On the Truncated Weibull Distribution and its Usefulness in Evaluating the Theoretical Capacity Factor of Potential Wind (or Wave) Energy Sites

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Abstract - *A shortcoming of renewable resources, such as wind or water wave power, is their inherent intermittence. This lack of predictability is a major problem for energy markets. Given the nature of wind and wave power, it is critical that sites of this kind are chosen carefully. This paper attempts to develop analytical expressions, based on a single parameter of the univariate Weibull distribution, to describe the relative power deviation and capacity factor of prospective wind and wave energy sites. A truncated form of Weibull distribution is utilised to model the effect of the cut-in wind speed, or cut-in wave height of the power generator.*

I. Introduction

Renewable resources, such as wind or water wave power, could play an important role in meeting the world's burgeoning demand for electricity; however, a shortcoming of nearly all renewable resources is their intermittent nature. This intermittence is a major problem for energy markets, which require predictable (preferably continuous) input sources [1]. Wind and wave power are no exception to this same drawback. Wind power is a function of wind speed cubed; wave power is principally a function of wave height squared. Both are derivatives of the solar resource, inherently discontinuous and, therefore, open to criticism. Given the nature of wind and wave power, it is crucial that sites of this kind are chosen carefully.

In order to compare prospective energy conversion sites, wind speed and wave height data can be fitted to probability distributions and the corresponding power distributions can then be analysed [1, 2]. This paper endeavours to develop a mathematical method of evaluating the predictability of wind speed and wave height data fitted to Weibull distributions. Here the level of predictability is quantified in dimensionless terms of relative power deviation and capacity factor (defined as the ratio of average power to rated power). This kind of approach could be used as part of commercial site feasibility studies or regional mapping exercises.

Traditionally, studies of prospective wind or wave energy sites have focused on average wind or wave power levels; some of these studies have utilised known generator characteristics to also predict the site capacity factor (see for example [3, 4, 5, 6]). Unlike methods found in the existing literature, the analytical expressions developed in this work do not rely on specific generator characteristics, and are of a purely theoretical (idealized) quality.

The paper is organised as follows. Before proceeding with the major body of work, a short overview of wind and water wave power is provided. In the next section, the standard univariate Weibull model used to analyse wind speed and wave height datasets is described. This includes a derivation of the standard Weibull properties of statistical mean, variance, standard deviation, and relative deviation. Then the truncated Weibull distribution is introduced as a tool by which higher order (power) distributions can be assessed. Truncated higher order properties of the statistical mean, through to the relative deviation are also derived. Lastly, an expression to describe the theoretical capacity factor of Weibull power distributions is developed.

II. Preliminaries

Ideally, the instantaneous power in a flow of wind is given by (1) where the wind speed, swept area (normal to the flow), and air density are represented by V, A and ρ_a respectively [1, 2, 7].

$$\frac{P_{wind}}{A} = \frac{1}{2}\rho_a V^3 \qquad (W/m^2) \tag{1}$$

Since the only variable on the right hand side of (1) is the wind speed, V, the available power in a flow of wind is proportional to the wind speed cubed.

$$P_{wind} \propto V^3 \tag{2}$$

Hence, it follows that dimensionless statistical properties of wind power, such as the relative power deviation and capacity factor can be evaluated by simply taking into account the value of the wind speed cubed.

The theoretical power per metre crest length of a simplistic (sinusoidal) water wave is given by (3) where the trough to crest wave height, crest width, group velocity, gravitational acceleration, and water density are represented by H, L, c_g, g, and ρ_w respectively [1, 8].

$$\frac{P_{wave}}{L} = \frac{1}{8}c_g \rho_w g H^2 \qquad (W/m) \tag{3}$$

There are two variables on the right hand side of (3). However, the first of these, group velocity, c_g, can be treated as a constant since its effect on wave power is only minimal [1]. Ignoring variations in group velocity leaves the expression with one remaining variable, wave height, H, which suggests that the available wave power is proportional to the wave height squared.

$$P_{wave} \propto H^2 \tag{4}$$

To evaluate the predictability of wave power distributions, one only needs to consider the value of the wave height squared.

III. The standard Weibull model

The Weibull probability density of a single and continuous variable, x, such as wind speed or wave height data, is given by (5) where the subscript, c, indicates a complete range of a distribution [2, 3, 7]. This popular two-parameter function has a very wide range of applicability. The first parameter, k, is usually referred to as the distribution 'shape', and the second parameter, λ, as the distribution 'scale'.

$$f_c(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left(-\left(\frac{x}{\lambda}\right)^k\right) \quad \begin{matrix} 0 \le x \le \infty \\ k, \lambda > 0\,[\Re] \end{matrix} \tag{5}$$

The cumulative distribution for the two-parameter Weibull function is given by (6). As will be shown in subsequent sections, this cumulative distribution function is particularly easy to work with.

$$F(x) = 1 - \exp\left(-\left(\frac{x}{\lambda}\right)^k\right) \tag{6}$$

III.a. Properties of the standard model

To find an expression for the relative deviation of wind speed or wave height data fitted to Weibull distributions, we must initially find an expression to describe the mean, variance and standard deviation.

Moving on, the first expectation of a distribution, or in other words, the mean, is also the first moment about the origin and given by (7) where the moment superscript, m, is eventually replaced with a 1 [1, 7].

$$\overline{x_c^m} = \int_0^\infty x^m f_c(x)\, dx \qquad m = 1, 2, 3... \tag{7}$$

By substituting $t = (x/\lambda)^k$ and performing a few algebraic manipulations, the resulting generalised (m^{th} moment) expression for the standard Weibull mean is given by (9) where Γ represents the gamma function.

$$\overline{x_c^m} = \int_0^\infty x^m \left[\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left(-\left(\frac{x}{\lambda}\right)^k\right)\right] dx \tag{8}$$

$$= \lambda^m \Gamma\left(1 + \frac{m}{k}\right) \tag{9}$$

The variance of the distribution, the second central moment, is given by (10) [9].

$$\sigma^2_{x_c^m} = \overline{x_c^{2m}} - (\overline{x_c^m})^2 \tag{10}$$

Using a straightforward process of substitution and simplification it can be shown that the generalised (m^{th} moment) variance of wind speed or wave height data fitted to the standard Weibull distribution is given by (11).

$$\sigma^2_{x_c^m} = \lambda^{2m}\left[\Gamma\left(1 + \frac{2m}{k}\right) - \Gamma\left(1 + \frac{m}{k}\right)^2\right] \tag{11}$$

Therefore, the generalised expression for the standard deviation of the standard Weibull distribution, which is defined as the square root of the variance, is given by (12).

$$\sigma_{x_c^m} = \lambda^m\left[\Gamma\left(1 + \frac{2m}{k}\right) - \Gamma\left(1 + \frac{m}{k}\right)^2\right]^{\frac{1}{2}} \tag{12}$$

When comparing the predictability of wind speed or wave height data taken from different locations, the imperative measure is the ratio of the standard deviation to the mean of the distribution, otherwise known as the relative variance or relative deviation (13) [1].

$$\psi_{m,c} = \frac{\sigma_{x_c^m}}{\overline{x_c^m}} \tag{13}$$

Therefore, the generalised expression for the relative deviation of the standard Weibull distribution is given by (14).

$$\psi_{m,c} = \frac{1}{\Gamma(1 + m/k)}\left[\Gamma\left(1 + \frac{2m}{k}\right) - \Gamma\left(1 + \frac{m}{k}\right)^2\right]^{\frac{1}{2}} \tag{14}$$

Substituting $m = 1$ for first order moments (i.e. wind speed or wave heights), results in expression (15).

$$\psi_{1,c} = \frac{1}{\Gamma(1 + 1/k)}\left[\Gamma\left(1 + \frac{2}{k}\right) - \Gamma\left(1 + \frac{1}{k}\right)^2\right]^{\frac{1}{2}} \tag{15}$$

Conveniently, the Weibull scale parameter has been eliminated, and the relative deviation of the distribution is solely determined by the order of the moment, and the Weibull shape parameter, k.

IV. The left-truncated Weibull model

Up to this point, the expressions to evaluate the predictability of wind speed and wave height data have been developed using the standard (or 'complete') Weibull probability distribution. However, an analysis of higher order (power) distributions, such as wind speed cubed and wave height squared, requires a slight modification to the models developed so far.

In reality, the total power in the wind and the water waves cannot be converted by the power generator. Most significantly, there is a point ($x > 0$) known as the cut-in speed (or cut-in height) at which the generator begins to operate. To model the effect of the cut-in point, the higher order distribution is left-truncated. This form of truncated Weibull distribution is given by (16) where the subscript, t, indicates a partial range of distribution, and x_0 represents the cut-in wind speed or cut-in wave height. Given that the cut-in point represents some finite value greater than zero, it is important to note that this distribution is not a normalised probability density function.

$$f_t(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left(-\left(\frac{x}{\lambda}\right)^{k}\right) \quad \begin{matrix} x_0 \le x \le \infty \\ k, \lambda > 0\,[\Re] \end{matrix} \tag{16}$$

For example, assume that the generator has a cut-in point below which 20% of the least energetic wind speeds or wave heights occur. The resulting cumulative probability of the truncated area is given by (17).

$$0.2 = 1 - \exp\left(-\left(\frac{x_0}{\lambda}\right)^{k}\right) \tag{17}$$

By manipulation, it can be seen that the cut-in point, x_0, is given by (18).

$$x_{0,20\%} = \lambda \ln\left(\frac{1}{1-0.2}\right)^{1/k} = \lambda(0.2231)^{1/k} \tag{18}$$

This definition for the generator cut-in point will be used throughout the sections that follow.

IV.a. Properties of the truncated model

The generalised expression for the truncated Weibull expectation, which can be found using a similar process as before, is given by (19) where γ represents the incomplete gamma function.

$$\overline{x_t^m} = \lambda^m \left[\Gamma\left(1+\frac{m}{k}\right) - \gamma\left(\left(1+\frac{m}{k}\right), \left(\frac{x_0}{\lambda}\right)^{k}\right)\right] \tag{19}$$

Then substituting (18) into (19) results in a more simplified expression (20).

$$\overline{x_t^m} = \lambda^m \left[\Gamma\left(1+\frac{m}{k}\right) - \gamma\left(\left(1+\frac{m}{k}\right), 0.2231\right)\right] \tag{20}$$

The generalised expression for the truncated distribution variance and standard deviation are given by (21) and (22), respectively.

$$\sigma^2_{x_t^m} = \lambda^{2m}\left[\left[\Gamma\left(1+\frac{2m}{k}\right) - \gamma\left(\left(1+\frac{2m}{k}\right), 0.2231\right)\right] - \left[\Gamma\left(1+\frac{m}{k}\right) - \gamma\left(\left(1+\frac{m}{k}\right), 0.2231\right)\right]^2\right] \tag{21}$$

$$\sigma_{x_t^m} = \lambda^{m}\left[\left[\Gamma\left(1+\frac{2m}{k}\right) - \gamma\left(\left(1+\frac{2m}{k}\right), 0.2231\right)\right] - \left[\Gamma\left(1+\frac{m}{k}\right) - \gamma\left(\left(1+\frac{m}{k}\right), 0.2231\right)\right]^2\right]^{1/2} \tag{22}$$

Therefore, the generalised expression for the relative deviation of the truncated distribution is given by (23). Once again, the scale parameter has been eliminated, and the relative deviation depends solely on the order of the moment and the shape parameter, k.

$$\psi_{m,t} = \left[\left[\Gamma\left(1+\frac{2m}{k}\right) - \gamma\left(\left(1+\frac{2m}{k}\right), 0.2231\right)\right] - \left[\Gamma\left(1+\frac{m}{k}\right) - \gamma\left(\left(1+\frac{m}{k}\right), 0.2231\right)\right]^2\right]^{1/2} \div \left[\Gamma\left(1+\frac{m}{k}\right) - \gamma\left(\left(1+\frac{m}{k}\right), 0.2231\right)\right] \tag{23}$$

The relative *power* deviation of the wind speed cubed or wave height squared distribution, which is desirably small, is deduced by substituting either $m = 3$ or $m = 2$, respectively.

V. Probabilistic 'capacity factor'

The 'capacity factor' of a generator is defined by the ratio of its mean power output to its rated power output; this value, usually expressed as a percentile, is desirably close to unity [1, 2].

$$CF_{m,t} = \frac{P_{AVG}}{P_R} \tag{24}$$

The capacity factor of wind speed or wave height data fitted to Weibull distributions can be evaluated using the relative power deviation of the truncated higher order distribution and an upper cut-off limit (rated wind speed, or rated wave height) applied to the first order cumulative integral. Limiting the sum probability from zero through 90% is known as applying the 10% rule [1]. In terms of the Weibull cumulative distribution function, this rule is given by (25).

$$x_{r,90\%} = \lambda \ln\left(\frac{1}{1-0.9}\right)^{1/k} = \lambda(2.3026)^{1/k} \tag{25}$$

This type of theoretical generator model attempts to balance the priorities of maximum generator output, and minimal part load losses [1]. While useful when comparing data from different sites, it should be noted that according to this approach, all available energy above the cut-in point (and through to the rated point) is converted into electricity at an equal level of efficiency.

The distance (α) between the mean of the power distribution and the rated point is expressed in units of standard deviation.

$$\alpha_{m,t} = \frac{(x_{r,90\%})^m - \overline{x_t^m}}{\sigma_{x_t^m}} \tag{26}$$

Therefore, the theoretical capacity factor can be defined in the form given by (28).

$$CF_{m,t} = \frac{P_{AVG}}{P_{AVG}(1+\psi_{m,t}\cdot\alpha_{m,t})} \tag{27}$$

$$= \frac{1}{1+\psi_{m,t}\cdot\alpha_{m,t}} \tag{28}$$

By a slightly tedious process of substitution and simplification, it can be shown that the capacity factor of the truncated higher order Weibull distribution is given by (29) where $m = 3$ for wind power distributions, and $m = 2$ for wave power distributions.

$$CF_{m,t} = \frac{\Gamma\left(1+\frac{m}{k}\right) - \gamma\left(\left(1+\frac{m}{k}\right), 0.2231\right)}{(2.3026)^{m/k}} \tag{29}$$

At this stage, it is important to recall that this particular expression for the capacity factor incorporates a cut-in point where the sum probability of the wind speed or wave height distribution equals 20%, and a rated point where the sum probability of the wind speed or wave height distribution equals 90%. In other words, on 20% of occasions the wind speeds or wave heights are below the generator cut-in point, and on 10% of occasions the generator runs above its rated capacity. Many other variations are possible.

The relationship between the Weibull distribution shape parameter, k, and the relative deviation (15), relative *power* deviation (23), and the theoretical capacity factor (29) is illustrated in Figures 1 and 2 for a shape parameter range: ($1 \leq k \leq 3.0$).

VI. Conclusion

Mathematical expressions to describe the relative power deviation and the capacity factor of wind speed or water wave height data fitted to univariate Weibull distributions have been developed. The author has utilised a truncated form of Weibull distribution to model the effect of the cut-in speed, or the cut-in height of the power generator. These expressions, which are a function of the Weibull distribution shape parameter, k, and the order of the moment, provide a quick and simple way in which the predictability of prospective wind or wave energy sites can be quantified and evaluated.

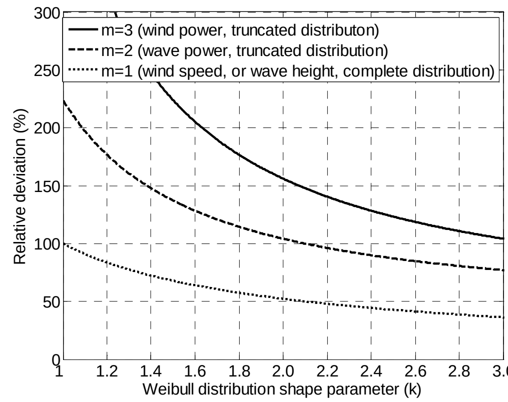

Figure 1. Relative deviation (%) as a function of the Weibull shape parameter.

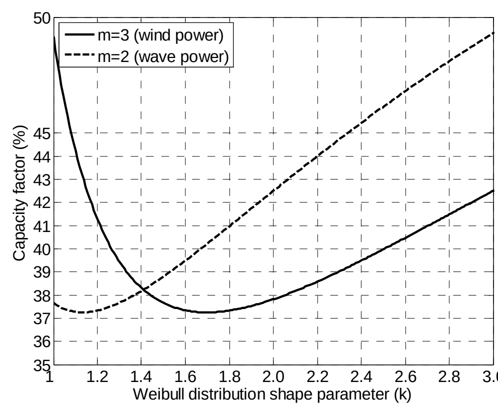

Figure 2. Theoretical capacity factor (%) as a function of the Weibull shape parameter.

VII. References

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