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A variable metric extension of the forward–backward–forward algorithm for monotone operators *

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Abstract

We propose a variable metric extension of the forward–backward-forward algorithm for finding a zero of the sum of a maximally monotone operator and a monotone Lipschitzian operator in Hilbert spaces. In turn, this framework provides a variable metric splitting algorithm for solving monotone inclusions involving sums of composite operators. Monotone operator splitting methods recently proposed in the literature are recovered as special cases.

Keywords: variable metric, composite operator, duality, monotone inclusion, monotone operator, operator splitting, primal-dual algorithm

Mathematics Subject Classifications (2010) 47H05, 49M29, 49M27, 90C25

1 Introduction

A basic problem in applied monotone operator theory is to find a zero of a maximally monotone operator A on a real Hilbert space $\mathcal{H}$. This problem can be solved by the proximal point algorithm proposed in [17] which requires only the resolvent of A, provided it is easy to implement numerically. In order to get more efficient proximal algorithms, some authors have proposed the use of variable metric or preconditioning in such algorithms [3, 5, 6, 10, 13, 15, 16].

This problem was then extended to the problem of finding a zero of the sum of a maximally monotone operator A and a cocoercive operator B (i.e., B^{-1} is strongly monotone). In such

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instances, the forward-backward splitting algorithm [1, 8, 12, 18] can be used. Recently, this algorithm has been investigated in the context of variable metric [11]. In the case when B is only Lipschitzian and not cocoercive, the problem can be solved by the forward-backward-forward splitting algorithm [4, 19]. New applications of this basic algorithm to more complex monotone inclusions are presented in [4, 9].

In the present paper, we propose a variable metric version of the forward-backward-forward splitting algorithm. In Section 2, we recall notation and background on convex analysis and monotone operator theory. In Section 3, we present our variable metric forward-backward-forward splitting algorithm. In Section 4, the results of Section 3 are used to develop a variable metric primal–dual algorithm for solving the type of composite inclusions considered in [9].

2 Notation and background

Throughout, $\mathcal{H}$, $\mathcal{G}$, and $(\mathcal{G}_i)_{1\leq i\leq m}$ are real Hilbert spaces. Their scalar products and associated norms are respectively denoted by $\langle\cdot\mid\cdot\rangle$ and $\|\cdot\|$. We denote by $\mathcal{B}(\mathcal{H},\mathcal{G})$ the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$. The adjoint of $L\in\mathcal{B}(\mathcal{H},\mathcal{G})$ is denoted by L^*. We set $\mathcal{B}(\mathcal{H})=\mathcal{B}(\mathcal{H},\mathcal{H})$. The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence, and Id denotes the identity operator, and $B(x;\rho)$ denotes the closed ball of center $x\in\mathcal{H}$ and radius $\rho\in\left]0,+\infty\right[$. The interior of $C\subset\mathcal{H}$ is denoted by $\operatorname{int}C$. We denote by $\ell_+^1(\mathbb{N})$ the set of summable sequences in $\left[0,+\infty\right[$.

Let M_1 and M_2 be self-adjoint operators in $\mathcal{B}(\mathcal{H})$, we write $M_1\succcurlyeq M_2$ if and only if $(\forall x\in\mathcal{H})\ \langle M_1x\mid x\rangle\geq\langle M_2x\mid x\rangle$. Let $\alpha\in\left]0,+\infty\right[$. We set

$$\mathcal{P}_\alpha(\mathcal{H})=\big\{M\in\mathcal{B}(\mathcal{H})\mid M^*=M\quad\text{and}\quad M\succcurlyeq\alpha\,\mathrm{Id}\big\}. \tag{2.1}$$

Moreover, for every $M\in\mathcal{P}_\alpha(\mathcal{H})$, we define respectively a scalar product and a norm by

$$(\forall x\in\mathcal{H})(\forall y\in\mathcal{H})\quad\langle x\mid y\rangle_M=\langle Mx\mid y\rangle\quad\text{and}\quad\|x\|_M=\sqrt{\langle Mx\mid x\rangle}. \tag{2.2}$$

Let $A\colon\mathcal{H}\to2^{\mathcal{H}}$ be a set-valued operator. The domain is $\operatorname{dom}A=\big\{x\in\mathcal{H}\mid Ax\neq\varnothing\big\}$, and the graph of A is $\operatorname{gra}A=\big\{(x,u)\in\mathcal{H}\times\mathcal{H}\mid u\in Ax\big\}$. The set of zeros of A is $\operatorname{zer}A=\big\{x\in\mathcal{H}\mid 0\in Ax\big\}$, and the range of A is $\operatorname{ran}A=\big\{u\in\mathcal{H}\mid(\exists\,x\in\mathcal{H})\,u\in Ax\big\}$. The inverse of A is $A^{-1}\colon\mathcal{H}\mapsto2^{\mathcal{H}}\colon u\mapsto\big\{x\in\mathcal{H}\mid u\in Ax\big\}$, and the resolvent of A is

$$J_A=(\mathrm{Id}+A)^{-1}. \tag{2.3}$$

Moreover, A is monotone if

$$(\forall(x,y)\in\mathcal{H}\times\mathcal{H})(\forall(u,v)\in Ax\times Ay)\quad\langle x-y\mid u-v\rangle\geq0, \tag{2.4}$$

and maximally monotone if it is monotone and there exists no monotone operator $B\colon\mathcal{H}\to2^{\mathcal{H}}$ such that $\operatorname{gra}A\subset\operatorname{gra}B$ and $A\neq B$. We say that A is uniformly monotone at $x\in\operatorname{dom}A$ if there exists an increasing function $\phi_A\colon\left[0,+\infty\right[\to\left[0,+\infty\right]$ vanishing only at 0 such that

$$\big(\forall u\in Ax\big)\big(\forall(y,v)\in\operatorname{gra}A\big)\quad\langle x-y\mid u-v\rangle\geq\phi_A(\|x-y\|). \tag{2.5}$$

3 Variable metric forward-backward-forward splitting algorithm

The forward-backward-forward splitting algorithm was first proposed in [19] to solve inclusion involving the sum of a maximally monotone operator and a monotone Lipschitzian operator. In [4], it was revisited to include computational errors. Below, we extend it to a variable metric setting.

Theorem 3.1 *Let $\boldsymbol{\mathcal{K}}$ be a real Hilbert space with the scalar product $\langle\langle\langle \cdot \mid \cdot \rangle\rangle\rangle$ and the associated norm $|||| \cdot ||||$. Let α and β be in $]0,+\infty[$, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^1_+(\mathbb{N})$, and let $(\boldsymbol{U}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}(\boldsymbol{\mathcal{K}})$ such that*

$$\mu = \sup_{n\in\mathbb{N}} \|\boldsymbol{U}_n\| < +\infty \quad \text{and} \quad (1+\eta_n)\boldsymbol{U}_{n+1} \succcurlyeq \boldsymbol{U}_n \in \mathcal{P}_\alpha(\boldsymbol{\mathcal{K}}). \tag{3.1}$$

Let $\boldsymbol{A}\colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}}$ be maximally monotone, let $\boldsymbol{B}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ be a monotone and β-Lipschitzian operator on $\boldsymbol{\mathcal{K}}$ such that $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$. Let $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ be absolutely summable sequences in $\boldsymbol{\mathcal{K}}$. Let $\boldsymbol{x}_0 \in \boldsymbol{\mathcal{K}}$, let $\varepsilon \in]0, 1/(\beta\mu+1)[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/(\beta\mu)]$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n \boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{p}_n = J_{\gamma_n \boldsymbol{U}_n \boldsymbol{A}} \boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n \boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{p}_n + \boldsymbol{c}_n) \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n. \end{array} \right. \tag{3.2}$$

Then the following hold for some $\overline{\boldsymbol{x}} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$.

(i) $\sum_{n\in\mathbb{N}} ||||\boldsymbol{x}_n - \boldsymbol{p}_n||||^2 < +\infty$ *and* $\sum_{n\in\mathbb{N}} ||||\boldsymbol{y}_n - \boldsymbol{q}_n||||^2 < +\infty$.

(ii) $\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$ *and* $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{x}}$.

(iii) *Suppose that one of the following is satisfied:*

  (a) $\underline{\lim}\, d_{\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})}(\boldsymbol{x}_n) = 0$.

  (b) $\boldsymbol{A}+\boldsymbol{B}$ *is demiregular (see [1, Definition 2.3]) at* $\overline{\boldsymbol{x}}$.

  (c) $\boldsymbol{A}$ *or* $\boldsymbol{B}$ *is uniformly monotone at* $\overline{\boldsymbol{x}}$.

  (d) $\operatorname{int}\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$ *and there exists* $(\nu_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$ *such that* $(\forall n \in \mathbb{N})\ (1+\nu_n)\boldsymbol{U}_n \succcurlyeq \boldsymbol{U}_{n+1}$.

  Then $\boldsymbol{x}_n \to \overline{\boldsymbol{x}}$ *and* $\boldsymbol{p}_n \to \overline{\boldsymbol{x}}$.

Proof. It follows from [11, Lemma 3.7] that the sequences $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{q}_n)_{n\in\mathbb{N}}$ are well defined. Moreover, using [10, Lemma 2.1(i)(ii)] and (3.1), we obtain

$$\big(\forall (\boldsymbol{z}_n)_{n\in\mathbb{N}} \in \boldsymbol{\mathcal{K}}^{\mathbb{N}}\big) \quad \sum_{n\in\mathbb{N}} ||||\boldsymbol{z}_n|||| < +\infty \quad \Leftrightarrow \quad \sum_{n\in\mathbb{N}} ||||\boldsymbol{z}_n||||_{\boldsymbol{U}_n^{-1}} < +\infty \tag{3.3}$$

and

$$\big(\forall (\boldsymbol{z}_n)_{n\in\mathbb{N}} \in \boldsymbol{\mathcal{K}}^{\mathbb{N}}\big) \quad \sum_{n\in\mathbb{N}} ||||\boldsymbol{z}_n|||| < +\infty \quad \Leftrightarrow \quad \sum_{n\in\mathbb{N}} ||||\boldsymbol{z}_n||||_{\boldsymbol{U}_n} < +\infty. \tag{3.4}$$

Let us set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \widetilde{\boldsymbol{y}}_n = \boldsymbol{x}_n - \gamma_n \boldsymbol{U}_n \boldsymbol{B} \boldsymbol{x}_n \\ \widetilde{\boldsymbol{p}}_n = J_{\gamma_n \boldsymbol{U}_n \boldsymbol{A}} \widetilde{\boldsymbol{y}}_n \\ \widetilde{\boldsymbol{q}}_n = \widetilde{\boldsymbol{p}}_n - \gamma_n \boldsymbol{U}_n \boldsymbol{B} \widetilde{\boldsymbol{p}}_n \\ \widetilde{\boldsymbol{x}}_{n+1} = \boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n, \end{cases} \quad \text{and} \quad \begin{cases} \boldsymbol{u}_n = \gamma_n^{-1} \boldsymbol{U}_n^{-1} (\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n) + \boldsymbol{B} \widetilde{\boldsymbol{p}}_n - \boldsymbol{B} \boldsymbol{x}_n \\ \boldsymbol{e}_n = \widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}_{n+1} \\ \boldsymbol{d}_n = \boldsymbol{q}_n - \widetilde{\boldsymbol{q}}_n + \widetilde{\boldsymbol{y}}_n - \boldsymbol{y}_n. \end{cases} \tag{3.5}$$

Then (3.5) yields

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{u}_n = \gamma_n^{-1} \boldsymbol{U}_n^{-1} (\ \widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n) + \boldsymbol{B} \widetilde{\boldsymbol{p}}_n \in \boldsymbol{A} \widetilde{\boldsymbol{p}}_n + \boldsymbol{B} \widetilde{\boldsymbol{p}}_n, \tag{3.6}$$

and (3.5), (3.2), Lemma [11, Lemma 3.7(ii)], and the Lipschitzianity of $\boldsymbol{B}$ on $\boldsymbol{\mathcal{K}}$ yield

$$(\forall n \in \mathbb{N}) \quad \begin{cases} ||||\boldsymbol{y}_n - \widetilde{\boldsymbol{y}}_n||||_{\boldsymbol{U}_n^{-1}} \leq (\beta\mu)^{-1} ||||\boldsymbol{a}_n||||_{\boldsymbol{U}_n} \\ ||||\boldsymbol{p}_n - \widetilde{\boldsymbol{p}}_n||||_{\boldsymbol{U}_n^{-1}} \leq ||||\boldsymbol{b}_n||||_{\boldsymbol{U}_n^{-1}} + (\beta\mu)^{-1} ||||\boldsymbol{a}_n||||_{\boldsymbol{U}_n} \\ ||||\boldsymbol{q}_n - \widetilde{\boldsymbol{q}}_n||||_{\boldsymbol{U}_n^{-1}} \leq 2\Big(||||\boldsymbol{b}_n||||_{\boldsymbol{U}_n^{-1}} + (\beta\mu)^{-1} ||||\boldsymbol{a}_n||||_{\boldsymbol{U}_n} \Big) + (\beta\mu)^{-1} ||||\boldsymbol{c}_n||||_{\boldsymbol{U}_n}. \end{cases} \tag{3.7}$$

Since $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ are absolutely summable sequences in $\boldsymbol{\mathcal{K}}$, we derive from (3.3), (3.4), (3.5), and (3.7) that

$$\begin{cases} \sum_{n\in\mathbb{N}} ||||\boldsymbol{p}_n - \widetilde{\boldsymbol{p}}_n|||| < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} ||||\boldsymbol{p}_n - \widetilde{\boldsymbol{p}}_n||||_{\boldsymbol{U}_n^{-1}} < +\infty \\ \sum_{n\in\mathbb{N}} ||||\boldsymbol{q}_n - \widetilde{\boldsymbol{q}}_n|||| < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} ||||\boldsymbol{q}_n - \widetilde{\boldsymbol{q}}_n||||_{\boldsymbol{U}_n^{-1}} < +\infty \\ \sum_{n\in\mathbb{N}} ||||\boldsymbol{d}_n|||| < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} ||||\boldsymbol{d}_n||||_{\boldsymbol{U}_n^{-1}} < +\infty. \end{cases} \tag{3.8}$$

Now, let $\boldsymbol{x} \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B})$. Then, for every $n \in \mathbb{N}$, $(\boldsymbol{x}, -\gamma_n \boldsymbol{U}_n \boldsymbol{B} \boldsymbol{x}) \in \operatorname{gra}(\gamma_n \boldsymbol{U}_n \boldsymbol{A})$ and (3.5) yields $(\widetilde{\boldsymbol{p}}_n, \widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n) \in \operatorname{gra}(\gamma_n \boldsymbol{U}_n \boldsymbol{A})$. Hence, by monotonicity of $\boldsymbol{U}_n \boldsymbol{A}$ with respect to the scalar product $\langle\langle\langle \cdot \mid \cdot \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}}$, we have $\langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \widetilde{\boldsymbol{p}}_n - \widetilde{\boldsymbol{y}}_n - \gamma_n \boldsymbol{U}_n \boldsymbol{B} \boldsymbol{x} \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \leq 0$. Moreover, by monotonicity of $\boldsymbol{U}_n \boldsymbol{B}$ with respect to the scalar product $\langle\langle\langle \cdot \mid \cdot \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}}$, we also have $\langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \gamma_n \boldsymbol{U}_n \boldsymbol{B} \boldsymbol{x} - \gamma_n \boldsymbol{U}_n \boldsymbol{B} \widetilde{\boldsymbol{p}}_n \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \leq 0$. By adding the last two inequalities, we obtain

$$(\forall n \in \mathbb{N}) \quad \langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \widetilde{\boldsymbol{p}}_n - \widetilde{\boldsymbol{y}}_n - \gamma_n \boldsymbol{U}_n \boldsymbol{B} \widetilde{\boldsymbol{p}}_n \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \leq 0. \tag{3.9}$$

In turn, we derive from (3.5) that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad & 2\gamma_n \langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \boldsymbol{U}_n \boldsymbol{B} \boldsymbol{x}_n - \boldsymbol{U}_n \boldsymbol{B} \widetilde{\boldsymbol{p}}_n \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \\ & = 2\langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \widetilde{\boldsymbol{p}}_n - \widetilde{\boldsymbol{y}}_n - \gamma_n \boldsymbol{U}_n \boldsymbol{B} \widetilde{\boldsymbol{p}}_n \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \\ & \quad + 2\langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \gamma_n \boldsymbol{U}_n \boldsymbol{B} \boldsymbol{x}_n + \widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \\ & \leq 2\langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \gamma_n \boldsymbol{U}_n \boldsymbol{B} \boldsymbol{x}_n + \widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \\ & = 2\langle\langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n \rangle\rangle\rangle_{\boldsymbol{U}_n^{-1}} \\ & = ||||\boldsymbol{x}_n - \boldsymbol{x}||||^2_{\boldsymbol{U}_n^{-1}} - ||||\widetilde{\boldsymbol{p}}_n - \boldsymbol{x}||||^2_{\boldsymbol{U}_n^{-1}} - ||||\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n|||^2_{\boldsymbol{U}_n^{-1}}. \end{aligned} \tag{3.10}$$

Hence, using (3.5), (3.10), the β-Lipschitz continuity of $\boldsymbol{B}$, (3.1), and [10, Lemma 2.1(ii)], for every $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
|||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}} &= |||\widetilde{\boldsymbol{q}}_n + \boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n - \boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}} \\
&= |||(\widetilde{\boldsymbol{p}}_n - \boldsymbol{x}) + \gamma_n \boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n)|||^2_{\boldsymbol{U}_n^{-1}} \\
&= |||\widetilde{\boldsymbol{p}}_n - \boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}} + 2\gamma_n \langle\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n \rangle\rangle \\
&\qquad + \gamma_n^2 |||\boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n)|||^2_{\boldsymbol{U}_n^{-1}} \\
&\le |||\boldsymbol{x}_n - \boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}} - |||\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n|||^2_{\boldsymbol{U}_n^{-1}} + \gamma_n^2 \mu \beta^2 |||\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n|||^2 \\
&\le |||\boldsymbol{x}_n - \boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}} - \mu^{-1} |||\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n|||^2 + \gamma_n^2 \mu \beta^2 |||\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n|||^2. \qquad (3.11)
\end{aligned}$$

Hence, it follows from (3.1) and [10, Lemma 2.1(i)] that

$$(\forall n \in \mathbb{N}) \quad |||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}|||^2_{\boldsymbol{U}_{n+1}^{-1}} \le (1+\eta_n)|||\boldsymbol{x}_n - \boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}} - \mu^{-1}(1 - \gamma_n^2 \beta^2 \mu^2)|||\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n|||^2. \quad (3.12)$$

Consequently,

$$(\forall n \in \mathbb{N}) \quad |||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}|||_{\boldsymbol{U}_{n+1}^{-1}} \le (1+\eta_n)|||\boldsymbol{x}_n - \boldsymbol{x}|||_{\boldsymbol{U}_n^{-1}}. \quad (3.13)$$

For every $n \in \mathbb{N}$, set

$$\varepsilon_n = \sqrt{\mu\alpha^{-1}}\Big(2\big(|||\boldsymbol{b}_n|||_{\boldsymbol{U}_n^{-1}} + (\beta\mu)^{-1}|||\boldsymbol{a}_n|||_{\boldsymbol{U}_n}\big) + (\beta\mu)^{-1}|||\boldsymbol{c}_n|||_{\boldsymbol{U}_n} + (\beta\mu)^{-1}|||\boldsymbol{a}_n|||_{\boldsymbol{U}_n}\Big). \quad (3.14)$$

Then $(\varepsilon_n)_{n\in\mathbb{N}}$ is summable by (3.3) and (3.4). We derive from [10, Lemma 2.1(ii)(iii)], and (3.8) that

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad |||\boldsymbol{e}_n|||_{\boldsymbol{U}_{n+1}^{-1}} &= |||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}_{n+1}|||_{\boldsymbol{U}_{n+1}^{-1}} \\
&\le \sqrt{\alpha^{-1}}|||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}_{n+1}||| \\
&\le \sqrt{\mu\alpha^{-1}}|||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}_{n+1}|||_{\boldsymbol{U}_n^{-1}} \\
&\le \sqrt{\mu\alpha^{-1}}(|||\widetilde{\boldsymbol{y}}_n - \boldsymbol{y}_n|||_{\boldsymbol{U}_n^{-1}} + |||\widetilde{\boldsymbol{q}}_n - \boldsymbol{q}_n|||_{\boldsymbol{U}_n^{-1}}) \\
&\le \varepsilon_n. \qquad (3.15)
\end{aligned}$$

In turn, we derive from (3.13) that

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad |||\boldsymbol{x}_{n+1} - \boldsymbol{x}|||_{\boldsymbol{U}_{n+1}^{-1}} &\le |||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}|||_{\boldsymbol{U}_{n+1}^{-1}} + |||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}_{n+1}|||_{\boldsymbol{U}_{n+1}^{-1}} \\
&\le |||\widetilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}|||_{\boldsymbol{U}_{n+1}^{-1}} + \varepsilon_n \\
&\le (1+\eta_n)|||\boldsymbol{x}_n - \boldsymbol{x}|||_{\boldsymbol{U}_n^{-1}} + \varepsilon_n. \qquad (3.16)
\end{aligned}$$

This shows that $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is $|\cdot|$–quasi-Fejér monotone with respect to the target set $\mathrm{zer}(\boldsymbol{A}+\boldsymbol{B})$ relative to $(\boldsymbol{U}_n^{-1})_{n\in\mathbb{N}}$. Moreover, by [10, Proposition 3.2], $(|||\boldsymbol{x}_n - \boldsymbol{x}|||_{\boldsymbol{U}_n^{-1}})_{n\in\mathbb{N}}$ is bounded. In turn, since $\boldsymbol{B}$ and $(J_{\gamma_n \boldsymbol{U}_n \boldsymbol{A}})_{n\in\mathbb{N}}$ are Lipschitzian, and $(\forall n \in \mathbb{N})$ $\boldsymbol{x} = J_{\gamma_n \boldsymbol{U}_n \boldsymbol{A}}(\boldsymbol{x} - \gamma_n \boldsymbol{U}_n \boldsymbol{B}\boldsymbol{x})$, we deduce from (3.5) that $(\widetilde{\boldsymbol{y}}_n)_{n\in\mathbb{N}}, (\widetilde{\boldsymbol{p}}_n)_{n\in\mathbb{N}}$, and $(\widetilde{\boldsymbol{q}}_n)_{n\in\mathbb{N}}$ are bounded. Therefore,

$$\tau = \sup_{n\in\mathbb{N}}\{|||\boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n - \boldsymbol{x}|||_{\boldsymbol{U}_n^{-1}}, |||\boldsymbol{x}_n - \boldsymbol{x}|||_{\boldsymbol{U}_n^{-1}}, 1+\eta_n\} < +\infty. \quad (3.17)$$

Hence, using (3.5), Cauchy-Schwarz for the norms $(|||\cdot|||_{\boldsymbol{U}_n^{-1}})_{n\in\mathbb{N}}$, and (3.11), we get

$$
\begin{aligned}
(\forall n\in\mathbb{N})\ \ |||\boldsymbol{x}_{n+1}-\boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}} &= |||\boldsymbol{x}_n-\boldsymbol{y}_n+\boldsymbol{q}_n-\boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}}\\
&= |||\widetilde{\boldsymbol{q}}_n+\boldsymbol{x}_n-\widetilde{\boldsymbol{y}}_n-\boldsymbol{x}+\boldsymbol{d}_n|||^2_{\boldsymbol{U}_n^{-1}}\\
&\leq |||\widetilde{\boldsymbol{q}}_n+\boldsymbol{x}_n-\widetilde{\boldsymbol{y}}_n-\boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}}+2\tau|||\boldsymbol{d}_n|||_{\boldsymbol{U}_n^{-1}}+|||\boldsymbol{d}_n|||^2_{\boldsymbol{U}_n^{-1}}\\
&\leq |||\boldsymbol{x}_n-\boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}}-\mu^{-1}(1-\gamma_n^2\beta^2\mu^2)|||\boldsymbol{x}_n-\widetilde{\boldsymbol{p}}_n|||^2+\varepsilon_{1,n},
\end{aligned}
\tag{3.18}
$$

where $(\forall n\in\mathbb{N})$ $\varepsilon_{1,n}=2\tau|||\boldsymbol{d}_n|||_{\boldsymbol{U}_n^{-1}}+|||\boldsymbol{d}_n|||^2_{\boldsymbol{U}_n^{-1}}$. In turn, for every $n\in\mathbb{N}$, by (3.1) and [10, Lemma 2.1(i)],

$$
\begin{aligned}
|||\boldsymbol{x}_{n+1}-\boldsymbol{x}|||^2_{\boldsymbol{U}_{n+1}^{-1}} &\leq (1+\eta_n)|||\boldsymbol{x}_{n+1}-\boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}}\\
&\leq |||\boldsymbol{x}_n-\boldsymbol{x}|||^2_{\boldsymbol{U}_n^{-1}}-\mu^{-1}(1-\gamma_n^2\beta^2\mu^2)|||\boldsymbol{x}_n-\widetilde{\boldsymbol{p}}_n|||^2+\tau\varepsilon_{1,n}+\tau^2\eta_n.
\end{aligned}
\tag{3.19}
$$

Since $(\tau\varepsilon_{1,n}+\tau^2\eta_n)_{n\in\mathbb{N}}\in\ell^1_+(\mathbb{N})$ by (3.8), it follows from [7, Lemma 3.1] that

$$
\sum_{n\in\mathbb{N}}|||\boldsymbol{x}_n-\widetilde{\boldsymbol{p}}_n|||^2<+\infty. \tag{3.20}
$$

(i): It follows from (3.20) and (3.8) that

$$
\sum_{n\in\mathbb{N}}|||\boldsymbol{x}_n-\boldsymbol{p}_n|||^2\leq 2\sum_{n\in\mathbb{N}}|||\boldsymbol{x}_n-\widetilde{\boldsymbol{p}}_n|||^2+2\sum_{n\in\mathbb{N}}|||\boldsymbol{p}_n-\widetilde{\boldsymbol{p}}_n|||^2<+\infty. \tag{3.21}
$$

Furthermore, we derive from (3.8) and (3.5) that

$$
\begin{aligned}
\sum_{n\in\mathbb{N}}|||\boldsymbol{y}_n-\boldsymbol{q}_n|||^2 &= \sum_{n\in\mathbb{N}}|||\widetilde{\boldsymbol{q}}_n-\widetilde{\boldsymbol{y}}_n+\boldsymbol{d}_n|||^2\\
&= \sum_{n\in\mathbb{N}}|||\widetilde{\boldsymbol{p}}_n-\boldsymbol{x}_n+\gamma_n\boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{x}_n-\boldsymbol{B}\widetilde{\boldsymbol{p}}_n)+\boldsymbol{d}_n|||^2\\
&\leq 3\Big(\sum_{n\in\mathbb{N}}|||\boldsymbol{x}_n-\widetilde{\boldsymbol{p}}_n|||^2+|||\gamma_n\boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{x}_n-\boldsymbol{B}\widetilde{\boldsymbol{p}}_n)|||^2+|||\boldsymbol{d}_n|||^2\Big)\\
&< +\infty.
\end{aligned}
\tag{3.22}
$$

(ii): Let $\boldsymbol{x}$ be a weak cluster point of $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$. Then there exists a subsequence $(\boldsymbol{x}_{k_n})_{n\in\mathbb{N}}$ that converges weakly to $\boldsymbol{x}$. Therefore $\widetilde{\boldsymbol{p}}_{k_n}\rightharpoonup\boldsymbol{x}$ by (3.20). Furthermore, it follows from (3.5) that $\boldsymbol{u}_{k_n}\to 0$. Hence, since $(\forall n\in\mathbb{N})$ $(\widetilde{\boldsymbol{p}}_{k_n},\boldsymbol{u}_{k_n})\in\operatorname{gra}(\boldsymbol{A}+\boldsymbol{B})$, we obtain, $\boldsymbol{x}\in\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$ [2, Proposition 20.33(ii)]. Altogether, it follows [10, Lemma 2.3(ii)] and [10, Theorem 3.3] that $\boldsymbol{x}_n\rightharpoonup\overline{\boldsymbol{x}}$ and hence that $\boldsymbol{p}_n\rightharpoonup\overline{\boldsymbol{x}}$ by (i).

(iii)(a): Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are maximally monotone and $\operatorname{dom}\boldsymbol{B}=\boldsymbol{\mathcal{K}}$, $\boldsymbol{A}+\boldsymbol{B}$ is maximally monotone [2, Corollary 24.4(i)], $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$ is therefore closed [2, Proposition 23.39]. Hence, the claims follow from (i), (3.16), and [10, Proposition 3.4].

(iii)(b): By (i), $\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$, and hence (3.20) implies that $\widetilde{\boldsymbol{p}}_n \rightharpoonup \overline{\boldsymbol{x}}$. Furthermore, it follows from (3.5) that $\boldsymbol{u}_n \to 0$. Hence, since $(\forall n \in \mathbb{N})\ (\widetilde{\boldsymbol{p}}_n, \boldsymbol{u}_n) \in \operatorname{gra}(\boldsymbol{A}+\boldsymbol{B})$ and since $\boldsymbol{A}+\boldsymbol{B}$ is demiregular at $\overline{\boldsymbol{x}}$, by [1, Definition 2.3], $\widetilde{\boldsymbol{p}}_n \to \overline{\boldsymbol{x}}$, and therefore (3.20) implies that $\boldsymbol{x}_n \to \overline{\boldsymbol{x}}$.

(iii)(c): If $\boldsymbol{A}$ or $\boldsymbol{B}$ is uniformly monotone at $\overline{\boldsymbol{x}}$, then $\boldsymbol{A}+\boldsymbol{B}$ is uniformly monotone at $\overline{x}$. Therefore, the result follows from [1, Proposition 2.4(i)].

(iii)(d): Suppose that $\boldsymbol{z} \in \operatorname{int}\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$ and fix $\rho \in \,]0,+\infty[$ such that $B(\boldsymbol{z};\rho) \subset \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$. It follows from (3.16) and [10, Proposition 3.2] that

$$\varepsilon = \sup_{\boldsymbol{x}\in B(\boldsymbol{z};\rho)} \sup_{n\in\mathbb{N}} ||||\boldsymbol{x}_n - \boldsymbol{x}||||_{\boldsymbol{U}_n^{-1}} \leq (1/\sqrt{\alpha})\big(\sup_{n\in\mathbb{N}} ||||\boldsymbol{x}_n - \boldsymbol{z}|||| + \sup_{\boldsymbol{x}\in B(\boldsymbol{z};\rho)} ||||\boldsymbol{x}-\boldsymbol{z}||||\big) < +\infty \quad (3.23)$$

and from (3.16) that

$$(\forall n \in \mathbb{N})(\forall \boldsymbol{x} \in B(\boldsymbol{z};\rho))\ ||||\boldsymbol{x}_{n+1} - \boldsymbol{x}||||^2_{\boldsymbol{U}_{n+1}^{-1}} \leq ||||\boldsymbol{x}_n - \boldsymbol{x}||||^2_{\boldsymbol{U}_n^{-1}} + 2\varepsilon(\varepsilon\eta_n + \varepsilon_n) + (\varepsilon\eta_n + \varepsilon_n)^2. \quad (3.24)$$

Hence, the claim follows from (i), [10, Lemma 2.1], and [10, Proposition 4.3]. □

Remark 3.2 Here are some remarks.

(i) In the case when $(\forall n \in \mathbb{N})\ \boldsymbol{U}_n = \mathbf{Id}$, the standard forward-backward-forward splitting algorithm (3.2) reduces to algorithm proposed in [4, Eq. (2.3)], which was proposed initially in the error-free setting in [19].

(ii) An alternative variable metric splitting algorithm proposed in [14] can be used to find a zero of the sum of a maximally monotone operator $\boldsymbol{A}$ and a Lipschitzian monotone operator $\boldsymbol{B}$ in instance when $\boldsymbol{\mathcal{K}}$ is finite-dimensional. This algorithm uses a different error model and involves more iteration-dependent variables than (3.2).

Example 3.3 Let $\boldsymbol{f}\colon \boldsymbol{\mathcal{K}} \to [-\infty,+\infty]$ be a proper lower semicontinuous convex function, let $\alpha \in \,]0,+\infty[$, let $\beta \in \,]0,+\infty[$, let $\boldsymbol{B}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ be a monotone and β-Lipschitzian operator, let $(\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$, and let $(\boldsymbol{U}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\boldsymbol{\mathcal{K}})$ that satisfies (3.1). Furthermore, let $\boldsymbol{x}_0 \in \boldsymbol{\mathcal{K}}$, let $\varepsilon \in \,]0, \min\{1, 1/(\mu\beta+1)\}[$, where μ is defined as in (3.1), let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/(\beta\mu)]$. Suppose that the variational inequality

$$\text{find} \quad \bar{\boldsymbol{x}} \in \boldsymbol{\mathcal{K}} \quad \text{such that} \quad (\forall \boldsymbol{y} \in \boldsymbol{\mathcal{K}}) \quad \langle \bar{\boldsymbol{x}} - \boldsymbol{y} \mid \boldsymbol{B}\bar{\boldsymbol{x}}\rangle + \boldsymbol{f}(\bar{\boldsymbol{x}}) \leq \boldsymbol{f}(\boldsymbol{y}) \quad (3.25)$$

admits at least one solution and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n \boldsymbol{U}_n \boldsymbol{B}\boldsymbol{x}_n \\ \boldsymbol{p}_n = \arg\min_{\boldsymbol{x}\in\boldsymbol{\mathcal{K}}} \big(\boldsymbol{f}(\boldsymbol{x}) + \frac{1}{2\gamma_n} ||||\boldsymbol{x} - \boldsymbol{y}_n||||^2_{\boldsymbol{U}_n^{-1}}\big) \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n \boldsymbol{U}_n \boldsymbol{B}\boldsymbol{p}_n \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n. \end{array} \right. \quad (3.26)$$

Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges weakly to a solution $\bar{\boldsymbol{x}}$ to (3.25).

Proof. Set $\boldsymbol{A} = \partial \boldsymbol{f}$ and $(\forall n \in \mathbb{N})\ \boldsymbol{a}_n = 0, \boldsymbol{b}_n = 0, \boldsymbol{c}_n = 0$ in Theorem 3.1(ii). □

4 Monotone inclusions involving Lipschitzian operators

The applications of the forward-backward-forward splitting algorithm considered in [4, 9, 19] can be extended to a variable metric setting using Theorem 3.1. As an illustration, we present a variable metric version of the algorithm proposed in [9, Eq. (3.1)]. Recall that the parallel sum of $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ is [2]

$$A \,\square\, B = (A^{-1} + B^{-1})^{-1}. \tag{4.1}$$

Problem 4.1 Let $\mathcal{H}$ be a real Hilbert space, let m be a strictly positive integer, let $z \in \mathcal{H}$, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone operator, let $C\colon \mathcal{H} \to \mathcal{H}$ be monotone and ν_0-Lipschitzian for some $\nu_0 \in \,]0, +\infty[$. For every $i \in \{1, \dots, m\}$, let $\mathcal{G}_i$ be a real Hilbert space, let $r_i \in \mathcal{G}_i$, let $B_i\colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone operator, let $D_i\colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be monotone and such that D_i^{-1} is ν_i-Lipschitzian for some $\nu_i \in \,]0, +\infty[$, and let $L_i\colon \mathcal{H} \to \mathcal{G}_i$ is a nonzero bounded linear operator. Suppose that

$$z \in \operatorname{ran}\left(A + \sum_{i=1}^{m} L_i^*\big((B_i \,\square\, D_i)(L_i \cdot - r_i)\big) + C\right). \tag{4.2}$$

The problem is to solve the primal inclusion

$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } z \in A\overline{x} + \sum_{i=1}^{m} L_i^*\big((B_i \,\square\, D_i)(L_i\overline{x} - r_i)\big) + C\overline{x}, \tag{4.3}$$

and the dual inclusion

$$\text{find } \overline{v}_1 \in \mathcal{G}_1, \dots, \overline{v}_m \in \mathcal{G}_m \text{ such that } (\exists x \in \mathcal{H}) \begin{cases} z - \sum_{i=1}^{m} L_i^*\overline{v}_i \in Ax + Cx, \\ (\forall i \in \{1, \dots, m\})\ \overline{v}_i \in (B_i \,\square\, D_i)(L_i x - r_i). \end{cases} \tag{4.4}$$

As shown in [9], Problem 4.1 covers a wide class of problems in nonlinear analysis and convex optimization problems. However, the algorithm in [9, Theorem 3.1] is studied in the context of a fixed metric. The following result extends this result to a variable metric setting.

Corollary 4.2 *Let α be in $]0, +\infty[$, let $(\eta_{0,n})_{n\in\mathbb{N}}$ be a sequence in $\ell_+^1(\mathbb{N})$, let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$, and for every $i \in \{1, \dots, m\}$, let $(\eta_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\ell_+^1(\mathbb{N})$, let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_i)$ such that $\mu = \sup_{n\in\mathbb{N}}\{\|U_n\|, \|U_{1,n}\|, \dots, \|U_{m,n}\|\} < +\infty$ and*

$$(\forall n \in \mathbb{N}) \quad (1 + \eta_{0,n})U_{n+1} \succcurlyeq U_n, \quad \text{and} \quad (\forall i \in \{1, \dots, m\}) \quad (1 + \eta_{i,n})U_{i,n+1} \succcurlyeq U_{i,n}. \tag{4.5}$$

Let $(a_{1,n})_{n\in\mathbb{N}}, (b_{1,n})_{n\in\mathbb{N}}$, and $(c_{1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, and for every $i \in \{1, \dots, m\}$, let $(a_{2,i,n})_{n\in\mathbb{N}}, (b_{2,i,n})_{n\in\mathbb{N}}$, and $(c_{2,i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$. Furthermore, set

$$\beta = \max\{\nu_0, \nu_1, \dots, \nu_m\} + \sqrt{\sum_{i=1}^{m} \|L_i\|^2}, \tag{4.6}$$

let $x_0 \in \mathcal{H}$*, let* $(v_{1,0}, \ldots, v_{m,0}) \in \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_m$*, let* $\varepsilon \in \left]0, 1/(1+\beta\mu)\right[$*, let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (1-\varepsilon)/(\beta\mu)]$*. Set*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_n - \gamma_n U_n\big(Cx_n + \sum_{i=1}^m L_i^* v_{i,n} + a_{1,n}\big) \\ p_{1,n} = J_{\gamma_n U_n A}(y_{1,n} + \gamma_n U_n z) + b_{1,n} \\ \text{for } i = 1, \ldots, m \\ \left\lfloor \begin{array}{l} y_{2,i,n} = v_{i,n} + \gamma_n U_{i,n}\big(L_i x_n - D_i^{-1} v_{i,n} + a_{2,i,n}\big) \\ p_{2,i,n} = J_{\gamma_n U_{i,n} B_i^{-1}}(y_{2,i,n} - \gamma_n U_{i,n} r_i) + b_{2,i,n} \\ q_{2,i,n} = p_{2,i,n} + \gamma_n U_{i,n}\big(L_i p_{1,n} - D_i^{-1} p_{2,i,n} + c_{2,i,n}\big) \\ v_{i,n+1} = v_{i,n} - y_{2,i,n} + q_{2,i,n} \end{array} \right. \\ q_{1,n} = p_{1,n} - \gamma_n U_n\big(Cp_{1,n} + \sum_{i=1}^m L_i^* p_{2,i,n} + c_{1,n}\big) \\ x_{n+1} = x_n - y_{1,n} + q_{1,n}. \end{array} \right. \tag{4.7}$$

Then the following hold.

(i) $\sum_{n\in\mathbb{N}} \|x_n - p_{1,n}\|^2 < +\infty$ *and* $(\forall i \in \{1, \ldots, m\})$ $\sum_{n\in\mathbb{N}} \|v_{i,n} - p_{2,i,n}\|^2 < +\infty$.

(ii) *There exist a solution* $\overline{x}$ *to* (4.3) *and a solution* $(\overline{v}_1, \ldots, \overline{v}_m)$ *to* (4.4) *such that the following hold.*

  (a) $x_n \rightharpoonup \overline{x}$ *and* $p_{1,n} \rightharpoonup \overline{x}$.

  (b) $(\forall i \in \{1, \ldots, m\})$ $v_{i,n} \rightharpoonup \overline{v}_i$ *and* $p_{2,i,n} \rightharpoonup \overline{v}_i$.

  (c) *Suppose that* A *or* C *is uniformly monotone at* $\overline{x}$*, then* $x_n \to \overline{x}$ *and* $p_{1,n} \to \overline{x}$.

  (d) *Suppose that* B_j^{-1} *or* D_j^{-1} *is uniformly monotone at* $\overline{v}_j$*, for some* $j \in \{1, \ldots, m\}$*, then* $v_{j,n} \to \overline{v}_j$ *and* $p_{2,j,n} \to \overline{v}_j$.

Proof. All sequences generated by algorithm (4.7) are well defined by [11, Lemma 3.7]. We define $\boldsymbol{\mathcal{K}} = \mathcal{H} \oplus \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$ the Hilbert direct sum of the Hilbert spaces $\mathcal{H}$ and $(\mathcal{G}_i)_{1\le i\le m}$, the scalar product and the associated norm of $\boldsymbol{\mathcal{K}}$ respectively defined by

$$\langle\langle\langle \cdot \mid \cdot \rangle\rangle\rangle \colon ((x, \boldsymbol{v}), (y, \boldsymbol{w})) \mapsto \langle x \mid y\rangle + \sum_{i=1}^m \langle v_i \mid w_i\rangle \quad \text{and} \quad |||\cdot||| \colon (x, \boldsymbol{v}) \mapsto \sqrt{\|x\|^2 + \sum_{i=1}^m \|v_i\|^2}, \tag{4.8}$$

where $\boldsymbol{v} = (v_1, \ldots, v_m)$ and $\boldsymbol{w} = (w_1, \ldots, w_m)$ are generic elements in $\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$. Set

$$\begin{cases} \boldsymbol{A} \colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}} \colon (x, v_1, \ldots, v_m) \mapsto (-z + Ax) \times (r_1 + B_1^{-1} v_1) \times \ldots \times (r_m + B_m^{-1} v_m) \\ \boldsymbol{B} \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (x, v_1, \ldots, v_m) \mapsto \Big(Cx + \sum_{i=1}^m L_i^* v_i, D_1^{-1} v_1 - L_1 x, \ldots, D_m^{-1} v_m - L_m x\Big) \\ (\forall n \in \mathbb{N}) \quad \boldsymbol{U}_n \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (x, v_1, \ldots, v_m) \mapsto \big(U_n x, U_{1,n} v_1, \ldots U_{m,n} v_m\big). \end{cases} \tag{4.9}$$

Since $\boldsymbol{A}$ is maximally monotone [2, Propositions 20.22 and 20.23], $\boldsymbol{B}$ is monotone and β-Lipschitzian [9, Eq. (3.10)] with $\operatorname{dom} \boldsymbol{B} = \boldsymbol{\mathcal{K}}$, $\boldsymbol{A}+\boldsymbol{B}$ is maximally monotone [2, Corollary 24.24(i)]. Now set $(\forall n \in \mathbb{N})$ $\eta_n = \max\{\eta_{0,n}, \eta_{1,n}, \ldots, \eta_{m,n}\}$. Then $(\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$. Moreover, we derive from our assumptions on the sequences $(U_n)_{n\in\mathbb{N}}$ and $(U_{1,n})_{n\in\mathbb{N}}, \ldots, (U_{m,n})_{n\in\mathbb{N}}$ that

$$\mu = \sup_{n\in\mathbb{N}} \|\boldsymbol{U}_n\| < +\infty \quad \text{and} \quad (1+\eta_n)\boldsymbol{U}_{n+1} \succcurlyeq \boldsymbol{U}_n \in \mathcal{P}_\alpha(\boldsymbol{\mathcal{K}}). \tag{4.10}$$

In addition, [2, Propositions 23.15(ii) and 23.16] yield $(\forall\gamma\in\,]0,+\infty[)(\forall n\in\mathbb{N})(\forall(x,v_1,\ldots,v_m)\in\boldsymbol{\mathcal{K}})$

$$J_{\gamma\boldsymbol{U}_n\boldsymbol{A}}(x,v_1,\ldots,v_m)=\Big(J_{\gamma U_nA}(x+\gamma U_nz),\big(J_{\gamma U_{i,n}B_i^{-1}}(v_i-\gamma U_{i,n}r_i)\big)_{1\le i\le m}\Big). \tag{4.11}$$

It is shown in [9, Eq. (3.12)] and [9, Eq. (3.13)] that under the condition (4.2), $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})\neq\varnothing$. Moreover, [9, Eq. (3.21)] and [9, Eq. (3.22)] yield

$$(\overline{x},\overline{v}_1,\ldots,\overline{v}_m)\in\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})\Rightarrow\overline{x}\text{ solves (4.3) and }(\overline{v}_1,\ldots,\overline{v}_m)\text{ solves (4.4)}. \tag{4.12}$$

Let us next set

$$(\forall n\in\mathbb{N})\quad\begin{cases}\boldsymbol{x}_n=(x_n,v_{1,n},\ldots,v_{m,n})\\ \boldsymbol{y}_n=(y_{1,n},y_{2,1,n},\ldots,y_{2,m,n})\\ \boldsymbol{p}_n=(p_{1,n},p_{2,1,n},\ldots,p_{2,m,n})\\ \boldsymbol{q}_n=(q_{1,n},q_{2,1,n},\ldots,q_{2,m,n})\end{cases}\quad\text{and}\quad\begin{cases}\boldsymbol{a}_n=(a_{1,n},a_{2,1,n},\ldots,a_{2,m,n})\\ \boldsymbol{b}_n=(b_{1,n},b_{2,1,n},\ldots,b_{2,m,n})\\ \boldsymbol{c}_n=(c_{1,n},c_{2,1,n},\ldots,c_{2,m,n}).\end{cases} \tag{4.13}$$

Then our assumptions imply that

$$\sum_{n\in\mathbb{N}}||||\boldsymbol{a}_n||||<\infty,\quad\sum_{n\in\mathbb{N}}||||\boldsymbol{b}_n||||<\infty,\quad\text{and}\quad\sum_{n\in\mathbb{N}}||||\boldsymbol{c}_n||||<\infty. \tag{4.14}$$

Furthermore, it follows from the definition of $\boldsymbol{B}$, (4.11), and (4.13) that (4.7) can be rewritten in $\boldsymbol{\mathcal{K}}$ as

$$(\forall n\in\mathbb{N})\quad\left\lfloor\begin{array}{l}\boldsymbol{y}_n=\boldsymbol{x}_n-\gamma_n\boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{x}_n+\boldsymbol{a}_n)\\ \boldsymbol{p}_n=J_{\gamma_n\boldsymbol{U}_n\boldsymbol{A}}\boldsymbol{y}_n+\boldsymbol{b}_n\\ \boldsymbol{q}_n=\boldsymbol{p}_n-\gamma_n\boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{p}_n+\boldsymbol{c}_n)\\ \boldsymbol{x}_{n+1}=\boldsymbol{x}_n-\boldsymbol{y}_n+\boldsymbol{q}_n,\end{array}\right. \tag{4.15}$$

which is (3.2). Moreover, every specific conditions in Theorem 3.1 are satisfied.

(i): By Theorem 3.1(i), $\sum_{n\in\mathbb{N}}||||\boldsymbol{x}_n-\boldsymbol{p}_n||||^2<\infty$.

(ii)(a)&(ii)(b): These assertions follow from Theorem 3.1(ii).

(ii)(c): Theorem 3.1(ii) shows that $(\overline{x},\overline{v}_1,\ldots,\overline{v}_m)\in\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$. Hence, it follows from [9, Eq (3.19)] that $(\overline{x},\overline{v}_1,\ldots,\overline{v}_m)$ satisfies the inclusions

$$\begin{cases}-\sum_{i=1}^m L_i^*\overline{v}_i-C\overline{x}\in -z+A\overline{x}\\ (\forall i\in\{1,\ldots,m\})\ L_i\overline{x}-D_i^{-1}\overline{v}_i\in r_i+B_i^{-1}\overline{v}_i.\end{cases} \tag{4.16}$$

For every $n\in\mathbb{N}$ and every $i\in\{1,\ldots,m\}$, set

$$\begin{cases}\widetilde{y}_{1,n}=x_n-\gamma_nU_n\big(Cx_n+\sum_{i=1}^m L_i^*v_{i,n}\big)\\ \widetilde{p}_{1,n}=J_{\gamma_nU_nA}(\widetilde{y}_{1,n}+\gamma_nU_nz)\end{cases}\quad\text{and}\quad\begin{cases}\widetilde{y}_{2,i,n}=v_{i,n}+\gamma_nU_{i,n}\big(L_ix_n-D_i^{-1}v_{i,n}\big)\\ \widetilde{p}_{2,i,n}=J_{\gamma_nU_{i,n}B_i^{-1}}(\widetilde{y}_{2,i,n}-\gamma_nU_{i,n}r_i).\end{cases} \tag{4.17}$$

Then, using [11, Lemma 3.7], we get

$$\widetilde{p}_{1,n} - p_{1,n} \to 0 \quad \text{and} \quad (\forall i \in \{1,\ldots,m\}) \quad \widetilde{p}_{2,i,n} - p_{2,i,n} \to 0, \tag{4.18}$$

in turn, by (i),(ii)(a), and (ii)(b), we obtain

$$\widetilde{p}_{1,n} - x_n \to 0, \quad \widetilde{p}_{1,n} \rightharpoonup \overline{x}, \quad \text{and} \quad (\forall i \in \{1,\ldots,m\}) \quad \widetilde{p}_{2,i,n} - v_{i,n} \to 0, \quad \widetilde{p}_{2,i,n} \rightharpoonup \overline{v}_i. \tag{4.19}$$

Furthermore, we derive from (4.17) that

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \gamma_n^{-1} U_n^{-1}(x_n - \widetilde{p}_{1,n}) - \sum_{i=1}^m L_i^* v_{i,n} - C x_n \in -z + A\widetilde{p}_{1,n} \\ (\forall i \in \{1,\ldots,m\})\ \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \widetilde{p}_{2,i,n}) + L_i x_n - D_i^{-1} v_{i,n} \in r_i + B_i^{-1}\widetilde{p}_{2,i,n}. \end{cases} \tag{4.20}$$

Since A is uniformly monotone at $\overline{x}$, using (4.16) and (4.20), there exists an increasing function $\phi_A\colon\ [0,+\infty[\, \to [0,+\infty]$ vanishing only at 0 such that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \phi_A(\|\widetilde{p}_{1,n} - \overline{x}\|) &\leqslant \left\langle \widetilde{p}_{1,n} - \overline{x} \mid \gamma_n^{-1} U_n^{-1}(x_n - \widetilde{p}_{1,n}) - \sum_{i=1}^m (L_i^* v_{i,n} - L_i^* \overline{v}_i) - (C x_n - C\bar{x}) \right\rangle \\ &= \left\langle \widetilde{p}_{1,n} - \overline{x} \mid \gamma_n^{-1} U_n^{-1}(x_n - \widetilde{p}_{1,n}) \right\rangle - \sum_{i=1}^m \langle \widetilde{p}_{1,n} - \overline{x} \mid L_i^* v_{i,n} - L_i^* \overline{v}_i \rangle - \chi_n, \end{aligned} \tag{4.21}$$

where we denote $\big(\forall n \in \mathbb{N}\big)\ \chi_n = \langle \widetilde{p}_{1,n} - \bar{x} \mid C x_n - C\bar{x} \rangle$. Since $(B_i^{-1})_{1 \le i \le m}$ are monotone, for every $i \in \{1,\ldots,m\}$, we obtain

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad 0 &\leqslant \left\langle \widetilde{p}_{2,i,n} - \overline{v}_i \mid L_i x_n + \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \widetilde{p}_{2,i,n}) - L_i \overline{x} - (D_i^{-1} v_{i,n} - D_i^{-1} \overline{v}_i) \right\rangle \\ &= \left\langle \widetilde{p}_{2,i,n} - \overline{v}_i \mid L_i(x_n - \overline{x}) + \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \widetilde{p}_{2,i,n}) \right\rangle - \beta_{i,n}, \end{aligned} \tag{4.22}$$

where $\big(\forall n \in \mathbb{N}\big)\ \beta_{i,n} = \langle \widetilde{p}_{2,i,n} - \overline{v}_i \mid D_i^{-1} v_{i,n} - D_i^{-1} \overline{v}_i \rangle$. Now, adding (4.22) from $i = 1$ to $i = m$ and (4.21), we obtain, for every $n \in \mathbb{N}$,

$$\begin{aligned} \phi_A(\|\widetilde{p}_{1,n} - \overline{x}\|) \le{}& \left\langle \widetilde{p}_{1,n} - \overline{x} \mid \gamma_n^{-1} U_n^{-1}(x_n - \widetilde{p}_{1,n}) \right\rangle + \left\langle \widetilde{p}_{1,n} - \overline{x} \mid \sum_{i=1}^m L_i^*(\widetilde{p}_{2,i,n} - v_{i,n}) \right\rangle \\ &+ \sum_{i=1}^m \left\langle \widetilde{p}_{2,i,n} - \overline{v}_i \mid L_i(x_n - \widetilde{p}_{1,n}) + \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \widetilde{p}_{2,i,n}) \right\rangle - \chi_n - \sum_{i=1}^m \beta_{i,n}. \end{aligned} \tag{4.23}$$

For every $n \in \mathbb{N}$ and every $i \in \{1,\ldots,m\}$, we expand χ_n and $\beta_{i,n}$ as

$$\begin{cases} \chi_n = \langle x_n - \overline{x} \mid C x_n - C\overline{x} \rangle + \langle \widetilde{p}_{1,n} - x_n \mid C x_n - C\overline{x} \rangle, \\ \beta_{i,n} = \left\langle v_{i,n} - \overline{v}_i \mid D_i^{-1} v_{i,n} - D_i^{-1} \overline{v}_i \right\rangle + \left\langle \widetilde{p}_{2,i,n} - v_{i,n} \mid D_i^{-1} v_{i,n} - D_i^{-1} \overline{v}_i \right\rangle. \end{cases} \tag{4.24}$$

By monotonicity of C and $(D_i^{-1})_{1 \le i \le m}$,

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \langle x_n - \overline{x} \mid C x_n - C\overline{x} \rangle \ge 0, \\ (\forall i \in \{1,\ldots,m\})\ \left\langle v_{i,n} - \overline{v}_i \mid D_i^{-1} v_{i,n} - D_i^{-1} \overline{v}_i \right\rangle \ge 0. \end{cases} \tag{4.25}$$

Therefore, for every $n \in \mathbb{N}$, we derive from (4.24) and (4.23) that

$$\begin{aligned}
\phi_A(\|\widetilde{p}_{1,n} - \overline{x}\|) &\leq \phi_A(\|\widetilde{p}_{1,n} - \overline{x}\|) + \langle x_n - \overline{x} \mid Cx_n - C\overline{x}\rangle + \sum_{i=1}^m \big\langle v_{i,n} - \overline{v}_i \mid D_i^{-1} v_{i,n} - D_i^{-1}\overline{v}_i\big\rangle \\
&\leq \big\langle \widetilde{p}_{1,n} - \overline{x} \mid \gamma_n^{-1} U_n^{-1}(x_n - \widetilde{p}_{1,n})\big\rangle + \left\langle \widetilde{p}_{1,n} - \overline{x} \mid \sum_{i=1}^m L_i^*(\widetilde{p}_{2,i,n} - v_{i,n})\right\rangle \\
&\quad + \sum_{i=1}^m \Big\langle \widetilde{p}_{2,i,n} - \overline{v}_i \mid L_i(x_n - \widetilde{p}_{1,n}) + \gamma_n^{-1} U_{i,n}^{-1}(v_{i,n} - \widetilde{p}_{2,i,n})\Big\rangle \\
&\quad - \langle \widetilde{p}_{1,n} - x_n \mid Cx_n - C\overline{x}\rangle - \sum_{i=1}^m \big\langle \widetilde{p}_{2,i,n} - v_{i,n} \mid D_i^{-1} v_{i,n} - D_i^{-1}\overline{v}_i\big\rangle. \qquad (4.26)
\end{aligned}$$

We set

$$\zeta = \max_{1\leq i\leq m} \sup_{n\in\mathbb{N}} \{\|x_n - \overline{x}\|, \|\widetilde{p}_{1,n} - \overline{x}\|, \|v_{i,n} - \overline{v}_i\|, \|\widetilde{p}_{2,i,n} - \overline{v}_i\|\}. \qquad (4.27)$$

Then it follows from (ii)(a), (ii)(b), and (4.19) that $\zeta < \infty$, and from [10, Lemma 2.1(ii)] that $(\forall n \in \mathbb{N})$ $\|\gamma_n^{-1} U_n^{-1}\| \leq (\varepsilon\alpha)^{-1}$ and $(\forall i \in \{1, \ldots, m\})$ $\|\gamma_n^{-1} U_{i,n}^{-1}\| \leq (\varepsilon\alpha)^{-1}$. Therefore, using the Cauchy-Schwarz inequality, and the Lipschitzianity of C and $(D_i^{-1})_{1\leq i\leq m}$, we derive from (4.26) that

$$\begin{aligned}
\phi_A(\|\widetilde{p}_{1,n} - \overline{x}\|) &\leq (\varepsilon\alpha)^{-1}\zeta\|x_n - \widetilde{p}_{1,n}\| + \zeta\sum_{i=1}^m \big(\|L_i\|\,\|x_n - \widetilde{p}_{1,n}\| + (\varepsilon\alpha)^{-1}\|v_{i,n} - \widetilde{p}_{2,i,n}\|\big) \\
&\quad + \zeta\Big(\sum_{i=1}^m \|L_i^*\|\|\widetilde{p}_{2,i,n} - v_{i,n}\| + \nu_0\|\widetilde{p}_{1,n} - x_n\| + \sum_{i=1}^m \nu_i\|\widetilde{p}_{2,i,n} - v_{i,n}\|\Big) \\
&\to 0. \qquad (4.28)
\end{aligned}$$

We deduce from (4.28) and (4.19) that $\phi_A(\|\widetilde{p}_{1,n} - \overline{x}\|) \to 0$, which implies that $\widetilde{p}_{1,n} \to \overline{x}$. In turn, $x_n \to \overline{x}$ and $p_n \to \overline{x}$. Likewise, if C is uniformly monotone at $\overline{x}$, there exists an increasing function $\phi_C \colon \left[0, +\infty\right[\to [0, +\infty]$ that vanishes only at 0 such that

$$\begin{aligned}
\phi_C(\|x_n - \overline{x}\|) &\leq (\varepsilon\alpha)^{-1}\zeta\|x_n - \widetilde{p}_{1,n}\| + \zeta\sum_{i=1}^m \big(\|L_i\|\,\|x_n - \widetilde{p}_{1,n}\| + (\varepsilon\alpha)^{-1}\|v_{i,n} - \widetilde{p}_{2,i,n}\|\big) \\
&\quad + \zeta\Big(\sum_{i=1}^m \|L_i^*\|\|\widetilde{p}_{2,i,n} - v_{i,n}\| + \nu_0\|\widetilde{p}_{1,n} - x_n\| + \sum_{i=1}^m \nu_i\|\widetilde{p}_{2,i,n} - v_{i,n}\|\Big) \\
&\to 0, \qquad (4.29)
\end{aligned}$$

in turn, $x_n \to \overline{x}$ and $p_n \to \overline{x}$.

(ii)(d): Proceeding as in the proof of (ii)(c), we obtain the conclusions. □

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