of levels. These range from those closely tied to the design and enactment of problem solving activities on a day-to-day basis to levels more distanced from classroom practice. This research paradigm consists of both instructional development and classroom-based research. In particular, it draws on heuristics for instructional development proposed by the theory of RME and on an interpretative framework known as the emergent perspective (Cobb & Yackel, 1996). The emergent perspective places the students' and teacher's activity in social context by explicitly coordinating sociological and psychological perspectives. The psychological perspective is constructivist and treats mathematical development as a process of self-organization in which the learner reorganizes his or her activity in an attempt to achieve purposes or goals. The sociological perspective is interactionist and views communication as a process of mutual adaptation wherein individuals negotiate mathematical meanings. From this perspective, learning is characterized as the personal reconstruction of societal means and models through negotiation in interaction. Together, the two perspectives treat mathematical learning as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society. Analyses of individual students' activity are therefore coordinated with analyses of the collective or communal classroom processes in which they participate.

We would argue that the continual negotiations between the teacher and students support the emergence of taken-as-shared meanings when certain social and sociomathematical norms are in place (Cobb & Yackel 1996; Yackel & Cobb, 1996; McClain, 1995). Conversations in which students discuss the adequacy of their mathematical interpretations and solutions to problem situations can serve as instances in which they might reflect on their own and others' mathematical reasoning. This acknowledges the teacher's role by highlighting the importance of initiating reflective shifts in discourse such that what is said and done in action subsequently becomes an explicit topic of discussion (cf. Cobb, Boufi, McClain, & Whitenack, 1997). In line with many current reform recommendations, our point of departure is that the starting points for mathematics instruction consist of settings in which students can immediately engage in informal, personally meaningful mathematical problem solving. The challenge then becomes that of supporting the collective learning of the classroom community during which taken-as-shared mathematical meanings emerge as the teacher and students negotiate interpretations and solutions. We view instructional design as offering part of the answer to this challenge. In planning a classroom teaching experiment, we find it helpful to formulate a conjectured "local instruction theory." Such a theory can be seen as a set of hypothetical learning trajectories though formulated on a more global level, and together encompassing a longer learning trajectory than Simon (1995) had in mind when he coined the term. To construe conjectured local instruction theories, we conduct anticipatory thought experiments by envisioning both how proposed problem solving activities might be realized in interaction in the classroom, and what students might learn as they participate in them. While we acknowledge the teacher's proactive role in this interaction, we contend that an instructional approach in which the teacher builds proactively on students' contributions is only possible if the sequence of problem-solving activities consists of tasks that give rise to interpretations and solutions that can advance the instructional agenda. By proactive, we mean that the teacher planfully attempts to capitalize on students' reasoning to achieve her pedagogical agenda. This necessarily involves an inherent tension between individual students' expressive creativity while problem solving and their enculturation into established mathematical ways of knowing.

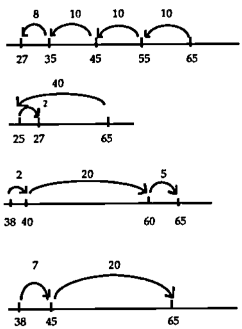
The challenge for the designer then becomes that of designing sets of problem-solving activities that will support the gradual emergence of increasingly sophisticated ways of reasoning and symbolizing. In doing so, we acknowledge that the teacher has to introduce particular ways of symbolizing at certain points in the instructional sequence while the students are expected to develop personally-meaningful ways of reasoning with these symbolizations. In this regard, designers who follow the RME approach have to cope with the inevitable tension between the ideal of building on students' contributions and deciding in advance which symbolizations students might come to use. The aspect of RME theory that makes this tension tractable is that the selected ways of symbolizing are designed to fit with the ways of reasoning that students have developed at particular points in the instructional sequence. Laying out a conjectured developmental route for the classroom community in which students first model situations in an informal way (called a *model of* the situation) and then mathematize their informal modeling activity (producing a *model for* mathematical reasoning) is an element of the RME heuristic. A hypothesized local instruction theory of this type involves the conjecture that the model, which emerges as students formalize their reasoning, will gradually take on a "life of its own" independent of situation-specific imagery. The benefit of such a local instruction theory is that it outlines both a learning agenda as well as a possible means for attaining it. In this way, it can serve as a resource for the teacher attempting to proactively support the collective development of taken-as-shared symbolizations and meanings.

In order to clarify our program of research and the role that problem solving plays in supporting students' mathematical development we address the question: How can the design and enactment of instructional activities support this development? In doing so, we focus on the role that students' models of their informal mathematical problem solving play in supporting their transition to more formal yet personally-meaningful mathematical activity. As an example, we will outline a teaching experiment involving an instructional sequence aimed at measuring and mental arithmetic. This twelve-week teaching experiment was conducted in a first-grade classroom (age six) in the spring semester of 1996. In the following sections of the paper, we will start by explaining and justifying the conjectured local instruction theory that formed the point of departure for the teaching experiment. Against this background, the students' mathematical activity that emerged will be analyzed. These analyses are intended to document (1) the role of problem solving and symbolizing in the learning of the classroom community, and (2) the role of the teacher in supporting this learning.[i]

Overview of the Instructional Sequence

The instructional sequence we discuss was intended to support students' continued construction of increasingly sophisticated concepts of ten, and their development of flexible mental computation strategies. The intent can be further clarified by noting a distinction made in the research literature between two general types of computational strategies. In one type, the child partitions numbers separately into a "tens part" and a "ones part" and adds or subtracts them separately (e.g., 39 + 53 is found by reasoning 30 + 50 = 80, 9 + 3 = 12, 80 + 12 = 92). In the second type, the child increments or decrements directly from one number (e.g., 39 + 53 found by reasoning 39 + 50 = 89, 89 + 3 = 92). These two types of strategies appear to reflect two different ways of interpreting two-digit numerals and number words that Cobb and Wheatley (1988) called collection-based and counting-based interpretations respectively, and that Fuson (1990) called collected multiunits and sequence multiunits. A point of departure for the design of the sequence under discussion was the Empty Number Line sequence that is specifically designed to support the development of counting-based rather than collection-based conceptions and strategies. The metaphor implicit in these conceptions is that of counting activity that can be curtailed or chunked.

The primary means of symbolizing developed during the instructional sequence was that of an empty number line on which the children recorded their arithmetical reasoning. Gravemeijer (1994) gives the following examples of strategies for solving 65 - 38 that have been notated by students:



The conjecture underlying the instructional sequence is that the empty number line will initially emerge as a *model of* children's informal arithmetical reasoning, but that it will subsequently become a *model for* increasingly abstract yet personally-meaningful arithmetical activity (Gravemeijer, 1994).

The initial instructional activities in the Empty Number Line sequence as it was originally outlined by its developers involved the use of a bead string composed of 100 beads (Treffers, 1991). The beads were of two colors and were arranged in groups of ten. We decided to modify the sequence by omitting these instructional activities because the bead string did not serve as a means by which children might explicitly model their prior problem-solving activity. Instead, we attempted to develop the Empty Number Line sequence by building measurement practices. Speculating about the genesis of the ruler in history, we imagined that the ruler came about as a curtailment of iterating a measurement unit. As a result, we conjectured that instructional activities could be designed in which the ruler would come to the fore as a *model of* iterating some measurement unit. The connection between measuring and the number line then would be in the relation between measurement as accumulating distance, and a cardinal interpretation of distance on the number line. In the context of measurement, distances on a ruler/number line would first have the status of magnitudes. Then, with tasks involving incrementing and decrementing, the context of measurement would drop into the background and the interpretations would shift towards quantities. Finally, even the direct reference to specific quantities consisting of identifiable objects would become less prominent as the numbers on the empty number line start to derive their meaning from a framework of number relations.

We envisioned something like the following trajectory: (1) The students start with measuring various lengths by iterating some measurement unit. (2) This activity of iterating is modeled with a ruler. At the same time the activity of measuring is extended to incrementing, decrementing and comparing. (3) These activities are represented with the help of arcs on a schematized ruler. (4) This schematized representation is used as a way of scaffolding, and as a way of communicating solution methods for all sorts of addition and subtraction problems. In light of the instructional intent of the sequence — to support students' continued construction of increasingly sophisticated concepts of ten, and their development of flexible mental computation strategies — special attention would have to be given to the decimal structure of our number system. In the context of measurement, this can be done by curtailing the iterating activity by means of a larger measurement unit, i.e. a unit of ten. Measuring with "tens" and "ones" can form the basis for the development of a ruler that is structured by tens and ones. At the same time, it may support the development of strategies that use doubles as reference points.

A feature that could cause problems in such a set up is the rigid character of a ruler whereas on one hand, and the intended more global character of the empty number line on the other. On a ruler, positions have to correspond to exact distances, on the number line the distances between the numbers do not have to correspond exactly with their numeric values. The latter is essential since otherwise, it would be impossible to represent unknown numbers or unknown differences. In reflection, moreover, it makes sense to differentiate between measuring and representing strategies. On the empty number line you would want to express how you would increment 64 with 28. For instance, by first measuring 64, then add 6 ones which would get you to 70, then measure two times ten which would get you to 80, and 90 respectively, and finally add two ones which gets you to 92. When describing this method, it would be sufficient to show that you start at 64, add 6, arrive at 70, then add 10 arriving at 80, another 10 arriving at 90, and 2 arriving at 92. To try to strive for an exact proportional representation of all the jumps would severely hamper a flexible use of the number line. Therefore, instead of schematizing the ruler, we would have to distinguish between the ruler and the empty number line. This fits with our contention that, in practice, the model may take various shapes (Cobb et. al., 1997). In relation to this we should differentiate between "the model" as an overarching concept, and the symbolic representations that are used in the instructional sequence. In our measurement/mental arithmetic sequence, "the model" is the concept of the ruler. In the concrete elaboration in the sequence, this overarching model takes on various manifestations. These manifestations then relate to the emergence in practice of a chain of signification (Walkerdine, 1988).

From a mathematics education perspective, the notion of a chain of signification is helpful in that it shows how formal mathematical signs can be rooted in concrete activities of the students. The dynamic character of the chain of signification as such justifies the term "emergent models." However, the meaning of the label emergent is broader. It refers both to the process by which models emerge within RME, and to the process by which these models support the emergence of formal mathematical knowledge. In our view, a crucial aspect of the emergent-models heuristic is that the shift from *model of* to *model for* is reflexively related with the creation of mathematical reality. This can be elucidated with the role of the ruler as a model. What is expected is, that in the course of the sequence, a shift is taking place in which the student's view of numbers transitions from referents of distances to numbers as mathematical objects. Crudely stated, this shift can be seen as a transition from viewing numbers as adjectives (i.e. constituends of magnitudes: "37 feet") to viewing numbers as nouns ("37"). For the student, a number viewed as a mathematical object still has quantitative meaning, but this meaning is no longer dependent upon its connection with an identifiable magnitude or quantity. In the students' experienced world, numbers viewed as mathematical objects derive their meaning from their place in a network of number relations. Such a network may include relations such as 37=30+7, 37=3x10+7, 37=20+17, 37=40-3. The critical aspect of this network is that the students' understanding of these relations transcends individual cases. That is, when students form notions of mathematical objects, they view these relations as holding for *any* quantity of 37 objects (including a magnitude of 37 units). In our view, this shift from numbers as referents to numbers as mathematical objects is reflexively related to the *model of* to *model for* transition described earlier. On the one hand, the students' actions with "the model" foster the constitution of a framework of number relations. On the other hand, through the students' development of this framework of number relations, "the model" can take its role as a model *for* mathematical reasoning. In this setting, the teacher will have to introduce particular ways of symbolizing at certain points in the instructional sequence, while the students are expected to develop personally-meaningful ways of reasoning with these symbolizations. The question then is how this ideal is realized in the classroom. In answering this question, we will provide classroom episodes that can be viewed as paradigmatic instances of supporting students' development of increasingly sophisticated ways of reasoning that emerged through problem solving.

Classroom Setting

Ms. Smith's class comprised seven girls and nine boys and was one of three first-grade (age six) classes in an elementary school in a large suburban area. The majority of the students were from middle- to upper-middle-class families. The teacher, Ms. Smith, was a highly motivated and very dedicated teacher in her sixth year in the classroom. She worked to create a classroom environment which supported her students' learning and had developed teaching practices that were very different from those of most other teachers at the school. Although she highly valued students' ability to communicate, explain, and justify, Ms. Smith indicated that she had previously found it difficult to enact an instructional approach that both met her students' needs and enabled her to achieve her own pedagogical agenda. She voiced frustration with traditional mathematics textbooks and had attempted to reform her practice prior to our collaboration. The relationship we established with Ms. Smith was such that she soon viewed the members of the research team as peers with whom she reflected daily about events in her classroom.

Although the instructional activities Ms. Smith used contributed to her effectiveness, it was the activities as they were realized in interaction in the classroom that supported the students' mathematical development. The way in which the instructional sequences were enacted was influenced in part by the general classroom social norms. Whole-class discussions typically involved genuine discussions about mathematically-significant issues that the students appeared to view as important. The emphasis on understanding in this classroom was such that the students rarely said they disagreed but instead challenged solutions they considered invalid by saying that they did not understand and then explaining why the solution did not make sense to them. The two general values that characterize the microculture established in Ms. Smith's classroom are those of attempting to understand and of active participation at all times including when others were speaking.

Classroom Episodes

It is important to note that the instruction in this teaching experiment was designed to build on prior mathematical practices so that when students encountered new problem situations, they would have a way to act. Further, as part of her role, Ms. Smith attempted to highlight solutions in whole-class discussion that would give rise to problematic situations. The format of the mathematics class entailed students typically working individually or in pairs on instructional tasks and then returning to whole-class setting to discuss their activity. During their small group work, Ms. Smith would circulate among the groups to monitor their activity. At these times she was attempting to identify mathematically significant issues that were emerging so that these could then become the focus of whole-class discussions. Her decisions were informed by her understanding of the intent of the instructional sequence, her knowledge of her students' mathematical background, and her collaboration with the research team.

In the part of the instructional sequence that dealt with measurement, our initial goal was that students might come to reason mathematically about measurement and not merely measure accurately. This approach differs significantly from many that are frequently used in American schools in that the focus was on the development of understanding rather than the correct use of tools. In particular, we hoped that the students would come to interpret the activity of measuring as the accumulation of distance (cf. Thompson & Thompson, 1996). For instance, as the students were measuring by pacing heel-to-toe, we hoped that the number words they said as they paced would each come to signify the measure of the *distance* paced thus far rather than the *single pace* that they made as they said a particular number word (e.g. saying "twelve" as students paced the twelfth step would indicate a distance that was twelve paces long instead of just the twelfth step). Further, our intent was that the results of measuring would be structured quantities of known measure. If this was the case, students would be able to think of a distance of, say, 20 steps that they had paced as a quantity itself composed of two distances of ten paces, or of distances of five paces and fifteen paces. We therefore hoped that the students would come to act in a spatial environment in which distances are structured quantities whose numerical measure can be specified by measuring. In such an environment, it would be self evident that while distances are invariant quantities, their measures vary according to the size of the measurement unit used.

The instructional activities used in the teaching experiment were typically posed in the context of an ongoing narrative. To accomplish this, the teacher engaged the students in a story in which the characters encountered various problems that the students were asked to solve. The narratives both served to ground the students' activity in imagery and provided a point of reference as they explained their reasoning. In addition, the problems were sequenced within the narratives so that the students developed increasingly effective measurement tools with the teacher's support. The narratives supported the emergence of tools out of students' problem-solving activity.

The first narrative involved a kingdom in which the king's foot was used as the unit of measure. The initial instructional activities involved students acting as the king and measuring by pacing. On the day that the king's foot scenario was introduced, the teacher and the students discussed how the king might use his feet to measure the length of items in the kingdom. In their discussions, the teacher and students decided that the king could pace the length of an item by placing his feet heel-to-toe without leaving any space in between paces. As students worked in pairs on measuring tasks, two distinct ways of counting paces emerged. Some students placed their first foot such that the heel was aligned with the beginning of the item and counted "one" with the placement of their first foot (see Figure 1a). Other students placed their first foot down such that their heel was aligned with the beginning of the item to be measured and counted "one" with the placement of the second step (see Figure 1b).

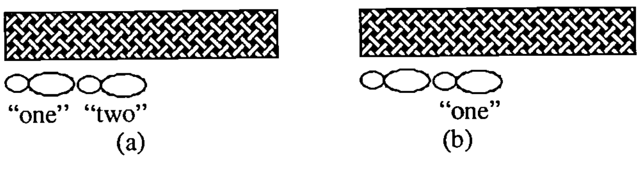


Figure 1(a-b). Two methods of counting as students paced.

In observing the students' activity, we conjectured that a discussion of these two ways of measuring would provide an opportunity for the mathematically significant issue of *measuring as filling space* to become a topic of conversation. As a consequence, the teacher initiated a whole-class discussion in which the two different ways of counting paces were contrasted.

Teacher: I was also really watching how a couple of you were measuring. Who wants to show us how you'd start off measuring [the rug], how you'd think about it?

Sandra: Well, I started right here [places the heel of her first foot at the beginning of the rug] and went 1 [starts counting with the placement of her second foot as in Figure 1b] 2, 3, 4, 5, 6, 7, 8.

Teacher: Were people looking at how she did it? Did you see how she started? Who thinks they started a different way? Or did everybody start like Sandra did? Angie, did you start a different way or the way she did it?

Angie: Well, when I started, I counted right here, [places the heel of her first foot at the beginning of the rug and counts it as one as in Figure 1a], 1, 2, 3.

Teacher: Why is that different to what she did?

Angie: She put her foot right here [places it next to rug] and went 1 [counts one as she places her second foot], 2, 3, 4, 5.

Teacher: How many people understand? Angie says that what she did and what Sandra did was different? How many people think they understand? Do you think you agree they've got different ways?

In order to support a conceptual discussion of these two different methods, the teacher asked Melanie to begin pacing the length of the rug so that she could place a piece of masking tape at the beginning and end of each pace. Once this record was made, the students who counted their paces Angie's way began to argue that Sandra's method of counting would lead to a smaller result because she did not count the first foot. In the excerpt below, Melanie differentiates between the two ways of counting paces while other students justify their particular method.

Melanie: Sandra didn't count this one [puts foot in first taped space], she just put it down and then she started counting 1, 2. She didn't count this one, though [points to the space between the first two pieces of tape].

Teacher: So she would count 1, 2 [refers to the first three spaces since the first space is not being counted by Sandra]. How would Angie count those [points to the first three taped spaces]?

Melanie: Angie counted them 1, 2, 3.

Teacher: So for Angie there's 1, 2, 3 there and for Sandra there's 1, 2.

Melanie: Because Angie counted this one [points to the first taped space] and Sandra didn't, but if Sandra would have counted it, Angie would have counted three and Sandra would have too. But Sandra didn't count this one so Sandra has one less than her.

Teacher: What do you think about those two different ways, Sandra, Angie or anybody else? Does it matter? Or can we do it either way? Hilary?

Hilary: You can do it Angie's way or you can do it Sandra's way.

Teacher: And it won't make any difference?

Hilary: Yeah, well they're different. But it won't make any difference because they're still measuring but just a different way and they're still using their feet. Sandra's leaving the first one out and starting with the second one, but Angie does the second one and Sandra's just calling it the first.

Preston: They're both different ways. I thought Sandra's way would go higher than Angie's. Cause Angie started by ones and got 3 and Sandra only got two. Sandra would go higher cause she was lesser than Angie. She's 15 [refers to the total number of feet Sandra counted when she paced]. Angie went to the end of the carpet [he means the beginning of the carpet]. Sandra started after the carpet. Hers is lesser 'cause there's lesser more carpet. Angie started here and there's more carpet. It's the same way but she's ending up with a lesser number than everybody else.

Alex: She's missing one right there. She's missing this one right here [points to the first taped space]. She's going one but this should be one cause you're missing a foot so it would be shorter.

Teacher: So he thinks that's really important. What do other people think?

Alex: Since you leave a spot, it's gonna be a little bit less carpet.

Throughout this episode, the teacher continued to raise issues that would contribute to a mathematical discussion about the different ways to measure. In the process, the students' prior activity of measuring became the topic of discussion as students clarified their differing interpretations of the task. As the discussion continued, the students seem to accept Angie's method. As a result, the teacher was able to support shifts in the students' understanding through their own efforts at problem solving.

As the students continued to use their own feet in measuring, the issue of different size feet yielding different measures emerged. In the subsequent discussion, the teacher and students decided to create a "standard" measure that could be used by all students. The result was called a *footstrip* and consisted of five shoe-prints taped together heel-to-toe. A number of mathematically significant issues emerged in whole-class discussions involving students measuring with the footstrip including that of describing distances measured in terms of footstrips (e.g. five footstrips versus 25 paces). (For a detailed analysis see Stephan, 1998).

The second narrative developed during the measurement sequence involved a community of Smurfs who often encountered problems that required finding the length or height of certain objects. The teacher explained that they decided to measure by stacking cans the height of the object to be measured. In the classroom, the students used unifix cubes as substitutes for cans and measured numerous objects for the Smurfs such as the height of the wall around the Smurf village, the length of the animal pens, and the depth of the water in the river. After several measuring activities, the teacher explained to the students that the Smurfs were getting tired of carrying around the large number of cans needed for measuring. The students agreed that this was cumbersome and discussed alternative approaches. Several suggested iterating a bar of cubes (cans), presumably influenced by the prior activity of measuring with the footstrip. This discussion seemed to influence their decision to measure with a bar of ten cubes that they called a *Smurf bar.*

When measuring with the Smurf bar, most of the students measured by iterating the bar along the length of the item to be measured and counting by tens. However, some students counted the last cubes of the measure *within* the last iterated decade instead of *beyond* it. Solutions of this type became the focus of discussions as can be seen in an incident that occurred two weeks after the measurement sequence began. The teacher had posed the following task: *The Smurfs are building a shed. They need to cut some planks out of a long piece of board. Each plank must be 23 cans long. Show on the board where they would cut to get a plank 23 cans long.* Students had been given long pieces of adding machine tape as the board and were asked to use the Smurf bar to measure a plank the length of 23 cans. Angie was the first student to share her solution process with the class. She showed how she had measured a length of 23 cans by iterating the bar twice and then counting 21, 22, and 23 *beyond* the second iteration. When she finished, Evan disagreed.

Evan: I think it's 33 because ten (places down bar as in Figure 2a), 20 (moves bar as in Figure 2b), 21, 22, 23 (points to the cubes *within* the second iteration, thus measuring a length that was actually 13 cubes).

Angie: Um, well, see, look, if we had ten (moves bar as shown in Figure 2a) that would be 11, 12, 13.

Evan: But, it's 23.

Figure 2a. Showing the first iteration in measuring 23 cans.

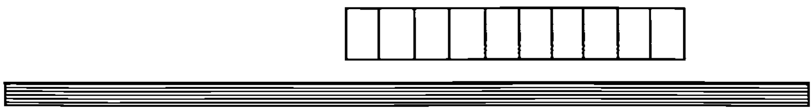


Figure 2b. Showing the second iteration in measuring 23 cans.

At this point, Evan and Angie appeared to be miscommunicating. Although Angie appeared to understand how Evan found a length that differed from hers, she was unable to explain her reasoning to him. This miscommunication continued as Andy explained why he agreed with Evan. The teacher then asked both Evan and Angie to share their solution methods again.

Teacher: Let's be sure all the Smurfs can understand 'cause we have what Angie had measured and what Evan had measured. We need to be sure everybody understands what each of them did so Evan, why don't you go ahead and show what it is to measure 23 cans.

Evan: Ten (places bar as shown in Figure 2a), 20 (moves bar as shown Figure 2b). (Pause) I changed by mind. She's right.

Teacher: What do you mean?

Evan: This would be 20 (points to end of second iteration).

Teacher: What would be 20?

Evan: This is 20 right here (places one hand at the beginning of the "plank" and the other at the end of the second iteration). This is the 20.

Teacher: So that where your fingers are shows a plank that would be 20 cans long? Is that what you mean? Any questions for Evan so far?

Evan: Then if I move it up just three more. There. (Breaks the bar to show 3 cans and places the 3 cans beyond 20). That's 23.

It appeared that in the course of re-explaining his solution, Evan reflected on Angie's method and reconceptualized what he was doing when he iterated the bar. Initially, for Evan, placing the Smurf bar down the second time as he said "20" meant the twenties decade. Therefore, for him, 21, 22, and 23 lay within the second iteration. However, he subsequently reconceptualized "20" as referring to the distance measured by iterating the bar twice and realized that 21, 22, and 23 must lie beyond the distance whose measure was 20. This type of reasoning was supported both by the teacher asking Evan to explain his method so that everyone would understand and by Evan at least implicitly counting the cans that he iterated when moving the bar (i.e. the measure of the first two iterations was 20 because he would count 20 cans).

As the episode continued, Angie continued to explain her thinking:

Angie: I have a way to help Andy because he thinks like Evan before he changed his mind. You see like this is ten, but you know that right? (Andy nods in agreement as Angie places down the first iteration). 11, 12, 13, . . ., 19, 20 (moves bar to second iteration and counts each cube individually, pointing to the cubes as she counts. She then moves the bar to the third iteration as she continues counting). 21, 22, 23. So it goes two tens and three more.

Here, in grounding her explanation in the counting of the individual cans that comprised the Smurf bar, Angie attempted to clarify her reasoning to Andy. As her explanation indicates, measuring involved the accumulation of distance in that iterating the bar while counting by tens was a curtailment of counting individual cans. As a consequence, it was self evident to her that she needed to measure beyond the second iteration to specify a length of 23 cans.

It is important to note that the teacher's overriding concern in this episode was not to ensure that all the students measured correctly. In fact, as noted earlier, the teacher frequently called on students who had reasoned differently about problems in order to make it possible for the class to reflect on and discuss the quantities being established by measuring. Her goal was that measuring with the Smurf bar would come to signify the measure of the distance iterated thus far rather than the single iteration that they made as they said a particular number word. Her focus was therefore on the development of mathematical reasoning that would make it possible for the students to measure correctly with understanding.

After the students had measured several planks and other items with the Smurf bar, the teacher explained that the Smurfs decided they needed a new measurement tool so they would not have to carry any cans around with them each time they wanted to measure. In the ensuing discussion, several students proposed creating a paper strip that would be the same length as a Smurf bar and marked with the increments for the cans. Students then made their own *ten-strips* and used them to solve a range of problems grounded in the Smurf narrative. During a discussion about the meaning of measuring by iterating a ten-strip, the teacher taped several of the students' strips end-to-end on the white board to show successive placements of the strip. In doing so, she created a *measurement strip* 100 cans long. Crucially, this new tool emerged from and was consistent with the students' current ways of measuring. As a consequence, they could all immediately use prepared measurement strips with little difficulty. They simply placed the strip along the dimensions of the object to be measured and counted along the strip by ten or, sometimes by five.

All the instructional activities we have discussed thus far involved measuring the lengths or heights of physical objects. The transition from the measurement sequence to the mental computation and estimation sequence occurred when the students began to use the measurement strip to reason about the relationship between the lengths or heights of objects that were not physically present. One of the first instructional activities in the mental computation and estimation sequence involved an experiment the Smurfs were conducting with sunflower seeds. The teacher explained that the Smurfs typically grew sunflowers that were 51 cans tall. However, in one of the experiments, the sunflowers grew only 45 cans tall. Students were then asked to find the difference in heights and were given only a measurement strip. As a consequence, they could not measure sunflowers 45 cans and 51 cans long, but instead had to specify the spatial extension of the sunflower on the strip and then use the strip to reason about how to solve the task.

Students first worked in pairs and then discussed their solution in the whole-class setting. The teacher began by placing a vertical measurement strip on the wall and asking students to mark both 51 cans and 45 cans (see Figure 3).

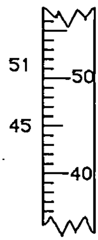


Figure 3. Measurement Strip marked to show 51 and 45.

An issue that emerged almost immediately in the discussion was that of whether to count the lines or the spaces on the strip.

Teacher: Think about how you would show us how much shorter that seed (points to the 45) grew than the regular seed. Preston, how would you show that?
Preston: Here (points to 51) all the way down to here (points to 45) would be seven.
Teacher: Can you show me the seven?
Preston: Here is 51 and here is 45 and here is 1, 2, 3, 4, 5, 6, 7 (points to lines as he counts).
Pat: I have a question. You are supposed to count the spaces not the lines.

At this point, the teacher asked Pat to explain why he thought you use the spaces and not the lines.

Pat: The cans of food are bigger than the lines and you are trying to figure out how many cans not lines.
Teacher: So when you say space you think of this space as a can of food (points)?
Pat: And that's how much and you're trying to figure out how much that is.

Preston seemed to be relating his counting activity to counting spaces when he said "I'm counting the lines because they're the same as spaces." In fact, he argued that either counting lines or spaces would give the same number. Thus, it seemed that Preston and others were indicating the number of cans or the measure of the spatial extent between two heights by counting what was perceptually available without taking the extra line into account. In the excerpt above, we were trying to encourage students to justify their particular method in terms of what each line or space signified to them. For Pat, the lines signified the top and bottom of a cube and the space signified the distance covered by one whole cube. Thus, counting the lines gave an extra number because "there's two lines in one space." As a result of this and other conversations, counting spaces to specify the measure of the spatial extension between two lengths became taken-as-shared.

Immediately after the exchange between Pat and Preston, Andy gave an explanation that involved reasoning about the quantities in a different way.

Andy: If you went from 50 down five you'd get to 45 cans. Think 5 less than 50. But you are really one more so it's six since it's one more than 50.

Andy's explanation indicates that for him, the task was to find the difference between these two quantities and he did so by reasoning with the strip. We would in fact argue that the strip supported the shift he made from a counting to a thinking strategy solution in which he reasoned that 50 to 45 was five, so 51 to 45 is six.

It is important to note that the solution method offered by Andy fit with the teacher's pedagogical agenda of supporting students' development of increasingly sophisticated strategies. While she both redescribed and notated Andy's solution, she was also aware of differences in her students' reasoning and did not want to create a situation where students simply imitated strategies that they did not understand. As a result, she continued to acknowledge the differing ways that students reasoned about tasks while highlighting solution methods that fit with her agenda. This diversity continued to support students' ability to construct personally-meaningful ways to reason mathematically. (An analysis of the students' activity with the Empty Number Line can be found in McClain, Cobb, Gravemeijer, & Estes, in press).

Discussion

In looking back at the role of symbolizations in students' development of increasingly sophisticated ways of reasoning, we would argue that the symbolizations created the opportunity for the students to discus mathematically relevant issues that otherwise would not have become a topic of discussion. In addition to this, the development of mathematical meaning that resulted from such discussions laid the basis for the introduction of more sophisticated ways of symbolizing.

By showing how they paced, for instance, the students created an opportunity to discus their interpretations of the act of pacing. In this discussion, the mathematically significant issue of measuring space emerged as a topic of conversation. Subsequently, their interpretations of pacing were integrated in the symbolization with the footstrip. Acting with the footstrip in turn created an opportunity to discuss the mathematical meaning of this activity. In a similar way, when measuring with the Smurf bar, the way of counting the last cubes became the focus of discussions. Here, someone like Angie grounded her explanation in the counting of the individual cans that comprised the Smurf bar as she attempted to clarify her reasoning. In terms of Pirie & Kieren (1994), we could say she folded back. This shows that for her measuring with the Smurf bar symbolizes a curtailment of counting individual cans. Next, the teacher created a measurement strip 100 cans long by taping several Smurf bars end-to-end to show successive placements of the bar. As before, the new symbolization emerged from and was consistent with the students' current ways of measuring. The students then started reasoning with the strip to solve tasks like finding the difference between two quantities. We would in fact argue that the strip supported shifts from counting strategies to thinking strategies.

In reflection, we can discern a dialectic process of symbolizing and the development of mathematical meaning as suggested by Meira (1995). Here we should emphasize the role of the teacher. Throughout this episode, the teacher continued to raise issues that would contribute to a mathematical discussion about the different ways to measure. In the process, the students' prior activity of measuring became the topic of discussions as students clarified their differing interpretations of the tasks. We would argue that it is in the combination of (1) instructional activities that comprise symbolizations that are an elaboration of the model/of-model/for design heuristic, and (2) in the teacher's activity of framing mathematical issues that are relevant in light of the endpoints, that the ideal of problem solving can be integrated with reaching commonly accepted endpoints.

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SUPPORTING STUDENTS' REASONING THROUGH PROBLEM SOLVING — IMPLICATIONS FOR CLASSROOM PRACTICE: A REACTION TO GRAVEMEIJER, McCLAIN AND STEPHAN

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This reaction to the paper by Gravemeijer, McClain and Stephan discusses the theoretical underpinnings of the research program in which they are engaged, highlights the need to address the question of how important findings from research such as this can be translated into a model of classroom practice accessible to the wider teaching community and looks at learning through problem solving in Japanese classrooms.

An overview of Gravemeijer, McClain and Stephan's paper

In their paper *Supporting students' construction of increasingly sophisticated ways of reasoning through problem solving*, Gravemeijer, McClain and Stephan (this volume) describe research arising from an ongoing collaboration with Paul Cobb and Erna Yackel, whose classroom experiments have formed the basis for a program of research which has now spanned approximately a decade (see, for example, Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Cobb, Wood & Yackel, 1991).

While focussing their paper on the topic of this research forum, *Learning through problem solving*, Gravemeijer, McClain and Stephan provide a valuable summary of the key features of this research program. These include three critical aspects which are inextricably interwoven in their attempts to develop in students powerful ways of reasoning and problem solving: the role of the teacher, the "design and enactment" of sequences of instructional tasks, and the development of a classroom culture which supports students in explaining and justifying their thinking.

The paper in this volume focuses on the role of instructional activities in achieving their aim. In developing instructional activities, the authors use an approach which comes from the tradition of Realistic Mathematics Education (see Treffers, 1991). In this approach, students are presented with "experientially-real" settings in which they can immediately engage in informal, personally meaningful, mathematical activity. This activity and its discussion in the classroom are used as the vehicles through which students develop increasingly sophisticated ways of symbolizing and understanding. The teacher's role in this approach is critical in "initiating reflective shifts in discourse such that what is said and done in action subsequently becomes an explicit topic of discussion". Thus a key element of the theoretical framework is the linking of a constructivist psychological perspective with an interactionist sociological perspective, which together "treat mathematical learning as both a process of active individual construction and a process of enculturation".

Another key feature of the "developmental research paradigm" used by this group of researchers is the "cyclic processes of thought experiments and teaching experiments". Among other things, this involves the researchers in formulating a conjectured "local instruction theory" which can be thought of as "a set of hypothetical learning trajectories" of a more global nature than those formulated by Simon (1995). The researchers conduct "thought experiments" which attempt to anticipate how the problem solving activities might be realised and the contributions which might be made by the students. In order to enable the teacher to purposefully build on student contributions and hence advance the learning agenda, activities have to be sufficiently robust to support "the gradual emergence of increasingly sophisticated ways of reasoning and symbolizing". The dual aims embodied in this approach provide an inherent tension between building on students' contributions and achieving pre-determined mathematical goals.

Part of the theoretical underpinning for the conjectured local instruction theories is based on Realistic Mathematics Education's heuristic of moving from situations in which students firstly model situations informally (i.e. produce a *model of* the situation) and then, later, formalize their modelling activity in such a way that it produces a *model for* their mathematical reasoning.

Gravemeijer, McClain and Stephan address the question of how the design and enactment of instructional activities can support students' mathematical development through a discussion of an instructional sequence which was used in a first grade class in order to "support students' sophisticated concepts of ten, and their development of flexible computation strategies". The role that students' *models of* their informal mathematical activity play in supporting the transition to more formal mathematics is emphasised.

During 1994, I spent six weeks at one of Erna Yackel's research sites just when the bead string was being used prior to the use of the *empty number line*. Having seen the bead string in use, I was somewhat surprised that the instructional sequence described in this paper rejects its use because it did not serve as a model of children's prior problem solving. Instead, the authors develop the empty number line through an extended sequence of measurement activities based on a series of narrative settings. In this sequence, the ruler is intended to be used initially as a *model of* the activity of iterating some measurement unit. As the sequence of activities progresses, the overarching model of a ruler is seen as becoming a *model for* the mathematical reasoning which will be symbolised through the use of the empty number line. The paper provides a detailed analysis of classroom episodes within this instructional sequence. These not only illustrate the students' increasingly sophisticated strategies for the solution of problems within the sequence, but also show how the teacher was able to acknowledge students' differing strategies, while at the same time highlighting those methods which advanced her "pedagogical agenda of supporting students' development of increasingly sophisticated strategies".

The authors argue that it is the combination of instructional activities, which have the potential to allow the use of the "*model of* to *model for*" design heuristic, and the teacher's ability to orchestrate the mathematical discussion, in such a way as to raise mathematically significant issues to fit her agenda, which enables problem solving to be used to achieve commonly accepted mathematical goals.

This perhaps excessively long overview already reveals some of the orientation which I bring to this reaction, in that it highlights certain aspects of the paper at the expense of others. It is not my intention to analyse the paper in any further detail, but rather to react to it (and the research program which it represents here) in a more global sense. But perhaps I should firstly say that, although I do not discuss it in detail here, I find the authors' analyses of the classroom episodes powerful confirmation of their theoretical standpoint and their final conclusion.

The need for a new model of mathematics teaching

Smith (1996) and Simon (1997) argue that attempts to reform school mathematics have undermined teachers' sense of efficacy by condemning the traditional expository model of teaching without replacing it with a clear new alternative. In the current climate of accountability in education, with its emphasis on test results and the advent of "Math Wars" in the USA, there is a danger that, rather than seriously explore what a new model of classroom practice might look like, there will be an attempt to return to the traditional expository model — the "failed lecture-to-class variety that was so characteristic of primary schools historically" (Reynolds, 1998, p. 41). Hence the articulation of a new model of mathematics teaching is an imperative for research in mathematics education.

The extensive, coherent and strongly theoretically grounded research program, which is exemplified by this paper of Gravemeijer, McClain and Stephan, provides an *existence proof* that students' mathematical knowledge can be developed in powerful ways through problem solving. Among the many important features of this research program are the simultaneous critical analyses of the design of the instructional sequences, the role of teacher and the classroom culture — including the socio-mathematical norms for what counts as acceptable explanations and justifications (see, for example, Yackel & Cobb, 1995).

Such empirically based research, and the work of others such as Lampert (1990) and Ball (1993), attempts to address a previous lack of connection between research on learning (particularly research based on constructivist views of learning) and research on teaching (Simon, 1995). However, when teachers in the wider education community attempt to implement ideas such as classrooms operating as communities of inquiry, they often only adopt superficial features (see, for example, Stigler, 1996; Knuth, 1997).

The paper by Gravemeijer, McClain and Stephan focuses on the role of the instructional activities in supporting students' development of increasingly sophisticated reasoning. The teacher in whose class the teaching experiment is carried out is described as highly motivated, having already "developed teaching practices that were very different from those of most other teachers at the school". Nevertheless, "she had previously found it difficult to enact an instructional approach that both met her students' needs and enabled her to achieve her own pedagogical agenda". Importantly, she is able to "view the members of the research team as peers with whom she reflected daily about events in her classroom".

In the paper, we see how this teacher is able to achieve her agenda by making the instructional sequence central to the learning process. Developing the instructional sequence draws on both the researchers' knowledge and experience (for example, their work with Realistic Mathematic Education) and on their joint experiences of enacting the sequence in the classroom, thus highlighting what Simon (1995, p. 138) calls "the experimental nature of mathematics teaching ... the ongoing cycle of hypothesis generation (or modification) and data collection".

The intellectual effort which is required for this process should not be underestimated. Simon (1995, p. 136) views the hypothetical learning trajectory as being "made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities". Considering this aspect alone, the predictions of student thinking and understanding need to be based on experience — the result of data collection in previous experiments as part of the "ongoing cycle". Simon points out that, while individual student learning is idiosyncratic, his hypothetical learning trajectory assumes "that an individual's learning has some regularity to it ... , that the classroom community constrains mathematical activity in often predictable ways, and that many of the students in the same class can benefit from the same mathematical task" (1995, p. 135).

The issue for me then is this: How can the important findings of research such as that of Gravemeijer, McClain and Stephan be translated into a model of classroom practice which is accessible to the wider teaching community? What could a new, widely applicable, model of mathematics teaching which adopts the aim of fostering students' constructions of powerful mathematical ideas through problem solving and inquiry look like?

I have no answers to these questions — in fact, questions such as these form the basis for a proposed research project, arising from previous work in the *Practical Mechanics in Primary Mathematics* project (see Doig, 1997; Groves, 1997; Williams, 1997), and Splitter's work in *Philosophy for Children* (see Splitter & Sharp, 1995).

Learning through problem solving in Japanese mathematics classes

While results from the recent Third International Mathematics and Science Study (TIMMS), in which Singapore, Korea, Japan and Hong Kong all performed significantly better in mathematics than Australia and the United States, have already led to calls for Australian and American schools to return to traditional models of classroom practice, there is extensive research evidence to show that, at least in Japanese and Korean schools, teaching is not characterised by rote learning but instead involves a considerable amount of whole class, teacher orchestrated discussion building on students' ideas (see, for example, Stevenson & Stigler, 1992). Moreover, Stigler's TIMMS study of video data from 100 German, 81 United States and 50 Japanese year 8 classrooms led him to conclude that "Japanese teachers come closer to implementing the spirit of current ideas advanced by American reformers than do American teachers" (Stigler, 1996, p. 7). I would therefore like to look at Japanese classroom practice through the window of the theoretical framework established by Gravemeijer, McClain and Stephan.

By way of contrast with German and American lessons, Stigler describes the Japanese lessons as being characterised by the fact that "problem solving comes first, followed by a time in which students reflect on the problem, share the solution methods they have generated, and jointly work to develop explicit understandings of the underlying mathematical concepts" (1996, p. 6). This description fits well with Japanese lessons which I have observed in Japan and Australia, as well as the sample video lessons produced as part of Stigler's video survey. To illustrate this, I will briefly describe a lesson I observed recently in a grade 5 class of Japanese children. Although the lesson was conducted entirely in Japanese and I speak no Japanese, I feel able to make a few limited comments on the lesson as I am well acquainted with the *Paper Folding* problem used (see Stacey & Groves, 1985, p. 53) and we had the benefit of a translation of parts of the chalkboard writing by an English speaking mathematics teacher from the secondary part of the school.

The lesson began with the teacher displaying the problem statement and a diagram on a "poster" which he stuck on the chalkboard. The diagram showed a rectangular piece of paper being folded in half along a vertical line four times in a row, a process which the teacher demonstrated using a piece of A4 paper. He then asked the children to predict the number of creases after a certain number (possibly 4 or 7) folds and wrote children's answers on the board. Children were given a piece of A4 paper each, as well as a worksheet which included the first two rows of the table below, but with the entries for the number of creases left blank. The worksheet also contained a space for a written explanation of any patterns which they observed.

Number of folds	1	2	3	4	5	6	7
Number of creases	1	3	7	15	31	63	127
Number of thicknesses	2	4	8	16	32	64	128

The children, who were seated in pairs, worked individually for about ten minutes. The teacher then asked children one by one to come to the board and fill in the table up to 4 folds. He then challenged the children to find the number of creases for 6 folds. The children continued working, discussing their work with their partners now, while the teacher added the third row to the table, with the entries left blank. After a further ten minutes the teacher selected four children to (simultaneously) reproduce their tables and written explanations on the board. These children then verbally explained their patterns to the class.

The first child, whose written solution contained no numeral other than 2, saw the pattern as doubling the previous number of creases and adding 1. The next child looked at the differences between successive terms, writing +2, +4, +8, etc, which the teacher added underneath his (two row) table. The other children were clearly impressed by this pattern. Before the next child explained his pattern, the teacher completed the third row of the large table. The child then explained his pattern as adding the previous number of creases to the previous number of thicknesses. The first child who had demonstrated his solution had many comments and questions related to this pattern. The fourth child explained her pattern as the number of creases being one less than the number of thicknesses.

The four explanations were displayed from left to right on the board — ordered by their increasing levels of sophistication in the reasoning. Explaining why the different patterns arise was not part of the agenda for the lesson.

After some further discussion and writing by the children, the teacher asked them to turn to page 83 of their textbook, the first page of a chapter devoted to number patterns. The problem they had just completed is described in some length, together with a series of related problems gradually moving towards tables of numbers divorced from any real context. The second problem asked the children to fold a square piece of paper in half along a diagonal and then repeatedly fold the resulting triangles in half. As before, the children were asked to find the number of creases. After working individually for a couple of minutes, the children were asked to provide answers for the number of creases and explain the patterns they found. Children mainly responded from their desks, but at one point a child needed to come to the board to illustrate his pattern. Throughout this interchange, the length of the spontaneous explanations provided by the children suggested, even to non-Japanese speakers, that the children were extremely fluent in explaining their reasoning. The discussion continued even after the bell went for lunch and it was only after the discussion ended that the teacher wrote a summary on the board.

If one is to believe the reports of large-scale studies such as Stigler (1996), then one can assume that most Japanese lessons operate on the same model — one which Ito-Hino (1995) describes in terms of involving students in three stages: "using informal approaches and [building] on previously learned concepts and procedures when attempting to solve problems, ... [acquiring] insight into mathematical ideas based on their own interpretations, and ... [seeking] algorithmic procedures" (p. 233). While according to Ito-Hino it is not uncommon for a whole class period to be devoted to a single problem, the sequence of instructional tasks span more than one lesson and make different uses of the different stages. So, for example, "teachers always enjoy seeing how students attempt to solve the very first problem in a chapter ... using only their previous knowledge" (as we witnessed in the grade 5 class) but it is difficult for "students to deal with an idea or concept after they were informed about algorithmic procedures" (Ito-Hino, 1995, p. 235; 244).

While Ito-Hino comments that "some of the students' problem-solving procedures appeared unexpectedly and showed unique connections", the Japanese model makes use of hypothetical learning trajectories in the sense that teachers have available to them potential solution strategies that students may use. These have been developed through a lengthy cycle of "thought experiments and teaching experiments", at least partly through the use of "lesson study groups".

In the lesson we observed, the teacher identified children whose explanations covered the four major patterns and selected these children to present their explanations in order of increasing mathematical sophistication. The lively and fluent discussion (which I could not understand unfortunately) appeared to make the children's solution methods the "explicit topic of discussion". The teacher supported the "emergence of increasingly sophisticated ways of reasoning and symbolizing" by his orchestration of the explanations given, by adding the extra row in the table, and by translating the second child's +2, +4, +8, etc into arrow notation on the table.

A major difference between the two models of practice is the apparent lack in the Japanese model of a "local instruction theory" which depends on a "*model of* to *model for*" design heuristic. What local instruction theory exists, appears to apply to an individual chapter in a textbook, which typically starts with an extended problem and gradually moves to more algorithmic tasks.

While discussion in the Japanese classroom is seen to move the instructional agenda forward by building students' mathematical insight on their own interpretations, the tension between pursuing student contributions and moving along a hypothesised learning trajectory to a pre-determined endpoint is skewed towards the achievement of clearly defined mathematical goals of a finer grain than those apparent in the work of Gravemeijer, McClain and Stephan.

Thus the Japanese model of learning through problem solving is an existence proof of a different kind to that of the research of Gravemeijer, McClain and Stephan. It is one which demonstrates the possibility of wide-scale classroom practice based on learning through problem solving and incorporating at least the beginnings of treating "mathematical learning as both a process of active individual construction and a process of enculturation". But it lacks the rich texture which has resulted from the application of Gravemeijer, McClain and Stephan's developmental research paradigm.

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RESEARCH FORUM

Theme 3: ***Learning and teaching data-handling***
Co-ordinator: Paul Laridon

Presentation 1:
Building the meaning of statistical association through data analysis activities
Carmen Batanero, Juan D. Godina & Antonio Estepa
Reaction: Michael Glencross

Presentation 2:
Graphing as a computer-mediated tool
Janet Ainley, Elena Nardi & Dave Pratt
Reaction: Ricardo Nemirovsky

BUILDING THE MEANING OF STATISTICAL ASSOCIATION THROUGH DATA ANALYSIS ACTIVITIES

Carmen Batanero, Juan D. Godino, *University of Granada, Spain*
& Antonio Estepa, *University of Jaén, Spain*

Abstract

In this research forum we present results from a research project concerning students' understanding of statistical association and its evolution after teaching experiments using computers. This research has been carried out at the Universities of Granada and Jaén over the years 1991-98. We have identified different incorrect preconceptions and strategies to assess statistical association and performed two different teaching experiments designed to overcome these difficulties and to identify the critical points, which arise in attempting to do this.

The concept of association has great relevance for mathematics education, because it extends functional dependence and it is essential for many statistical methods, allowing us to model numerous phenomena in different sciences. This topic has significant connections with functional thinking and other areas of mathematics education, such as probability and proportional reasoning. The main goal in studying association is to find causal explanations, which help us to understand our environment. However, association does not necessarily imply a causal relationship. "Spurious correlation" sometimes exists when a high coefficient of association among variables arises because of the influence of concurrent factors without there being any causal link.

Besides this epistemological difficulty, psychological research has shown that the ability to judge association well is not developed intuitively. Adults sometimes base their judgement on their previous beliefs about the type of association that ought to exist between the relevant variables rather than on the empirical contingencies between them presented in the data. The existence of such preconceptions in applied situations is another difficulty for the teaching of association. Despite these epistemological and psychological issues, mathematics education researchers have carried out very little research on this topic, and most psychological research has only concentrated on 2x2 contingency tables.

THEORETICAL FRAMEWORK

Our research has been based on a theoretical framework concerning the *meaning* and *understanding* of mathematical objects, for which we distinguish between its personal and institutional dimensions, and which has been described in Godino (1996), and Godino & Batanero (1998). The epistemological assumptions behind this theory are that mathematical objects emerge from the subjects' activity of solving problems, mediated by semiotic instruments provided by institutional contexts. Consequently, the meaning of mathematical objects is conceived as the system of practices linked to specific *problem fields*, where three different types of *elements* need to be considered:

(1) Extensional element of the meaning: The different prototype situations and problems where the object is used, that is, the *problem field* from which the object emerges.

(2) Instrumental/ representational elements of meaning: The different semiotic tools available to deal with or to represent the problems and objects involved.

(3) Intensional elements of meaning: Different characteristic properties and relationships with other entities: definitions, propositions, procedural descriptions, etc.

We postulate a relativity of the emergent mathematical objects, as regards the specific institutional contexts involved in the solution of the problems, as well as the tools and expressive forms available.

According to this model, understanding a mathematical concept will imply the appropriation of the different elements that compose the institutional meaning of the concept, and therefore, has a systemic nature.

Psychological research into association

The research design, as well as the interpretation of its results has been guided by the theoretical framework described above, and by previous research on statistical association, in the scope of Psychology, which we summarise below.

Research into reasoning about association has developed from the work by Inhelder & Piaget (1955), who considered that the evolutionary development of the concepts of association and probability are related, and that understanding association requires prior comprehension of proportionality, probability, and combinatorics. They investigated the understanding of association with children aged 13 or older, posing the problem of association between two dichotomous variables. In Table 1 we describe the data in this type of problem, where a, b, c and d represent absolute frequencies.

Table 1: *Typical format for a 2x2 contingency table*

	B	Not B	Total
A	A	B	a+b
Not A	C	D	c+d
Total	a+c	b+d	a+b+c+d

Piaget and Inhelder found that some adolescents who are able to compute single probabilities only analyse the favourable positive cases in the association (cell [a] in Table 1). In other cases they only compare the cells two by two. This fact is explained by observing that understanding association requires considering quantities (a+d) as favourable to the association and (b+c) as opposed to it and also that it is necessary to consider the difference between cases confirming the association (a+d) and cases opposed to it (b+c) compared to all the possibilities.

According to Piaget and Inhelder, recognition of this fact only happens at 15 years of age.

Following Piaget and Inhelder, several psychologists have studied the judgement of association in 2x2 contingency tables in adults, using various kinds of tasks and, as a consequence, it has been noted that subjects perform poorly when establishing judgements about association (see Crocker, 1981; Beyth-Maron, 1982 for a survey). Other researchers, such as Chapman & Chapman (1969) and Jennings, Amabile and Ross (1982), have studied the effect that previous theories about the context of a problem have on judging association. Their general conclusion is that when data do not coincide with these expectations there is a cognitive conflict which affects the accuracy in the perception of covariation.

INTUITIVE STRATEGIES IN GENERAL CONTINGENCY TABLES, SCATTER PLOTS AND COMPARISON OF SAMPLES. PRECONCEPTIONS OF ASSOCIATION

Extensional meaning of association

Judging association in a 2x2 contingency table, such as that presented in item 1, is a particular case of the *problem field* from which statistical association has emerged. The processes of judging association in a contingency table require relating or operating with the different frequencies in the table.

Item 1: In a medical centre 250 people have been observed to determine whether the habit of smoking has some relationship with a bronchial disease. The following results were obtained:

	Bronchial disease	No bronchial disease	Total
Smoke	90	60	150
Don't smoke	60	40	100
Total	150	100	250

Using the information contained in this table, would you think that, for this sample of people, bronchial disease depends on smoking? Explain your answer.

Another related problem is assessing the existence of correlation between two quantitative variables (e.g., item 2), where we could compute the covariance or the correlation coefficient, or fit a line to the scatter plot. A third type of problem is inquiring whether a numerical variable has the same distribution in two different samples (item 3),where it is possible to compare the differences in means or medians, or the tabular or graphical representations of the two distributions. All these problems and activities are essential for progressively building up the concept of statistical association and form part of the *institutional meaning* of the concept within an introductory University course on data analysis. More specifically, the three types of problems described are prototypical *extensional elements of the meaning* of association within this institution.

Item 2. In a sociological study, data relative to daily consumption of animal protein and birth rate of different countries were collected (Figure 1). The data were represented in the attached scatter plot. Do you think that the relationship between daily consumption of animal protein and birth rate of these different countries is direct, inverse or that there is no relationship at all? (Explain your answer).

Item 3. The following data were obtained when measuring the blood pressure for a group of 10 women, before and after applying a medical treatment. Using the information contained in this table, do you think that the blood pressure in this sample depends on the time of measurement (before or after the treatment)? Explain your answer.

Blood pressure	Mrs. A	Mrs. B	Mrs.C	Mrs. D	Mrs. E	Mrs. F	Mrs. G	Mrs.H	Mrs. I	Mrs J
Before treatment	115	112	107	119	115	138	126	105	104	115
After treatment	128	115	106	128	122	145	132	109	102	117

Personal meaning of association

As a first step towards understanding how students develop this institutional meaning, our research project started by extending earlier research into the *personal intuitive meaning* given by students to the concept of association: What type of strategies do they use to solve the problems, and what conceptions about association can be deduced from their strategies. With this aim, an extensive study was carried out from 1992 to 1994. After some revisions with pilot samples, a questionnaire was developed and given to 213 students in their last year of secondary school (17-18 year-old students), which is the level at which association is introduced into the Spanish curriculum. The questionnaire was given to the students before the teaching of association began.

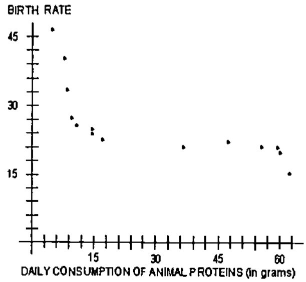

Figure 1

The 10 items in the questionnaire were similar to items 1 to 3 presented before and included 2x2, 2x3 and 3x3 contingency tables (e.g., item 1), scatter plots (e.g. item 3), and comparing a numerical variable in two samples (e.g., item 2). Sign and strength of the association and the relationship between the problem context and students' prior beliefs were taken into account in designing the questionnaire.

For each item, we analysed the type of association perceived by the students (direct association, inverse association or independence) and a scheme was developed for classifying students' solution strategies from a mathematical point of view. This allowed us to identify the incorrect strategies previously reported by Inhelder and Piaget (1955) and later psychological research, as well as intuitively correct strategies pointing out correct or partially correct conceptions concerning statistical association (Estepa et al., 1994; Batanero et al., 1996; Estepa, & Batanero, 1996; Estepa, & Sánchez-Cobo, 1996).These are some examples:

(1) Using the increasing, decreasing or constant trend of points in the scatter plot to justify the type of association (negative, positive or null): *"Because when you increase the daily consumption of proteins the birth rate is going down"* (Item 3). As in the case of independence, there would be no joint variation of the two variables, this student is using a correct conception about association.

(2) Using means or totals for comparing the distribution of one variable in two different samples: *"Because the sum of all the values of the blood pressure before the treatment is lower than the sum of blood pressure values after treatment* (Item 2). Here, the

student implicitly uses the correct idea that a difference in totals implies association between the variables.

(3) Comparing either a) the frequencies of cases in favour of and against each value of the response variable or b) the ratio of these frequencies in each value of the explanatory variable in 2xr contingency tables: *It is not dependent, because the odds for having bronchial desease in smokers is 3/2, and there are the same odds in non smokers"* (Item 1). This points to a correct conception, as the odds ratio can be used to assess association in a contingency table.

On the basis of the students' incorrect strategies, which lead them to incorrect judgements of association in the problems, we have described the following misconceptions concerning statistical association:

(1) *Determinist* conception of association. Some students do not admit exceptions to the existence of a relationship between the variables and expect a correspondence that assigns only a single value in the dependent variable for each value of the independent variable. When this is not so, they consider there is no dependency between the variables. That is, the correspondence between the variables must be, from the mathematical point of view, a function. An example is given in the following answer to item 2:

"There is not much influence from the treatment, because in some women the blood pressure increases, whereas in other cases it decreases" (item 3).

(2) *Unidirectional* conception of association. Sometimes students perceive the dependence only when the sign is positive (direct association), so that they consider an inverse association (negative sign) as independence. The following response illustrates the case of inverse association being interpreted as independence, as well as difficulties with proportional reasoning in item 1:

"I personally believe there is no dependence, because if you look at the table there is a higher proportion of people with bronchial disease in smokers".

(3) *Local* conception of association. Students form their judgements using only part of the data provided in the problem. If this partial information serves to confirm a given type of association, they adopt this type of association in their answer:

"There is dependence on smoking in having bronchial disease, because if we observe the table, there are more smokers with bronchial disease than non-smokers: 90>60" (Item 1).

Often this partial information is reduced to only one conditional distribution (as in the example, where the student only uses the distribution of people with bronchial disease) or even only one cell, frequently the cell for which the frequency is maximum. These strategies are similar to those reported by Inhelder and Piaget (1955). Konold et al. (1997) also described similar reasoning, which they attributed to lack of transition from thinking about individual cases to thinking about group propensities. The latter is of course essential in statistical ...ng.

(4) *Causal* conception of association: Some students only considered the association between the variables if this could be attributed to a causal relationship between them. This was particularly found in one problem concerning the ranking of a group of students by two different judges, even when there was a moderate correlation:

"Because one judge cannot influence the other. Each one has a preference and therefore there can not be much relation between the order given by them both".

RESEARCHING THE POTENTIAL OF COMPUTER-BASED ENVIRONMENTS FOR TEACHING AND LEARNING ASSOCIATION

After carrying out the study on students' initial conceptions about association, our research was aimed at assessing the impact of computer-based teaching experiments on the same. The use of computers in the teaching of Statistics is receiving increasing attention from teachers and researchers as is shown in Shaughnessy et al., (1996) and the IASE Round Table Conference on the impact of new technologies in teaching and learning statistics (Garfield, & Burrill, 1997). In this research it is suggested that computers may not only extend what statistics is taught, but may also affect how statistics is learnt, because technology provides students with powerful resources and multiple representations, which can help them to widen the meaning of statistical concepts. The aim of our research was to assess this impact for the specific case of statistical association.

Building instructional strategies

Biehler (1997) classified different types of statistical software according to their educational functions. *Tools* allow students to practice statistics as current statisticians do. In the particular case of exploratory data analysis, these tools should enable students to do interactive, exploratory and open-ended work, utilising flexible software, which is easy to use and to learn. *Microworlds* summarises interactive experiments, simulations and exploratory visualisations, which help students to conceptualise statistics. In our two experiments, we have considered c*omputer learning environments,* as an integrated instructional setting which allows the teacher and a group of students to work with tools, microworlds, data sets and related problems as well as with a selection of statistical concepts and procedures.

The content of the two courses included the fundamentals of descriptive statistics, using an exploratory data analysis approach. The specific statistical contents were the following:

(1) Random and deterministic experiments. Population and samples. Measurement scales. Type of variables. Collecting and organising data.

(2) Frequency, cumulative frequency, grouping data. Graphical representation: bar chart, pie charts, histograms, stem and leaf, graphical representation for cumulative frequencies.

(3) Parameters and statistics. Location; Spread. Order statistics. Skewness and kurtosis.

(4)Two-dimensional statistical variables: Contingency tables. Association in contingency tables. Statistical association in numerical variables. Covariance and correlation. Linear regression.

(5) In the second experiment students were also introduced to inference. The concepts of sampling, sampling distribution, interval confidence and hypotheses testing of means for one and two samples and Chi-square test were introduced.

The planning of the teaching involved the organisation of an instructional sequence to meet the learning goals and contents, the selection of appropriate data sets to contextualize statistical knowledge, and the design of a graded problem sequence including the main task variables relevant to understanding association (Godino, et. al, 1991).

All of this was planned to provide the students with a representative sample of extensional, intensional and representational elements for the meaning of association. We adopted a "multivariate perspective", even though only univariate or bivariate techniques were taught at a formal level. Therefore, students explored data files with the support of an interactive computer software package. The number of sessions (90 minutes long) were 21 in the first experiment and 40 in the second. In seven of these sessions in the first experiment and 20 sessions in the second, the students worked in the statistical laboratory, solving problems whose solutions required them to analyse different data sets provided by the lecturer or collected by themselves. In the remaining sessions students were introduced to statistical concepts and solved related problems.

First experiment: Identifying resistant conceptions and analysing the learning process

The first experimental sample consisted of nineteen 20 year-old University students in a first year course of exploratory data analysis and descriptive statistics. The students worked with the statistical software PRODEST, which had been developed by the research team some years before. Although this software had rather limited capability compared with modern statistical packages, it included all the different tools needed in a course of exploratory data analysis at undergraduate level: frequency tables and graphical representation for grouped and non grouped data; computation of statistics; stem and leaf plots; box and whiskers plots; cross tabulation; linear regression and correlation. In addition, data file facilities and possibilities of selecting part of the data set were available.

To assess the changes in students' conceptions, two equivalent versions of a questionnaire were given to the students as a pre-test and post-test. We found general improvement in students' strategies as well as the persistence of unidirectional and causal misconceptions concerning statistical association for some students after the teaching process. All these results have been described in greater detail, see Batanero et al. (1997).

One pair of students was observed throughout their work in the laboratory sessions to trace their learning process. As stated by Biehler (1994), when working with the computer, an adequate solution to statistical problems is only found through a feedback process with the specific problem and data. The students do not just choose an algorithm, but have more freedom, because they have a system of options available that they can combine and select according to their strategies and partial solutions when solving the problem.

A member of the research team observed the students' work, gathering their written responses to the different problems. This observation also included the recording of their discussions, of their interaction with the lecturer and the computer. These students were also interviewed at the beginning and at the end of the experiment. When we studied in detail the observations made on these students, some recurring difficulties relating to the idea of association were identified. Some of them were finally solved, either by the students themselves, when discussing and looking at the results of several computer programs, or with the lecturer's help, although they reappeared from time to time. At other times the difficulty was not solved, in spite of the lecturers' explanations. Occasionally, the lecturer did not realise the students' confusion.

Intensional elements of meaning of association

In the following, we describe the learning process for these two students, commenting on nine key intensional elements of the institutional meaning of association (Godino & Batanero, 1998). We found evidence in our data that students' understanding of these elements of meanings seemed to develop at specific moments in time throughout the understanding process.

1. To study the association between two variables, the comparison of two or more samples has to be made in terms of relative frequencies. However, in the first session the students compared absolute frequencies of the same variables in two samples. Although the lecturer commented on this mistake at the end of that session, the same incorrect procedure appeared again in sessions 2, 3 and 5. Afterwards, the students seemed to overcome this difficulty.

2. The complete distribution in the different samples should be used to assess the differences in the same variable between two or more samples. Finding local differences is not sufficient, but rather the association should be deduced from the complete data set. In spite of this, the students started solving the problems by comparing isolated values in the two samples. For example, they only compared the values with maximum and minimum frequencies for both samples in the first session. Although these differences pointed to a possible association, they were not sufficient to quantify its intensity. This difficulty reappeared in Sessions 2 and 3 and finally disappeared.

3. *From the same absolute frequency in a contingency table cell two different relative conditional frequencies may be computed, depending on which one is the conditioning variable. The role of condition and conditioned in the conditional relative frequency is not interchangeable.* Falk (1986) and other authors have pointed out that students have difficulties in the interpretation of conditional probabilities, because they do not discriminate between the probabilities P(A/B) and P(B/A). Many students in our sample showed a similar confusion referring to the conditional relative frequencies in the pre-test and throughout the experimental sessions. This confusion was noticed in the students observed during Session 5, although they solved it with the lecturer's help. They did not show this confusion during the rest of the sessions.

4. *Two variables are independent if the distribution of one of these variables does not change when conditioning by values of the other variable.* Until Session 5, the students did not discover that a condition for independence is the invariance of the conditional relative frequency distribution when varying the value of the conditioning variable.

5. *The decision about what size of differences should be considered to admit the existence of association is, to some extent, subjective. It is difficult to obtain either perfect association or independence. The problem of association should be set in terms of intensity instead of in terms of existence.* Though the students had not studied hypothesis testing, in Session 5 they discovered that judging association implies taking a decision about whether to attribute small differences to sampling fluctuations or to real association between the variables. They also realised that there are different grades of association, from perfect independence to functional relationship.

6. *When studying association both variables play a symmetrical role. However, when studying regression the role played by the variables is not symmetrical.* The fact that correlation ignores the distinction between explanatory and response variables, whilst in regression this difference is essential (Moore, 1995), caused a great deal of confusion for the students. When they had to select the explanatory variable for computing the regression line, in Sessions 5, 6 and 7, they did not know which variable ought to be chosen.

For example, when computing the regression line between height and weight, the students were misled by the fact that there was a mutual dependence of the two variables. A great amount of discussion follows in which the students were not capable of solving this confusion. The lecturer did not notice the problem and finally, the students computed the regression lines by choosing the explanatory variable at random. At the end of the teaching period these students had not discovered that two different regression lines can be computed.

7. *A positive correlation points to a direct association between the variables.* Although, in Session 6, students could interpret the size of the correlation coefficient, they did not discuss the type of association (direct or inverse). At the end of the session, they noticed that when the correlation coefficient is positive,

and there is a linear relationship, the variables are positively associated and above-average values for one tend to accompany above-average values for the other. However, they did not explicitly use the term "direct association."

8. *A negative correlation points to an inverse association between the variables.* When, in Session 6, students came across a negative correlation coefficient for the first time, they were so surprised that they asked their lecturer if this was possible. They also had trouble when comparing two negative correlation coefficients.

The students knew that a negative number with a high absolute value is smaller than a negative number with a low absolute value. However, a negative correlation coefficient with a high absolute value points to a higher degree of dependence than a negative correlation coefficient with a lower absolute value. This fact caused much misinterpretation in the problems in which a negative correlation occurred. Therefore, the knowledge of the properties of negative number ordering acted as an obstacle to dealing with negative correlation.

Although, with the lecturers' assistance, they observed that a negative correlation coefficient corresponded to a negative slope of the regression line and that this meant that the y value decreased when the x value increased, they did not explicitly use the term "inverse association", neither did they differentiate between the two types of association at the end of their learning.

9. *The absolute value of the correlation coefficient shows the intensity of association.* Although the students related the absolute value of the correlation coefficient with the intensity of association, they did not relate this idea to the spread of the scatter plot around the regression line.

Second experiment: Improving instrumental/representational elements of meaning

In the second experiment, we concentrate on analysing the use that students make of different statistical tools to solve the problems following teaching. 36 students took part in the second experiment, working with some procedures in the statistics package *Statgraphics*, where new statistical instrumental/ representational procedures, as well as a more user-friendly and dynamic environment were available to students for studying association. These statistical tools allow students to operate or represent bivariate data in different ways, leading them to progressively build a more complete meaning of association. We could classify these tools by their level of data reduction, numerical or graphic nature, and their analytical approach (descriptive or inferential):

(1) Numerical representations at a descriptive level: Contingency tables (e.g., Item 1), and different associated frequencies. Unidimensional frequency tables of conditional distributions and their statistics, correlation and determination coefficients, and parameters of the regression line.

(2) *Numerical representations at an inferential level:* Confidence intervals for means or for the difference of means. Hypothesis tests on the means or medians. Chi – squared test of association between variables.

(3) Unidimensional representations of conditional distributions: Steam and leaf plots, bar graphs, box plots (Fig. 2), pie charts, histograms (Fig.4), quantile plots (Fig 3), density traces.

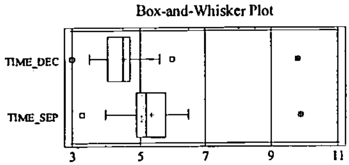

Figure 2

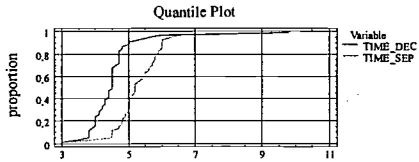

Figure 3

(4) Bidimensional graphics: Three-dimensional histograms, mosaic graphics (Fig. 5) and scatter plots (Fig. 4).

As shown in the figures, each one of these representations provides different data summaries and therefore different elements of *meaning* for association. For example, in Fig. 2 and 3 the two distributions of the time, in seconds, taken to run 30 meters by a group of students in September and December are represented. However, in Fig. 2 the outliers and slight asymmetry in the distributions, as well as the interval of 50% central values are more clearly shown, whereas in Fig. 3 it is easier to notice that the probability of obtaining a given value of the time is always greater in one distribution (September), which means that the time in September is stochastically larger than the time in December.

As well as the usual possibilities for selecting data and variables, it was also possible with Statgraphics to display several tabular and graphical representations on the screen at once, at the same time as the student handled features like width, format scale of graphics, etc. The *Statfolio* file also allowed the research report to be written and partial results to be included in the text at the same time as the analysis was being carried out.

Personal meaning of association following teaching

The students' previous statistical knowledge was wide-ranging, because the course was optional and students from different backgrounds, such as Education, Psychology and Business, took the course. During the course, generative and interpretative skills (Gal, 1997) were evaluated from students' solutions to data analysis activities, paper and pencil tests, and from their individual projects. In addition, at the end of the course a test was given to the students concerning the analysis of a new data set to assess the final meaning given to association by students. Each student worked alone with the computer and his/her solutions were recorded individually on a disk file, using the "Statfolio", which included the ...tions and graphics used, together with his/her comments and solutions. The test consisted of the analysis of a new data set concerning the scores of 48 pupils in a physical education course, for which the students were given some related questions. Below we analyse the solutions to four association problems.

Problem 1. Do you think that, in this data set, practising sport depends on a person's sex.

Problem 2. Is there any relationship between practising sport and number of heartbeats after 30 press-up?

Problem 3. The teacher wants to assess the improvement in the physical preparation of his pupils. Do you think there has been any improvement in the time pupils took to run 30 meters between September and December?

Problem 4. Do you believe that the number of heartbeats after 30 press-up depends on the pupils' heartbeat when resting?

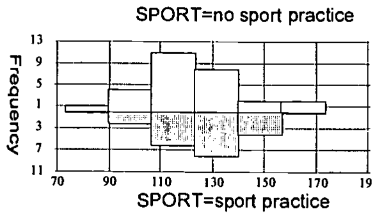

Figure 4

The main difference between the problems is the type of variables involved: two qualitative variables (problem 1), two quantitative variables (problem 4) and a variable of each type (problem 2 and 3).

Another important variable is the strength of association: Independence (problem 4), weak dependency (problem 2), moderate association highly significant test t value in problem 3), (significant value of the Chi–squared test in problem 1).

The students should identify these differences (they need to discriminate between different extensional elements of meaning). Then they should select from the different tools available (instrumental/representational elements of meaning) those that are adequate to solve the problem, such as contingency tables and related statistics, mosaic chart (Fig. 5), or comparing bar graphs in problem 1, comparing histograms (Figure 4), quantile plots (Fig.3), density traces, box plots (Fig.2) or different statistics in problem 2 and 3, and studying the correlation coefficient or the scatter plot (Fig. 6) in problem 4. Finally, intensional elements of meaning should be used to interpret the outputs of the different programs and to make an accurate judgement of association.

By printing out each student's statfolio, we categorised the students' procedures and solutions. We summarise the results below, which will be presented in detail in Batanero, & Godino (in press). As a rule, the students reached a correct association judgement, yet some correct solutions were obtained through a procedure unsuitable for the type of problem, because students did not always correctly link the association extensional and instrumental/ representational elements of meaning.

Moreover, among the tools suitable for the problems, the students did not always choose what a statistician analysing the data would have chosen. Consequently, the students' solutions did not always coincide with the "standard" solution. For example, the best solution to problem 1 would have been using the chi-squared test to compare the proportions of males and females practising sport that gives a significant result. Seven students, however, computed the correlation coefficient instead, which provides a value of 0.28. As this value is very small, they interpreted that there was no relationship between the variables. Another example is problem 3, where a student tried to display the relationship using a scatter plot that is not adequate.

This selection of a correct though non optimum procedure points to the students' lack of flexibility for changing representations of association and the greater facility to interpret the correlation coefficient as compared to contingency tables. For example, students S11, S23, S26, S35, S36 solved 3 problems using the correlation coefficient; student S12 solved 3 problems by comparing bar graphs, student S10 solved all 4 problems by comparing graphical representations of marginal distributions and S27 solved all the problems by comparing double frequencies in contingency tables. Another example is not taking as an explanatory variable the one which allows the simplest interpretation of the analysis, like 3 students who reached an incorrect conclusion when computing conditional relative frequencies for practising sports with regards to the number of heartbeats, a strategy that makes it difficult to display the relationships.

Other students misinterpreted results of correctly selected and carried out

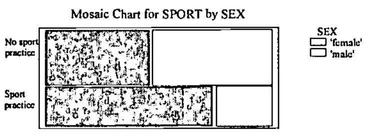

Figure 5

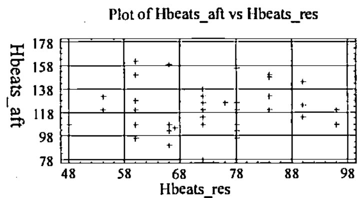

Figure 6

analysis due to failures in grasping some intensional elements in the meaning of association. The main difficulties were the following:

a) Confusing relative conditional with double frequencies (9 students) or with marginal frequencies (4 students) in contingency tables;

b) using only a marginal distribution (1 student in problem 1) or comparing the marginal distributions of the two variables involved (1 student in problem 4);

c) comparing the correlation coefficient for each variable implied with a different variable, because of a lack of understanding of entry parameters in the programs (1 student in problem 4);

d) using previous theories without taking the data into account (problem 4);

e) believing that the dependence of heartbeats at the time when they are registered would imply a constant value of the number of heartbeat at each given time, that is to say, interpreting association in a deterministic way (1 student in problem 3);

f) using the correlation coefficient between the same variable in two related samples to study the differences in the two samples (1 student in problem 3) or the coefficient of determination (1 student in problem 3);

h) comparing absolute frequencies in bar charts, instead of using relative frequencies (1 student in problem 2).

As a rule, we observe that students preferred numerical to graphical representations of association, especially in problems 1 to 3. This is possibly due to the fact that each available graph requires its own interpretation that the students do not always master. They also preferred numerical summaries, because of the difficulty with the idea of distribution, which was also shown in the confusion about the different types of frequency. We finally point out the scant use of inferential procedures. Possibly students require a longer period of study to understand these concepts before deciding to employ them in solving their problems.

FINAL REFLECTIONS

Our research results reveal the complexity of the *meaning* and *understanding* of association, which should be conceived in a systemic and complex way (Godino, 1996; Godino & Batanero, 1998). In constructing this meaning, three different types of *elements* need to be grasped and linked for students to master the concept of association and use it in problem solving:

(1) *Extensional element of the meaning*: The different situations and problems whose solution requires the study of association and that have been described in this paper.

(2) *Instrumental/ Representational elements of meaning*: The use of different tools to deal with or represent the concept, such as mosaic charts, scatter plots, two-way tables or a series of parallel box-plot, cumulative plots, histograms or bar charts.

(3) *Intensional elements of meaning*: such as the difference between statistical dependence and functional dependence, the different relative frequencies that can be deduced from a contingency table, the role of independent and dependent variables, and the parameters in the regression equation, the interpretation of the correlation coefficient, and the difference between correlation and causality.

A second consequence of our research is the distinction between the *personal* (subjective) and *institutional* (mathematical) dimensions of knowledge, meaning and understanding in mathematics. This is particularly shown when comparing the meaning of association that was presented in the teaching experiments (institutional meaning within the particular institution of a course on exploratory data analysis) and the personal meaning that the students have finally acquired, where they have only built part of the intended meaning, and some of the incorrect conceptions on association still remain.

Our results finally show that analysing data is a highly skilled activity even at an exploratory level, requiring a wide knowledge about the problems and concepts underlying graphical, numerical, descriptive and inferential procedures to deal with association. It requires selecting the best data instruments and representations, flexibility in changing the selected procedure, adequate interpretation of results (intensional elements), the ability to relate them to the problem (extensional elements) and to assess the validity and reliability of the conclusions drawn. Even when many of our students achieved correct solutions to the problems, we could observe their difficulties in each step of the process described.

Being able to master this complex activity, beyond routine or elementary tasks, or being capable to teach it to a group of students with different prior knowledge and capacities is not a simple task and certainly requires greater time and experience that what is possible to provide in an introductory statistics course. The research carried out supports the view of mathematical objects as signs of cultural units, whose systemic and complex nature cannot be described merely by formal definitions when the perspective taken is that of the study of teaching and learning processes. Based on this viewpoint, we might explain some learning misconceptions and difficulties, not only in terms of mental processes, but by recognising the complexity of meaning of mathematical concepts and the necessarily incomplete teaching processes found in teaching institutions.

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BUILDING THE MEANING OF STATISTICAL ASSOCIATION THROUGH DATA ANALYSIS ACTIVITIES

A reaction paper to Batenero et. al.

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University of Transkei, South Africa

In this reaction, a critique of the Batanero, et al., paper is presented, its strengths and weaknesses are highlighted and suggestions are made for further research.

Statistical association is a central concept in scientific thinking, if only because scientific progress depends on finding out which things are associated and which things are not. "Without the knowledge of how one thing varies with another, it would be impossible to make predictions. And whenever causal relationships are involved, without knowledge of covariation we should be unable to control one thing by manipulating another" (Guilford & Fruchter, 1978, p. 77). It is not surprising, then, that the concept of association is regarded as a fundamental idea in any worthwhile statistics curriculum.

With the recent world wide explosion of interest in statistics and its role in the school curriculum, the use of computers for data analysis and for the teaching of statistics is receiving more and more attention from both teachers and researchers (Shaughnessy, Garfield & Greer, 1996). At the same time, the phenomenal developments in information technology and the increasing availability of statistical software make it clear that the possibilities of superficial and inadequate analysis of data have increased correspondingly (Chatterjee, Handcock & Simonoff, 1995).

It is in this context that the authors of this interesting and timeous paper have brought together the results of their substantial research and show that a knowledge of students' understanding of statistical association is important for our own understanding of a topic that is paradoxically simple in appearance, yet decidedly complex under the surface.

Theoretical Framework

The brief description of the theoretical framework underpinning the research, together with its concomitant epistemological assumptions, distinguishes between the personal, or subjective, and the institutional, or mathematical, dimensions of knowledge, meaning and understanding in mathematics. It is further argued that three different types of elements need to be considered if the meaning of a mathematical object is to be understood. These elements, the extensional, instrumental and intensional elements of meaning provide the context for the existence of a particular mathematical object, in this case, the concept of association, while it is made clear that the three types of elements co-exist in a systemic manner to provide the institutional meaning of the concept. The model is certainly reasonable and coherent, serving to address the situation where there is a difference between the personal meaning acquired by an individual and the institutional meaning presented in a given course of instruction. In such a situation the model is valuable for making explicit the nature of a misconception and has much to commend it, especially when translated into the closely related teaching situation.

The psychological research into the concept of association, which dates back to Inhelder and Piaget (1955), is described succinctly, the major point being that for many people, students and adults alike, judgements about association are often poor. In an earlier paper (Batanero, Estepa, Godino & Green, 1996), the authors noted that psychological research has shown that the ability to judge association is not intuitive, while the existence of certain preconceptions about empirical relationships, some of which are resistant to change, presents difficulties for teaching the concept. This is an important example of a more general principle which has value for research in other areas.

Results and Discussion

The extensional meaning of association is articulated by presenting three types of prototypical problems. These problems, which form part of the institutional meaning of the concept within the specified context, are regarded as essential for progressively building up the concept of association. The steps needed to understand how students develop such institutional meaning begin with attempting to identify the personal intuitive meaning students give to the idea of association. Several incorrect strategies were identified from the analysis of questionnaire responses which led to four specific misconceptions, namely, determinist, unidirectional, local and causal. The first three are similar to those reported in the literature (Inhelder & Piaget, 1955; Konold, Pollatsek, Well & Gagnon, 1997). The latter is a common misconception among inexperienced users of statistics and may be related to those situations which give rise to the well known Simpson's paradox (see, for example, Weldon, 1986, pp. 35-36).

The potential of computer-based environments was explored by means of two experiments. The first was designed to assess changes in students' conceptions before and after working with statistical software developed by the research team. No details of the software are provided and there are no details of whether or not the software itself was evaluated. A combination of quantitative and qualitative methods was used, which together support the validity of the study. In addition to a general improvement of student performance, the researchers noted the persistence of some misconceptions, although few details are provided.

In highlighting the intentional elements of meaning of association, a particular strength of this research lies in tracing the learning process of just two students by observation of their work in the laboratory. To carry out such detailed observational work successfully is time-consuming and arduous, even with computer assistance and the researchers are to be commended on their attention to detail in this vital phase of the research. The results provided evidence that students' understanding developed at specific moments in time throughout the process.

The second experiment focused on improving students' instrumental element of meaning using Statgraphics, the well known commercially available statistics package. Students were enabled to build a more complete meaning of association by this means. The assessment of each student's personal meaning of association was accomplished by examining the detailed records available through 'Statfolio' and categorising the procedures used and the solutions obtained. A summary of the main outcomes is provided, it being noted that the students generally reached a correct association judgement, despite not using the most suitable procedures or the most appropriate statistical tools. The selection of a correct, but not optimum procedure, is suggested by the researchers as being due to the students' lack of flexibility. An alternative explanation would be the students' lack of experience with statistical procedures and situations.

The students' main difficulties are summarised clearly, an important finding being that students preferred numerical to graphical representations of association. It is suggested that this might have been due to students not being able to master the graphical interpretation. Consistent with their preference for numerical representations, the students also preferred numerical summaries, apparently because of difficulties with the idea of distribution. The students' apparent reliance on numerical aspects of association at the expense of the graphical may also be due to a greater facility with numerical procedures.

In the concluding remarks it is argued that even at the exploratory level the activity of data analysis is a highly skilled and sophisticated endeavour. It requires a wide knowledge of a variety of problems, concepts and procedures related to association, as well as a mature flexibility in terms of adopted approaches. The authors conclude by arguing that the research results support their view of mathematical objects whose nature is systemic and complex and that it is possible to explain learning misconceptions and difficulties in terms of the complexity of meaning of the mathematical concepts.

Further Reflections

In providing a critique of any research endeavour, it is sometimes difficult to keep a balance between remarks about the strengths and benefits of the research on the one hand and the weaknesses and shortcomings on the other. The strengths of this research should be obvious to perceptive readers. First, there is a sound theoretical base which is supported by a number of related research studies. Second, the paper reports on research activities that have been carried out over an extended period of time, 1991-1998, with the more recent work building on and developing that carried out earlier. Third, there is a good balance of large group, quantitative data and analysis and small group/individual, qualitative data and analysis. Such carefully managed research methodology is to be encouraged and admired. Finally, the conclusions, which, in the best research tradition, are starting points for yet further research, focus on the students' learning and understanding and point to the need for an examination of the related teaching processes within the teaching institutions.

The authors have made only a loose connection between the students' learning outcomes and the 'incomplete teaching process' related to them. Since teaching and learning are effectively two sides of the same coin, it would now be opportune to extend the present focus of the research to embrace these teaching processes. For example, the work of Crawford and her co-workers in Australia (Crawford, Gordon, Nicholas & Prosser, 1994, 1998), which examined students' conceptions of mathematics and how it is learned, is both interesting an innovative and could sensibly be used to inform future related research in statistics education. Some related Australian research (Prosser, Trigwell & Taylor, 1994), which examined the conceptions of teaching and learning held by first year chemistry and physics teachers, pointed clearly to the necessity to relate these conceptions to student learning. If "helping teachers change their conceptions of teaching (is) likely to improve the quality of student learning" (Prosser, et al., 1994, p. 230), then there is merit in trying to do the same in the statistics arena. Further, if students' misconceptions are to be addressed, then teachers need to be sensitive to their own conceptions about teaching and to be prepared to make changes, however large or small. Ultimately it is the quality of student learning which will be affected.

If one accepts that two distinguishing features of good research are its replicability and generalizability, then one looks for ways to replicate and generalise. In the South African context, replicating this research would immediately involve language issues, because a majority of South African students learn through the medium of a second language. Batanero and her colleagues have made no remarks about language matters, although the specialist use of statistical words and terms is one that all statistics students face, irrespective of whether they are learning through their mother tongue or not. This would be a fruitful are for further research, especially in South Africa and other multilingual countries.

Another area worthy of consideration arises from the finding that students preferred numerical rather than graphical representations and summaries. The origins of such an outcome may be traced back to the initial learning situation in which students may not have been provided with the necessary prerequisite visual and visualization experiences. It is widely accepted that the most effective way to learn statistics is by actively engaging in doing the statistical analysis. In particular, learning to master data analysis involves the process of visualization, a process which reveals the "intricate structure in data that cannot be absorbed in any other way" (Cleveland, 1993, p. 1). Visualizing data involves, among other things, graphing and this is facilitated in today's modern age by the computer and such statistical packages as Statgraphics. Graphs are vital "because visualization implies a process in which information is encoded on visual displays" (Cleveland, 1993, p. 1). It is further argued that "when a graph is made, quantitative and categorical information is encoded by a display method. Then the information is visually decoded. This visual perception is a vital link. No matter how clever the choice of information, and no matter how technologically impressive the encoding, a visualization fails if the decoding fails. Some display methods lead to efficient, accurate decoding, and others lead to inefficient, inaccurate decoding" (Cleveland, 1993, p. 2). This is another potential area for innovative research and the recent work of Gorgorio (1998) on visual and non-visual strategies, albeit in a different context, is relevant and informative.

It is appropriate to conclude with a comment on the authors' use of the terms 'mathematics education' and 'mathematical objects' in preference to 'statistics education' and 'statistical objects'. Such terminology in a paper whose focus is stated clearly as 'statistical association' through 'data analysis' suggests that the authors view statistics as a branch of mathematics and statistics education as a branch of mathematics education. This is disappointing, at least for those deeply involved in the field of statistics education, and while this may not be the place to debate the issue fully, some brief remarks are called for. The term statistics has been used to refer to "the branch of scientific method that deals with the study of the theory and practice of data collection, data description and analysis, and the making of statistical inferences. It follows ... that statistics education refers to the art of teaching and learning these statistical activities" (Glencross & Binyavanga, 1997, p. 302). Statistics education itself has come of age (Vere-Jones, 1995) and is recognized internationally as an identifiable and important field of knowledge in its own right (Glencross & Binyavanga, 1997). Thus it would have been more appropriate for the authors to make use of a theoretical model from within mathematics and mathematics education and adapt it for use in statistics and statistics education. There can surely be no loss of generality to describe 'association' as a statistical object or concept in preference to regarding it as a mathematical one.

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GRAPHING AS A COMPUTER-MEDIATED TOOL

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This paper looks back over a number of exploratory studies, which have researched young children's construction of meanings for graphs, personally generated from spreadsheets. These early studies led to the notion of a pedagogic approach which we have termed active graphing. We report on a systematic study into the questions raised by that initial work. We have carried out a detailed analysis of children's actions when engaged in specially designed active graphing tasks. In this paper, we discuss children's construction of meanings for trend and formal notation.

Introduction

Our research on graphing has developed in the context of a number of studies from both mathematical and scientific perspectives (Kerslake 1981, Johnson and Welford 1988, Sharma 1993) which have revealed relatively low levels of graphing skills amongst secondary school children, and highlighted particular areas of difficulty. Two studies (Swatton and Taylor, 1994, Padilla, Mckenzie and Shaw, 1986) show relatively low success rates in interpretative skills (interpolation and particularly reading relationships between variables), but higher rates in some construction skills (such as plotting points). These findings serve to strengthen the widely-held perception of interpretation skills as being of a higher order than construction skills.

We offer an alternative explanation of these poor levels of performance; traditional teaching results in the production of graphs frequently being seen as the end point, indeed the purpose, of the activity, with little attention focused on interpretation (beyond rather superficial 'reading' of data), or on the use of graphing as a problem solving tool. We suggest that a mastery of graphing requires three separate, though clearly related, capabilities: the practical skills required to produce graphs by hand, knowledge of the conventions and technicalities of graphs (such as the use of scale) and an understanding of how to interpret and use graphs.

In a conventional classroom situation, it is difficult to separate the physical and intellectual demands of producing graphs from the intellectual demands of interpreting them. The demands of learning conventions and technicalities are high, and it is easy to make the assumption that the experience of drawing graphs, and 'knowing' the conventions, are necessary prerequisites for being able to interpret graphs effectively.

Our preliminary studies using computer-based activities with 8 to 11 year-olds has led us to challenge this conventional view of graphing, and assumptions which tend to stem from it, about the types of graph which are appropriate for children in primary schools. When graphs are produced on the computer, there is no need for the child to have the practical skills required to produce graphs by hand, or to know the conventions of graphing. Attention can be focused directly on the interpretation of the graph. Bryant and Somerville's (1986) claim that even 6 to 8 year-olds do not find the spatial demands of plotting and reading points on ready made line graphs difficult, is reflected in our own observations of the relatively high levels of success in interpretative skills shown by young children working with the computer. Phillips (1997) in his review of a number of studies related to young children's graphical interpretation skills, including the use of motion sensors and other real-time datalogging devices, concludes that there is evidence of a "surprising proficiency" demonstrated by some young students. These students are "capable of a wide range of operations with graphs that include... the use of scattergraphs to see a trend".

We conjecture that by allowing the development of interpretation skills before explicit teaching of the conventions and technicalities of graphing and the practical skills of drawing graphs, the computer may allow us to re-evaluate the progression traditionally applied to the teaching and learning of graphing skills.

In this paper, we report on three related areas of study:

(i) An experiment in which we explored children's understanding of line graphs.

(ii) Initial work on children's construction of meanings for scattergraphs, followed by a recent systematic study in this field.

(iii) Initial exploration and subsequent systematic study of children's construction of meanings for formal notation.

We postpone discussion until each of these aspects of our research has been summarised. First, we describe the setting in which the research has taken place.

The Research Setting

Our studies of children's use of spreadsheets to construct new meanings for graphing took place as part of the ongoing research of the Primary Laptop Project, in which we are studying the effects on young children's mathematical learning when they have continuous and immediate access to portable computers. The computers are seen as part of a complex working environment, where many integrated factors support the children's learning. The machines were generally shared between two children. Ownership of the machines by the children, and parental involvement, is encouraged by a number of strategies.

- The children to a large extent control the machine — they put the computer away, re-charge the battery, maintain the desktop and folders, organise who takes the machine home overnight, and often decide when the computer should be used in class (though activities would often be specially designed with computer use in mind).
- The children are encouraged to demonstrate and use their growing knowledge by showing their work to their parents and peer tutoring other children in the school.

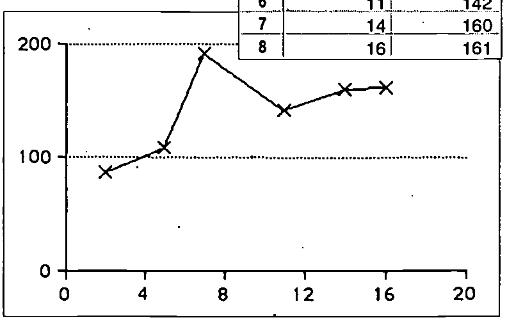
The teachers and the researchers involved in the project team plan activities which seem to be rich in mathematical potential. The Primary Laptop Project continues to be exploratory in nature, though when clearly focused research questions emerge we carry out more systematic study. Indeed, in this paper, we will present an example of how early exploratory work into young children's use of spreadsheets to produce graphs developed into a careful study of the meanings that children construct for graphing. In all of this work, the children are using *ClarisWorks*, a software package which integrates word processing, graphics, database and spreadsheet facilities.

Line Graphs

Our study of working with line graphs arose from a classroom activity for 8-9 year-olds, in which the children were presented with tables showing the heights of four imaginary children at various ages (Ainley 1994, 1995a). Their task was to use the spreadsheet to produce a line graph from each set of data, and to find out from it how tall that child was at age 3 and age 13.

The children were able to engage quickly with this task. For example, Emma and Philipa were surprised at their line graph (Figure 1) of Danny's heights; they said that it looked as though Danny had shrunk as he got older. Philipa decided that they had probably put in the wrong data. They went away to check, and found that they had put in 191 instead of 119 for the height at age 7. They quickly changed this entry, and were satisfied with their new graph.

This brief incident illustrates the intuitive ease with which the children seemed to handle line graphs, despite never having been formally taught how to draw or read them. Structured interviews with a cross-section of the children revealed that some were able to produce hand-drawn graphs from similar sets of data, showing a reasonable grasp of the technical skills of scaling axes and plotting points, even though these skills had not been explicitly taught.



	A	B
1	Danny	
2	Age	Height
3	2	87
4	5	108
5	7	191
6	11	142
7	14	160
8	16	161

Fig. 1 : Emma and Philipa's line graph

These initial observations led us [illegible]stion which features of the activity as a whole, and the use of the spreadsheet in particular, contributed to these levels of facility. We designed a comparative study with two matched groups of 9-10 year olds, one group following a teaching programme using spreadsheets, the other following a similar programme, but using pencil and paper. The post-test showed no apparent differences in performance between the two groups, with the majority of children (15 out of 25 across both groups) being able to draw, and interpolate from, line graphs. A retrospective analysis of the design of the teaching programmes revealed that, in an attempt to avoid favouring either group, significant elements had been kept as similar as possible in both programmes. We believe that the children's success was supported by these common elements: the familiarity of the context (growth), the presentation of a complete image of a graph (avoiding decomposition into a series of small techniques), and the repeated use of a number of similar graphs.

The incident above with Emma and Philipa illustrates another significant aspect of the activity. We were struck by the way in which the two girls identified their input error. The children made sense of a strange looking graph by cueing into prior knowledge about growth. The meaningfulness of the activity enabled them to use the line graph as an error-spotting tool. We carried this notion with us when we began to think about the design of activities which might involve the production of scattergraphs.

Scattergraphs

Our experience of observing children interpreting line graphs allowed us to conjecture that the computer might open up new problems and new ways of learning. We developed a new computer-based pedagogic approach, termed *active graphing*, which may help young children to develop interpretative skills (Pratt 1994, 1995a, 1995b).

In an active graphing task (Figure 2), children enter data directly into a spreadsheet, and produce scattergraphs as part of an ongoing experiment; the physical experiment, the tabulated data and the graph are brought into close proximity. The ability to produce graphs during the course of an experiment enables the graph to be used as an analytical tool; the children make decisions about future trials in their experiment on the basis of their interpretation of the graph.

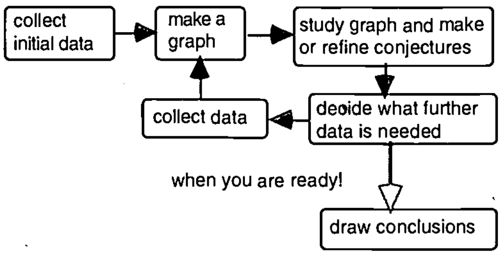


Fig. 2 : A model of the active graphing approach

Exploratory Work

In exploratory studies using this approach, we observed children as young as 8 years old learning to interpret scattergraphs confidently, making general statements about the relationships of the variables, and extrapolating and interpolating to predict results.

For example, Andrew, Ben and Sam, aged 8 to 9 years, were exploring the effect on the time of flight of paper spinners when they made changes to the wing length, using an active graphing approach. They produced scattergraphs of wing length against time. Initially they focused on the variation from point to point. With some teacher support, the boys began to focus on an apparent linear trend and to identify points which did not seem to fit. As in the previous study with line graphs, we observed Andrew, Ben and Sam using their graph to identify errors. As the boys collected more data, the assumed linear pattern broke down and eventually they decided that the trend appeared to go up and then down (Figure 3).

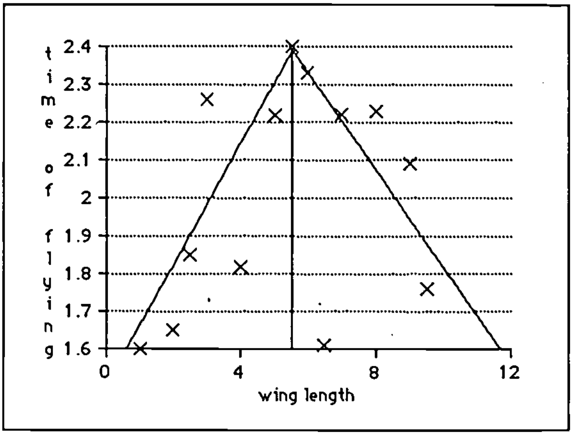


Fig. 3 : Andrew, Ben and Sam recognised the trend went up and down

A Systematic Study

Incidents such as that with Andrew. Ben and Sam suggested that, in active graphing, the children constructed meanings through making connections between three separable modalities: the Experiment, the Data and the Graph, which we came to term the EDG triangle.

In a recent study (funded by the ESRC), we have attempted to trace the emergence of meanings as children negotiate the three vertices of the EDG triangle. The study has focused on a number of research questions.

- How do children make sense of their interactions with three modalities of Experiment, Data and Graph to support their understanding of graphs?
- What difficulties and confusions are presented by the computer medium?
- How does the computer mediate the development of the children's construction of meanings for graphing?

- How do specific aspects of the activity design shape the children's use of the computer-based graphing tools?

Method

The class taking part had not had any previous exposure to the active graphing approach, or to the use of scattergraphs. We worked with their teacher to design a teaching programme of four active graphing tasks.

(i) Bridges

In *Bridges,* the children were asked to construct a bridge by folding a piece of paper and then to measure the maximum weight that the bridge could hold (Figure 4). The children experimented by altering the number of folds. The children's aim for the investigation was to find a bridge which could hold a heavy ornament, a marble egg, that the teacher showed the class in her introduction to the activity. The children carried out the experiment by gradually increasing the weights until the bridge collapsed and were encouraged to use an active graphing approach to study the data from their experiment.

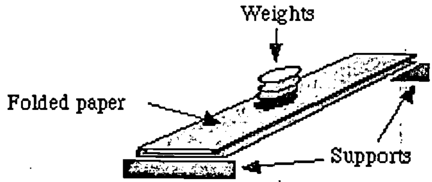


Fig. 4 : The Bridges apparatus

(ii) Display Area

The children were asked to make a rectangular frame from a 75 cm piece of string, into which they could fit as many pictures as possible: i.e. they were asked to find the rectangle with maximum area for a perimeter of 75 cm.

(iii) Helicopters

The helicopters activity was identical to the paper spinners task discussed above. The children were challenged to design paper spinners to investigate how the design parameters influence the flight of the spinner. The activity became one of finding the 'best' spinner, the one which had the longest time of flight.

(iv) Sheep Pen

The children were asked to design a rectangular sheep pen using 39 m of fencing to be set against a wall. The sheep pen should hold as many sheep as possible: i.e. they were asked to find the rectangle with maximum area when one length and two widths of this rectangle total 39 m. They modelled this using straws cut to 39 cm.

Each activity was used for one week with a class of 8 and 9 year olds, lead by the class teacher, generally with a two-week gap between activities. The class was organised in two halves, each working in small groups on the activity for a period of up to two hours on alternate days: thus each group worked for 2 sessions on each activity. Four girl/boy pairs (two from each half of the class) were selected for close observation of working sessions by the researcher (as participant observer), including regular interviews; the sessions and the interviews, in which the children reviewed their work to date and discussed their plans for the next stage, were audio-recorded. On day 4, there was a closing session, which was also recorded, in which each group presented their work to the class. The data consists of the recorded sessions and interviews, the children's work (spreadsheets and graphs) and fieldnotes.

Selected parts of the recordings were transcribed and these were combined with the other sources of data to produce extended narrative accounts, the *Stories*, describing the work of each pair on each activity. Significant learning incidents, varying in size and content, were then extracted and categorised in terms of the main themes that had emerged at the various stages of data processing:

Analysis

Discourse in the four activities was characterised by intermittent phases of sense-making, at times located within a single modality (E, D or G) and at other times connected across two of these modalities. We pick out incidents from four types of discourse in order to illustrate the variety and construction of meanings for trend during the active graphing approach. Our examples are taken from the Bridges and Helicopters activities (reserving the data from the other two activities for the section on formal notation) but the reader should bear in mind that these children carried out all four activities in the sequence described above.

(i) G-only discourse

The children, fascinated by the visual impact of the graph, were sometimes attracted towards the identification of familiar shapes (triangles, circles) and pictures (of animals, houses). In the following illustrative extract, Laura, Daniel and Claire discuss the latest graph of their helicopter data (Figure 5). (All boxed excerpts are extracts either from transcripts or from condensed versions of the recorded events. The first person refers to the researcher, the second named author of this paper.)

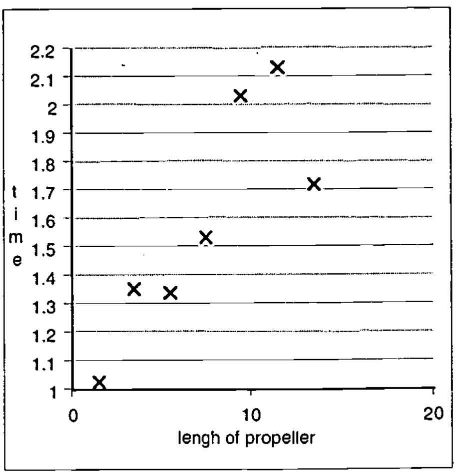


Fig. 5 : "it goes like a ramp"

Lau:	It's going like in a pattern.
Cl:	It's in steps...this is a really sloppy step and it's going like that.
Lau:	And then it looks as if it's going like a ramp. It first goes like that and then it goes like that - in a ramp.
Dan:	It's going like that, I think. And then its going to go down there.
Res:	Can you talk to me a bit more about this?
Lau:	I think it goes like mmm and then down again. If we do high ones it could still go up.
Cl:	Looks like some sort of animal lying down.
Dan:	It looks like a lion. There is the tail and the head. There is its body and ...
Res:	Right, and what would that mean for our experiment? I mean how...?
Lau:	It looks as if its going lower and then its going up higher again. And then its going lower again - oh no, the biggest ones are going slower cos that one is going.

Such *shape-spotting* was often transient and did sometimes indicate that the children were looking *through* the random fluctuations in the data to observe trends (as distinct from looking at the data in a point-to-point fashion). These shapes were clearly meaningless in relation to the experiment (though the children would sometimes invent humorous stories which in their minds linked the animals to the experiment!).

(ii) E-D discourse

Quite early in the activities the children were naturally drawn by the impact of the experiment — visual or otherwise. An intuitive feel for the experiment often informed their thinking when operating in other modalities. In the following extract. Claire objected to Chris' mathematical 'prediction', which was based on his observing the data and beginning to believe that the weight on the bridge was increasing proportionally with the number of folds.

The 5-bridge has just held 28 gr. Chris suggests they try the 10-bridge next and he wants to give a prediction:

Ch: If that's 28, two 28, because it's on the five, two 28 are... [the researcher says 56] so if it's 56 or if it goes over, I am predicting that it's 56, so I'll put 56 on the ten folds.

Claire is not pleased with Chris' proportional prediction because she expects a 10-bridge to hold more. She is right. It holds 68 g. Claire suggests they try a 20 and then try the egg. Chris wants to put the egg on the 10. I get the egg and Chris asks how much the egg weighs. We don't know but Claire says we can try it first and then find out about its weight. When the first day's work is completed theteacher asks Claire and Chris to talk about their work. In the discussion both children recall the incident:

Folds	Weight g	
1	1	
2	3	
3	7	
4	12	
5	28	
10		56

Cl: ... and then on five we got the 28...

Ch: ... and then for the 10 I thought it was 56 and it's actually 68.

We observe how the processing of information in one modality (the numerical representation of the data in the above extract) by means of information in another (Claire's intuitions about the experiment) supports their attempts to predict the course of the experiment. When graphing is involved in the discourse, such predictions can lead to the construction of meanings for trend.

(iii) G-D discourse

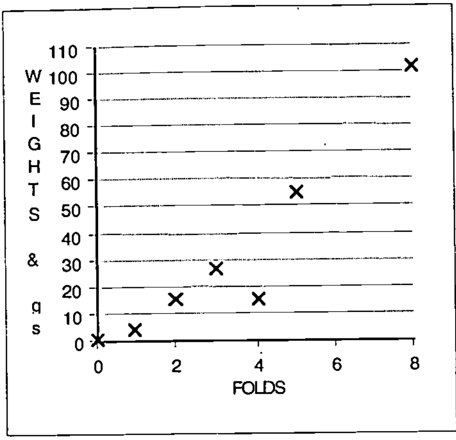


Fig. 6 : The cross for 4 is falling

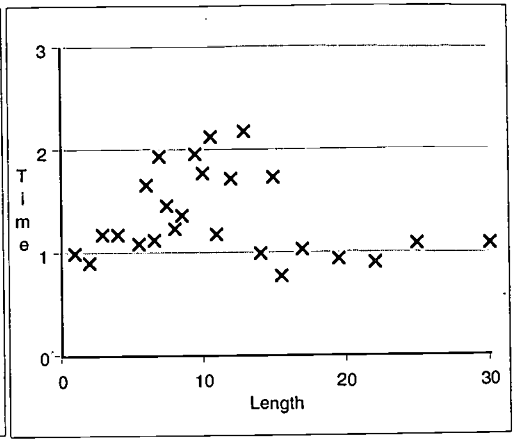


Fig. 7 : Tarquin and Eli's scattergraph

In the following extract, we observe a typical incident in which Tarquin and Eli were concerned that a particular point did not seem to fit the rest of the emerging picture in the Bridges activity (Figure 6).

> Tarquin points at the cross for 4 for which he says 'the cross is falling' being the same as 2. He wishes to try the 4 again. Eli joins us and I repeat to her what we have just been saying with Tarquin.
> **Res:** So, where exactly would you expect it [the cross for 4] to be?
> She is pointing along 40, in between the results for 3 and 5.

Such incidents would usually lead to re-testing. We refer to processes in which the appearance of the graph is 'corrected' as *normalising*.

(iv) G-E discourse

In the following extract, Tarquin and Eli discuss with the researcher the scattergraph that they have produced from the helicopters activity (Figure 7).

> Tarquin and Eli begin by describing how the left side of the graph goes up and the right side is "all on the same line". They carry on to describe how the graph goes up and down.
> **Ta:** They are going up instead of down because there they are going down.
> **Eli:** Sort of ... they are climbing like that and they are going like that.
> **Ta:** I think the under 10 15 under 15 are going are the best.
> **Res:** Okay, so less than 15 and more than ... what is this?
> **Ta:** Actually 12 and 9 they are all coming up.
> **Res:** Yes, and then what's happening?
> **Ta:** Over 12 they are all going down.
> **Res:** They are going down and becoming more or less the same lower down - okay.
> **Ta:** They are just falling instead of spinning.
> Oh, you say falling instead of spinning.
> Yeah, going straight down.

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Res: Right, and they are going fast because they are going down fast. Then the flight is quite short, isn't it? They don't stay up in the air for a long time, do they?
Eli: No, they are just falling down.
Res: Is that what we are seeing here?
Eli: These are like staying in the air - longer sort of time.
Res: Are these staying longer?
Eli: Well, not longer but they are staying in the air the same sort of time.
Jen: They are all the same, aren't they?

We observe the children constructing connections between their intuitions of the experiment and features of the scattergraph, a process we term *feature-spotting*.

Brief Summary of Findings

Our analysis suggests:

(i) shape-spotting was an important category of activity associated with early attempts to look through, rather than at, the data;

(ii) active graphing provided an effective strategy for the construction of meanings for trend through the encouragement of normalising and feature spotting;

(iii) when normalising and feature spotting were acted out as part of E/G and D/G discourse, we observed the articulation of meanings for trend;

(iv) shape-spotting, normalising and feature-spotting appeared and re-appeared intermittently, presumably cued by surface features of the activity.

Formal Notation

Exploratory Work

The Sheep Pen activity, described above, had also been used in our early exploratory work; we had observed children extending the active graphing approach to construct meanings for the formal algebraic notation in the spreadsheet.

We have discussed this work elsewhere (Ainley, 1995b & 1996) using a case study of two 11 year old boys to illustrate how the cell reference seemed to take on increasingly sophisticated meanings:

- initially as little more than a name for the width of the pen,
- then more as a placeholder for a potential number,
- and eventually as a placeholder for a range of numbers (i.e. a variable).

By moving towards the use of formal notation within an active graphing approach, the children seemed to construct a sense of utility for what is often seen as a remote and disconnected aspect of mathematics; the Sheep Pen task had begun experimentally with the children making sheep pens out of straws whereas the formal notation of the spreadsheet enabled the children to generate much more quickly large sets of data and produce 'perfect' scattergraphs.

A Systematic Study

The Sheep Pen task had raised so many questions that we decided to include it as the last in our four activities for systematic study. At the same time, we wanted to introduce the much younger children in our systematic study to the possibilities of formal notation in a more straight forward situation. Hence we included the display area activity as the second task in the set of four. Both the Display Area and Sheep Pen activities produce similar graphs, in which the maximum value is found from a parabola. In each, the relationship between the length and the width of the rectangle is accessible to the children and amenable to algebraic modelling (e.g. $l = \frac{75 - 2w}{2}$ for Display Area , and $l = 39 - 2w$ for Sheep Pen).

The children recorded the results from each practical experiment on spreadsheets, which (with help) they had set up to calculate the area of the rectangle, and made x-y scattergraphs of the width or length of the display area or sheep pen against its area.

Analysis

We use the term *formalising* to encapsulate three categories of activity which we observed as contributing to the construction of meanings for formal notation:

- connecting a pattern based on the data with the experiment,
- connecting a pattern based on the data with a rule, and
- connecting a rule based on the data with a formula.

We will exemplify formalising by providing extracts from the data which illustrate these three categories of activity.

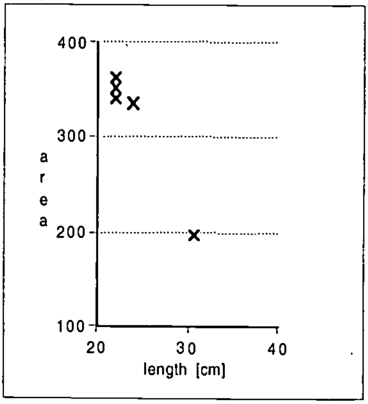
Connecting a pattern based on the data with the experiment

In both activities, the formula for the area was introduced in the beginning with the help of the teacher, so the notion of such a formula was familiar to the children. A focus on the relationship between the length and the width gradually emerged during the activities. The children went through a phase during which there was a vague, growing realisation that the length and the width are somehow interrelated but it was not clear how. Consider this incident involving Laura and Daniel working on the display area activity:

I ask Laura and Daniel whether their measurements have become more accurate. I observe how they are doing it and notice they are fixing the length. I ask them whether they can find out the width given the length. Daniel does the 23-length case: he doubles 23 and then the pair notice that their measurement of 16 cm for the width makes the perimeter 76 cm. They measure the 23 cm sides and the other comes out as 15.5 cm, so they record 15.5 cm as the width. *(Numerical inaccuracies in these extracts are due to difficulties in making measurements, or to hasty calculations.)*

The children choose (fix) the length, but still measure the width. However they are ...ally unaware of their interdependence. When asked to predict the width if the length is given, Daniel starts by doubling the given length but then is carried away by the realisation that the previous measurement has been inaccurate. However, after several trials of the experiment, the children gradually perceive the width-length interdependence more clearly and begin to connect patterns based on the data with a rule.

Connecting patterns based on the data with a rule.
The following example begins with an attempt to normalise the graph. For Claire, this evolves into a realisation of how to calculate the width of a rectangle when given the length and the perimeter. Although Claire uses this calculation as a correcting procedure, applied to experimental data, she is still unaware of its potential as a data-generating tool.



width (cm)	length (cm)	area
14	24	336
6.5	30.5	198.25
16	22	352
15.5	22	341
16.5	22	363

Fig. 8 : Claire and Chris's graph and spreadsheet

While Claire and Chris look at the graph (Figure 8), they identify three crosses one on top of the other, that is all measurements corresponding to the same length of 22 cm. We decide to find the accurate measurement: so if the one is 22 how much is the other? 'Let's measure', suggests Claire. I ask them to do it in their head. '22+22 is 44', says Claire 'and then you have to try and make 75'. 'So how much from 44 up to 75?' '31', says Claire. 'So each of the two lengths is how much when both are 31?' '15 times 2 is 30, 16 times 2 is 32', says Claire, and then: 'I don't know'.

Res: This is 22 and this 22 and this is 44, so these two are 31 both of them. So how much each?

Cl: Divide it - ah ... so it's 15.5.

Res: Excellent. So this is how to choose which one ...

Cl: Now I get it! That one is right but these two aren't!

In a very similar normalising incident, Jenny offers the following method when asked to check the accuracy of her measurement and make sure that her widths and lengths add up to 75 cm:

Jen: You add those [widths] together and then [to find the length] you like try to make 75 ...

In the second session she gives a similar description, but adds:

Jen: ...and then you have to halve...

During the activity the children demonstrated knowledge of the width/length relationship especially when asked to check the accuracy of their measurements. They added two widths and two lengths and if the result was 75 they accepted the measurement: if not, they didn't.

The construction of an inverse process (given the length and the perimeter, find the width) is less straightforward. The following example is from session 2.

> While trying to make the 19x19 square out of 75 cm, Laura realises that there doesn't seem to be enough string for the four angles of their quadrilateral to be right angles.
> **Lau**: So we need to make it smaller! Instead of 19, make it 18!
> **Dan**: Then the width has to be 20.

Using their experience from *Display Area*, most of the children move swiftly towards constructing the formula and using it to generate data in *Sheep Pen*. The following example is only a few minutes into the activity for this child.

> I ask them to prepare a sheep pen for demonstration and save their file. Chris shows me why the width-29.5 sheep pen is impossible.
> **Ch**: You can't because you have to double the 29.5.

Daniel observed at about the same time.

> **Dan**: We don't really need these [the straws], do we?

Laura and Daniel both claim at this stage: "we don't have to make them" because "we've got a pattern". The following incident confirms the clarity with which Laura and Daniel 'see' the relationship between the length and the width of the sheep pen. When the teacher set up the activity initially, a length of 41m of fencing was chosen for the sheep pen, but this was later changed to 39m.

> The children instantly suggest "taking 2 out of 41" because "If it's 41, if it's going to be 39 we are going to go 41, 40, 39 we are going to take two off everything so that makes that ...".

The passage from an initial connection between the experiment and the data to connecting this pattern with a rule becomes gradually more overt, as in the following examples, taken from the first sessions of activities 2 and 4 respectively.

> Eli and Tarquin are busy correcting their measurements. The teacher joins in and asks what they have been doing. Eli explains the 26 case and also the previous 31 case. They explain how they get from width to two widths minus from the perimeter and then divided by two. After a discussion in which Tarquin confuses area with perimeter in his calculations, the teacher says she wonders whether they can find any other way of doing the calculation automatically. Tarquin mentions 'formula' but cannot say more. The teacher lets them go on.

WIDTH	LENGTH	AREA
15	23	345
7	30.5	213.5
14	23.5	329
16	21	336
6.5	31	201.5
11.5	26	299

It is the beginning of Activity 4 for Laura and Daniel and they have chosen to start from width of 4 cm for a perimeter of 41 cm. Laura then suggests their choice of widths follows a pattern: 'increase it by a half cm. So the next is 4.5 cm'.

Res: So, if you have 4.5 what's gonna happen then?
Lau: It's gonna be thirty the...in the 3s...
Res: So this is 4.5 and this is going to be?
Dan: That's going to be 32!
Res: Clever, how did you do that?
Dan: The first one was 4 and then it's 4.5 then both sides you take off a whole.

Connecting a rule based on the data with a formula

The extracts of the discussions in which, for instance in Display Area, a pair of children construct the width-length formula and put it into use are very extended and cannot be included here. Indicatively though we summarise how Laura and Daniel rather quickly decided to work systematically in the Sheep Pen activity, introducing formulae for the generation of widths and lengths.

Laura and Daniel, having noticed that there is a pattern in the way the length changes when the width changes — increasing the width by a half cm causes the decreasing of the length by 1 cm — propose the use of two formulas one in the width column and one on the length column. Starting from 4 cm for the width they introduce the +0.5 formula and they fill down. Similarly they introduce the -1 formula for the lengths. Later in the day, and once the children have introduced the spreadsheet formula, =39-2*F2, in order to connect the lengths in column E to the widths in column F, they decide to refine their measurements and generate data for an increment of a quarter for the width. Laura says that all they have to do is replace the 'sign for a half' with 'the sign for a quarter'. The discussion leads to deciding that .25 is the sign for a quarter and Laura says:

Length	Width	Area
31	4	124
30.5	4.25	129.625
30	4.5	135
29.5	4.75	140.125
29	5	145
28.5	5.25	149.625
28	5.5	154
27.5	5.75	158.125
▪	▪	▪
▪	▪	▪
▪	▪	▪
1.5	18.75	28.125
1	19	19
0.5	19.25	9.625
0	19.5	0

Lau: So we have to put plus .25.
Res: I think you have got the idea very well so can we see it: please, shall we start from 4?
Dan: Can we do the wholes?
Lau: We've done the wholes!
Res: Because you start with halves: if you put half and half again you have a whole.

They insert the .25 formula. Daniel looks as if he is really clear about it but Laura does not. I ask her and she says:

Lau: When the pattern comes up it will come to me.
Dan: It's a different language, Laura, it's a different language!

They fill down the width column.

Res: Can you tell me now, are you filling down? Good.
Lau: 4.25, 4.5, 4.75 and then it does it again, it's repeating itself!
Res: What do you mean repeating itself?
Lau: 4.25, 4.5, 4.75, 5, 5.25, 5.5, 5.75 so it's like in a pattern.

Brief Summary of Findings

The above analysis suggests:

(i) new meanings for the formal notation emerged out of seeing patterns in the data (supported by the construction of meanings for trend in the graph);

(ii) a utility was constructed in which formal notation helps to normalise data;

(iii) a further utility was constructed in which formal notation can generate data.

Discussion

These studies were motivated by the suggestion that the use of technology might facilitate a pedagogical change of emphasis from the sub-skills of drawing graphs to their use and interpretation. The work with line graphs produced from a spreadsheet indicated that they could be used as a problem-solving tool in purposeful enquiry even when the children had not been explicitly taught those sub-skills. Through the pedagogic approach of active graphing, children have expressed new meanings in which scattergraphs have utility in helping to identify abnormalities in the data and in guiding the direction of the ongoing experiment. Some types of active graphing task contain embedded accessible algebraic models, which offer opportunities for children to engage with formal notation. Under these circumstances, children have expressed new meanings in which formal notation has utility in helping to smooth out experimental errors and to generate large sets of data.

We now see the construction of meanings for trend and formal notation in active graphing as interrelated; both fall into a more general domain of constructing patterns. The purposefulness of an active graphing task draws the children into the need to construct meanings for the data which will help them to decide what to do next in the experiment. They are therefore inclined to make predictions and to identify discrepancies. These tendencies manifest themselves as feature-spotting and normalising. Indeed, as their sense of trend becomes more firmly established, the identification of perceived abnormalities in the graph becomes more efficient. In their efforts to normalise the graph or the data, they consolidate the notion of dependence between the two variables in question.

When the activity is amenable to an extension into formal notation, this consolidation can lead to the identification of patterns in the data. The teacher is able to recruit normalising as an intervention strategy, drawing children's attention towards discrepancies in the hope of accelerating formalising. Thus, the pay-off for normalising is seen as an increased sense of trend (and consequent ability to spot features), and the pay-off for formalising is seen initially as increased efficiency in normalising. Subsequently, a new utility for formal notation as generator of new data is constructed, which has the pay-off that perfect graphs are created.

The richest episodes in our data exemplifying the construction of meanings for trend are characterised by intense interaction between the graph and the data and between the graph and the experiment, while the richest in our data relating to constructing meanings for formal notation are characterised by the intensity of the data-formula interaction. In both cases the most powerful learning incidents seem to occur where there is more interaction among the different modalities.

We conclude that meanings do not emerge exclusively within any one of the four modalities, but are gradually established through repeated interaction between them (Nemirovsky and Rubin's (1991) work offers another perspective on this idea). The appearance and re-appearance of shape-spotting, normalising and feature-spotting indicate the intermittent nature of this sense-making as they constantly revisit each of the E/D/G modalities and make connections between them. As active graphing clearly encourages the interaction between different representational modalities, it emerges as an approach with high pedagogical potential.

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Symbol-Use, Fusion, and Logical Necessity: On the Significance of Children's Graphing

A reaction paper to Ainley et.al.

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The work of Ainley, Nardi, and Pratt is in many ways parallel to the work we are developing at TERC on the learning of graphing. In this reaction I will examine the data described in their paper and strive to articulate its significance as seen from our own experience. I will organize my comments in terms of three themes common in our work: Symbol-Use, Fusion, and Logical Necessity.

Symbol-Use

Ainley, Nardi, and Pratt report how competently children learn to interpret and make use of graphs when they are created actively for a purpose and in relation to a situation that is close and familiar to them. For example, children easily learn to recognize anomalies and formulate expectations. I think that the overall significance of this finding, highly consistent with our own observations, is that fluency with symbols is developed through their use, rather than by mastering a set of isolated skills. Symbol-use is not only a matter of having purposes but also of being acquainted with the context that is symbolized as well as considering symbolic notations as a malleable and personal means of expression (Nemirovsky, 1994). In Carraher, Schliemann, and Nemirovsky (1995) we report on an interview that Analucia Schliemann conducted with Zefinha, a Brazilian woman in her fifties with only three years of elementary schooling, in which they discussed a graph that had appeared in the front page of a major Brazilian newspaper. The graph described the results of several months of opinion polls concerning impending presidential elections. Throughout the conversation Zefinha started to make both month by month comparisons between the approval ratings of the main candidates and overall estimations of trends for approval ratings over time. Observing that toward the end of the graph her favorite candidate was trailing behind, she said: [Moving her hand up twice as she speaks] "This little line has to give more votes, it has to go up more numbers." In all probability Zefinha had never been taught the conventions of graphing, plotting points, and so forth, and yet, through her conversation with Analucia, she started to see in the graph systematic changes in public opinion over time; changes that she cared very much about.

I include this example because it highlights the centrality of symbol-use with or without the use of computers. Because of the paper's title and examples, the reader of Ainley, Nardi, and Pratt (1998) will tend to think that the key of their approach resides in the computer-mediated tools. If that were the case we would lose what I see as the most significant aspect of their contribution. As part of the "Investigations" curriculum we have developed a curricular unit for 4th grade (Tierney, Weinberg, Nemirovsky, 1994) which includes activities with plant growth. Every day each child grows a plant and measures its height. The students record the measurements on graph paper by hand. The fact that they produced the graphs by hand on paper did not preclude them from discussing trends, identifying anomalies, and developing mathematical narratives (Nemirovsky, 1996). For example, the class discussed the "trouble with the weekend" created by the fact that the children had graphed subsequently the heights measured on Friday and Monday and therefore the graphs appeared to show a spurt of growth over the weekends. (Tierney, Weinberg, Nemirovsky, 1992). Or they found that the scale that they had used to record the measurements of their own plant would not work as well when graphing someone else's plant heights. What the activities with these students or with Zefinha had in common with the ones reported by Ainley, Nardi, and Pratt, was not the use of computers but the use of graphs by people reflecting in flexible ways about situations that were accessible and relevant to them.

Fusion

Zefinha's comment: "This little line has to give more votes, it has to go up more numbers," exemplifies a phenomenon that we have repeatedly observed in many contexts of symbol-use and that we have called "fusion" (Nemirovsky, Tierney, Wright, 1998). Fusion is talking and acting without distinguishing between symbols and referents. Lines do not "give votes" and Zefinha was perfectly aware that votes are really given by people out there in the country; but she, as well as Analucia, found it expressive and meaningful to talk about the line as "giving votes." This was not a sloppy way of talking or a confusion about what graphs do. It was a natural expression of directly seeing social events in a symbol and, conversely, of recognizing symbolic behavior in a social process (e.g.: numeric and graphical up-ness in changing people's opinion toward her candidate). Examples of fusion can also be found in Ainley, Nardi, and Pratt (1998). For example, they report Tarquin's assertion that "the cross for 4 is *falling*" for a point in a graph that described the weight that a folded paper could bear. Eli's description of a scattergraph graph showing the time that different paper spinners remained flying in the air exemplifies fusion when he says: "They [the points on the scattergraph] are *climbing* like that."

Fusion is an expression of fluency. While fusion is an inherent aspect of all symbol-use (Nemirovsky and Monk, forthcoming), there are particular features of graph-use that we have described which relate directly to the findings reported by Ainley, Nardi, and Pratt (1998). Specifically, we have identified the following traits in fusion associated with graphs (Nemirovsky, Tierney, Wright, 1998):

1. Cycles of interplay between the graph as a shape and the graph as a response to actions
2. The interplay between graphing and drawing
3. Imaginary traveling along trajectories on the graphical plane

1. Cycles of interplay between the graph as a shape and the graph as a response to actions

Ainley, Nardi, and Pratt included examples of children describing a graph in terms of appearing figures: "looks like some sort of animal lying down," "It looks like a lion," and so forth. We have documented (Nemirovsky, Tierney, Wright, 1998) similar descriptions by students who were working with a motion detector, generating graphs of position vs. time in real time as they moved in front of the detector: "They look like, kind of like mountains," "I was thinking we could have made an elephant if we went straight down." The conversations seemed to go back and forth between figural attributes and actions that had produced graphical responses: "They [the mountains] kept on getting smaller [because] I didn't walk as far." As suggested by Ainley, Nardi, and Pratt, we recognize important elements in perceiving the graph as a shape. It helps the graph-user to develop an overall sense of the graph, to share salient features with others, and to plan actions.

2. The interplay between graphing and drawing

Ainley, Nardi, and Pratt comments about children figural interpretations of the graphs also relate to our thinking that drawing is a major source of expectations and meanings that students bring to graphing. To a certain extent learning graphing is learning in which sense it is similar to and different from drawing. An episode with a 9-year old girl (Nemirovsky, Tierney, Wright, 1998) who was trying to generate position vs. time graphs that looked like letters help us to understand this. Noticing that the line cannot come backwards, she commented: "A 'T' I don't think would work either. It [capital T] would be like an upside down L. It wouldn't have the stick in the middle of the straight one at the top. So it can't be a 'T' or an 'a.'" She was pondering the fact that something that it is easy to draw, such as a 'T', is impossible to graph. Another aspect that drawing brings in to graphing is gesturing. We often gesture shapes as if we were drawing them. Gesturing shapes is an ever present aspect in interpreting graphs.

3 Imaginary traveling along trajectories on the graphical plane

A fundamental way in which students and experts talk about graphs is by imagining trajectories on the graphical space. These trajectories are equally marked by graphical shapes and by symbolized events or actions. For example, Eli describes the points of a scattergraph saying: "they [points on the graph] are climbing like that and they are *going* like that," or Tarquin says "Over 12 they [points on the graph] are all *going* down." After their ethnographic study of professional physicists, Ochs, Jacoby, and Gonzales (1994) describe this notion of graphical trajectories as "interpretive journeys" in which the symbolized physical events coalesce with the shape of the trajectory and the personal intentions of the story reflected by the imaginary journey.

Empirical observation and logical necessity

The last section of Ainley, Nardi, and Pratt 's paper is entitled "Formal Notation." In this section they describe how children approached problems of area and perimeter subject to algebraic constraints (e.g.: maximize the area of a rectangle given a fixed perimeter). The focus of their analysis is how the children gradually replaced actual measurements with arguments of logical necessity. Students' views on when one needs to measure and when it "has" to be a certain way, even if the measurements seem to indicate the contrary, is a major topic for mathematics education. It is at the root of algebraic reasoning and of the notion of proof. What I find particularly valuable in the paper is that rather than positing the relationship between empirical observation and logical necessity as if it were determined by the task, Ainley, Nardi, and Pratt try to trace how students think of this relationship in light of the tools that they had available and the specific problem that they were dealing with.

Pondering the relationship between empirical observation and logical necessity is grounded in everyday experience. For example, we discussed with students of different grades whether in the movement of one's hand back and forth there is a moment in which one actually stops the hand. A 9-year old girl looked at her hand in motion several times and concluded that she was not sure, sometimes it seemed to her that she did and some other times that she did not. Later she used the motion detector to create velocity vs. time graphs by recording the motion of her hand. The sign of the velocity graph corresponded to her moving her hand away from the sensor (positive) and toward the sensor (negative). She looked at the computer screen and commented with the tone of having an insight: "You can see that you *have* to stop when you change direction, you have to cross that line [the line of zero velocity]." Note how symbolizing the everyday experience with body motion prompted her to envision the necessity of stopping moment.

We have related students' sense of empirical observation and logical necessity to their use of tools (Nemirovsky, Tierney, Wright, 1998). It seemed to us that the changing ways in which they related these two aspects were intimately related to how they conceived of the nature of the measurement and calculation tools they were using. The girl's noticing the necessity of stopping one's hand is a case in point; her sense of necessity was intimately related to how she thought of the motion detector and the computer display. A similar connection seems to emerge from the work of Ainley, Nardi, and Pratt. Children's ideas about when to measure and when to calculate appeared to change together with their notions of what a formula does in a spreadsheet.

A subject that is absent in the paper is the role of students' invention of graphs and of systems to organize data and numbers. This is an area that could enrich their work on children's graphing and formal notations (diSessa et al., 1991; Tiemey and Nemirovsky, 1995). Overall, I think that the convergence of ideas between Ainley, Nardi, and Pratt and our own work expresses a shared commitment to pay close attention to students' ideas, describe the situations in which they emerge and learn from them.

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WORKING GROUPS

WG1: *Advanced mathematical thinking*

WG2: *Algebra: Epistemology, cognition and new technologies*

WG3: *Classroom research*

WG4: *Cultural aspects in the learning of mathematics*

WG5: *Geometry*

WG6: *Research on the psychology of mathematics teacher development*

WG7: *Research in the social aspects of mathematics education*

WG8: *The teaching and learning of stochastics*

WG9: *Understanding of multiplicative concepts*

WG10: *Embodiment, enactivism and mathematics education*

WORKING GROUP: ADVANCED MATHEMATICAL THINKING

Co-ordinator:

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The AMT working group is concerned with all kinds of mathematical thinking, developing and extending theories of the Psychology of Mathematics Education to cover the full range of ages. This interest includes gathering information on current research, discussions of both the mathematical and psychological aspects of advanced mathematical thinking and research into thinking in specific subject areas within mathematics. This will be the thirteenth meeting of the working group.

We are concerned to understand how people think about mathematical and statistical ideas after they have already acquired the use of a significant set of relevant skills; how they form mental models and cognitive constraints in the deductive reasoning and what are the nature, structure and proof of the theorems. The AMT is concerned with studying how one learns to understand and to internalise notions in advanced mathematics. We seek to provide a forum for discussion of mathematical, psychological, and pedagogical issues (and the ways in which they interact) in the area of advanced mathematics. We believe that language facilitates thinking and the symbolic language of mathematics helps facilitate advanced mathematical thinking.

Our PME-22 sessions will be centred on the book project that is being developed since the meeting in Recife. Participants will have a chance to discuss with the authors who have submitted their contributions prior to the dead line set to May 1st. Three themes have been selected as unifying threads to connect sections of the book: *The role of technology*, *Teaching the nature of mathematics* and *Teaching a mathematical attitude*. Fulvia Furinghetti, Thomas Lingefjärd and David Reid constitute the editorial board. This book intends to complement the group's last major publication *Advanced Mathematical Thinking*, D. O. Tall (Ed.).

According to the interests of participants a parallel section may be organised in order that each one can present their own ideas through short oral communications about *What is (or should be) Advanced Mathematical Thinking?* This central question was discussed in Lahti and produced the following sub-questions:

What does it entail?
What is the experience?
How do we research it?
How do we enable it?
How does technology affect it?

This discussion is being co-ordinated by Manya Raman at
http://socrates.berkeley.edu/~manya/AMT/index.html

WORKING GROUP

ALGEBRA: EPISTEMOLOGY, COGNITION AND NEW TECHNOLOGIES.

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Aims of the group:

The idea of a new Working Group arose as a shared necessity, from people attending PME-XXI Working Group *Algebraic Processes and Structures*, in order to exploit the richness of the work made during the Recife, Valencia and Lahti meetings, and to address some important issues emerging from the usage of new technologies in Algebra Teaching. A relevant use of these new technologies indeed, requires a deeper understanding of the epistemological and cognitive aspects involved in the teaching and learning of algebra, and raises new problems about it.

Then, we intend to reconsider the contributions brought during the previous meetings (partly published in the booklets prepared for them) and to focus the first meeting of the new working group, around the two following points of interest:

1. The *epistemological* status of algebra *signs*, *processes* and *concepts* in both *paper-and-pencil* and *computer environment* activities
2. The tools and methods of *observation* and *interpretation* about both *paper-and-pencil* and *computer environment* activities

Planned activities for the Stellenbosch meeting

Before the meeting: collect (preferably by e-mail, drouhard@unice.fr) contributions on the two sub-topics above, edit them in a booklet (which will be available to the participants) and in a website devoted to the theme:

http://math.unice.fr/~iremnice/pme_wg_aecnt/index.html

During the meeting: alternate short presentations and larger discussion times (the Working Group being by no ways a kind of second "short oral presentations" time). In order to be more efficient, the whole Discussion Group may be split into subgroups devoted to sub-themes. Short reports of the discussions will be written during the conference.

After the meeting: edit the informal reports of the discussions (in a addendum to the booklet and in the website). Prepare the 1999 discussion, in the perspective of a collective book on the topic.

Working Group on Classroom Research

Coordinators:

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The purpose of this group is to examine emerging issues and techniques relating to research in mathematics education which is conducted in 'natural' (Lincoln & Guba, 1985) classroom settings.

Over the past several years, most of the presentations made to the Working Group on Classroom Research have touched on one or both of the following themes: innovations in research methodology; and, new uses of *technology* for the collection and analysis of data. More recently, we have also used a portion of the Working Group Meetings to give each participant a brief chance to note the strategies and methods that they are employing in their *current* classroom-based research.

The group as a whole does not focus on a particular content area in mathematics, and it does not focus on the relative merits of different theoretical frameworks; these are interests that are better served by other working groups. With a focus on research methodology and technology, the Working Group aims to present and discuss the "state of the art" for classroom-based research techniques. The meetings are intended to give researchers an opportunity to hear about methodologies and research technologies that may be different from those in their own repertoire. Furthermore, the meetings are intended to give participants an opportunity to be introduced to colleagues who may be currently using -- or planning to use -- methodologies or research technologies that are similar to their own. One particular interest of the Working Group organizers is in collaborative cross-cultural studies that involve classrooms of students in multiple countries. The Working Group may play a facilitating role in helping researchers identify and organize possible collaborations.

The detailed agenda for the meetings will be developed in discussions hosted at our Working Group's Web site (http://www.webcom.com/explorer/classroomresearch). Past, present and future members of the Working Group are invited to participate in this planning process, and in discussions concerning future publications.

WORKING GROUP ON CULTURAL ASPECTS IN THE LEARNING OF MATHEMATICS

Coordinators: Norma Presmeg, Florida State University
Marta Civil, University of Arizona
Judit Moschkovich, Institute for Research on Learning
Phil Clarkson, Australian Catholic University

The aim of this Working Group is to provide an open and exploratory forum for the sharing of issues and ideas which concern cultural aspects in the learning of mathematics. Newcomers to the group are welcomed. New and ongoing themes at the meetings in South Africa will build on the issues discussed at PME-21 in Lahti, Finland, in 1997. These themes were grouped under two broad headings, as follows.

1. Language and classroom culture.
2. Power relations, social practices, and theoretical frameworks.

The sessions in South Africa will aim to broaden and deepen these themes, for the purpose of increased understanding of issues which are relevant in a world context, as well as those which relate specifically to diversity and equity in South Africa. Relevant short papers will be presented, with opportunities to learn about the work and perspectives of participants. A focal point of the group is rich discussion and sharing based on the diverse cultures of participants.

Geometry Working Group

Co-ordinators:

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A. L.Mesquita
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The meeting of Geometry Working Group in 1997, at Lahti, was devoted to the discussion of some central aspects on geometrical representations. The complexity of the question raised and the interest shared by participants suggested the possibility of preparing a collective publication of the Geometry Working Group, in which a kind of "state of the art" (or at least a trace) about PME discussions on Geometry could be done.
Previous contacts we have had with scientific editors were positive and encouraging, suggesting an effective interest by this kind of publication.

In this sense, the proposed agenda for discussion at PME23 at Stellenbosch concerns mainly the implementation of this project:

1. Presentation of new papers;
2. Discussion (on new papers and previous papers);
3. Organization of the group: theme of the WG for the next years; choice of new co-ordinators.

Research on the Psychology of Mathematics Teacher Development

Coordinators:

Andrea Peter-Koop
University of Muenster
Germany

Vania Santos-Wagner
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Between 1986 and 1989, a Discussion Group on *Research on the Psychology of Mathematics Teacher Development* met at the annual PME conferences. In 1990, this Discussion Group was accepted as a Working Group and since then has continued in this format at PME. A strong feature of the Working Group has been its cohesiveness and its wide representation across many countries.

Aims of the Working Group

The Working Group aims to

- develop, communicate and examine paradigms and frameworks for research in the psychology of mathematics teacher development;
- collect, develop, discuss and critique tools and methodologies for conducting research concerning the development of mathematics teachers' knowledge, beliefs, actions and reflections in order to better understand the process of teachers' professional learning;
- implement collaborative research projects;
- foster and develop communication between participants;
- produce a joint publication on international research perspectives with respect to mathematics teacher development.

Plans for Working Group Activities at PME 22

The Working Group's discussions during the previous meetings were concerned with the following questions: How can we facilitate meaningful preservice training and effective professional development for mathematics teachers in our countries? What kind of research is desirable in order to inform our knowledge about teachers' professional learning?

During the Working Group meetings in 1998 we will have a 30 min. invited presentation during each of the three sessions addressing one aspect of these issues from international perspectives, each focussing on individual experiences and research interests. *Dylan Wiliam* (UK) will put forward an approach to facilitate change in the mathematics classroom through the development and implementation of alternative assessment criteria and procedures. The opportunities and difficulties encountered during the development and implementation of an inservice teacher education program in response to a revised national mathematics curriculum will be the focus of *Zahra Gooya's* (Iran) presentation, while *Regina Möller* (Germany) will introduce her interdisciplinary as well as interactive approach with respect to the training of prospective mathematics teachers. In addition, all previous Working Group participants have been invited to report on their collaborative work with mathematics teachers (see Working Group Report in PME NEWS, Nov. 1997, p. 9-10) during the scheduled sessions. Furthermore, we want to explore the possibility of a joint publication of our work in the first half of 1999 and the development of a world wide web home page. New participants are always welcome!

Research in the Social Aspects of Mathematics Education

Jo Boaler *King's College, London* *United Kingdom*
Paola Valero *"una empresa docente"* *Colombia*
Royal Danish School of Educational Studies *Denmark*

One important issue that interests this group is the question of what is considered to be relevant knowledge in the area of research inquiring about the social aspects of the teaching and learning of mathematics. Answering this implies considering both the questions that may be informed by a social perspective on mathematics education and interpretations of the types of knowledge that may be used to answer such questions.

Issues that may be discussed in the first instance include:

- ▲ equity and social justice,
- ▲ selection and 'ability' grouping (tracking/setting),
- ▲ the role played by mathematics education in creating and maintaining social divisions,
- ▲ the sociology of mathematics classrooms.

Such issues raise questions about what is considered to be "relevant knowledge" in the area of mathematics education research. In exploring these questions we intend to reflect upon:

- ▲ the role of interdisciplinarity (mathematics education and other social disciplines) in this area,
- ▲ the nature of the conceptual frameworks shaping research,
- ▲ the nature of research methodologies employed,
- ▲ the criteria that are used for judging research produced,

In order to consider these questions, the sessions will cover a wide range of activities, including:

- ▲ presentation of the initial proposal for the working group,
- ▲ work in small groups, according to members' interests,
- ▲ plenary discussion of the main points tackled in the small-group work,
- ▲ agreement on agendas for further co-operation and work,
- ▲ discussion about the preparation of preliminary papers on the topics, in order to begin preparing a publication about the concerns of the group.

WORKING GROUP ON THE TEACHING AND LEARNING OF STOCHASTICS

This Working Group exists as a focus for members interested in the psychology of the teaching and learning of probability, statistics and combinatorics. It maintains an informal network between Conferences by means of an electronically distributed newsletter. It particularly seeks to bring together interested people from all language groups, and does its best to provide translation facilities as appropriate.

At PME in 1997 a proposal was raised for developing a book which presents a survey of the main research done in statistical education within both Education and Psychology. The working group co-ordinators were asked to prepare a first draft of the possible orientation and structure of the book. Planning has proceeded since then and by the time of the 1998 meeting we hope that a first framework will be ready for discussion.

Part of our Working Group meetings will also be devoted to ensuring that all of us have an opportunity to talk informally about our work. People who wish to be involved in this Working Group are invited, if they wish, to make a 10–15 minute presentation on their interests which might be supported by two or three overhead transparencies and perhaps some handouts of work which they think will be of interest for others.

Part of one session will also be devoted to the making of plans for developing electronic ways of developing data bases which will provide researchers into stochastics understanding to have efficient access to authoritative previous work.

John Truran, University of Adelaide
Kathleen Truran, University of South Australia
Carmen Batanero, University of Granada

UNDERSTANDING OF MULTIPLICATIVE CONCEPTS

Convenors: Tom Cooper, Queensland University of Technology, Australia
Tad Watanabe, Towson State University, U.S.A.

Mathematical concepts tied to "multiplicative conceptual fields" include multiplication and division, ratio, rate, fraction and rational number, linear functions, and linear mappings (Vergnaud, 1988). These concepts play a central role in school mathematics and their developments are closely connected (Harel & Confrey, 1994). This working group continues from 1997, follows up activities set from that year, and brings together researchers who have studied children's understandings of these concepts to share their findings and inform each other's future investigations. It is hoped to expand the framework of research suggested at the 1997 meeting.

This year, the intention is for the working group to have three foci. The first is to look at theoretical perspectives which connect and provide a common or comprehensive framework for the various multiplicative concepts (e.g., extending Greer, 1992) and which relate to learning metaschemes such as abstract schema (Ohlsson, 1993). The second is to explore findings from a common study of children's understanding of the multiplication operation from across the world. The third is to share research findings in relation to multiplicativity projects. It is hoped that some of these will relate to topics which do not appear directly related to multiplicativity (e.g., the numeration system) and others to instructional programs that offer potential for effective learning in the conceptual field.

Therefore, the working group will offer participants the opportunity to present the findings of their research, to discuss the findings of other participants and to form groups for collaborative research. It is hoped to do this in conjunction with the website set up for this working group (www.fed.qut.edu.au/projects/mult_pme). It is hoped that findings from the common research project can be placed on the site before the conference to be read by participants, that the site will provide a forum for participants' research after the conference, and that papers will be collected into a publication.

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PME 22 Working Group:

Embodiment, Enactivism, and Mathematics Education

Coordinators: Rafael Núñez, University of California at Berkeley, USA
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Enactivism is a newly emerging perspective on cognition, evolution and human experience that draws upon both phenomenology and pragmatism. It can be characterized in a general sense as a conceptual framework for investigating the interrelations between our embodied experience of being in the world and our understanding of that world. Research conducted from an enactivist perspective in mathematics education has investigated the systems of conceptual mappings that may underlie specific mathematical knowledge, as well as the emergence of mathematical cognition within individuals, groups and human culture.

Due to an increasing level of interest and research activity in the application of theories of embodiment and enactivism to mathematical thinking, learning and teaching, the main objectives of this working group are to help to discuss, integrate, support, and disseminate efforts in this area. Overall, the purpose of the working group will be to help clarify and elaborate some theoretical, methodological and educational issues of enactivism in the context of research in mathematics education.

The working group will focus on a number of questions such as:

- What are some of the theoretical and methodological implications for enactivist research in mathematics education?
- In what ways is enactivism similar or different from constructivism?
- What would/should an enactivist account of such things as objectivity, identity, difference, and agency look like?
- How is mathematical cognition embodied and grounded within a shared biological and physical context?
- What are some of the ways in which an embodied view of human cognition can help to determine the nature of mathematical thinking and understanding?
- What are some of the implications of enactivism for learning and teaching mathematics in both individual and group settings?

DISCUSSION GROUPS

DG1: *Learning and teaching number theory*

DG2: *Under-represented countries in PME: National mathematics education research communities and priorities*

DG3: *Frameworks for research on open-ended mathematical tasks*

DG4: *Exploring different ways of working with videotape in research and inservice work*

Discussion Group: Learning and Teaching Number Theory

Stephen (Sen) Campbell, University of California, Irvine
Rina Zazkis, Simon Fraser University

The Number Theory discussion group convened for the first time at PME 21 in Lahti, Finland. A central mandate and guiding motivation for this group is to explore the extent to which elementary number theory can serve as a gateway leading from concrete arithmetic understanding to more abstract levels of algebraic understanding. Last year's discussion covered a broad spectrum of issues including:

- the nature of number theory, its influence on the historical development of mathematics, and its philosophical implications regarding the nature of mathematics more generally
- the role of number theory in the K-16 curriculum
- difficulties teachers and students alike may encounter in understanding introductory concepts of elementary number theory, such as factors, divisors, and prime decomposition
- the utility of number theory as a vehicle for teaching and learning mathematical concepts such as uniqueness, variables, and proof
- theories and methods for conducting research into these areas

Last year's participants enthusiastically recognised the need to encourage, conduct, and disseminate research into the learning and teaching of elementary number theory. Since that time, many participants have remained in contact via email and a website for the group has been established (http://www.gse.uci.edu/pme_ntdg/). Continued interest in this area on the part of last year's participants has helped to motivate this proposal for a continuation of the discussion group at PME 22.

This year, we propose to discuss in more detail some of the issues initially raised at PME 21, especially with respect to the utility of number theory in learning and teaching of more general abstract mathematical notions such as uniqueness, variables, and proof. Some of the participants from last year's discussion group have offered to give short 5-15 minute presentations of work that they have been conducting in this area. These presentations will serve as a focus for discussion and as an opportunity for feedback from other participants. Another objective this year is to identify and pursue specific items for further research and potential collaboration in the areas of learning and teaching introductory number theory.

Under-represented Countries in PME: National Mathematics Education Research Communities and Priorities

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Renuka Vithal	*University of Durban - Westville*	*South Africa*
Pedro Gómez	*"una empresa docente"*	*Colombia*

Last year, the discussion group explored the lack of solid national research communities in mathematics education as one of the reasons for under-representation of many countries in PME. The discussion around this postulate lead the group to conclude that, in fact, factors such as the little emphasis given to research to support current curricular changes, the scarce resources and little investment of them in research projects in the area, the weak communication infrastructures both intranationally and internationally, among others, are connected to the fact that in many cases there are few research studies produced, and that they can not easily have a place in PME. The connection between national research communities and participation –understood not only as "presentations" in the conferences– in the different activities of PME as an international study group are still worth exploring.

One of the questionings that emerged in the group was whether PME constitutes an attractive scenario of international academic debate in mathematics education, for many practitioners and researchers (see PME News, November 1997, p.14). Coming back to the reasoning about national research communities, it may be possible that the relevant areas, interests, concerns and research methodologies in many under-represented countries diverge from those of the group, so that under-represented researchers do not consider PME an adequate forum for their work.

The discussion during this conference aims at exploring the reasons why, within given national research communities, there is under-representation in PME. Then, the sessions will be devoted to deliberate on the following general issues:

- ▲ Which the current state of mathematics research communities in both represented and under-represented countries is, in respect of: 1. The existence of cultural, social, economic and political climate that supports the development of mathematics education, as both research and practice; and 2. The existence and social recognition of mathematics education as an academic field.
- ▲ Which the relevant foci for research in mathematics education are in those national communities and contexts.
- ▲ How the interests of research and academic debate of PME meet the needs and interests of different types of national research communities.

Special attention will be payed to the African context and situation, given the fact that this PME Conference is precisely an attempt to discuss "Diversity and change", considering mathematics education in the African Continent.

FRAMEWORK FOR RESEARCH ON OPEN-ENDED MATHEMATICAL TASKS

Coordinators:

Peter Sullivan
Australian Catholic University

Shukkwan Susan Leung
National Chiayi Teachers College
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There have been discussions at PME and elsewhere for some time on the use of open-ended tasks for stimulating mathematical investigations for students, and for engaging and assessing students in processes of mathematical problem solving. At PME 21, the Discussion Group publication, *Using open-ended problems in mathematics classroom,* was distributed. The lead article, by Erkki Pehkonen, proposed a classification structure arranged by starting and goal situations. There was some discussion of the application of this categorisation.

This discussion group will consider this classification further, and will discuss whether a framework can be development to guide research on open-ended problems. The framework will need to accommodate such issues as topic, level, contextualisation, assessment requirements, mode of posing format for responses, as well as cognitive dimension which acknowledges the different type of thinking required by open-ended problems.

The group will also examine what type of thinking is accessed by different kinds of open-ended tasks and questions and to what extent open-ended questions provide a way of accessing different styles of thinking than are normally available through conventional closed assessment items.

This discussion group will provide opportunities for sharing perspectives on the use of open-ended tasks, particularly for those participants who have been using open-ended problems in their research and/or practice.

Exploring Different Ways of Working with Videotape in Research and Inservice Work.

Susan Pirie, University of British Columbia, Canada
Chris Breen, University of Cape Town, South Africa

> *"Life here is like a movie. You wouldn't believe it and you won't be able to portray it either." (Tierney, 1997)*[1]

In the first session Susan Pirie will provide the stimulus for discussion by looking at how video data can be re-presented when endeavouring to communicate our research to others. How can we deal with the impoverishment of our research stories that occurs as we move into written reporting? How can we better allow the teachers and students that we study to speak for themselves? How do we integrate their perspectives with our own, in ways that are faithful to the data? A number of members of the mathematics education community, use portions of their video-data, in the form of short clips, to illustrate or enliven oral presentations. On the whole the use of such clips has been to serve exactly the same purpose as the verbal quotations that are taken from transcripts and added to written research reports. Yet this is to deny the richness and power of the original video data. In this technologically advancing world it behoves us to reconsider the ways in which we seek to present our research to others. In what ways, for example, can we exploit the opportunities offered by the Internet? For those of us working in the broad area of theory-building, there is a further question: how can we involve our audiences in the on-going analysis of our data? Susan will bring some of her video data for the group members to work with as we address these and many other questions.

In the second session, Chris Breen will frame the discussion around the work on canonical and universal images he presented at PME in Finland[2]. This paper contrasted a narrow and focused gaze used to explore mathematical films with an open-ended approach used in offering didactical images to a preservice mathematics method class. The group will be asked to apply the former focused discipline to a videotape sequence to create a common account which avoids any sense that a past event is being recreated.

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