F-THRESHOLDS, TIGHT CLOSURE, INTEGRAL CLOSURE, AND MULTIPLICITY BOUNDS

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Abstract. The F-threshold $c^J(\mathfrak{a})$ of an ideal $\mathfrak{a}$ with respect to the ideal J is a positive characteristic invariant obtained by comparing the powers of $\mathfrak{a}$ with the Frobenius powers of J. We show that under mild assumptions, we can detect the containment in the integral closure or the tight closure of a parameter ideal using F-thresholds. We formulate a conjecture bounding $c^J(\mathfrak{a})$ in terms of the multiplicities $e(\mathfrak{a})$ and $e(J)$, when $\mathfrak{a}$ and J are zero-dimensional ideals, and J is generated by a system of parameters. We prove the conjecture when J is a monomial ideal in a polynomial ring, and also when $\mathfrak{a}$ and J are generated by homogeneous systems of parameters in a Cohen-Macaulay graded k-algebra.

Introduction

Let R be a Noetherian ring of positive characteristic p. For every ideal $\mathfrak{a}$ in R, and for every ideal J whose radical contains $\mathfrak{a}$, one can define asymptotic invariants that measure the containment of the powers of $\mathfrak{a}$ in the Frobenius powers of J. These invariants were introduced in the case of a regular local F-finite ring in [MTW], where it was shown that they coincide with the jumping exponents for the generalized test ideals of Hara and Yoshida [HY]. In this paper we work in a general setting, and show that the F-thresholds still capture interesting and subtle information. In particular, we relate them to tight closure and integral closure, and to multiplicities.

If $\mathfrak{a}$ and J are as above, we define for every positive integer e

$$\nu_{\mathfrak{a}}^J(p^e) := \max\{r \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\},$$

where $J^{[q]}$ is the ideal generated by the p^e-powers of the elements of J. We put

$$c_+^J(\mathfrak{a}) := \limsup_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e},\ \ c_-^J(\mathfrak{a}) := \liminf_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e},$$

and if these two limits coincide, we denote their common value by $c^J(\mathfrak{a})$, and call it the *F-threshold of* $\mathfrak{a}$ *with respect to* J.

Our first application of this notion is to the description of the tight closure and of the integral closure for parameter ideals. Suppose that $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring of positive characteristic, and that J is an ideal in R generated by a full system of parameters. We show that under mild conditions, for every ideal $I \supseteq J$, we have $I \subseteq J^*$ if and only if $c_+^I(J) = d$ (and in this case $c_-^I(J) = d$, too). We similarly show that under suitable mild hypotheses, if $I \supseteq J$, then $I \subseteq \overline{J}$ if and

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only if $\mathrm{c}_+^J(I) = d$. For the precise statements, see Corollary 3.2 and Theorem 3.3 below.

As we have mentioned, if R is regular and F-finite, then it was shown in [MTW] that the F-thresholds of an ideal $\mathfrak{a}$ coincide with the jumping exponents for the generalized test ideals of [HY]. In order to recover such a result in a more general setting, we develop a notion of F-threshold for the ideal $\mathfrak{a}$ corresponding to a submodule N of a module M, such that $\mathfrak{a}^n N = 0$ for some n. We then show that under suitable hypotheses on a local ring R, one can again recover the jumping exponents for the generalized test ideals of an ideal $\mathfrak{a}$ in R from the F-thresholds of $\mathfrak{a}$ with respect to pairs (E, N), where N is a submodule of the injective hull E of the residue field (see Corollary 4.4).

We study the connection between F-thresholds and multiplicity, and formulate the following conjecture: if $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring of characteristic $p > 0$, $\mathfrak{a}$ and J are $\mathfrak{m}$-primary ideals in R, with J generated by a system of parameters, then

$$e(\mathfrak{a}) \geq \frac{d^d}{\mathrm{c}_-^J(\mathfrak{a})^d} e(J).$$

The case $J = \mathfrak{m}$ (when R is in fact regular) was proved in [TW]. We mention that in this case $\mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$ is related via reduction mod p to a fundamental invariant in birational geometry, the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ (see *loc. cit.* for the precise relation between these two invariants). The corresponding inequality between the multiplicity and the log canonical threshold of $\mathfrak{a}$ was proved in [dFEM], and plays a key role in proving that for small values of n, no smooth hypersurface of degree n in $\mathbb{P}^n$ is rational (see [Cor] and [?]).

We prove our conjecture when both $\mathfrak{a}$ and J are generated by homogeneous systems of parameters in a graded Cohen-Macaulay k-algebra (cf. Corollary 5.9). Moreover, we prove it also when R is regular and $J = (x_1^{a_1}, \ldots, x_n^{a_n})$, for a regular system of parameters $x_1, \ldots, x_n$. The proof of this latter case follows the ideas in [TW] and [dFEM], reducing to the case of a monomial ideal $\mathfrak{a}$, and then using the explicit interpretation of the invariants involved in terms of the Newton polyhedron of $\mathfrak{a}$.

On the other hand, the proof of the homogeneous case is based on new ideas that we expect to be useful also in attacking the general case of the conjecture. In fact, we prove the following stronger statement. Suppose that $\mathfrak{a}$ and J are ideals generated by homogeneous systems of parameters in a d-dimensional graded Cohen-Macaulay k-algebra, where k is a field of *arbitrary* characteristic. If $\mathfrak{a}^N \subseteq J$ for some N, then

$$e(\mathfrak{a}) \geq \left(\frac{d}{d+N-1}\right)^d e(J).$$

The paper is structured as follows. In the first section we recall some basic notions of tight closure theory, and review the definition of generalized test ideals from [HY]. In §2 we introduce the F-thresholds and discuss some basic properties. The third section is devoted to the connections with tight closure and integral closure. We introduce the F-thresholds with respect to pairs of modules in §4, and relate them to the jumping exponents for the generalized test ideals. In the last section we discuss inequalities involving F-thresholds and multiplicities. In particular, we state here our conjecture and prove the above-mentioned special cases.

1. Preliminaries

In this section we review some definitions and notation that will be used throughout the paper. All rings are Noetherian commutative rings with unity. For a ring R, we denote by R° the set of elements of R that are not contained in any minimal prime ideal. Elements $x_1, \ldots, x_r$ in R are called *parameters* if they generate an ideal of height r. The integral closure of an ideal $\mathfrak{a}$ is denoted by $\overline{\mathfrak{a}}$. The order of a nonzero element f in a Noetherian local ring $(R, \mathfrak{m})$ is the largest r such that $f \in \mathfrak{m}^r$. For a real number u, we denote by $\lfloor u \rfloor$ the largest integer $\leq u$, and by $\lceil u \rceil$ the smallest integer $\geq u$.

Let R be a ring of characteristic $p > 0$, and let $F\colon R \to R$ denote the Frobenius map which sends $x \in R$ to $x^p \in R$. The ring R viewed as an R-module via the e-times iterated Frobenius map $F^e\colon R \to R$ is denoted by ${}^eR$. We say that R is *F-finite* if ${}^1R$ is a finitely generated R-module. We also say that R is *F-pure* if the Frobenius map is pure, that is, $F_M = 1_M \otimes F\colon M = M \otimes_R R \to M \otimes_R {}^1R$ is injective for any R-module M. For every ideal I in R, and for every $q = p^e$, we denote by $I^{[q]}$ the ideal generated by the q^{th} powers of all elements of I.

If M is an R-module, then we put $\mathbb{F}^e(M) := {}^eR \otimes_R M$. Hence in $\mathbb{F}^e(M)$ we have $u \otimes (ay) = ua^{p^e} \otimes y$ for every $a \in R$. Note that the e-times iterated Frobenius map $F^e_M\colon M \to \mathbb{F}^e(M)$ is an R-linear map. The image of $z \in M$ via this map is denoted by $z^q := F^e_M(z)$. If N is a submodule of M, then we denote by $N^{[q]}_M$ (or simply by $N^{[q]}$) the image of the canonical map $\mathbb{F}^e(N) \to \mathbb{F}^e(M)$ (note that if $N = I$ is a submodule of $M = R$, then this is consistent with our previous notation for $I^{[q]}$).

First, we recall the definitions of classical tight closure and related notions. Our references for classical tight closure theory and for F-rational rings are [HH] and[FW], respectively; see also the book [Hu].

Definition 1.1. Let I be an ideal in a ring R of characteristic $p > 0$.

(i) The *Frobenius closure* I^F of I is defined as the ideal of R consisting of all elements $x \in R$ such that $x^q \in I^{[q]}$ for some $q = p^e$. If R is F-pure, then $J = J^F$ for all ideals $J \subseteq R$. The *tight closure* I^* of I is defined to be the ideal of R consisting of all elements $x \in R$ for which there exists $c \in R^{\circ}$ such that $cx^q \in I^{[q]}$ for all large $q = p^e$.

(ii) We say that $c \in R^{\circ}$ is a *test element* if for all ideals $J \subseteq R$ and all $x \in J^*$, we have $cx^q \in I^{[q]}$ for all $q = p^e \geq 1$. Every excellent and reduced ring R has a test element.

(iii) If $N \subseteq M$ are R-modules, then the *tight closure* N_M^* of N in M is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists $c \in R^\circ$ such that $cz^q \in N_M^{[q]}$ for all large $q = p^e$. The *test ideal* $\tau(R)$ of R is defined to be $\tau(R) = \bigcap_M \mathrm{Ann}_R(0_M^*)$, where M runs over all finitely generated R-modules. If $M = R/I$, then $\mathrm{Ann}_R(0_M^*) = (I : I^*)$. That is, $\tau(R)J^* \subseteq J$ for all ideals $J \subseteq R$. We say that R is *F-regular* if $\tau(R_P) = R_P$ for all prime ideals P of R.

(iv) R is called *F-rational* if $J^* = J$ for every ideal $J \subseteq R$ generated by parameters. If R is an excellent equidimensional local ring, then R is F-rational if and only if $I = I^*$ for *some* ideal I generated by a full system of parameters for R.

We now recall the definition of $\mathfrak{a}^t$-tight closure and of the generalized test ideal $\tau(\mathfrak{a}^t)$. The reader is referred to [HY] for details.

Definition 1.2. Let $\mathfrak{a}$ be a fixed ideal in a reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let I be an arbitrary ideal in R.

(i) Let $N \subseteq M$ be R-modules. Given a rational number $t \geq 0$, the *$\mathfrak{a}^t$-tight closure* $N_M^{*\mathfrak{a}^t}$ of N in M is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists $c \in R^\circ$ such that $cz^q\mathfrak{a}^{\lceil tq \rceil} \subseteq N_M^{[q]}$ for all large $q = p^e$.

(ii) The *generalized test ideal* $\tau(\mathfrak{a}^t)$ is defined to be $\tau(\mathfrak{a}^t) = \bigcap_M \mathrm{Ann}_R(0_M^{*\mathfrak{a}^t})$, where M runs through all finitely generated R-modules. If $\mathfrak{a} = R$, then the generalized test ideal $\tau(\mathfrak{a}^t)$ is nothing but the test ideal $\tau(R)$.

(iii) Assume that R is an F-regular ring and that J is an ideal containing $\mathfrak{a}$ in its radical. The *F-jumping exponent* of $\mathfrak{a}$ with respect to J is defined by

$$\xi^J(\mathfrak{a}) = \sup\{c \in \mathbb{R}_{\geq 0} \mid \tau(\mathfrak{a}^c) \not\subseteq J\}.$$

If $(R, \mathfrak{m})$ is local, then we call the smallest F-jumping exponent $\xi^{\mathfrak{m}}(\mathfrak{a})$ the *F-pure threshold* of $\mathfrak{a}$ and denote it by $\mathrm{fpt}(\mathfrak{a})$.

In characteristic zero, one defines multiplier ideals and their jumping exponents using resolution of singularities (see Ch. 9 in [La]). It is known that for a given ideal in characteristic zero and for a given t, the reduction mod $p \gg 0$ of the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ coincides with the generalized test ideal $\tau(\mathfrak{a}_p^t)$ of the reduction $\mathfrak{a}_p$ of $\mathfrak{a}$. Therefore the F-jumping exponent $\xi^J(\mathfrak{a})$ is a characteristic p analogue of jumping exponent of multiplier ideals. We refer to [BMS2], [HM], [HY], [MTW] and [TW] for further discussions.

2. Basic properties of F-thresholds

The F-thresholds are invariants of singularities of a given ideal $\mathfrak{a}$ in positive characteristic, obtained by comparing the powers of $\mathfrak{a}$ with the Frobenius powers of other ideals. They were introduced and studied in [MTW] in the case when we work in a regular ring. In this section, we recall the definition of F-thresholds and study their basic properties when the ring is not necessarily regular.

Let R be a Noetherian ring of dimension d and of characteristic $p > 0$. Let $\mathfrak{a}$ be a fixed proper ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. To each ideal J of R such that $\mathfrak{a} \subseteq \sqrt{J}$, we associate an F-threshold as follows. For every $q = p^e$, let

$$\nu_{\mathfrak{a}}^J(q) := \max\{r \in \mathbb{N} | \mathfrak{a}^r \not\subseteq J^{[q]}\}.$$

Since $\mathfrak{a} \subseteq \sqrt{J}$, this is a nonnegative integer (if $\mathfrak{a} \subseteq J^{[q]}$, then we put $\nu_{\mathfrak{a}}^J(q) = 0$). We put

$$\mathrm{c}_+^J(\mathfrak{a}) = \limsup_{q\to\infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}, \quad \mathrm{c}_-^J(\mathfrak{a}) = \liminf_{q\to\infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}.$$

When $\mathrm{c}_+^J(\mathfrak{a}) = \mathrm{c}_-^J(\mathfrak{a})$, we call this limit the *F-threshold* of the pair $(R, \mathfrak{a})$ (or simply of $\mathfrak{a}$) with respect to J, and we denote it by $\mathrm{c}^J(\mathfrak{a})$.

Remark 2.1. (1) (cf. [MTW, Remark 1.2]) One has

$$0 \le \mathrm{c}_-^J(\mathfrak{a}) \le \mathrm{c}_+^J(\mathfrak{a}) < \infty.$$

In fact, if $\mathfrak{a}$ is generated by l elements and if $\mathfrak{a}^N \subseteq J$, then

$$\mathfrak{a}^{N(l(p^e-1)+1)} \subseteq (\mathfrak{a}^{[p^e]})^N = (\mathfrak{a}^N)^{[p^e]} \subseteq J^{[p^e]}.$$

Therefore $\nu_{\mathfrak{a}}^J(p^e) \le N(l(p^e-1)+1) - 1$. Dividing by p^e and taking the limit gives $\mathrm{c}_+^J(\mathfrak{a}) \le Nl$.

(2) Question 1.4 in [MTW] asked whether the F-threshold $\mathrm{c}^J(\mathfrak{a})$ is a rational number (when it exists). A positive answer was given in [BMS2] and [BMS1] for a regular F-finite ring, essentially of finite type over a field, and for every regular F-finite ring, if the ideal $\mathfrak{a}$ is principal. For a proof in the case of a principal ideal in a complete regular ring (that is not necessarily F-finite), see [KLZ]. However, this question remains open in general.

Recall that a ring extension $R \hookrightarrow S$ is *cyclic pure* if for every ideal I in R, we have $IS \cap R = I$.

Proposition 2.2 (cf. [MTW, Proposition 1.7])**.** *Let $\mathfrak{a}$, J be ideals as above.*

(1) *If $I \supseteq J$, then $\mathrm{c}_\pm^I(\mathfrak{a}) \le \mathrm{c}_\pm^J(\mathfrak{a})$.*
(2) *If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\mathrm{c}_\pm^J(\mathfrak{b}) \le \mathrm{c}_\pm^J(\mathfrak{a})$. Moreover, if $\mathfrak{a} \subseteq \overline{\mathfrak{b}}$, then $\mathrm{c}_\pm^J(\mathfrak{b}) = \mathrm{c}_\pm^J(\mathfrak{a})$.*
(3) *$\mathrm{c}_\pm^J(\mathfrak{a}^r) = \frac{1}{r}\,\mathrm{c}_\pm^J(\mathfrak{a})$ for every integer $r \ge 1$.*
(4) *$\mathrm{c}_\pm^{J^{[q]}}(\mathfrak{a}) = q\,\mathrm{c}_\pm^J(\mathfrak{a})$ for every $q = p^e$.*

(5) *If $R \hookrightarrow S$ is a cyclic pure extension, then*

$$\mathrm{c}^J_{\pm}(\mathfrak{a}) = \mathrm{c}^{JS}_{\pm}(\mathfrak{a}S).$$

(6) *Let $R \hookrightarrow S$ be an integral extension. If the conductor ideal $\mathfrak{c}(S/R) := \mathrm{Ann}_R(S/R)$ contains the ideal $\mathfrak{a}$ in its radical, then*

$$\mathrm{c}^J_{\pm}(\mathfrak{a}) = \mathrm{c}^{JS}_{\pm}(\mathfrak{a}S).$$

(7) $\mathrm{c}^J_+(\mathfrak{a}) \le c$ *(resp.* $\mathrm{c}^J_-(\mathfrak{a}) \ge c$*) if and only if for every power q_0 of p, we have $\mathfrak{a}^{\lceil cq \rceil + q/q_0} \subseteq J^{[q]}$ (resp. $\mathfrak{a}^{\lceil cq \rceil - q/q_0} \not\subseteq J^{[q]}$) for all $q = p^e \gg q_0$.*

Proof. For (1)–(4), see [MTW] (the proofs therein do not use the fact that R is regular). If $R \hookrightarrow S$ is cyclic pure, then $\nu^{JS}_{\mathfrak{a}S}(q) = \nu^J_{\mathfrak{a}}(q)$ for every q, and we get (5).

For (6), we fix a positive integer m such that $\mathfrak{a}^m \subseteq \mathfrak{c}(S/R)$. By the definition of the conductor ideal $\mathfrak{c}(S/R)$, if $(\mathfrak{a}S)^n \subseteq (JS)^{[q]}$ for some $n \in \mathbb{N}$ and some $q = p^e$, then $\mathfrak{a}^{m+n} \subseteq J^{[q]}$. This implies that

$$\nu^{JS}_{\mathfrak{a}S}(q) \le \nu^J_{\mathfrak{a}}(q) \le \nu^{JS}_{\mathfrak{a}S}(q) + m.$$

These inequalities imply (6).

In order to prove (7), suppose first that $\mathrm{c}^J_+(\mathfrak{a}) \le c$. It follows from the definition of $\mathrm{c}^J_+(\mathfrak{a})$ that for every power q_0 of p, we can find q_1 such that $\nu^J_{\mathfrak{a}}(q)/q < c + \frac{1}{q_0}$ for all $q = p^e \ge q_1$. Thus, $\nu^J_{\mathfrak{a}}(q) < \lceil cq \rceil + \frac{q}{q_0}$, that is,

$$\mathfrak{a}^{\lceil cq \rceil + q/q_0} \subseteq J^{[q]} \tag{1}$$

for all $q = p^e \ge q_1$. Conversely, suppose that (1) holds for every $q \ge q_1$. This implies $\nu^J_{\mathfrak{a}}(q) \le \lceil cq \rceil + \frac{q}{q_0} - 1$. Dividing by q and taking the limit gives $\mathrm{c}^J_+(\mathfrak{a}) \le c + \frac{1}{q_0}$. If this holds for every q_0, we conclude that $\mathrm{c}^J_+(\mathfrak{a}) \le c$. The assertion regarding $\mathrm{c}^J_-(\mathfrak{a})$ follows from a similar argument. □

We now give a variant of the definition of F-threshold. If $\mathfrak{a}$ and J are ideals in R, such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and $\mathfrak{a} \subseteq \sqrt{J}$, then we put

$$\widetilde{\nu}^J_{\mathfrak{a}}(q) := \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq (J^{[q]})^F\}.$$

It follows from the definition of Frobenius closure that if $u \notin (J^{[q]})^F$, then $u^p \notin (J^{[pq]})^F$. This means that

$$\frac{\widetilde{\nu}^J_{\mathfrak{a}}(pq)}{pq} \ge \frac{\widetilde{\nu}^J_{\mathfrak{a}}(q)}{q}$$

for all $q = p^e$. Thus,

$$\lim_{q \to \infty} \frac{\widetilde{\nu}^J_{\mathfrak{a}}(q)}{q} = \sup_{q = p^e} \frac{\widetilde{\nu}^J_{\mathfrak{a}}(q)}{q}.$$

We denote this limit by $\widetilde{\mathrm{c}}^J(\mathfrak{a})$. Note that we have $\widetilde{\mathrm{c}}^J(\mathfrak{a}) \le \mathrm{c}^J_-(\mathfrak{a})$.

The F-threshold $\mathrm{c}^J(\mathfrak{a})$ exists in many cases.

Lemma 2.3. *Let $\mathfrak{a}$, J be as above.*

(1) *If* $J^{[q]} = (J^{[q]})^F$ *for all large* $q = p^e$, *then the F-threshold* $\mathrm{c}^J(\mathfrak{a})$ *exists, that is,* $\mathrm{c}^J_+(\mathfrak{a}) = \mathrm{c}^J_-(\mathfrak{a})$. *In particular, if* R *is F-pure, then* $\mathrm{c}^J(\mathfrak{a})$ *exists.*
(2) *If the test ideal* $\tau(R)$ *contains* $\mathfrak{a}$ *in its radical, then the F-threshold* $\mathrm{c}^J(\mathfrak{a})$ *exists and* $\mathrm{c}^J(\mathfrak{a}) = \mathrm{c}^{J^*}(\mathfrak{a})$.
(3) *If* $\mathfrak{a}$ *is principal, then* $c^J(\mathfrak{a})$ *exists.*

Proof. (1) follows from the previous discussion since in that case we have $\widetilde{\nu}^J_{\mathfrak{a}}(q) = \nu^J_{\mathfrak{a}}(q)$ for all $q \gg 0$.

In order to prove (2), we take an integer $m \geq 1$ such that $\mathfrak{a}^m \subseteq \tau(R)$. Then, by the definition of $\tau(R)$, one has $\mathfrak{a}^{2m}((J^*)^{[q]})^F \subseteq \mathfrak{a}^m(J^*)^{[q]} \subseteq J^{[q]}$ for all $q = p^e$. This means that

$$\widetilde{\nu}^{J^*}_{\mathfrak{a}}(q) \leq \nu^{J^*}_{\mathfrak{a}}(q) \leq \nu^J_{\mathfrak{a}}(q) \leq \widetilde{\nu}^{J^*}_{\mathfrak{a}}(q) + 2m.$$

Since $\widetilde{\mathrm{c}}^{J^*}(\mathfrak{a})$ always exists, $\mathrm{c}^J(\mathfrak{a})$ and $\mathrm{c}^{J^*}(\mathfrak{a})$ also exist and these three limits are all equal.

For (3), note that if $\mathfrak{a}$ is principal and $\mathfrak{a}^r \subseteq J^{[q]}$, then $a^{pr} \subseteq J^{[pq]}$. Therefore we have

$$\frac{\nu^J_{\mathfrak{a}}(pq) + 1}{pq} \leq \frac{\nu^J_{\mathfrak{a}}(q) + 1}{q}$$

for every $q = p^e$. This implies that

$$\lim_{q \to \infty} \frac{\nu^J_{\mathfrak{a}}(q)}{q} = \lim_{q \to \infty} \frac{\nu^J_{\mathfrak{a}}(q) + 1}{q} = \inf_{q = p^e} \frac{\nu^J_{\mathfrak{a}}(q)}{q}.$$

□

As shown in [MTW, Proposition 2.7], the F-threshold $\mathrm{c}^J(\mathfrak{a})$ coincides with the F-jumping exponent $\xi^J(\mathfrak{a})$ when the ring is F-finite and regular. The statement in *loc. cit.* requires the ring to be local, however the proof easily generalizes to the non-local case (see [BMS1]). More precisely, we have the following

Proposition 2.4. *Let* R *be an F-finite regular ring of characteristic* $p > 0$. *If* $\mathfrak{a}$ *is a nonzero ideal contained in the radical of* J, *then* $\tau(\mathfrak{a}^{\mathrm{c}^J(\mathfrak{a})}) \subseteq J$. *Going the other way, if* $\alpha \in \mathbb{R}_+$, *then* $\mathfrak{a}$ *is contained in the radical of* $\tau(\mathfrak{a}^\alpha)$ *and* $\mathrm{c}^{\tau(\mathfrak{a}^\alpha)}(\mathfrak{a}) \leq \alpha$. *In particular, the F-threshold* $\mathrm{c}^J(\mathfrak{a})$ *coincides with the F-jumping exponent* $\xi^J(\mathfrak{a})$.

Remark 2.5. The F-threshold $\mathrm{c}^J(\mathfrak{a})$ sometimes coincide with the F-jumping exponent $\xi^J(\mathfrak{a})$ even when R is singular. For example, let $R = k[[X, Y, Z, W]]/(XY - ZW)$, and let $\mathfrak{m}$ be the maximal ideal of R. Then the F-threshold $\mathrm{c}^{\mathfrak{m}}(\mathfrak{m})$ of $\mathfrak{m}$ with respect to $\mathfrak{m}$ and the F-pure threshold (that is, the smallest F-jumping exponent) $\mathrm{fpt}(\mathfrak{m})$ of $\mathfrak{m}$ are both equal to two.

However, $\mathrm{c}^J(\mathfrak{a})$ does not agree with $\xi^J(\mathfrak{a})$ in general. For example, let $R = k[[X, Y, Z]]/(XY - Z^2)$ be a rational double point of type A_1 over a field k of characteristic $p > 2$ and let $\mathfrak{m}$ be the maximal ideal of R. Then $\mathrm{fpt}(\mathfrak{m}) = 1$ (see [TW, Example 2.5]), whereas $\mathrm{c}^{\mathfrak{m}}(\mathfrak{m}) = 3/2$.

Remark 2.6. Suppose that $\mathfrak{m}$ is a maximal ideal in any Noetherian ring R, and that J is an $\mathfrak{m}$-primary ideal. For every $q = p^e$ we have $J^{[q]}R_{\mathfrak{m}} \cap R = J^{[q]}$, hence for every ideal $\mathfrak{a} \subseteq \mathfrak{m}$ we have $\nu_{\mathfrak{a}}^J(q) = \nu_{\mathfrak{a}R_{\mathfrak{m}}}^{JR_{\mathfrak{m}}}(q)$. In particular, $\mathrm{c}_{\pm}^J(\mathfrak{a}) = \mathrm{c}_{\pm}^{JR_{\mathfrak{m}}}(\mathfrak{a}R_{\mathfrak{m}})$.

Example 2.7. (i) Let R be a Noetherian local ring of characteristic $p > 0$, and let $J = (x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ form a full system of parameters in R. It follows from the Monomial Conjecture (which is a theorem in this setting, see [Ho, Prop. 3]) that $(x_1 \cdots x_d)^{q-1} \not\in J^{[q]}$ for every q. Hence $\nu_J^J(q) \geq d(q-1)$ for every q, and therefore $\mathrm{c}_{-}^J(J) \geq d$. On the other hand, $\mathrm{c}_{+}^J(J) \leq d$ by Remark 2.1 (1), and we conclude that $\mathrm{c}^J(J) = d$.

(ii) Let $R = k[x_1, \ldots, x_d]$ be a d-dimensional polynomial ring over a field k of characteristic $p > 0$, and let $\mathfrak{a}, J \subseteq R$ be zero-dimensional ideals generated by monomials. In order to compute $c^J(\mathfrak{a})$ we may assume that k is perfect, hence we may use Proposition 2.4.

Let $P(\mathfrak{a}) \subseteq \mathbb{R}^d_{\geq 0}$ denote the Newton polyhedron of $\mathfrak{a}$, that is $P(\mathfrak{a})$ is the convex hull of those $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$ such that $x^u = x_1^{u_1} \cdots x_n^{u_n} \in \mathfrak{a}$. It follows from [HY, Thm. 6.10] that

$$\tau(\mathfrak{a}^c) = (x^u \mid u + e \in \mathrm{Int}(c \cdot P_{\mathfrak{a}})),$$

where $e = (1, 1, \ldots, 1)$. We deduce that if $\lambda(u)$ is defined by the condition $u + e \in \partial(\lambda(u) \cdot P(\mathfrak{a}))$, then

$$\mathrm{c}^J(\mathfrak{a}) = \max\{\lambda(u) \mid u \in \mathbb{N}^n, x^u \not\in J\}$$

(note that since J is zero-dimensional, this maximum is over a finite set). In particular, we see that if $J = (x_1^{a_1}, \ldots, x_n^{a_n})$, then $\mathrm{c}^J(\mathfrak{a})$ is characterized by $a = (a_1, \ldots, a_n) \in \partial(c^J(\mathfrak{a}) \cdot P(\mathfrak{a}))$.

(iii) Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring of characteristic $p > 0$, and let $J \subset R$ be an $\mathfrak{m}$-primary ideal. We claim that

$$\mathrm{c}^J(\mathfrak{m}) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \mathfrak{m}^r \not\subseteq J\} + d. \tag{2}$$

In particular, $\mathrm{c}^J(\mathfrak{m})$ is an integer $\geq d$.

Indeed, if $u \not\in J$, then $(J\colon u) \subseteq \mathfrak{m}$, hence $J^{[q]}\colon u^q = (J\colon u)^{[q]} \subseteq \mathfrak{m}^{[q]}$, and therefore $u^q\mathfrak{m}^{d(q-1)} \not\subseteq J^{[q]}$. If $u \in \mathfrak{m}^r$, it follows that $\nu_{\mathfrak{m}}^J(q) \geq rq + d(q-1)$. Dividing by q and passing to the limit gives $c^J(\mathfrak{m}) \geq r + d$, hence we have "$\geq$" in (2). For the reverse inequality, note that if $\mathfrak{m}^{r+1} \subseteq J$, then

$$\mathfrak{m}^{(r+d)q} \subseteq (\mathfrak{m}^{r+1})^{[q]} \subseteq J^{[q]}$$

for every $q = p^e$. Hence $\nu_{\mathfrak{m}}^J(q) \leq (r+d)q-1$ for all q, and we get $c^J(\mathfrak{m}) \leq r+d$.

3. Connections with tight closure and integral closure

Theorem 3.1. *Let $(R, \mathfrak{m})$ be an excellent analytically irreducible Noetherian local domain of positive characteristic p. Set $d = \dim(R)$, and let $J = (x_1, \ldots, x_d)$ be* *an ideal generated by a full system of parameters in R, and let $I \supseteq J$ be another ideal. Then I is not contained in the tight closure J^* of J if and only if there exists $q_0 = p^{e_0}$ such that $x^{q_0-1} \in I^{[q_0]}$, where $x = x_1x_2\cdots x_d$.*

Proof. After passing to completion, we may assume that R is a complete local domain. Suppose first that $x^{q_0-1} \in I^{[q_0]}$, and by way of contradiction suppose also that $I \subseteq J^*$. Let $c \in R^\circ$ be a test element. Then for all $q = p^e$, one has $cx^{q(q_0-1)} \in cI^{[qq_0]} \subset J^{[qq_0]}$, so that $c \in J^{[qq_0]} : x^{q(q_0-1)} \subseteq (J^{[q]})^*$, by colon-capturing [HH, Theorem 7.15a]. Therefore c^2 lies in $\bigcap_{q=p^e} J^{[q]} = (0)$, a contradiction.

Conversely, suppose that $I \not\subseteq J^*$, and choose an element $f \in I \smallsetminus J^*$. We choose a coefficient field k, and let $B = k[[x_1, \ldots, x_d, f]]$ be the complete subring of R generated by $x_1, \ldots, x_d, f$. Note that B is a hypersurface singularity, hence Gorenstein. Furthermore, by persistence of tight closure [HH, Lemma 4.11a], $f \notin ((x_1, \ldots, x_d)B)^*$. If we prove that there exists $q_0 = p^{e_0}$ such that $x^{q_0-1} \in ((x_1, \ldots, x_d, f)B)^{[q_0]}$, then clearly x^{q_0-1} is also in $I^{[q_0]}$. Hence we can reduce to the case in which R is Gorenstein. Since $I \not\subseteq J^*$, it follows from a result of Aberbach [Ab] that $J^{[q]} \colon I^{[q]} \subseteq \mathfrak{m}^{n(q)}$, where $n(q)$ is a positive integer with $\lim_{q\to\infty} n(q) = \infty$. In particular, we can find $q_0 = p^{e_0}$ such that $J^{[q_0]} : I^{[q_0]} \subseteq J$. Therefore $x^{q_0-1} \in J^{[q_0]} \colon J \subseteq J^{[q_0]} \colon (J^{[q_0]} \colon I^{[q_0]}) = I^{[q_0]}$, where the last equality follows from the fact that R is Gorenstein. □

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a d-dimensional excellent analytically irreducible Noetherian local domain of characteristic $p > 0$, and let $J = (x_1, \ldots, x_d)$ be an ideal generated by a full system of parameters in R. Given an ideal $I \supseteq J$, we have $I \subseteq J^*$ if and only if $\mathrm{c}_+^I(J) = d$ (and in this case $\mathrm{c}^I(J)$ exists). In particular, R is F-rational if and only if $\mathrm{c}_+^I(J) < d$ for every ideal $I \supsetneq J$.*

Proof. Note first that by Remark 2.1 (1), for every $I \supseteq J$ we have $\mathrm{c}_+^J(I) \leq d$. Suppose now that $I \subseteq J^*$. It follows from Theorem 3.1 that $J^{d(q-1)} \not\subseteq I^{[q]}$ for every $q = p^e$. This gives $\nu_J^I(q) \geq d(q-1)$ for all q, and therefore $\mathrm{c}_-^I(J) \geq d$. We conclude that in this case $\mathrm{c}_+^I(J) = \mathrm{c}_-^I(J) = d$.

Conversely, suppose that $I \not\subseteq J^*$. By Theorem 3.1, we can find $q_0 = p^{e_0}$ such that

$$\mathfrak{b} := (x_1^{q_0}, \ldots, x_d^{q_0}, (x_1\cdots x_d)^{q_0-1}) \subseteq I^{[q_0]}.$$

If $(x_1, \ldots, x_d)^r \not\subseteq \mathfrak{b}^{[q]}$, then

$$r \leq (qq_0 - 1)(d-1) + q(q_0-1) - 1 = qq_0d - q - d.$$

Therefore $\nu_J^{\mathfrak{b}}(q) \leq qq_0d - q - d$ for every q, which implies $\mathrm{c}^{\mathfrak{b}}(J) \leq q_0d - 1$. Since q_0 is a fixed power of p, we deduce

$$\mathrm{c}_+^I(J) = \frac{1}{q_0}\,\mathrm{c}_+^{I^{[q_0]}}(J) \leq \frac{1}{q_0}\,\mathrm{c}^{\mathfrak{b}}(J) \leq d - \frac{1}{q_0} < d.$$

□

Theorem 3.3. *Let $(R,\mathfrak{m})$ be a d-dimensional formally equidimensional Noetherian local ring of characteristic $p>0$. If I and J are ideals in R, with J generated by a full system of parameters, then*

(1) $\mathrm{c}_+^J(I) \le d$ *if and only if* $I \subseteq \overline{J}$.
(2) *If, in addition,* $J \subseteq I$, *then* $I \subseteq \overline{J}$ *if and only if* $\mathrm{c}_+^J(I) = d$. *Moreover, if these equivalent conditions hold, then* $\mathrm{c}^J(I) = d$.

Proof. Note that if $J \subseteq I$, then $\mathrm{c}_-^J(I) \ge \mathrm{c}_-^J(J) = \mathrm{c}^J(J) = d$, by Example 2.7 (i). Hence both assertions in (2) follow from the assertion in (1).

One implication in (1) is easy: if $I \subseteq \overline{J}$, then by Proposition 2.2 (2) we have $\mathrm{c}_+^J(I) \le \mathrm{c}_+^J(\overline{J}) = \mathrm{c}^J(J) = d$. Conversely, suppose that $\mathrm{c}_+^J(I) \le d$. In order to show that $I \subseteq \overline{J}$, we may assume that R is complete and reduced. Indeed, first note that the inverse image of $\overline{J\widehat{R}_{\mathrm{red}}}$ in R is contained in $\overline{J}$, hence it is enough to show that $I\widehat{R}_{\mathrm{red}} \subseteq \overline{J\widehat{R}_{\mathrm{red}}}$. Since $J\widehat{R}_{\mathrm{red}}$ is again generated by a full system of parameters, and since we trivially have

$$\mathrm{c}^{J\widehat{R}_{\mathrm{red}}}(I\widehat{R}_{\mathrm{red}}) \le \mathrm{c}^J(I) \le d,$$

we may replace R by $\widehat{R}_{\mathrm{red}}$.

Since R is complete and reduced, we can find a test element c for R. By Proposition 2.2 (7), the assumption $\mathrm{c}_+^J(I) \le d$ implies that for all $q_0 = p^{e_0}$ and for all large $q = p^e$, we have

$$I^{q(d+(1/q_0))} \subseteq J^{[q]}.$$

Hence $I^q J^{q(d-1+(1/q_0))} \subseteq J^{[q]}$, and thus

$$I^q \subseteq J^{[q]} \colon J^{q(d-1+(1/q_0))} \subseteq \left(J^{q-d+1-(q/q_0)}\right)^*,$$

where the last containment follows from the colon-capturing property of tight closure [HH, Theorem 7.15a]. We get $cI^q \subseteq cR \cap J^{q-d+1-(q/q_0)} \subseteq cJ^{q-d+1-(q/q_0)-l}$ for some fixed integer l that is independent of q, by the Artin-Rees lemma. Since c is a non-zero divisor in R, it follows that

$$I^q \subseteq J^{q-d+1-(q/q_0)-l}. \tag{3}$$

If ν is a discrete valuation with center in $\mathfrak{m}$, we may apply ν to (3) to deduce $q\nu(I) \ge \left(q-d+1-\frac{q}{q_0}-l\right)\nu(J)$. Dividing by q and letting q go to infinity gives $\nu(I) \ge \left(1-\frac{1}{q_0}\right)\nu(J)$. We now let q_0 go to infinity to obtain $\nu(I) \ge \nu(J)$. Since this holds for every ν, we have $I \subseteq \overline{J}$. □

Example 3.4. Let $(R,\mathfrak{m})$ be a regular local ring of characteristic $p>0$ with $\dim(R) = d$, and J be an ideal of R generated by a full system of parameters. We define a to be the maximal integer n such that $\mathfrak{m}^n \not\subseteq J$. Then $\mathfrak{m}^s \subseteq \overline{J}$ if and only if $s \ge \frac{a}{d}+1$ since $\mathrm{c}^J(\mathfrak{m}^s) = \frac{a+d}{s}$ by Example 2.7 (iii) and Proposition 2.2 (3).

Question 3.5. Does this statement hold in a more general setting ? Can we replace "regular" by "Cohen-Macaulay" ?

4. F-thresholds of modules

In the section we give a generalization of the notion of F-thresholds, in which we replace the auxiliary ideal in the definition by a submodule of a given module. We have seen in Proposition 2.4 that in a regular F-finite ring, the F-thresholds of an ideal $\mathfrak{a}$ coincide with the F-jumping exponents of $\mathfrak{a}$. This might fail in non-regular rings, and in fact, it is often the case that $\mathrm{fpt}(\mathfrak{a}) < \mathrm{c}^J(\mathfrak{a})$ for every ideal J. However, as Corollary 4.4 below shows, we can remedy this situation if we consider the following more general notion of F-thresholds.

Suppose now that $\mathfrak{a}$ is a fixed ideal in a Noetherian ring R of characteristic $p > 0$. Let M be an R-module, and $N \subseteq M$ a submodule such that $\mathfrak{a}^n N = 0$ for some $n > 0$. We define

(1) For $q = p^e$, let $\nu^N_{M,\mathfrak{a}}(q) = \max\{r \in \mathbb{N} \mid \mathfrak{a}^r N^{[q]}_M \neq 0\}$ (we put $\nu^N_{M,\mathfrak{a}}(q) = 0$ if $\mathfrak{a} N^{[q]}_M = 0$).

(2) $\mathrm{c}^N_{M,+}(\mathfrak{a}) = \limsup_{q\to\infty} \frac{\nu^N_{M,\mathfrak{a}}(q)}{q}$ and $\mathrm{c}^N_{M,-}(\mathfrak{a}) = \liminf_{q\to\infty} \frac{\nu^N_{M,\mathfrak{a}}(q)}{q}$. When $\mathrm{c}^N_{M,+}(\mathfrak{a}) = \mathrm{c}^N_{M,-}(\mathfrak{a})$, we call this limit the *F-threshold* of $\mathfrak{a}$ with respect to (N, M), and we denote it by $\mathrm{c}^N_M(\mathfrak{a})$.

Remark 4.1. If J is an ideal of R with $\mathfrak{a} \subseteq \sqrt{J}$, then it is clear that $\nu^{A/J}_{\mathfrak{a},A/J}(q) = \nu^J_{\mathfrak{a}}(q)$, hence $\mathrm{c}^{A/J}_{A/J,\pm}(\mathfrak{a}) = \mathrm{c}^J_{\pm}(\mathfrak{a})$. Thus the notion of F-threshold with respect to modules extends our previous definition of F-thresholds with respect to ideals.

Lemma 4.2. *Let R, $\mathfrak{a}$, M and N be as in the above definition.*

(1) *If $\mathfrak{b} \subseteq \mathfrak{a}$ is an ideal, then $\mathrm{c}^N_{M,\pm}(\mathfrak{b}) \leq \mathrm{c}^N_{M,\pm}(\mathfrak{a})$.*
(2) *If $N' \subseteq N$, then $\mathrm{c}^{N'}_{M,\pm}(\mathfrak{a}) \leq \mathrm{c}^N_{M,\pm}(\mathfrak{a})$.*
(3) *If $\phi\colon M \to M'$ is a homomorphism of R-modules, and if $N' = \phi(N)$, then $\mathrm{c}^{N'}_{M',\pm}(\mathfrak{a}) \leq \mathrm{c}^N_{M,\pm}(\mathfrak{a})$. If R is regular and ϕ is injective, then $\mathrm{c}^{N'}_{M',\pm}(\mathfrak{a}) = \mathrm{c}^N_{M,\pm}(\mathfrak{a})$.*
(4) *If R is F-pure, then $\dfrac{\nu^N_{M,\mathfrak{a}}(q)}{q} \leq \dfrac{\nu^N_{M,\mathfrak{a}}(qq')}{qq'}$ for every q, q'. Hence in this case the limit $\mathrm{c}^N_M(\mathfrak{a})$ exists and it is equal to $\sup_q \dfrac{\nu^N_{M,\mathfrak{a}}(q)}{q}$.*

Proof. The assertions in (1) and (2) follow from definition. For (3), note that ϕ induces a surjection $N^{[q]} \to N'^{[q]}$, which gives the first statement. Moreover, if R is regular and ϕ is injective, then the flatness of the Frobenius morphism implies $N^{[q]} \simeq N'^{[q]}$, and we have equality.

Suppose now that R is F-pure, hence $M \otimes_R {}^e R$ is a submodule of $M \otimes_R {}^{ee'} R$. If $q = p^e$ and $q' = p^{e'}$, and if $\mathfrak{a}^r N^{[q]} \neq 0$, then $\mathfrak{a}^{q'r} N^{[qq']} \supseteq (\mathfrak{a}^r)^{[q']} N^{[qq']} \neq 0$. Therefore $\nu^N_{M,\mathfrak{a}}(qq') \geq q' \cdot \nu^N_{M,\mathfrak{a}}(q)$. $\square$

Our next proposition gives an analogue of Proposition 2.4 in the non-regular case.

Proposition 4.3. *Let $\mathfrak{a}$ be a proper nonzero ideal in a local normal $\mathbb{Q}$-Gorenstein ring $(R, \mathfrak{m})$. Suppose that R is F-finite and F-pure, and that the test ideal $\tau(R)$ is $\mathfrak{m}$-primary. We denote by E the injective hull of $R/\mathfrak{m}$.*

1. *If N is a submodule of E such that $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}_R(N)}$, and if $\alpha = \mathrm{c}_E^N(\mathfrak{a})$, then $N \subseteq (0)_E^{*\mathfrak{a}^\alpha}$.*
2. *If α is a non-negative real number, and if we put $N = (0)_E^{*\mathfrak{a}^\alpha}$, then $\mathrm{c}_E^N(\mathfrak{a}) \leq \alpha$.*
3. *There is an order-reversing bijection between the F-thresholds of $\mathfrak{a}$ with respect to the submodules of E and the ideals of the form $\tau(\mathfrak{a}^\alpha)$.*

Proof. For (1), note that since R is F-pure, we have $\nu_E^N(q) \leq \alpha q$ for every $q = p^e$. This implies

$$\mathfrak{a}^{\lceil \alpha q \rceil + 1} N_E^{[q]} = 0,$$

hence for every nonzero $d \in \mathfrak{a}$ we have $d\mathfrak{a}^{\lceil \alpha q \rceil} N_E^{[q]} = 0$ for all q. By definition, $N \subseteq (0)_E^{*\mathfrak{a}^\alpha}$.

Suppose now that $\alpha \geq 0$, and that $N = (0)_E^{*\mathfrak{a}^\alpha}$. By hypothesis, we can find m such that $\mathfrak{a}^m \subseteq \tau(R)$. It follows from [HT, Cor. 2.4] that every element in $\tau(R)$ is an $\mathfrak{a}^\alpha$-test element. Therefore $\mathfrak{a}^{m + \lceil \alpha q \rceil} N_E^{[q]} = 0$, hence $\nu_{E,\mathfrak{a}}^N(q) < m + \alpha q$ for all $q \gg 0$. Dividing by q and taking the limit as q goes to infinity, gives $\mathrm{c}_E^N(\mathfrak{a}) \leq \alpha$.

We assume that R is F-finite, normal and $\mathbb{Q}$-Gorenstein, hence for every non-negative t we have $\tau(\mathfrak{a}^t) = \mathrm{Ann}_R(0_E^{*\mathfrak{a}^t})$. Note also that by [HT, Prop. 3.2], taking the generalized test ideal commutes with completion. This shows that the set of ideals of the form $\tau(\mathfrak{a}^\alpha)$ is in bijection with the set of submodules of E of the form $(0)_E^{*\mathfrak{a}^\alpha}$. Hence in order to prove (3) it is enough to show that the map

$$\{(0)_E^{*\mathfrak{a}^\alpha} \mid \alpha \geq 0\} \to \{\mathrm{c}_E^N(\mathfrak{a}) \mid N \subseteq E, \mathfrak{a} \subseteq \sqrt{\mathrm{Ann}_R(N)}\}$$

that takes N to $\mathrm{c}_E^N(\mathfrak{a})$ is bijective, the inverse map taking α to $(0)_E^{*\mathfrak{a}^\alpha}$.

Suppose first that $N = (0)_E^{*\mathfrak{a}^\alpha}$, and let $\beta = \mathrm{c}_E^N(\mathfrak{a})$. It follows from (2) that $\beta \leq \alpha$, hence $(0)_E^{*\mathfrak{a}^\beta} \subseteq N$. On the other hand, (1) gives $N \subseteq (0)_E^{*\mathfrak{a}^\beta}$, hence we have equality.

Let us now start with $\alpha = \mathrm{c}_E^N(\mathfrak{a})$, and let $N' = (0)_E^{*\mathfrak{a}^\alpha}$. We deduce from (1) that $N \subseteq N'$, hence $\mathrm{c}_E^{N'}(\mathfrak{a}) \geq \alpha$. Since (2) implies $\mathrm{c}_E^{N'}(\mathfrak{a}) \leq \alpha$, we get $\alpha = \mathrm{c}_E^{N'}(\mathfrak{a})$, which completes the proof of (3). □

Corollary 4.4. *Let $\mathfrak{a}$ be a proper nonzero ideal in a local normal $\mathbb{Q}$-Gorenstein ring $(R, \mathfrak{m})$. If R is F-finite and F-regular, then for every ideal J in R we have*

$$\xi^J(\mathfrak{a}) = \mathrm{c}_E^N(\mathfrak{a}),$$

where E is the injective hull of $R/\mathfrak{m}$ and $N = \mathrm{Ann}_E(J)$. In particular, the F-pure threshold $\mathrm{fpt}(\mathfrak{a})$ is equal to $\mathrm{c}_E^Z(\mathfrak{a})$, where $Z = (0 :_E \mathfrak{m})$ is the socle of E.

Proof. Let $\beta := \mathrm{c}_E^N(\mathfrak{a})$. Given $\alpha \geq 0$, Matlis duality implies that $\tau(\mathfrak{a}^\alpha) \subseteq J$ if and only if $N \subseteq (0)_E^{*\mathfrak{a}^\alpha}$. If this holds, then part (2) in the proposition gives

$$\alpha \geq \mathrm{c}_E^{(0)_E^{*\mathfrak{a}^\alpha}}(\mathfrak{a}) \geq \mathrm{c}_E^N(\mathfrak{a}) = \beta.$$

Conversely, if $\alpha \geq \beta$, then

$$(0)_E^{*\mathfrak{a}^\alpha} \supseteq (0)_E^{*\mathfrak{a}^\beta} \supseteq N,$$

by part (1) in the proposition. This shows that $\mathrm{c}_E^N(\mathfrak{a}) = \xi^J(\mathfrak{a})$, and the last assertion in the corollary follows by taking $J = \mathfrak{m}$. □

Remark 4.5. Let $\mathfrak{a}$ be an ideal in the local ring $(R, \mathfrak{m})$. We have seen that $\mathrm{c}^I(\mathfrak{a}) \geq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$ for every proper ideal I. Note also that applying Prop 4.2 (3) to the embedding $R/m \simeq Z \hookrightarrow E = E_R(R/\mathfrak{m})$, we get $\mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) = \mathrm{c}_{R/\mathfrak{m}}^{R/\mathfrak{m}}(\mathfrak{a}) \geq \mathrm{c}_E^Z(\mathfrak{a}) = \mathrm{fpt}(\mathfrak{a})$. Thus we always have $\mathrm{fpt}(\mathfrak{a}) \leq \mathrm{c}^I(\mathfrak{a})$, and equality is possible only if $\mathrm{fpt}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$. While this equality holds in some non-regular examples (see Remark 2.5), this seems to happen rather rarely.

5. Connections between F-thresholds and multiplicity

Given an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in a regular local ring $(R, \mathfrak{m})$, essentially of finite type over a field of characteristic zero, de Fernex, Ein and the second author proved in [dFEM] an inequality involving the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ and the multiplicity $e(\mathfrak{a})$. Later, the third and fourth authors gave in [TW] a characteristic p analogue of this result, replacing the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ by the F-pure threshold $\mathrm{fpt}(\mathfrak{a})$. We propose the following conjecture, generalizing this inequality.

Conjecture 5.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic $p > 0$. If $J \subseteq \mathfrak{m}$ is an ideal generated by a full system of parameters, and if $\mathfrak{a} \subseteq \mathfrak{m}$ is an $\mathfrak{m}$-primary ideal, then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{\mathrm{c}_-^J(\mathfrak{a})}\right)^d e(J).$$

Remark 5.2. (1) When R is regular and $J = \mathfrak{m}$, the above conjecture is precisely the above-mentioned inequality, see [TW, Proposition 4.5].

(2) When R is a d-dimensional regular local ring, essentially of finite type over a field of characteristic zero, we can consider an analogous problem: let $\mathfrak{a}, J$ be $\mathfrak{m}$-primary ideals in R such that J is generated by a full system of parameters. Does the following inequality hold

$$e(\mathfrak{a}) \geq \left(\frac{d}{\lambda^J(\mathfrak{a})}\right)^d e(J),$$

where $\lambda^J(\mathfrak{a}) := \max\{c > 0 \mid \mathcal{J}(\mathfrak{a}^c) \not\subseteq J\}$. This would generalize the inequality in [dFEM], which is the special case $J = \mathfrak{m}$. However, this version is also open in general.

(3) The condition in Conjecture 5.1 that J is generated by a system of parameters is crucial, as otherwise there are plenty of counterexamples. Suppose, for example, that $(R, \mathfrak{m})$ is a regular local ring of dimension $d \geq 2$ and of characteristic $p > 0$. Let $\mathfrak{a} = \mathfrak{m}^k$ and $J = \mathfrak{m}^\ell$ with $k \geq 1$, $\ell \geq 2$ integers. It follows from Example 2.7 (3) that $\mathrm{c}^J(\mathfrak{a}) = (d + \ell - 1)/k$. Moreover, we have $e(\mathfrak{a}) = k^d$ and $e(J) = \ell^d$, thus

$$e(\mathfrak{a}) = k^d < (dk\ell/(d + \ell - 1))^d = \left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d e(J).$$

Example 5.3. Let $R = k[\![X, Y, Z]\!]/(X^2 + Y^3 + Z^5)$ be a rational double point of type E_8, with k a field of characteristic $p > 0$. Let $\mathfrak{a} = (x, z)$ and $J = (y, z)$. Then $e(\mathfrak{a}) = 3$ and $e(J) = 2$. It is easy to check that $\mathrm{c}^J(\mathfrak{a}) = 5/3$ and $\mathrm{c}^{\mathfrak{a}}(J) = 5/2$. Thus,

$$e(\mathfrak{a}) = 3 > \frac{72}{25} = \left(\frac{2}{\mathrm{c}^J(\mathfrak{a})}\right)^2 e(J),$$
$$e(J) = 2 > \frac{48}{25} = \left(\frac{2}{\mathrm{c}^{\mathfrak{a}}(J)}\right)^2 e(\mathfrak{a}).$$

See Corollary 5.9 below for a general statement in the homogeneous case.

We now show that Conjecture 5.1 implies an effective estimate of the multiplicity of complete intersection F-rational rings.

Proposition 5.4. *Let $(R, \mathfrak{m})$ be a d-dimensional F-rational local ring of characteristic $p > 0$ with infinite residue field (resp. a rational singularity over a field of characteristic zero) which is a complete intersection. If Conjecture 5.1 (resp. Remark 5.2 (1)) holds true for the regular case, then $e(R) \leq 2^{d-1}$.*

Proof. Let $J \subseteq \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$. Note that J is generated by a full system of parameters for R. The Briançon-Skoda theorem for F-rational rings (or for rational singularities), see [HV] and [AH], gives $\mathfrak{m}^d \subseteq J$. Taking the quotient of R by J, we reduce the assertion in the proposition to the following claim:

Claim. Let $(A, \mathfrak{m})$ be a complete intersection Artinian local ring of characteristic $p > 0$ (resp. essentially of finite type over a field of characteristic zero). If s is the largest integer s such that $\mathfrak{m}^s \neq 0$, then $e(A) \leq 2^s$.

We now show that the regular case of Conjecture 5.1 implies the claim in positive characteristic (the argument in characteristic zero is entirely analogous). Write $A = S/I$, where $(S, \mathfrak{n})$ is an n-dimensional regular local ring and $I \subseteq S$ is an ideal generated by a full system of parameters $f_1, \ldots, f_n$ for S. For every i, we denote by α_i the order of f_i. We may assume that $\alpha_i \geq 2$ for all i.

Let $\mathfrak{n} = (y_1, \ldots, y_n)$, and let us write $f_i = \sum_j a_{ij} y_j$. A standard argument relating the Koszul complexes on the f_i and, respectively, the y_i, shows that $\det(a_{ij})$ generates the socle of A. In particular, if

$$s := \max\{r \in \mathbb{N} \mid \mathfrak{n}^r \not\subseteq I\},$$

then $s \geq \sum_{i=1}^{n}(a_i - 1) \geq n$. On the other hand, it follows from Example 2.7 (iii) that $\mathrm{c}^I(\mathfrak{m}) = s + n$ (the corresponding formula in characteristic zero is an immediate consequence of the description of the multiplier ideals of the ideal of a point). Applying Conjecture 5.1 to S, we get

$$1 = e(\mathfrak{n}) \geq \left(\frac{n}{\mathrm{c}^I(\mathfrak{m})}\right)^n e(I) = \left(\frac{n}{s+n}\right)^n e(I).$$

Note that $(n/(s+n))^n \geq (s/(s+s))^s = (1/2)^s$, because $s \geq n$. Thus, we have $e(A) = e(I) \leq 2^s$. □

Proposition 5.5. *If $(R, \mathfrak{m})$ is a one-dimensional analytically irreducible local domain of characteristic $p > 0$, and if $\mathfrak{a}, J$ are $\mathfrak{m}$-primary ideals in R, then*

$$\mathrm{c}^J(\mathfrak{a}) = \frac{e(J)}{e(\mathfrak{a})}.$$

In particular, Conjecture 5.1 holds in R.

Proof. By Proposition 2.2 (5), we may assume that R is a complete local domain. Since R is one-dimensional, the integral closure $\overline{R}$ is a DVR. Therefore we have

$$\mathrm{c}^{J\overline{R}}(\mathfrak{a}\overline{R}) = \mathrm{ord}_{\overline{R}}(J\overline{R})/\mathrm{ord}_{\overline{R}}(\mathfrak{a}\overline{R}).$$

On the other hand, $e(J\overline{R}) = \mathrm{ord}_{\overline{R}}(J\overline{R})$ and $e(\mathfrak{a}\overline{R}) = \mathrm{ord}_{\overline{R}}(\mathfrak{a}\overline{R})$. Thus, by Proposition 2.2 (6),

$$\mathrm{c}^J(\mathfrak{a}) = \mathrm{c}^{J\overline{R}}(\mathfrak{a}\overline{R}) = \frac{e(J\overline{R})}{e(\mathfrak{a}\overline{R})} = \frac{e(J)}{e(\mathfrak{a})}.$$

□

Theorem 5.6. *If $(R, \mathfrak{m})$ is a regular local ring of characteristic $p > 0$ and $J = (x_1^{a_1}, \ldots, x_d^{a_d})$, with $x_1, \ldots, x_d$ a full regular system of parameters for R, and with $a_1, \ldots, a_d$ positive integers, then the inequality given by Conjecture 5.1 holds.*

Proof. The proof follows the idea in [dFEM] and [TW], reducing the assertion to the case when $\mathfrak{a}$ is a monomial ideal, and then using the explicit description of the invariants involved. We have by definition $e(\mathfrak{a}) = \lim_{n\to\infty} \frac{d!\cdot \ell_R(R/\mathfrak{a}^n)}{n^d}$, hence it is enough to show that for every $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R,

$$\ell_R(R/\mathfrak{a}) \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d e(J). \tag{4}$$

After passing to completion and using Proposition 2.2 (5) and Remark 2.6, we see that it is enough to prove the inequality (4) in the case when $R = k[x_1, \ldots, x_d]$, $\mathfrak{m} = (x_1, \ldots, x_d)$, $\mathfrak{a}$ is $\mathfrak{m}$-primary, and $J = (x_1^{a_1}, \ldots, x_d^{a_d})$.

Note that $e(J) = a_1 \cdots a_d$. We fix a monomial order λ on the monomials in the polynomial ring, and use it to take a Gröbner deformation of $\mathfrak{a}$, see [Eis, Ch. 15]. This is a flat family $\{\mathfrak{a}_s\}_{s\in k}$ such that $R/\mathfrak{a}_s \cong R/\mathfrak{a}$ for all $s \neq 0$, and such that $\mathfrak{a}_0 = \mathrm{in}_\lambda(\mathfrak{a})$, the initial ideal of $\mathfrak{a}$.

If I is an ideal generated by monomials, we denote by $P(I)$ the Newton polyhedron of I (see Example 2.7 (2) for definition). We also put $\mathrm{Vol}(P)$ for the volume of a region P in $\mathbb{R}^n$, with the Euclidean metric. Since the deformation we consider is flat, it follows that $\mathrm{in}_\lambda(\mathfrak{a})$ is also $\mathfrak{m}$-primary and

$$\ell_R(R/\mathfrak{a}) = \ell_R(R/\mathrm{in}_\lambda(\mathfrak{a})) \geq \mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus P(\mathrm{in}_\lambda(\mathfrak{a}))\right),$$

where the inequality follows from [dFEM, Lemma 1.3].

On the other hand, by [dFe, Prop. 5.3], we have $\tau(\mathrm{in}_\lambda(\mathfrak{a})^t) \subseteq \mathrm{in}_\lambda(\tau(\mathfrak{a}^t))$ for all $t > 0$. This implies that $\mathrm{c}^J(\mathfrak{a}) \geq \mathrm{c}^{\mathrm{in}_\lambda(J)}(\mathrm{in}_\lambda(\mathfrak{a}))$. Note also that since J is generated by monomials, we have $\mathrm{in}_\lambda(J) = J$. Thus, we can reduce to the case when $\mathfrak{a}$ is generated by monomials in $x_1, \ldots, x_d$. That is, it is enough to show that for every $\mathfrak{m}$-primary monomial ideal $\mathfrak{a} \subseteq R$,

$$\mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus P(\mathfrak{a})\right) \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d a_1 \cdots a_d.$$

It follows from the description of $\mathrm{c}^J(\mathfrak{a})$ in Example 2.7 (2) that we have $(a_1, \ldots, a_d) \in \partial(\mathrm{c}^J(a) \cdot P(\mathfrak{a}))$. We can find a hyperplane $H_q := u_1/b_1 + \cdots + u_d/b_d = 1$ passing through the point $(a_1, \ldots, a_d)$ such that

$$H^+ := \left\{(u_1, \ldots, u_d) \in \mathbb{R}^d_{\geq 0} \mid \frac{u_1}{b_1} + \cdots + \frac{u_d}{b_d} \geq 1\right\} \supseteq \mathrm{c}^J(\mathfrak{a}) \cdot P(\mathfrak{a}).$$

Therefore, we have

$$\mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus P(\mathfrak{a})\right) \geq \mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus \frac{1}{\mathrm{c}^J(\mathfrak{a})} H^+\right) = \frac{b_1 \ldots b_d}{d! \cdot \mathrm{c}^J(\mathfrak{a})^d}.$$

On the other hand, since H passes through $(a_1, \ldots, a_d)$, it follows that $a_1/b_1 + \cdots + a_d/b_d = 1$. Comparing the arithmetic and geometric means of $\{a_i/b_i\}_i$, we see that

$$b_1 \cdots b_d \geq d^d \cdot a_1 \cdots a_d.$$

Thus, combining these two inequalities, we obtain that

$$\mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus P(\mathfrak{a})\right) \geq \frac{b_1 \cdots b_d}{d! \cdot \mathrm{c}^J(\mathfrak{a})^d} \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d a_1 \cdots a_d,$$

as required. □

Remark 5.7. It might seem that in the above proof we have shown a stronger assertion than the one in Conjecture 5.1, involving the length instead of the multiplicity. However, the two assertions are equivalent: this follows from [Mu, Corollary 3.8] which says that for every zero-dimensional ideal $\mathfrak{a}$ in a d-dimensional regular local ring R, we have

$$\ell_R(R/\mathfrak{a}) \geq \frac{e(\mathfrak{a})}{d!}.$$

We can prove a graded version of Conjecture 5.1. In fact, we prove a more precise statement, which is valid independently of the characteristic.

Theorem 5.8. *Let $R = \bigoplus_{d\geq 0} R_d$ be an n-dimensional graded Cohen-Macaulay ring with R_0 a field of arbitrary characteristic. If $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters for R, and if $\mathfrak{a}^N \subseteq J$, then*

$$e(\mathfrak{a}) \geq \left(\frac{n}{n+N-1}\right)^n e(J).$$

Corollary 5.9. *Let R be as in the theorem, with $\mathrm{char}(R_0) = p > 0$. If $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters for R, then*

$$e(\mathfrak{a}) \geq \left(\frac{n}{\mathrm{c}_-^J(\mathfrak{a})}\right)^n e(J).$$

Proof. Note that each $J^{[q]}$ is again generated by a full homogeneous systems of parameters. It follows from the theorem and from the definition of $\nu_{\mathfrak{a}}^J(q)$ that for every $q = p^e$ we have

$$e(\mathfrak{a}) \geq \left(\frac{n}{n+\nu_{\mathfrak{a}}^J(q)}\right)^n e(J^{[q]}) = \left(\frac{qn}{n+\nu_{\mathfrak{a}}^J(q)}\right)^n e(J).$$

On the right-had side we can take a subsequence converging to $\left(\frac{n}{\mathrm{c}_-^J(\mathfrak{a})}\right)^n e(J)$, hence we get the inequality in the corollary. □

Proof of Theorem 5.8. Suppose that $\mathfrak{a}$ is generated by a full homogeneous system of parameters $x_1, \ldots, x_n$ of degrees $a_1 \leq \cdots \leq a_n$, and that J is generated by another homogeneous system of parameters $f_1, \ldots, f_n$ of degrees $d_1 \leq \cdots \leq d_n$. Define nonnegative integers $t_1, \ldots, t_{n-1}$ inductively as follows: t_1 is the smallest integer t such that $x_1^t \in J$. If $2 \leq i \leq n-1$, then t_i is the smallest integer t such that $x_1^{t_1-1} \cdots x_{i-1}^{t_{i-1}-1} x_i^t \in J$. Note that we have by assumption $N \geq t_1 + \cdots + t_{n-1} - n + 1$.

We first show the following inequality for every $i = 1, \ldots, n-1$:

$$t_1a_1 + \cdots + t_ia_i \geq d_1 + \cdots + d_i. \tag{5}$$

Let I_i be the ideal of R generated by $x_1^{t_1}, x_1^{t_1-1}x_2^{t_2}, \ldots, x_1^{t_1-1} \cdots x_{i-1}^{t_{i-1}-1} x_i^{t_i}$. Note that the definition of the integers t_j implies that $I_i \subseteq J$. The natural surjection of R/I_i onto R/J induces a comparison map between their free resolutions (we resolve R/J by the Koszul complex, and R/I_i by a Taylor-type complex). Note that the i^{th} step in the Taylor complex for the monomials $X_1^{t_1}, X_1^{t_1-1}X_2^{t_2}, \ldots, X_1^{t_1-1} \cdots X_{i-1}^{t_{i-1}-1} X_i^{t_i}$ in a polynomial ring with variables $X_1, \ldots, X_n$, is a free module of rank one, with a generator corresponding to the monomial

$$\mathrm{lcm}(X_1^{t_1}, X_1^{t_1-1}X_2^{t_2}, \ldots, X_1^{t_1-1} \cdots X_{i-1}^{t_{i-1}} X_i^{t_i}) = X_1^{t_1} \cdots X_{i-1}^{t_{i-1}} X_i^{t_i}$$

(see [Eis, Exercise 17.11]). It follows that the map between the i^{th} steps in the resolutions of R/I_i and R/J is of the form

$$R(-t_1a_1 - \cdots - t_ia_i) \to \bigoplus_{1\leq v_1 < \cdots < v_i \leq n} R(-d_{v_1} - \cdots - d_{v_i}).$$

In particular, unless this map is zero, we have

$$t_1a_1 + \cdots + t_ia_i \geq \min_{1\leq v_1<\cdots<v_i\leq n} (d_{v_1} + \cdots + d_{v_i}) = d_1 + \cdots + d_i.$$

We now show that this map cannot be zero. If it is zero, then also the induced map

$$\mathrm{Tor}_i^R(R/I_i, R/\mathfrak{b}_i) \to \mathrm{Tor}_i^R(R/J, R/\mathfrak{b}_i) \tag{6}$$

is zero, where $\mathfrak{b}_i$ is the ideal generated by $x_1, \ldots, x_i$. On the other hand, using the Koszul complex on $x_1, \ldots, x_i$ to compute the above Tor modules, we see that the map (6) can be identified with the natural map

$$(I_i : \mathfrak{b}_i)/I_i \to (J : \mathfrak{b}_i)/J.$$

Since $x_1^{t_1-1} \cdots x_i^{t_i-1} \in (I_i : \mathfrak{b}_i)$, it follows that $x_1^{t_1-1} \cdots x_i^{t_i-1}$ lies in J, a contradiction. This proves (5).

We next prove the following inequality:

$$t_1a_1 + \cdots + t_{n-1}a_{n-1} + (N - t_1 - \cdots - t_{n-1} + n - 1)a_n \geq d_1 + \cdots + d_n. \tag{7}$$

Since $\mathfrak{a}^N \subseteq J$, we have

$$(x_1^N, \ldots, x_n^N) : J \subseteq (x_1^N, \ldots, x_n^N) : \mathfrak{a}^N = (x_1^N, \ldots, x_n^N) + \mathfrak{a}^{(n-1)(N-1)}. \tag{8}$$

On the other hand, the ideal $(x_1^N, \ldots, x_n^N) : J$ can be described as follows. If we write $x_i^N = \sum_{j=1}^n b_{ij} f_j$, then using the Koszul resolutions of R/J and $R/(x_1^N, \ldots, x_n^N)$ one sees that multiplication by $D = \det(b_{ij})$ gives an injection $R/J \hookrightarrow R/(x_1^N, \ldots, x_n^N)$, hence $J = (x_1^N, \ldots, x_n^N) : D$. Moreover, we also get

$$(x_1^N, \ldots, x_n^N) : J = (x_1^N, \ldots, x_n^N, D)$$

(see, for example, [PS, Prop. 2.6]; note that the statement therein requires R to be regular, but this condition is not used). It follows from the above description that D is homogeneous, and $\deg(D) = N(a_1 + \cdots + a_n) - (d_1 + \cdots + d_n)$.

It follows from (8) that after possibly adding to D an element in $(x_1^N, \ldots, x_n^N)$, we may write

$$D = \sum_{m_1+\cdots+m_n=(n-1)(N-1)} c_{m_1\ldots m_n} x_1^{m_1} \ldots x_n^{m_n},$$

where all $c_{m_1,\ldots,m_n}$ are homogeneous. Since $x_1^{t_1-1} \cdots x_{n-1}^{t_{n-1}-1}$ is not in $J = (x_1^N, \ldots, x_n^N) : D$, we see that

$$D \notin (x_1^N, \ldots, x_n^N) : x_1^{t_1-1} \cdots x_{n-1}^{t_{n-1}-1} = (x_1^{N-t_1+1}, \ldots, x_{n-1}^{N-t_{n-1}+1}, x_n^N).$$

Thus there is some $(m_1, \ldots, m_n)$ with $\sum_j m_j = (n-1)(N-1)$ and $m_j \leq N - t_j$ for all $j \leq n-1$, such that $c_{m_1\ldots m_m} \neq 0$. We deduce that the degree of D is at least as large as the smallest degree of such a monomial $x_1^{m_1} \cdots x_n^{m_n}$, hence

$$\begin{aligned} \deg D &= N(a_1 + \cdots + a_n) - (d_1 + \cdots + d_n) \\ &\geq (N - t_1)a_1 + \cdots + (N - t_{n-1})a_{n-1} + (t_1 + \cdots + t_{n-1} - n + 1)a_n, \end{aligned}$$

which implies the inequality (7).

To finish the proof, we will use the following claim.

Claim. Let $\alpha_i, \beta_i, \gamma_i$ be real numbers, for $1 \leq i \leq n$. If $1 = \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$, and if $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i \geq \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for all $i = 1, \ldots, n$, then $\alpha_1 + \cdots + \alpha_n \geq \beta_1 + \cdots + \beta_n$.

Proof of Claim. Let $\lambda_i = \alpha_i - \beta_i$ for $1 \leq i \leq n$, so that $\gamma_1\lambda_1 + \cdots + \gamma_i\lambda_i \geq 0$ for all $i = 1, \ldots, n$. We prove that $\lambda_1 + \cdots + \lambda_n \geq 0$ by induction on n, the case $n = 1$ being trivial. Suppose that $n > 1$ and that there is i such that $\lambda_i < 0$ (otherwise the assertion to prove is clear). We must have $i \geq 2$, and since $\gamma_i \geq \gamma_{i-1}$, it follows that $\gamma_i\lambda_i \leq \gamma_{i-1}\lambda_i$. Let us put $\gamma'_j = \gamma_j$ for $1 \leq j \leq i-1$ and $\gamma'_j = \gamma_{j+1}$ for $i \leq j \leq n-1$. Define also $\lambda'_j = \lambda_j$ for $1 \leq j \leq i-2$, $\lambda'_{i-1} = \lambda_{i-1} + \lambda_i$ and $\lambda'_j = \lambda_{j+1}$ for $i \leq j \leq n-1$. It is straightforward to check that $\gamma'_1\lambda'_1 + \cdots + \gamma'_j\lambda'_j \geq 0$ for all $j = 1, \ldots, n-1$, hence the induction hypothesis implies $\lambda_1 + \cdots + \lambda_n = \lambda'_1 + \cdots + \lambda'_{n-1} \geq 0$. □

We now set $\alpha_i = t_i$ for $1 \leq i \leq n-1$ and $\alpha_n = N - t_1 - \cdots - t_{n-1} + n - 1$. We put $\beta_i = d_i/a_i$ and $\gamma_i = a_i/a_1$ for $1 \leq i \leq n$. Since $a_1 \leq \cdots \leq a_n$, we deduce $1 = \gamma_1 \leq \cdots \leq \gamma_n$. Moreover, using (5) and (7), we get $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i \geq \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for $1 \leq i \leq n$. Using the above claim, we conclude that

$$N + n - 1 = \alpha_1 + \cdots + \alpha_n \geq \beta_1 + \cdots + \beta_n = \left(\frac{d_1}{a_1} + \cdots + \frac{d_n}{a_n}\right).$$

Comparing the arithmetic and geometric means of $\{d_i/a_i\}_i$, we see that

$$(N + n - 1)^n a_1 \ldots a_n \geq n^n d_1 \ldots d_n.$$

Since $e(\mathfrak{a}) = a_1 \cdots a_n$ and $e(J) = d_1 \cdots d_n$, this concludes the proof. □

When J is not necessarily a parameter ideal, we can prove another inequality involving the F-threshold $\mathrm{c}^J(\mathfrak{a})$, generalizing the results in [dFEM] and [TW].

Proposition 5.10. *If $(R, \mathfrak{m})$ is a d-dimensional regular local ring of characteristic $p > 0$, and if $\mathfrak{a}, J$ are $\mathfrak{m}$-primary ideals in R, then we have the following inequality:*

$$e(\mathfrak{a}) \geq \left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d (\mathrm{c}^J(\mathfrak{m}) - d + 1).$$

Proof. As in the proof of Theorem 5.6, we do a reduction to the monomial case. We first see that it is enough to show that if R is the polynomial ring $k[x_1, \ldots, x_d]$ and $\mathfrak{m} = (x_1, \ldots, x_d)$, and $\mathfrak{a}$, J are $\mathfrak{m}$-primary ideals, then

$$\ell(R/\mathfrak{a}) \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d (\mathrm{c}^J(\mathfrak{m}) - d + 1). \tag{9}$$

Claim. We can find monomial ideals $\mathfrak{a}_1$ and J_1 such that

$$\ell_R(R/\mathfrak{a}) = \ell_R(R/\mathfrak{a}_1),\ \mathrm{c}^J(\mathfrak{a}) \geq \mathrm{c}^{J_1}(\mathfrak{a}_1),\ \text{and}\ \mathrm{c}^J(\mathfrak{m}) = \mathrm{c}^{J_1}(\mathfrak{m}). \tag{10}$$

This reduces the proof of (9) to the case when both $\mathfrak{a}$ and J are monomial ideals.

Proof of claim. We do a two-step deformation to monomial ideals. We consider first a flat deformation of $\mathfrak{a}$ and J to $\mathfrak{a}'$ and J', respectively, where for an ideal $I \subseteq R$, we denote by I' the ideal defining the respective tangent cone at the origin. We then fix a monomial order λ, and consider a Gröbner deformation of $\mathfrak{a}'$ and J' to $\mathfrak{a}_1 := \mathrm{in}_\lambda(\mathfrak{a}')$ and $J_1 := \mathrm{in}_\lambda(J')$, respectively. It follows as in the proof of Theorem 5.6 that the first two conditions in (10) are satisfied. For the third condition, in light of Example 2.7 (3) it is enough to show that

$$\mathfrak{m}^r \subseteq J \text{ iff } \mathfrak{m}^r \subseteq \mathrm{in}_\lambda(J).$$

It is clear that if $\mathfrak{m}^r \subseteq J$, then $\mathfrak{m}^r \subseteq J'$ and $\mathfrak{m}^r \subseteq J_1$. For the converse, suppose that $\mathfrak{m}^r \subseteq J_1$. Since J' and J_1 are both homogeneous ideals, and since $\dim_k(R/J_1)_r = \dim_k(R/J')_r$ (see [Eis, Ch. 15]), it follows that $\mathfrak{m}^r \subseteq J'$ (note that if I is a homogeneous ideal in R, then $\mathfrak{m}^r \subseteq I$ if and only if $(R/I)_r = 0$). We know that $\mathfrak{m}^s \subseteq J$ for some s, hence in order to prove that $\mathfrak{m}^r \subseteq J$ it is enough to show the following assertion: if $\mathfrak{m}^t \subseteq J'$ and $\mathfrak{m}^{t+1} \subseteq J$, then $\mathfrak{m}^t \subseteq J$. It is easy to check that $(J \cap \mathfrak{m}^t)' = J' \cap \mathfrak{m}^t$, and since $\mathfrak{m}^{t+1} \subseteq J$, we see that $J \cap \mathfrak{m}^t$ is homogeneous, hence

$$\mathfrak{m}^t \subseteq J' \cap \mathfrak{m}^t = (J \cap \mathfrak{m}^t)' = J \cap \mathfrak{m}^t.$$

$\square$

We return to the proof of Proposition 5.10. From now on we assume that $\mathfrak{a}$ and J are $\mathfrak{m}$-primary monomial ideals. Arguing as in the proof of Theorem 5.6, and using Example 2.7 (3), we see that it is enough to show

$$\mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus P(\mathfrak{a})\right) \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d (r+1),$$

where $r := \max\{s \in \mathbb{Z}_{\geq 0} \mid \mathfrak{m}^s \not\subseteq J\}$. By definition, we can choose a monomial $x_1^{r_1} \cdots x_d^{r_d}$ of degree r that is not contained in J. Since $\tau(\mathfrak{a}^{\mathrm{c}^J(\mathfrak{a})}) \subseteq J$ by Proposition 2.4, this monomial cannot belong to $\tau(\mathfrak{a}^{\mathrm{c}^J(\mathfrak{a})})$. Using the description of generalized test ideals of monomial ideals (see [HY, Theorem 4.8]), this translates as

$$(r_1+1, \ldots, r_d+1) \notin \mathrm{Int}(\mathrm{c}^J(\mathfrak{a}) \cdot P(\mathfrak{a})).$$

Therefore we can find a hyperplane $H : u_1/a_1 + \cdots + u_d/a_d = \mathrm{c}^J(\mathfrak{a})$ passing through the point $(r_1+1, \ldots, r_d+1)$ such that

$$H^+ := \left\{(u_1, \ldots, u_d) \in \mathbb{R}^d_{\geq 0} \mid \frac{u_1}{a_1} + \cdots + \frac{u_d}{a_d} \geq \mathrm{c}^J(\mathfrak{a})\right\} \supseteq \mathrm{c}^J(\mathfrak{a}) \cdot P(\mathfrak{a}). \tag{11}$$

Note that we get $\mathrm{c}^J(\mathfrak{a}) = (1+r_1)/a_1 + \cdots + (1+r_d)/a_d$. Comparing the arithmetic and geometric means of $\{(1+r_i)/a_i\}_i$, we see that

$$\left(\frac{\mathrm{c}^J(\mathfrak{a})}{d}\right)^d = \left(\frac{1+r_1}{da_1} + \cdots + \frac{1+r_d}{da_d}\right)^d \geq \frac{(1+r_1)\ldots(1+r_d)}{a_1 \ldots a_d} \geq \frac{1+r}{a_1 \ldots a_d}.$$

On the other hand, (11) implies

$$\begin{aligned}\mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus P(\mathfrak{a})\right) &\geq \mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \smallsetminus (1/\mathrm{c}^J(\mathfrak{a}))H^+\right) \\ &= \frac{a_1 \ldots a_d}{d!} \\ &\geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d (r+1)\,.\end{aligned}$$

□

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