EQUIVALENCES OF CLASSIFYING SPACES COMPLETED AT ODD PRIMES

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Abstract. We prove here the Martino-Priddy conjecture for an odd prime p: the p-completions of the classifying spaces of two groups G and G' are homotopy equivalent if and only if there is an isomorphism between their Sylow p-subgroups which preserves fusion. A second theorem is a description for odd p of the group of homotopy classes of self homotopy equivalences of the p-completion of BG, in terms of automorphisms of a Sylow p-subgroup of G which preserve fusion in G. These are both consequences of a technical algebraic result, which says that for an odd prime p and a finite group G, all higher derived functors of the inverse limit vanish for a certain functor $\mathcal{Z}_G$ on the p-subgroup orbit category of G.

In an earlier paper [BLO1] in collaboration with Carles Broto and Ran Levi, we reduced certain problems involving equivalences between p-completed classifying spaces of finite groups to a question of whether certain obstruction groups vanish. The main technical result of this paper is that these groups do always vanish when p is odd. The proof of this result depends on the classification theorem for finite simple groups.

Fix a prime p and a finite group G. For any pair of subgroups $P, Q \leq G$, let $N_G(P,Q)$ denote the *transporter*:

$$N_G(P,Q) = \{x \in G \,|\, xPx^{-1} \leq Q\}.$$

The *p-subgroup orbit category* of G is the category $\mathcal{O}_p(G)$ whose objects are the p-subgroups of G, and where

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q) = Q\backslash N_G(P,Q) \cong \mathrm{Map}_G(G/P, G/Q).$$

A p-subgroup $P \leq G$ is called *p-centric* if $Z(P)$ is a Sylow p-subgroup of $C_G(P)$, or equivalently if $C_G(P) = Z(P) \times C'_G(P)$ for some subgroup $C'_G(P)$ of order prime to p. Let

$$\mathcal{Z}_G\colon \mathcal{O}_p(G) \longrightarrow \mathsf{Ab}$$

denote the functor $\mathcal{Z}_G(P) = Z(P)$ if P is p-centric in G, and $\mathcal{Z}_G(P) = 0$ otherwise. We refer to [BLO1, §6] for more details on how this is made into a functor.

This paper is centered around the proof of the following theorem.

Theorem A. *For any odd prime p and any finite group G,*

$$\varprojlim_{\mathcal{O}_p(G)}{}^{i}(\mathcal{Z}_G) = 0 \qquad \text{for all } i \geq 1.$$

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Theorem A is proven as Theorem 4.5 below. It was motivated by applications for studying equivalences between p-completed classifying spaces of finite groups. Let G and G' be finite groups, and let $S \le G$ and $S' \le G'$ be Sylow p-subgroups. An isomorphism $\varphi\colon S \xrightarrow{\cong} S'$ is called *fusion preserving* if for all $P, Q \le S$ and all $P \xrightarrow[\cong]{\alpha} Q$, α is conjugation by an element of G if and only if $\varphi(P) \xrightarrow[\cong]{\varphi\alpha\varphi^{-1}} \varphi(Q)$ is conjugation by an element of G'.

The *Martino-Priddy conjecture* states that for any prime p, and any pair G, G' of finite groups, $BG^\wedge_p \simeq BG'^\wedge_p$ if and only if there is a fusion preserving isomorphism between Sylow p-subgroups of G and G'. The "only if" part of the conjecture was proved by Martino and Priddy [MP], and follows from the bijection

$$\operatorname{Rep}(P,G) \stackrel{\text{def}}{=} \operatorname{Hom}(P,G)/\operatorname{Inn}(G) \xrightarrow{\quad\cong\quad} [BP, BG^\wedge_p]$$

for any p-group P and any finite group G (cf. [BLO1, Proposition 2.1]). Conversely, by [BLO1, Proposition 6.1], given a fusion preserving isomorphism between Sylow p-subgroups of G and G', the obstruction to extending it to a homotopy equivalence $BG^\wedge_p \simeq BG'^\wedge_p$ lies in $\varprojlim^2(\mathcal{Z}_G)$. Hence Theorem A implies:

Theorem B (Martino-Priddy conjecture at odd primes)**.** *For any odd prime p, and any pair G and G' of finite groups with Sylow p-subgroups $S \le G$ and $S' \le G'$, $BG^\wedge_p \simeq BG'^\wedge_p$ if and only if there is a fusion preserving isomorphism $S \xrightarrow{\cong} S'$.*

We next turn to the question of self equivalences of $BG^\wedge_p$. For any space X, let $\operatorname{Out}(X)$ denote the group of homotopy classes of self homotopy equivalences of X. For any finite group G, any prime p, and any Sylow p-subgroup $S \le G$, let $\operatorname{Aut}_{\text{fus}}(S)$ be the group of fusion preserving automorphisms of S, let $\operatorname{Aut}_G(S)$ be the group of automorphisms induced by conjugation by elements of G (i.e., elements of $N_G(S)$), and set

$$\operatorname{Out}_{\text{fus}}(S) = \operatorname{Aut}_{\text{fus}}(S)/\operatorname{Aut}_G(S).$$

Theorem A, when combined with [BLO1, Theorem 6.2], gives the following description of $\operatorname{Out}(BG^\wedge_p)$.

Theorem C. *For any odd prime p and any finite group G, with Sylow p-subgroup $S \le G$,*

$$\operatorname{Out}(BG^\wedge_p) \cong \operatorname{Out}_{\text{fus}}(S).$$

Theorem A should be a special case of a more general vanishing result, formulated here as Conjecture 2.2, where orbit categories of groups are replaced by orbit categories of arbitrary "saturated fusion systems" in the sense of Puig [Pu]. We refer to [BLO2, §1], and to the summary in Section 2 below, for definitions of saturated fusion systems. Conjecture 2.2 would, in particular, imply the existence and uniqueness of linking systems, hence of classifying spaces, associated to an arbitrary saturated fusion system over a p-group. This has motivated us to state results here, as far as possible, in the context of abstract saturated fusion systems. It is only at the end that we translate our partial results to a condition on simple groups, which is then checked in the individual cases.

When $p=2$, Theorem A is not true, since $\varprojlim^1(\mathcal{Z}_G)$ can be nonzero. The simplest counterexamples occur for $G=PSL_2(q)$, when $q\equiv\pm1 \pmod 8$. Recently, we have proved that $\varprojlim^i(\mathcal{Z}_G)=0$ for all $i\geq 2$, when $p=2$ and G is an arbitrary finite group. This means that the Martino-Priddy conjecture does hold for $p=2$, but that Theorem C is not true (as formulated above) in this case. The proof for $p=2$ not only requires the classification theorem for finite simple groups, but also (in its current form) requires a long, detailed case-by-case check when handling the simple groups of Lie type in odd characteristic as well as the sporadic groups. For this reason, we have not tried to incorporate it into this paper, but will write it up separately.

Section 1 contains general material about higher limits over orbit categories of finite groups, and Section 2 some results about saturated fusion systems and higher limits over their orbit categories. Concrete criteria for proving the acyclicity of $\mathcal{Z}_\mathcal{F}$ are then set up in Section 3, where the problem is reduced to a question about "simple" fusion systems (Proposition 3.8). Also, at the end of Section 3, there is a discussion of what further results would be necessary to prove Conjecture 2.2 for odd primes. Finally, in Section 4, we restrict attention to fusion systems of finite groups, and apply the classification theorem for finite simple groups to finish the proof of Theorem A.

I would like to thank George Glauberman for his encouragement, and his efforts to prove a result about p-groups (Conjecture 3.9 below) which would have led to a proof of Conjecture 2.2 for odd p, and in particular to a "classification free" proof of Theorem A. I also want to point out the importance to this work of Jesper Grodal's techniques in [Gr] for computing higher limits of functors on orbit categories. His main theorem, while not used here directly, was used in many of the computations which led to this proof. Finally, I thank Carles Broto and Ran Levi, not only for their collaboration in the papers [BLO1] and [BLO2] which are closely connected to this one, but also for introducing me to this problem in the first place.

General notation: We list, for easy reference, the following notation which will be used throughout the paper.

- $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G
- $O_p(G)$ is the maximal normal p-subgroup of G
- $\Omega_n(P)=\langle g\in P\,|\,g^{p^n}=1\rangle$ (for a p-group P)
- $N_G(H,K)=\{x\in G\,|\,xHx^{-1}\leq K\}$ (for $H,K\leq G$)
- c_x denotes conjugation by x $(g\mapsto xgx^{-1})$
- $\mathrm{Hom}_G(H,K)$ (for $H,K\leq G$) is the set of homomorphisms from H to K induced by conjugation in G
- $\mathrm{Rep}(H,K)=\mathrm{Inn}(K)\backslash\mathrm{Hom}(H,K)$ and $\mathrm{Rep}_G(H,K)=\mathrm{Inn}(K)\backslash\mathrm{Hom}_G(H,K)$
- $\mathrm{Aut}_G(H)=\mathrm{Hom}_G(H,H)$, and $\mathrm{Out}_G(H)=\mathrm{Rep}_G(H,H)=\mathrm{Aut}_G(H)/\mathrm{Inn}(H)$
- A functor $F:\mathcal{C}^{\mathrm{op}}\longrightarrow\mathsf{Ab}$ is called *acyclic* if $\varprojlim^i(F)=0$ for all $i>0$.

1. Higher limits over orbit categories of groups

We first collect some tools for computing higher limits of functors over the orbit category of a finite group G. Very roughly, these reduce to two general techniques. One is to filter a functor by a sequence of subfunctors, such that each of the subquotients vanishes except on one conjugacy class of p-subgroups of G. Proposition 1.1 then gives some tools which are very effective when computing the higher limits of these subquotients. The other method is to reduce computations to a situation, described in Proposition 1.3, where the functor extends to a Mackey functor, and hence is acyclic by a theorem of Jackowski and McClure [JM].

Fix a prime p, a finite group G, and a $\mathbb{Z}_{(p)}[G]$-module M. Let F_M be the functor on $\mathcal{O}_p(G)$ defined by setting $F_M(P) = M^P$ (the fixed submodule), and define

$$\Lambda^*(G;M) = \varprojlim_{\mathcal{O}_p(G)}{}^*(F_M).$$

These graded groups were shown in [JMO] to be very effective tools when computing higher limits over functors on orbit categories. We first summarize the properties of the Λ^* which will be needed here.

Proposition 1.1. *Fix a prime p. Then the following hold.*

(a) *For any finite group G and any functor $F : \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}$-$\mathsf{mod}$ which vanishes except on subgroups conjugate to some given p-subgroup $P \le G$,*

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(F) \cong \Lambda^*(N_G(P)/P; F(P)).$$

(b) *If G is a finite group, $H \lhd G$ is a normal subgroup which acts trivially on the $\mathbb{Z}_{(p)}[G]$-module M, and $p \big| |H|$, then $\Lambda^*(G;M) = 0$.*

(c) *If G is a finite group, and $H \lhd G$ is a normal subgroup of order prime to p which acts trivially on the $\mathbb{Z}_{(p)}[G]$-module M, then*

$$\Lambda^*(G;M) \cong \Lambda^*(G/H;M).$$

(d) *If G is a finite group, and $O_p(G) \ne 1$ (if G contains a nontrivial normal p-subgroup), then $\Lambda^*(G;M) = 0$ for all $\mathbb{Z}_{(p)}[G]$-modules M.*

Proof. See [JMO, Propositions 5.4, 5.5, & 6.1]. □

The idea now is to filter an arbitrary functor $F : \mathcal{O}_p(G) \longrightarrow \mathbb{Z}_{(p)}$-$\mathsf{mod}$ in such a way that all quotient functors vanish except on one conjugacy class, and hence are described via Proposition 1.1(a).

We next look for some conditions on a pair of finite groups $H \le G$, and a functor F on $\mathcal{O}_p(G)$, which reduce the computation of $\varprojlim^*(F)$ to one of higher limits of a functor over $\mathcal{O}_p(N(H)/H)$. In general, for any small categories $\mathcal{C}$ and $\mathcal{D}$ and any functors

$$\mathcal{C} \xrightarrow{\ \Phi\ } \mathcal{D} \xrightarrow{\ F\ } \mathsf{Ab},$$

there is an induced homomorphism

$$\varprojlim_{\mathcal{D}}{}^*(F) \xrightarrow{\ \Phi^*\ } \varprojlim_{\mathcal{C}}{}^*(F \circ \Phi)$$

defined as follows. Let I^* be an injective resolution of F (in the category of functors $\mathcal{D} \longrightarrow \mathsf{Ab}$), and let $\widehat{I}^*$ be an injective resolution of $F \circ \Phi$. Since $I^* \circ \Phi$ is a resolution of $F \circ \Phi$ (though not injective), there is a chain homomorphism $I^* \circ \Phi \xrightarrow{\psi^*} \widehat{I}^*$ which extends the identity on $F \circ \Phi$, and which is unique up to chain homotopy. Then Φ^* is the homology of the composite homomorphism

$$\varprojlim_{\mathcal{D}}(I^*) \longrightarrow \varprojlim_{\mathcal{C}}(I^* \circ \Phi) \xrightarrow{\lim(\psi^*)} \varprojlim_{\mathcal{C}}(\widehat{I}^*),$$

where the first map is induced by the universal property of inverse limits over $\mathcal{C}$.

Lemma 1.2. *Fix a finite group G and a p-subgroup $Q \le G$. Then there is a well defined functor*

$$\Phi = \Phi_Q^G\colon \mathcal{O}_p(N_G(Q)/Q) \longrightarrow \mathcal{O}_p(G)$$

such that $\Phi(P/Q) = P$ for all $P/Q \le N_G(Q)/Q$. Let $\mathcal{T}$ be the set of all p-subgroups $P \le G$ with the property

$$Q \lhd P, \text{ and } Q \lhd xPx^{-1} \text{ for } x \in G \text{ implies } x \in N_G(Q). \tag{$*$}$$

Then for any functor $F\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ which vanishes except on subgroups G-conjugate to elements of $\mathcal{T}$, the induced homomorphism

$$\varprojlim_{\mathcal{O}_p(G)}{}^*(F) \xrightarrow[\cong]{\Phi^*} \varprojlim_{\mathcal{O}_p(N_G(Q)/Q)}{}^*(F \circ \Phi) \tag{1}$$

is an isomorphism.

Proof. Clearly, Φ is well defined on objects. To see that it is well defined on morphisms, recall first that

$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P, P') = P' \backslash N_G(P, P'),$$

where $N_G(P, P')$ is the set of all $x \in G$ such that $xPx^{-1} \le P'$. Hence for any pair of objects P/Q and P'/Q in $\mathcal{O}_p(N_G(Q)/Q)$,

$$\begin{aligned}\mathrm{Mor}_{\mathcal{O}_p(N_G(Q)/Q)}(P/Q, P'/Q) = (P'/Q)\backslash N_{N(Q)/Q}(P/Q, P'/Q) &\cong P'\backslash N_{N(Q)}(P, P') \\ &\subseteq P'\backslash N_G(P, P') = \mathrm{Mor}_{\mathcal{O}_p(G)}(P, P');\end{aligned}$$

and Φ is defined on morphism sets to be this inclusion.

Composition with Φ is natural in F and preserves short exact sequences of functors. Hence if $F' \subseteq F$ is a pair of functors from $\mathcal{O}_p(G)$ to $\mathbb{Z}_{(p)}\text{-}\mathsf{mod}$, and the lemma holds for F' and for F/F', then it also holds for F by the 5-lemma. Hence it suffices to prove that (1) is an isomorphism when F vanishes except on the G-conjugacy class of one subgroup $P \in \mathcal{T}$. When $P = Q$, then (1) is precisely the isomorphism $\varprojlim^*(F) \cong \Lambda^*(N(Q)/Q; F(Q))$ of Proposition 1.1(a).

Now let $P \in \mathcal{T}$ be arbitrary. By condition $(*)$, $Q \lhd P$, $N_G(P) \le N_G(Q)$, and $F \circ \Phi$ vanishes except on the $\mathcal{O}_p(N_G(Q)/Q)$-isomorphism class of P/Q. Let

$$\Psi = \Phi_{P/Q}^{N(Q)/Q}\colon \mathcal{O}_p(N_G(P)/P) \longrightarrow \mathcal{O}_p(N_G(Q)/Q)$$

be the functor $\Psi(R/P) = R/Q$ for p-subgroups $R \le N_G(P) \le N_G(Q)$ containing P. Then the following square commutes

$$\begin{CD}
\varprojlim_{\mathcal{O}_p(G)}{}^*(F) @>{\Phi^*}>> \varprojlim_{\mathcal{O}_p(N(Q)/Q)}{}^*(F\circ\Phi) \\
@V{(\Phi\circ\Psi)^*}V{\cong}V @V{\Psi^*}V{\cong}V \\
\Lambda^*(N_G(P)/P; F(P)) @= \Lambda^*(N_G(P)/P; F(P)),
\end{CD}$$

and the vertical maps are isomorphisms by Proposition 1.1(a) (see the proof of [JMO, Lemma 5.4] for the precise description of the isomorphisms). This shows that Φ^* is an isomorphism. $\square$

The next proposition describes a different condition which implies the acyclicity of a functor on the orbit category of a finite group.

Proposition 1.3. *Fix a finite group G, a prime p, and a $\mathbb{Z}_{(p)}[G]$-module M, and let*

$$H^0M\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod},$$

be the functor defined by setting

$$H^0M(P) = H^0(P;M) = M^P.$$

Let

$$F\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

be any subfunctor of H^0M (thus $F(P) \le M^P$ for all P) which satisfies the following "relative norm property": for each pair of p-subgroups $P \le Q \le G$,

$$\mathfrak{N}_P^Q\big(F(P)\big) \overset{\text{def}}{=} \Big\{ \sum_{gP\in Q/P} gx \,\Big|\, x \in F(P) \le M^P \Big\} \subseteq F(Q). \tag{$*$}$$

Then F is acyclic: $\varprojlim^i(F) = 0$ for all $i > 0$.

Proof. The relative norms $\mathfrak{N}_P^Q$ make F into a proto-Mackey functor in the sense of [JM], and hence it is acyclic by [JM, Proposition 5.14]. $\square$

The following application of Proposition 1.3 plays an important role in Section 3. If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then $\mathcal{O}_S(G) \subseteq \mathcal{O}_p(G)$ denotes the full subcategory whose objects are the subgroups of S (the inclusion is clearly an equivalence of categories). As usual, a subgroup $T \le S$ is called *strongly closed in S with respect to G* if no element of T is G-conjugate to any element of $S \smallsetminus T$.

Recall that for any p-group P and any $n \ge 1$, $\Omega_n(P)$ denotes the subgroup generated by all $x \in P$ such that $x^{p^n} = 1$.

Proposition 1.4. *Fix a finite group G, a Sylow subgroup $S \in \mathrm{Syl}_p(G)$, and a subgroup $T \le S$ which is strongly closed in S with respect to G. Let M be a finite $\mathbb{Z}_{(p)}[G]$-module, and let*

$$F\colon \mathcal{O}_S(G)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

be a subfunctor of H^0M (in particular, $F(P) \le M^P$ for all P), which satisfies the relative norm property:

$$\mathfrak{N}_P^Q\bigl(F(P)\bigr) \le F(Q) \tag{$*$}$$

for each pair of subgroups $P \le Q \le S$. Let $F_1 \subseteq F$ be the subfunctor

$$F_1(P) = \begin{cases} F(P) & \text{if } P \cap T = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Assume $\mathfrak{N}_{Z(T)} \cdot \Omega_1(M) = 0$. Then $\varprojlim_{\mathcal{O}_S(G)}^i (F_1) = 0$ for all $i \ge 1$.

Proof. Note first that F_1 is a functor: if $P, P' \le S$ are G-conjugate, then $P \cap T = 1$ if and only if $P' \cap T = 1$ since T is strongly closed.

Assume $\varprojlim^n(F_1) \ne 0$ for some $n \ge 1$. We must prove that $\mathfrak{N}_{Z(T)} \cdot \Omega_1(M) \ne 0$. This will be shown by induction on n.

Let $\Omega_k F \subseteq F$ be the p^k-torsion subfunctor in F; i.e., $(\Omega_k F)(P) = \Omega_k(F(P))$ for each P. Then for some k, $\varprojlim^n(\Omega_k F/\Omega_{k-1}F) \ne 0$. This functor $\Omega_k F/\Omega_{k-1}F$ satisfies all of the hypotheses of the proposition with respect to the $\mathbb{F}_p[G]$-module $\Omega_k(M)/\Omega_{k-1}(M) \subseteq \Omega_1(M)$. It thus suffices to prove the proposition when $M = \Omega_1(M)$; i.e., when $pM = 0$.

Without loss of generality, we can assume $M = F(1)$. Define a functor $\overline{F} \subseteq F$ by setting

$$\overline{F}(P) = F(P) \cap (\mathfrak{N}_{P\cap T} \cdot M)$$

for all $P \le S$. We claim that $\overline{F}$ still satisfies condition $(*)$ above. To see this, fix subgroups $P \le Q \le S$, set $P' = P \cap T \lhd P$ and $Q' = Q \cap T \lhd Q$, and set $Q'' = PQ'$. Then $P' = P \cap Q'$, so coset representatives for Q'/P' are also representatives for Q''/P. Hence

$$\begin{aligned} \mathfrak{N}_P^Q\bigl(\overline{F}(P)\bigr) = \mathfrak{N}_P^Q\bigl(F(P) \cap \mathfrak{N}_{P'} \cdot M\bigr) \subseteq \mathfrak{N}_P^Q(F(P)) \cap \mathfrak{N}_{P'}^{Q'}(\mathfrak{N}_{P'} \cdot M) \\ \subseteq F(Q) \cap (\mathfrak{N}_{Q'} \cdot M) = \overline{F}(Q). \end{aligned}$$

Thus, upon replacing F by $\overline{F}$ (without changing F_1), we can assume that $F(P) = \mathfrak{N}_P \cdot M$ for all $P \le T$.

By Proposition 1.3, $\varprojlim^i(F) = 0$ for all $i > 0$. Hence $\varprojlim^{n-1}(F/F_1) \ne 0$, since $\varprojlim^n(F_1) \ne 0$ by assumption. For each subgroup $Q \le T$, let F_Q be the functor on $\mathcal{O}_S(G)$ defined by

$$F_Q(P) = \begin{cases} F(P) & \text{if } P \cap T \text{ is } G\text{-conjugate to } Q \\ 0 & \text{otherwise.} \end{cases}$$

There is an obvious filtration of F/F_1 whose quotients are all isomorphic to F_Q for various $1 \ne Q \le T$. Hence there is some $Q \le T$ such that

$$\varprojlim_{\mathcal{O}_S(G)}^{n-1}(F_Q) \ne 0. \tag{1}$$

Since we can replace Q by any other subgroup of S in its G-conjugacy class, we can assume that $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$.

If $n=1$, then (1) implies that $F_Q(S)\neq 0$. Hence $S\cap T=Q$, so $Q=T$, and $0\neq F_T(S)\subseteq F_T(T)=\mathfrak{N}_T{\cdot}M$. In particular, $\mathfrak{N}_{Z(T)}{\cdot}M\neq 0$.

Now assume $n>1$. Set $G'=N_G(Q)/Q$, $S'=N_S(Q)/Q\in\operatorname{Syl}_p(G')$, and $T'=N_T(Q)/Q$. Then T' is strongly closed in S' with respect to G': no element of $N_T(Q)$ is $N_G(Q)$-conjugate to any element of $N_S(Q)\smallsetminus N_T(Q)$ since no element of T is G-conjugate to any element of $S\smallsetminus T$. Define functors

$$F',F_1'\colon \mathcal{O}_{S'}(G')\longrightarrow \mathsf{Ab}$$

by setting

$$F'(P/Q)=F(P)\qquad\text{and}\qquad F_1'(P/Q)=F_Q(P).$$

Thus $F_1'(P/Q)=F'(P/Q)$ whenever $P\cap T=Q$, equivalently whenever $(P/Q)\cap T'=1$; and $F_1'(P/Q)=0$ otherwise.

Consider the set

$$\mathcal{T}_0=\{P\le S\,|\,P\cap T=Q\}.$$

If $F_Q(P)\neq 0$ (and $P\le S$), then $P\cap T$ is G-conjugate to Q, P is G-conjugate to some P' such that $Q\le P'\le N_S(Q)$ (since $N_S(Q)\in\operatorname{Syl}_p(N_G(Q))$), $P'\cap T$ is G-conjugate to $P\cap T$ since T is strongly closed, and thus $P'\cap T=Q$ and $P'\in\mathcal{T}_0$. By a similar argument, if $P\in\mathcal{T}_0$ and $Q\lhd xPx^{-1}$, then $(yx)P(yx)^{-1}\le N_S(Q)$ for some $y\in N_G(Q)$, $(yx)P(yx)^{-1}\cap T=Q$, and so $x\in N_G(Q)$. This shows that $\mathcal{T}_0$ is contained in the set $\mathcal{T}$ defined in Lemma 1.2, and thus that each subgroup of G for which $F_Q(P)\neq 0$ is G-conjugate to a subgroup in $\mathcal{T}$. Hence by Lemma 1.2,

$$\varprojlim_{\mathcal{O}_{S'}(G')}{}^{*}(F_1')\cong\varprojlim_{\mathcal{O}_S(G)}{}^{*}(F_Q).$$

In particular, $\varprojlim^{n-1}(F_1')\neq 0$ by (1). All of the conditions of the proposition are satisfied (with G, S, T, F, and M replaced by G', S', T', F', and $M'\overset{\text{def}}{=}F(Q)=\mathfrak{N}_Q{\cdot}M$). So by the induction hypothesis, if we set $Z'/Q=Z(T')$, then

$$\mathfrak{N}_{Z'}{\cdot}M=\mathfrak{N}_{Z(T')}{\cdot}\bigl(\mathfrak{N}_Q{\cdot}M\bigr)=\mathfrak{N}_{Z(T')}{\cdot}M'\neq 0,$$

and hence $\mathfrak{N}_{Z(T)}{\cdot}M\neq 0$ since $Z(T)\le Z'$. $\square$

2. Higher limits over orbit categories of fusion systems

We first briefly recall some definitions. We refer to [BLO2, §1] or [Pu] for more details.

A *fusion system* over a finite p-group S is a category $\mathcal{F}$ whose objects are the subgroups of S, and whose morphisms satisfy the following conditions:

- $\operatorname{Hom}_S(P,Q)\subseteq\operatorname{Mor}_{\mathcal{F}}(P,Q)\subseteq\operatorname{Inj}(P,Q)$ for all $P,Q\le S$; and
- each morphism in $\mathcal{F}$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

To emphasize that the morphisms in $\mathcal{F}$ are all homomorphisms of groups, we write $\operatorname{Hom}_{\mathcal{F}}(P,Q)=\operatorname{Mor}_{\mathcal{F}}(P,Q)$ for the morphism sets. Two subgroups of $\mathcal{F}$ are called *$\mathcal{F}$-conjugate* if they are isomorphic in $\mathcal{F}$. A subgroup $P\le S$ is called *fully centralized* *in $\mathcal{F}$* (*fully normalized in $\mathcal{F}$*) if $|C_S(P)| \geq |C_S(P')|$ ($|N_S(P)| \geq |N_S(P')|$) for all P' in the $\mathcal{F}$-conjugacy class of P. The fusion system $\mathcal{F}$ is *saturated* if

(I) for each fully normalized subgroup $P \leq S$, P is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$; and

(II) for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ whose image is fully centralized in $\mathcal{F}$, if we set
$$N_\varphi = \{x \in N_S(P) \,|\, \varphi c_x \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$
then φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$.

If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then we let $\mathcal{F}_S(G)$ denote the category whose objects are the subgroups of S, and where
$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q) \cong N_G(P,Q)/C_G(P).$$
It is not hard to see [BLO2, Proposition 1.3] that $\mathcal{F}_S(G)$ is a saturated fusion system over S, and that a subgroup $P \leq S$ is fully centralized (fully normalized) if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$ ($N_S(P) \in \mathrm{Syl}_p(N_G(P))$).

By analogy with the orbit category of a finite group, when $\mathcal{F}$ is a saturated fusion system over a p-group S, we let $\mathcal{O}(\mathcal{F})$ (the *orbit category* of $\mathcal{F}$) be the category with the same objects, and with morphism sets
$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) \overset{\text{def}}{=} \mathrm{Inn}(Q)\backslash\mathrm{Hom}_{\mathcal{F}}(P,Q).$$
If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G, then $\mathcal{O}(\mathcal{F})$ is a quotient category of $\mathcal{O}_S(G)$ (the full subcategory of $\mathcal{O}_p(G)$ whose objects are the subgroups of S), but its morphism sets are much smaller in general. More precisely, if P and Q are two p-subgroups of G, then
$$\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q) \cong Q\backslash N_G(P,Q),$$
while
$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F}_p(G))}(P,Q) \cong Q\backslash N_G(P,Q)/C_G(P).$$
Thus, there is a natural projection functor $\mathcal{O}_p(G) \xrightarrow{\Phi} \mathcal{O}(\mathcal{F}_p(G))$ which is the identity on objects and a surjection on all morphism sets, but these maps of morphism sets are very far in general from being bijections. However, the next lemma shows that if one restricts to p-centric subgroups of G, then p-local functors over these two categories have the same higher limits.

If $\mathcal{F}$ is a saturated fusion system over S, then a subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S(P') = Z(P')$ for all P' $\mathcal{F}$-conjugate to P. If $\mathcal{F} = \mathcal{F}_S(G)$, then a subgroup is $\mathcal{F}$-centric if and only if it is p-centric in G (see [BLO1, Lemma A.5]). Let $\mathcal{F}^c \subseteq \mathcal{F}$ and $\mathcal{O}(\mathcal{F}^c) \subseteq \mathcal{O}(\mathcal{F})$ be the full subcategories whose objects are the $\mathcal{F}$-centric subgroups of S. Similarly, for any finite group G, and any $S \in \mathrm{Syl}_p(G)$, we let $\mathcal{O}_p^c(G) \subseteq \mathcal{O}_p(G)$ and $\mathcal{O}_S^c(G) \subseteq \mathcal{O}_S(G)$ be the full subcategories whose objects are the p-centric subgroups of G, and those contained in S, respectively.

Lemma 2.1. *Fix a prime p and a finite group G. Let*
$$F\colon \mathcal{O}(\mathcal{F}_p^c(G))^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$
be any functor, and let
$$\Phi\colon \mathcal{O}_p^c(G) \longrightarrow \mathcal{O}(\mathcal{F}_p^c(G))$$

be the projection functor. Define

$$\widehat{F}\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$$

by setting $\widehat{F}|_{\mathcal{O}_p^c(G)} = F\circ\Phi$*, and* $\widehat{F}(P)=0$ *if* P *is not* p*-centric in* G*. Then*

$$\varprojlim_{\mathcal{O}(\mathcal{F}_p^c(G))}{}^{*}(F) \cong \varprojlim_{\mathcal{O}_p^c(G)}{}^{*}(F\circ\Phi) \cong \varprojlim_{\mathcal{O}_p(G)}{}^{*}(\widehat{F}). \tag{1}$$

Proof. For any pair of p-centric subgroups $P,Q\le G$, write $C_G(P)=Z(P)\times C'_G(P)$ where $C'_G(P)$ has order prime to p. For any $x\in N_G(P,Q)$ and $a\in Z(P)$, $xa = (xax^{-1})x\in Qx$, since $xax^{-1}\in Q$ by definition of the transporter. Thus

$$\begin{aligned}\mathrm{Mor}_{\mathcal{O}(\mathcal{F}_p^c(G))}(P,Q) &= Q\backslash N_G(P,Q)/C_G(P)\\ &\cong Q\backslash N_G(P,Q)/C'_G(P) \cong \mathrm{Mor}_{\mathcal{O}_p^c(G)}(P,Q)/C'_G(P).\end{aligned}$$

Since $C'_G(P)$ has order prime to p, the first isomorphism in (1) now follows as an immediate consequence of [BLO1, Lemma 1.3]. The second isomorphism holds since $\widehat{F}$ vanishes on all p-subgroups which are not p-centric, and since every p-subgroup of G which contains a p-centric subgroup is also p-centric. □

For any saturated fusion system $\mathcal{F}$ over a finite p-group S, let

$$\mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathtt{Ab}$$

denote the functor $\mathcal{Z}_{\mathcal{F}}(P)=Z(P)$ for all $\mathcal{F}$-centric subgroups $P\le S$. If $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$, then $\mathcal{Z}_{\mathcal{F}}(\varphi)$ is the composite

$$Z(Q)\le Z(\varphi(P)) \xrightarrow{\ \varphi^{-1}\ } Z(P).$$

If $\mathcal{F}=\mathcal{F}_S(G)$ for some finite group G with Sylow p-subgroup S, then $\mathcal{Z}_G|_{\mathcal{O}_S(G)}$ is the composite of $\mathcal{Z}_{\mathcal{F}}$ with the projection between orbit categories. So Lemma 2.1 implies as a special case that

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(\mathcal{Z}_G) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(\mathcal{Z}_{\mathcal{F}}).$$

What we would like to prove is the following conjecture, of which Theorem A is just the special case where $\mathcal{F}=\mathcal{F}_S(G)$ and p is odd.

Conjecture 2.2. *Fix a prime* p*, and let* $\mathcal{F}$ *be a saturated fusion system over a* p*-group* S*. Then*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{i}(\mathcal{Z}_{\mathcal{F}}) = 0$$

if p *is odd and* $i\ge 1$*, or if* $p=2$ *and* $i\ge 2$*.*

Conjecture 2.2 would imply that each saturated fusion system over a p-group S has a unique associated "linking system", in the sense of [BLO2, §1], and hence a unique associated classifying space (see [BLO2, Proposition 3.1]). The vanishing of $\varprojlim^1(\mathcal{Z}_{\mathcal{F}})$ would also imply (when p is odd) a description of the group of homotopy classes of self equivalences of the classifying space, similar to the description of $\mathrm{Out}(BG^\wedge_p)$ in Theorem C.

Throughout this section and the next, we will be developping tools for computing higher limits of functors on centric orbit categories of saturated fusion systems; in particular, those with connections to Conjecture 2.2. Only in the last section do we again return to the special case of fusion systems of finite groups, and finish the proof of Theorem A.

If $\mathcal{F}$ is any saturated fusion system over a p-group S, and $Q \le S$ is fully normalized in $\mathcal{F}$, then $N_{\mathcal{F}}(Q)$ is defined to be the fusion system over $N_S(Q)$ whose morphisms are defined by the formula

$$\mathrm{Hom}_{N_{\mathcal{F}}(Q)}(P,P') = \bigl\{\alpha|_P \bigm| \alpha \in \mathrm{Hom}_{\mathcal{F}}(PQ,P'Q),\ \alpha(P)\le P',\ \alpha(Q)=Q\bigr\}.$$

By [BLO2, Proposition A.6], this is a saturated fusion system over $N_S(Q)$. We also let $\mathcal{O}^{\ge Q}(N_{\mathcal{F}}(Q))$ denote the full subcategory of the orbit category of $N_{\mathcal{F}}(Q)$ whose objects are the subgroups which contain Q.

Lemma 2.3. *Fix a saturated fusion system $\mathcal{F}$ over a p-group S, and a fully normalized $\mathcal{F}$-centric subgroup $Q \le S$. Consider the functor*

$$\Psi = \Psi_Q^{\mathcal{F}} \colon \mathcal{O}^{\ge Q}(N_{\mathcal{F}}(Q)) \longrightarrow \mathcal{O}_{\mathrm{Out}_S(Q)}(\mathrm{Out}_{\mathcal{F}}(Q))$$

defined by setting

$$\Psi(P) = \mathrm{Out}_P(Q) \quad\text{and}\quad \Psi\bigl(P \xrightarrow{\ \alpha\ } P'\bigr) = [\alpha|_Q].$$

Then Ψ is an isomorphism of categories. Hence there is a functor

$$\Phi = \Phi_Q^{\mathcal{F}} \colon \mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(Q)) \longrightarrow \mathcal{O}(\mathcal{F}^c),$$

unique up to natural isomorphism, whose restriction to $\mathcal{O}_{\mathrm{Out}_S(Q)}(\mathrm{Out}_{\mathcal{F}}(Q))$ is equal to Φ^{-1}.

Proof. Write $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$ and $S' = \mathrm{Out}_S(Q)$ for short. Since Q is fully normalized in $\mathcal{F}$, S' is a Sylow p-subgroup of Γ (condition (I) in the definition of a saturated fusion system), and so the inclusion $\mathcal{O}_{S'}(\Gamma) \subseteq \mathcal{O}_p(\Gamma)$ is an equivalence of categories.

Now, $S' \cong N_S(Q)/Q$ since Q is $\mathcal{F}$-centric in S. So Ψ defines a bijection between objects of $\mathcal{O}^{\ge Q}(N_{\mathcal{F}}(Q))$ and objects of $\mathcal{O}_{S'}(\Gamma)$, sending P to $\mathrm{Out}_P(Q) \cong P/Q$.

Fix subgroups $P, P' \le N_S(Q)$ containing Q, and consider the function

$$\Psi_{P,P'} \colon \mathrm{Rep}_{N_{\mathcal{F}}(Q)}(P,P') = \mathrm{Mor}_{\mathcal{O}(N_{\mathcal{F}}(Q))}(P,P') \longrightarrow \mathrm{Mor}_{\mathcal{O}_p(\Gamma)}(\mathrm{Out}_P(Q), \mathrm{Out}_{P'}(Q))$$

which sends the class $[\alpha]$, for $\alpha \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(P,P')$, to the class of $[\alpha|_Q] \in \mathrm{Out}_{\mathcal{F}}(Q) = \Gamma$. For any such α, the following square commutes

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "c_g"'] & P' \arrow[d, "c_{\alpha(g)}"] \\ P \arrow[r, "\alpha"] & P' \end{tikzcd}$$

for all $g \in P$, so α lies in the transporter $N_\Gamma(\mathrm{Out}_P(Q), \mathrm{Out}_{P'}(Q))$, and the map $\Psi_{P,P'}$ is well defined. If $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$ is such that conjugation by $[\beta] \in \Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$ sends $\mathrm{Out}_P(Q)$ into $\mathrm{Out}_{P'}(Q)$, then $\beta c_g \beta^{-1} \in \mathrm{Aut}_{P'}(Q)$ for all $g \in P$, so β extends to some $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P,P')$ by condition (II) in the definition of a saturated fusion system, and $\Psi_{P,P'}$ sends $[\alpha]$ to $[\beta]$. Thus $\Psi_{P,P'}$ is onto. If $\alpha_1, \alpha_2 \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(P,P')$ are such that $\Psi_{P,P'}([\alpha_1]) = \Psi_{P,P'}([\alpha_2])$ in $\mathcal{O}_p(\Gamma)$, then $\alpha_1|_Q = c_g \circ \alpha_2|_Q$ for some $g \in Q$, hence $\alpha_1 = c_g \circ \alpha_2 \circ c_z$ for some $z \in Z(Q)$ by [BLO2, Proposition A.8], and so $[\alpha_1] = [\alpha_2]$ in

$\mathrm{Rep}_{\mathcal{F}}(P,P')$. Thus, $\Psi_{P,P'}$ is a bijection for each pair of objects P,P', and this finishes the proof that Ψ is an isomorphism of categories.

The last statement now follows by letting Φ be the composite of a retraction of $\mathcal{O}_p(\Gamma)$ onto $\mathcal{O}_{S'}(\Gamma)$, followed by Ψ^{-1}, followed by the inclusion of $\mathcal{O}^{\geq Q}(N_{\mathcal{F}}(Q))$ into $\mathcal{O}(\mathcal{F}^c)$. □

The next proposition describes how higher limits over $\mathcal{O}(\mathcal{F}^c)$ can be reduced in certain cases to higher limits over the orbit category of $\mathrm{Out}_{\mathcal{F}}(Q)$ for some subgroup Q. Note its similarity with Lemma 1.2, in both the statement and the proof.

By analogy with the usual definition for subgroups of finite groups, for any saturated fusion system $\mathcal{F}$ over a p-group S, a subgroup $P \leq S$ is called *weakly $\mathcal{F}$-closed* (or weakly $\mathcal{F}$-closed in S) if P is not $\mathcal{F}$-conjugate to any other subgroup of S.

Proposition 2.4. *Fix a saturated fusion system $\mathcal{F}$ over a p-group S and a fully normalized $\mathcal{F}$-centric subgroup $Q \leq S$, and let*

$$\Phi = \Phi_Q^{\mathcal{F}} \colon \mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(Q)) \longrightarrow \mathcal{O}(\mathcal{F}^c)$$

be the functor of Lemma 2.3. Let $\mathcal{T}$ be the set of all subgroups $P \leq S$ such that

$$Q \lhd P, \text{ and } Q \lhd \alpha(P) \text{ for } \alpha \in \mathrm{Hom}_{\mathcal{F}}(P,S) \text{ implies } \alpha(Q) = Q. \tag{$*$}$$

Then for any functor $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ which vanishes except on subgroups $\mathcal{F}$-conjugate to elements of $\mathcal{T}$, the induced homomorphism

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(F) \xrightarrow{\ \Phi^*\ } \varprojlim_{\mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(Q))}{}^{*}(F\circ\Phi) \tag{1}$$

is an isomorphism. In particular, if Q is weakly $\mathcal{F}$-closed in S, then (1) holds for any functor F which vanishes except on subgroups which contain Q.

Proof. Composition with Φ is natural in F and preserves short exact sequences of functors. If $F' \subseteq F$ is a pair of functors from $\mathcal{O}(\mathcal{F}^c)$ to $\mathbb{Z}_{(p)}$-mod, and the lemma holds for F' and for F/F', then it also holds for F by the 5-lemma. Hence it suffices to prove that (1) is an isomorphism when F vanishes except on the $\mathcal{F}$-conjugacy class of one subgroup $P \in \mathcal{T}$.

Fix $P \in \mathcal{T}$, and set $\overline{P} = \mathrm{Out}_P(Q) \leq \mathrm{Out}_{\mathcal{F}}(Q)$. By condition $(*)$, $Q \lhd P$ (so $\overline{P} \cong P/Q$), and $F\circ\Phi$ vanishes except on the $\mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(Q))$-isomorphism class of $\mathrm{Out}_P(Q) \cong P/Q$. Also, by $(*)$ again,

$$\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{N_{\mathcal{F}}(Q)}(P),$$

and

$$\mathrm{Out}_{N_{\mathcal{F}}(Q)}(P) = \mathrm{Aut}_{\mathcal{O}_p(N_{\mathcal{F}}(Q))}(P) \cong \mathrm{Aut}_{\mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(Q))}(\overline{P}) = N_{\mathrm{Out}_{\mathcal{F}}(Q)}(\overline{P})/\overline{P}$$

by Lemma 2.3. Let

$$\Psi = \Phi_{\mathrm{Out}_P(Q)}^{\mathrm{Out}_{\mathcal{F}}(Q)} \colon \mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(P)) \longrightarrow \mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(Q))$$

be the functor $\Psi(R/P) = R/Q$ for p-subgroups $R \le N_G(P) \le N_G(Q)$ containing P. Then the following square commutes

$$\begin{tikzcd} \varprojlim\limits_{\mathcal{O}(\mathcal{F})}{}^*(F) \arrow[r, "\Phi^*"] \arrow[d, "(\Phi\circ\Psi)^*"', "\cong"] & \varprojlim\limits_{\mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(Q))}{}^* (F\circ\Phi) \arrow[d, "\Psi^*"', "\cong"] \\ \Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F(P)) \arrow[r, "\cong"] & \Lambda^*(N_{\mathrm{Out}_{\mathcal{F}}(Q)}(\overline{P})/\overline{P}; F(P)) \,, \end{tikzcd}$$

and the vertical maps are isomorphisms by [BLO2, Proposition 3.2] (and its proof) and Proposition 1.1(a). It follows that Φ^* is an isomorphism.

The last statement follows since if Q is weakly $\mathcal{F}$-closed in S, then $\mathcal{T} = \{P \le S \,|\, P \ge Q\}$: every subgroup which contains Q satisfies $(*)$. □

The following lemma describes how quotient fusion systems are obtained by dividing out by weakly $\mathcal{F}$-closed subgroups.

Lemma 2.5. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and let $Q \lhd S$ be a weakly $\mathcal{F}$-closed subgroup. Let $\mathcal{F}/Q$ be the fusion system over S/Q defined by setting*

$$\mathrm{Hom}_{\mathcal{F}/Q}(P/Q, P'/Q) = \{\varphi/Q \,|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, P')\}$$

for all $P, P' \le S$ which contain Q. Then $\mathcal{F}/Q$ is saturated. Also, for any $P/Q \le S/Q$, P/Q is fully normalized in $\mathcal{F}/Q$ if and only if P is fully normalized in $\mathcal{F}$, while P is fully centralized in $\mathcal{F}$ whenever P/Q is fully centralized in $\mathcal{F}/Q$.

Proof. For each $P \le S$ which contains Q, set

$$K_P = \mathrm{Ker}[\mathrm{Aut}_{\mathcal{F}}(P) \twoheadrightarrow \mathrm{Aut}_{\mathcal{F}/Q}(P/Q)].$$

and

$$K_P^0 = \mathrm{Ker}[\mathrm{Aut}_S(P) \twoheadrightarrow \mathrm{Aut}_{S/Q}(P/Q)].$$

Then

$$|C_{S/Q}(P/Q)| = |C_S(P)|{\cdot}|K_P^0|\big/|Q| \qquad \text{and} \qquad |N_{S/Q}(P/Q)| = |N_S(P)|\big/|Q|. \tag{1}$$

By the second formula, P/Q is fully normalized in $\mathcal{F}/Q$ if and only if P is fully normalized in $\mathcal{F}$.

Assume P/Q is fully normalized in $\mathcal{F}/Q$. Then P is fully normalized in $\mathcal{F}$, so by condition (I) in the definition of a saturated fusion system applied to $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. This last condition implies that

$$K_P^0 \in \mathrm{Syl}_p(K_P) \qquad \text{and} \qquad \mathrm{Aut}_{S/Q}(P/Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}/Q}(P/Q)).$$

Thus $|C_S(P)|$ and $|K_P^0|$ both take the largest possible values among subgroups in the $\mathcal{F}$-conjugacy class of P, and hence P/Q is fully centralized by (1). This finishes the proof that condition (I) holds for $\mathcal{F}/Q$. It also shows that if P/Q is fully centralized in $\mathcal{F}/Q$, then $|C_S(P)|$ and $|K_P^0|$ must both take the largest possible values among subgroups in the $\mathcal{F}$-conjugacy class of P, and in particular P is fully centralized in $\mathcal{F}$.

To prove condition (II), fix a morphism $\varphi/Q \in \mathrm{Hom}_{\mathcal{F}/Q}(P/Q, S/Q)$ such that $\varphi(P)/Q$ is fully centralized in $\mathcal{F}/Q$, and set

$$\widetilde{N}_\varphi = \{g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in K_{\varphi(P)}{\cdot}\mathrm{Aut}_S(\varphi(P))\}.$$

Then

$$\widetilde{N}_\varphi/Q = N_{\varphi/Q} \overset{\text{def}}{=} \bigl\{gQ \in N_S(P)/Q \bigm| (\varphi c_g \varphi^{-1})/Q \in \operatorname{Aut}_{S/Q}(\varphi(P)/Q)\bigr\},$$

and we must show that φ/Q extends to $\widetilde{N}_\varphi/Q$. Set $P' = \varphi(P)$ for short; P' is fully centralized in $\mathcal{F}$ since P'/Q is fully centralized in $\mathcal{F}/Q$. Since

$$\varphi\cdot \operatorname{Aut}_{\widetilde{N}_\varphi}(P)\cdot\varphi^{-1} \le K_{P'}\cdot \operatorname{Aut}_S(P'),$$

where $K_{P'} \lhd \operatorname{Aut}_{\mathcal{F}}(P')$, $\operatorname{Aut}_S(P') \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}}(P'))$, and the left hand side is a p-group, there is $\psi \in K_{P'}$ such that

$$(\psi\varphi)\cdot \operatorname{Aut}_{\widetilde{N}_\varphi}(P)\cdot(\psi\varphi)^{-1} \le \operatorname{Aut}_S(P').$$

So by condition (II) for the saturated fusion system $\mathcal{F}$, $\psi\varphi$ extends to a homomorphism $\bar{\varphi} \in \operatorname{Hom}_{\mathcal{F}}(\widetilde{N}_\varphi, S)$, and $\bar{\varphi}/Q$ is an extension of φ/Q to $N_{\varphi/Q}$. □

3. Reduction to simple fusion systems

In this section, we establish a sufficient condition for proving the acyclicity of $\mathcal{Z}_{\mathcal{F}}$: a criterion which in the case $\mathcal{F} = \mathcal{F}_S(G)$ will depend only on the simple components in the decomposition series of the finite group G.

Recall that for any p-group P and any $n \ge 1$, $\Omega_n(P)$ denotes the subgroup of P generated by p^n-torsion elements.

If H and K are two subgroups of a group G (usually normal subgroups) and $n \ge 1$, then we write $[H,K;n]$ for the n-fold iterated commutator: $[H,K;1] = [H,K]$, $[H,K;2] = [[H,K],K]$, and $[H,K;n{+}1] = [[H,K;n],K]$.

Definition 3.1. *For any p-group S, $\mathfrak{X}(S)$ denotes the largest subgroup of S for which there is a sequence*

$$1 = Q_0 \le Q_1 \le \cdots \le Q_n = \mathfrak{X}(S) \le S$$

of subgroups, all normal in S, such that

$$[\Omega_1(C_S(Q_{i-1})), Q_i; p{-}1] = 1 \tag{1}$$

for each $i = 1, \ldots, n$.

It is easy to see that there always is such a largest subgroup. If

$$1 = Q_0 \le Q_1 \le \cdots \le Q_n \qquad \text{and} \qquad 1 = Q'_0 \le Q'_1 \le \cdots \le Q'_m$$

are two sequences of normal subgroups of S which satisfy condition (1) in Definition 3.1, then the sequence

$$1 = Q_0 \le Q_1 \le \cdots \le Q_n = Q_n{\cdot}Q'_0 \le Q_n{\cdot}Q'_1 \le \cdots \le Q_n{\cdot}Q'_n$$

also satisfies the same condition.

When $p = 2$, $\mathfrak{X}(S) = C_S(\Omega_1(S))$ for any finite 2-group S. In particular, $\mathfrak{X}(S) = Z(S)$ if S is generated by elements of order 2. So these subgroups are not very interesting in that case.

We first note some elementary properties of these subgroups $\mathfrak{X}(S)$:

Lemma 3.2. *If p is odd and S is a p-group, then $\mathfrak{X}(S) \geq A$ for every normal abelian subgroup $A \lhd S$. In particular, $\mathfrak{X}(S)$ is centric in S.*

Proof. If $A \lhd S$ is abelian, then $[S, A; p{-}1] \leq [[S, A], A] = 1$, and so $A \leq \mathfrak{X}(S)$ by definition.

Now let A be maximal among the normal abelian subgroups of S. If $C_S(A) \nleq A$, then $A{\cdot}C_S(A)/A$ is a nontrivial normal subgroup of S/A, and hence contains an element $xA \in Z(S/A)$ of order p. But then $\langle A, x\rangle$ is a larger normal abelian subgroup of S, which is a contradiction. Thus A is centric in S, and in particular $\mathfrak{X}(S) \geq A$ is centric in S. □

The following lemma is useful when proving that certain subgroups of S are contained in $\mathfrak{X}(S)$.

Lemma 3.3. *Fix an odd prime p and a p-group S. Let $Q \lhd S$ be any normal subgroup such that*

$$[\Omega_1(Z(\mathfrak{X}(S))), Q; p{-}1] = 1. \tag{1}$$

Then $\mathfrak{X}(S) \geq Q$.

Proof. Set $\mathfrak{X} = \mathfrak{X}(S)$ for short. By definition, there is a sequence

$$1 = Q_0 \leq Q_1 \leq \cdots \leq Q_n = \mathfrak{X}$$

of subgroups normal in S, such that $[\Omega_1(C_S(Q_{i-1})), Q_i; p{-}1] = 1$ for each i. If $Q \lhd S$ is normal and satisfies condition (1), then since $Z(\mathfrak{X}) = C_S(\mathfrak{X})$ by Lemma 3.2, we can set $Q_{n+1} = Q{\cdot}Q_n$, and $Q \leq Q_{n+1} \leq \mathfrak{X}$ by definition. □

The purpose of these subgroups $\mathfrak{X}(S)$ is to provide a tool for applying Proposition 1.4, when trying to show that the functors $\mathcal{Z}_{\mathcal{F}}$ are acyclic. These are most useful when applied to a filtration of these functors, described as follows.

For any saturated fusion system $\mathcal{F}$ over a p-group S, a subgroup $P \leq S$ is *strongly $\mathcal{F}$-closed in S* if no element of P is $\mathcal{F}$-conjugate to any element of $S{\smallsetminus}P$. If $T \leq S$ is strongly $\mathcal{F}$-closed subgroup in S, let

$$\mathcal{Z}_{\mathcal{F}}^T\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

be the subfunctor of $\mathcal{Z}_{\mathcal{F}}$ defined by setting $\mathcal{Z}_{\mathcal{F}}^T(P) = Z(P) \cap T$.

When $\mathcal{F}$ is a saturated fusion system over a p-group S, and $T \lhd S$ is a strongly $\mathcal{F}$-closed subgroup, then a fully $\mathcal{F}$-normalized subgroup $P \leq T$ will be called *$\mathcal{F}|_T$-radical* if

$$O_p(\mathrm{Out}_{\mathcal{F}}(P)) \cap \mathrm{Out}_T(P) = 1.$$

Lemma 3.4. *Fix an odd prime p, a saturated fusion system $\mathcal{F}$ over a p-group S, and a pair $T_0 \lhd T \lhd S$ of subgroups strongly $\mathcal{F}$-closed in S. Write $\mathfrak{X}(T/T_0) = \mathfrak{X}/T_0$ for short. For any fully $\mathcal{F}$-normalized subgroup $Q \leq T$, define*

$$\mathcal{Z}_Q\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

by setting, for $\mathcal{F}$-centric $P \le S$,

$$\mathcal{Z}_Q(P) = \begin{cases} \big(\mathcal{Z}_{\mathcal{F}}^T/\mathcal{Z}_{\mathcal{F}}^{T_0}\big)(P) \cong \frac{Z(P)\cap T}{Z(P)\cap T_0} & \text{if } P\cap T \text{ is } \mathcal{F}\text{-conjugate to } Q \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $Q \not\geq \mathfrak{X}$, or that Q is not centric in T, or that Q is not $\mathcal{F}|_T$-radical. Then $\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^*(\mathcal{Z}_Q) = 0$.

Proof. Since Q is fully $\mathcal{F}$-normalized, for any $Q' \le S$ which is $\mathcal{F}$-conjugate to Q, there is some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q'), N_S(Q))$ such that $\varphi(Q') = Q$ [BLO2, Proposition A.2(c)]. Hence each subgroup $P' \le S$ for which $\mathcal{Z}_Q(P') \neq 0$ is $\mathcal{F}$-conjugate to a subgroup P such that $P \cap T = Q$.

Assume first $Q \not\geq T_0$. Then for each $\mathcal{F}$-centric subgroup P such that $P \cap T = Q$, $N_{PT_0}(P)/P \neq 1$ and acts trivially on $\mathcal{Z}_Q(P)$, and so $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); \mathcal{Z}_Q(P)) = 0$ by Proposition 1.1(b). Thus $\varprojlim^*(\mathcal{Z}_Q) = 0$ in this case.

If Q is not centric in T, then for each $\mathcal{F}$-centric subgroup P such that $P \cap T = Q$, $N_{P\cdot C_T(Q)}(P)/P \neq 1$ and acts trivially on $\mathcal{Z}_Q(P)$, and so $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); \mathcal{Z}_Q(P)) = 0$ by Proposition 1.1(b). Again, $\varprojlim^*(\mathcal{Z}_Q) = 0$ in this case.

Now assume Q is centric in T, but not $\mathcal{F}|_T$-radical. Set

$$\widehat{Q} = \{x \in N_T(Q) \mid c_x \in O_p(\mathrm{Out}_{\mathcal{F}}(Q))\};$$

$\widehat{Q}/Q \neq 1$ by assumption. Let $P \le S$ be a $\mathcal{F}$-centric subgroup such that $P \cap T = Q$. Each element of $\mathrm{Aut}_{\mathcal{F}}(P)$ leaves Q invariant (since $T \lhd S$), so we have a restriction map

$$\rho\colon\ \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q) \times \mathrm{Aut}(P/Q),$$

and $\mathrm{Ker}(\rho)$ is a p-group by [Go, Corollary 5.3.3]. Hence $\rho^{-1}(O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \times 1)$ is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. Also, P normalizes $\widehat{Q}$, since it normalizes Q, so

$$1 \neq Q_0/Q \overset{\text{def}}{=} (\widehat{Q}/Q)^P$$

since $\widehat{Q}/Q \neq 1$, and $Q_0 \le N_S(P)$ since $[Q_0, P] \le Q$ by definition. For any $x \in Q_0 \smallsetminus Q \subseteq N(P) \smallsetminus P$, $c_x \in \rho^{-1}(O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \times 1)$, its class in $\mathrm{Out}_{\mathcal{F}}(P)$ is nontrivial since P is $\mathcal{F}$-centric and $x \notin P$, and hence $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1$. Thus $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); \mathcal{Z}_Q(P)) = 0$ by Proposition 1.1(b). Since this holds for all $\mathcal{F}$-centric P with $P \cap T = Q$, $\varprojlim^*(\mathcal{Z}_Q) = 0$ in this case.

It remains to consider the case where $Q \ge T_0$, Q is centric in T, and $Q \not\geq \mathfrak{X}$. This will be done in three steps. In the first two steps, we show that $\varprojlim^*(\mathcal{Z}_{\mathfrak{X}})$ is isomorphic to the higher limits of a certain functor over an orbit category of a group. Only in Step 3 do we apply the assumption that $Q \not\geq \mathfrak{X}$.

Step 1: In this case, we set

$$\widehat{Q} = Q{\cdot}C_S(Q).$$

Then $\widehat{Q}$ is $\mathcal{F}$-centric, and $\widehat{Q} \cap T = Q$ does not contain $\mathfrak{X}$. Also, $\widehat{Q}$ is fully normalized in $\mathcal{F}$, since if Q' is $\mathcal{F}$-conjugate to $\widehat{Q}$ and fully normalized in $\mathcal{F}$, then there is some

$\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(\widehat{Q}), N_S(Q'))$ with $\alpha(\widehat{Q}) = Q'$ (see [BLO2, Proposition A.2(c)]). Hence $\alpha(Q) = Q' \cap T \lhd N_S(Q')$, and

$$|N_S(\widehat{Q})| = |N_S(Q)| \geq |N_S(\alpha(Q))| \geq |N_S(Q')|$$

since Q is fully normalized.

Let $\widehat{\mathcal{Z}}_Q$ be the quotient functor of $\mathcal{Z}_Q$ where

$$\widehat{\mathcal{Z}}_Q(P) = \begin{cases} \mathcal{Z}_Q(P) & \text{if } P \text{ contains a subgroup } \mathcal{F}\text{-conjugate to } \widehat{Q} \\ 0 & \text{otherwise.} \end{cases}$$

If $\widehat{\mathcal{Z}}_Q(P) \neq \mathcal{Z}_Q(P)$ (i.e., if $\widehat{\mathcal{Z}}_Q(P) = 0$ and $\mathcal{Z}_Q(P) \neq 0$), then up to conjugacy, P is $\mathcal{F}$-centric and $P \cap T = Q$, but $P \not\geq \widehat{Q}$. Then $N_{P\widehat{Q}}(P)/P$ is a nontrivial p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ which acts trivially on $\mathcal{Z}_Q(P)$, so $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); \mathcal{Z}_Q(P)) = 0$ in this case. Thus

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(\widehat{\mathcal{Z}}_Q) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(\mathcal{Z}_Q). \tag{1}$$

Step 2: Set

$$\Gamma = \mathrm{Out}_{\mathcal{F}}(\widehat{Q}), \qquad S' = \mathrm{Out}_S(\widehat{Q}) \in \mathrm{Syl}_p(\Gamma), \qquad \text{and} \qquad T' = \mathrm{Out}_T(\widehat{Q})$$

for short. Using the isomorphism

$$\mathcal{O}^{\geq \widehat{Q}}(N_{\mathcal{F}}(\widehat{Q})) \xrightarrow[\cong]{\Psi} \mathcal{O}_{S'}(\Gamma)$$

of Lemma 2.3, we see that $T' = \Psi(T\widehat{Q})$ is strongly closed in $S' = \Psi(S)$ with respect to Γ, since no element of $T\widehat{Q}$ can be $N_{\mathcal{F}}(Q)$-conjugate to any element of $S \smallsetminus T\widehat{Q}$.

By definition, each subgroup on which $\widehat{\mathcal{Z}}_Q$ is nonvanishing is $\mathcal{F}$-conjugate to some $P \geq \widehat{Q}$ such that $P \cap T = Q$. In particular, $Q \lhd P$ since $T \lhd S$, and so $\widehat{Q} = Q \cdot C_S(Q) \lhd P$. If P' is any subgroup $\mathcal{F}$-conjugate to P which contains $\widehat{Q}$, and $\alpha \in \mathrm{Iso}_{\mathcal{F}}(P, P')$ is any isomorphism, then

$$\alpha(Q) = \alpha(P \cap T) = P' \cap T \geq \widehat{Q} \cap T = Q,$$

and this is an equality since $|\alpha(Q)| = |Q|$. Hence $\alpha(\widehat{Q}) = \widehat{Q}$. Hypothesis $(*)$ of Proposition 2.4 is thus satisfied, and hence

$$\varprojlim_{\mathcal{O}_{S'}(\Gamma)}{}^{*}(\widehat{\mathcal{Z}}_Q \circ \Psi^{-1}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(\widehat{\mathcal{Z}}_Q). \tag{2}$$

Set

$$M_1 = Z(Q) = Z(\widehat{Q}) \cap T \qquad \text{and} \qquad M_0 = Z(Q) \cap T_0 = Z(\widehat{Q}) \cap T_0;$$

and set $M = M_1/M_0$. We regard these as $\mathbb{Z}_{(p)}[\Gamma]$-modules. Let

$$F \colon \mathcal{O}_{S'}(\Gamma)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$$

be the functor $F(P) = {M_1}^P/{M_0}^P$ for all $P \leq S'$. This is clearly a subfunctor of H^0M which satisfies the relative norm condition $(*)$ in Propositions 1.3 and 1.4. Also, for

$P' = \mathrm{Out}_P(\widehat{Q}) \le S'$ (i.e., $\widehat{Q} \lhd P$ and $P' \cong P/\widehat{Q}$),

$$(\widehat{\mathcal{Z}}_Q \circ \Psi^{-1})(P') = \widehat{\mathcal{Z}}_Q(P) = \begin{cases} \frac{Z(P)\cap T}{Z(P)\cap T_0} \cong F(P') & \text{if } P\cap T = Q \\ 0 & \text{otherwise;} \end{cases}$$

and $P \cap T = Q$ if and only if $P \cap T\widehat{Q} = \widehat{Q}$, if and only if $P' \cap T' = 1$. Hence by Proposition 1.4, together with (1) and (2),

$$\mathfrak{N}_{Z(T')}{\cdot}\Omega_1(M) = 0 \quad \text{implies} \quad \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^*(\mathcal{Z}_Q) \cong \varprojlim_{\mathcal{O}_{S'}(\Gamma)}{}^*(\widehat{\mathcal{Z}}_Q \circ \Phi) = 0. \tag{3}$$

(More precisely, Proposition 1.4 only tells us that $\mathcal{Z}_Q$ is acyclic. But $Q \lneqq T$ since it does not contain $\mathfrak{X}$, so $\mathcal{Z}_Q(S) = 0$, and this implies $\varprojlim{}^0(\mathcal{Z}_Q) = 0$.)

Step 3: By definition of $\mathfrak{X}(T/T_0)$, there are subgroups

$$1 = Q_0/T_0 \le Q_1/T_0 \le \cdots \le Q_n/T_0 = \mathfrak{X}(T/T_0),$$

all normal in T/T_0, such that

$$[\Omega_1(C_{T/T_0}(Q_{i-1}/T_0)), Q_i/T_0; p{-}1] = 1 \tag{4}$$

for all $i = 1, \ldots, n$. Let $i \le n$ ($i \ge 1$) be the smallest integer such that $Q \not\ge Q_i$. Then $QQ_i \gneqq Q$, so $N_{QQ_i}(Q)/Q$ is nontrivial, and is normal in $N_T(Q)/Q$ since $Q_i \lhd T$. Hence the fixed subgroup

$$\big(N_{QQ_i}(Q)/Q\big)^{N_T(Q)/Q} = \{xQ \mid x \in N_{QQ_i}(Q),\ [x, N_T(Q)] \le Q\}$$

is also nontrivial. Fix some $x \in N_{QQ_i}(Q) \smallsetminus Q$ such that $x \in Q_i$ and $[x, N_T(Q)] \le Q$. In particular,

$$xQ \in Z(N_T(Q)/Q). \tag{5}$$

Since $Q \ge Q_{i-1}$ by assumption,

$$[\Omega_1(M), x; p{-}1] \le [\Omega_1(Z(Q/T_0)), x; p{-}1] \le [\Omega_1(C_{T/T_0}(Q_{i-1}/T_0)), Q_i; p{-}1] = 1, \tag{6}$$

where the last equality holds by (4).

Now regard M additively as a $\mathbb{Z}_{(p)}[\Gamma]$-module. Then (6) translates to the statement that

$$\mathfrak{N}_{\langle x\rangle}{\cdot}\Omega_1(M) = (1-x)^{p-1}{\cdot}\Omega_1(M) = 0.$$

Also, $x \in Z(\mathrm{Out}_T(Q))$ by (5), so $x \in Z(T')$ by (3). Hence $\mathfrak{N}_{Z(T')}{\cdot}\Omega_1(M) = 0$, so $\varprojlim{}^*(\mathcal{Z}_Q) = 0$ by (3), and this finishes the proof. □

Using Proposition 2.4 and Lemma 3.4 (with $T_0 = 1$ and $T = S$), it is not hard to show that for any saturated fusion system $\mathcal{F}$ over a p-group S, $\mathcal{Z}_{\mathcal{F}}$ is acyclic if $\mathfrak{X}(S)$ contains a subgroup which is both centric and weakly $\mathcal{F}$-closed in S. Since we are unable to prove directly that this holds for all $\mathcal{F}$, we instead filter $\mathcal{Z}_{\mathcal{F}}$ via a maximal series of strongly $\mathcal{F}$-closed subgroups of S, and use the following more general result.

Proposition 3.5. *Fix a saturated fusion system $\mathcal{F}$ over a p-group S, and let $T_0 \lhd T \lhd S$ be a pair of subgroups strongly $\mathcal{F}$-closed in S. Assume there is a subgroup $\mathfrak{X}/T_0 \le \mathfrak{X}(T/T_0)$ which is centric in T/T_0 and weakly $\mathcal{F}/T_0$-closed. Then the quotient functor $\mathcal{Z}_{\mathcal{F}}^T/\mathcal{Z}_{\mathcal{F}}^{T_0}$ is acyclic.*

More generally, let $\mathfrak{X}_{\mathcal{F}}(T/T_0)$ be the intersection of all subgroups $Q/T_0 \le T/T_0$ containing $\mathfrak{X}(T/T_0)$ such that Q is fully $\mathcal{F}$-normalized and $\mathcal{F}|_T$-radical. Assume there is a subgroup $\mathfrak{X}/T_0 \le \mathfrak{X}_{\mathcal{F}}(T/T_0)$ which is centric in T/T_0 and weakly $\mathcal{F}/T_0$-closed. Then the quotient functor $\mathcal{Z}^T_{\mathcal{F}}/\mathcal{Z}^{T_0}_{\mathcal{F}}$ is acyclic.

Proof. Write $\mathcal{Z} = \mathcal{Z}^T_{\mathcal{F}}/\mathcal{Z}^{T_0}_{\mathcal{F}}$ for short. Assume $\mathfrak{X}/T_0 \le \mathfrak{X}_{\mathcal{F}}(T/T_0)$ is centric in T/T_0 and weakly $\mathcal{F}/T_0$-closed. In particular, $\mathfrak{X}$ is weakly $\mathcal{F}$-closed. Let $\mathcal{Z}_{\mathfrak{X}}$ be the functor on $\mathcal{O}(\mathcal{F}^c)$ defined by setting, for all $P \le S$,

$$\mathcal{Z}_{\mathfrak{X}}(P) = \begin{cases} \mathcal{Z}(P) & \text{if } P \not\ge \mathfrak{X} \\ 0 & \text{otherwise.} \end{cases}$$

(Note that since $\mathfrak{X}$ is weakly $\mathcal{F}$-closed, if $P \not\ge \mathfrak{X}$, then the same holds for all subgroups in its $\mathcal{F}$-conjugacy class.) We regard $\mathcal{Z}_{\mathfrak{X}}$ as a subfunctor of $\mathcal{Z}$. Define $\mathcal{Z}_Q$ as in Lemma 3.4; then $\varprojlim^*(\mathcal{Z}_Q) = 0$ for all fully $\mathcal{F}$-normalized $Q \le T$ such that $Q/T_0 \not\ge \mathfrak{X}(T/T_0)$, or such that Q is not $\mathcal{F}|_T$-radical. In particular, this applies to all $Q \not\ge \mathfrak{X}$. Thus via the obvious filtration of $\mathcal{Z}_{\mathfrak{X}}$, we get that $\varprojlim^*(\mathcal{Z}_{\mathfrak{X}}) = 0$, and hence that

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^*(\mathcal{Z}/\mathcal{Z}_{\mathfrak{X}}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^*(\mathcal{Z})\,. \tag{1}$$

Set $\mathfrak{X}^* = \mathfrak{X}{\cdot}C_S(\mathfrak{X})$. Then $\mathfrak{X}^*$ is $\mathcal{F}$-centric; and $\mathfrak{X}^* \cap T = \mathfrak{X}$ since $\mathfrak{X}$ is centric in T (since $\mathfrak{X}/T_0$ is centric in T/T_0). If $\mathfrak{X} \le P \le S$ and $P \not\ge \mathfrak{X}^*$, and P is $\mathcal{F}$-centric, then $N_{\mathfrak{X}^*P}(P)/P \cong \mathrm{Out}_{\mathfrak{X}^*}(P)$ is a nontrivial p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ which acts trivially on $(Z(P)\cap T)/(Z(P)\cap T_0)$, and so

$$\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); \mathcal{Z}(P)) = 0$$

for such P. Hence if we let F denote the functor

$$F(P) = \begin{cases} \mathcal{Z}(P) = \frac{Z(P)\cap T}{Z(P)\cap T_0} & \text{if } P' \ge \mathfrak{X}^* \text{ for some } P' \ \mathcal{F}\text{-conjugate to } P \\ 0 & \text{otherwise;} \end{cases}$$

then

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^*(F) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^*(\mathcal{Z}/\mathcal{Z}_{\mathfrak{X}}). \tag{2}$$

Set

$$M_1 = Z(\mathfrak{X}^*) \cap T = Z(\mathfrak{X}) \qquad \text{and} \qquad M_0 = Z(\mathfrak{X}^*) \cap T_0.$$

Since $\mathfrak{X}$ is weakly $\mathcal{F}$-closed, $\mathfrak{X}^* \overset{\text{def}}{=} \mathfrak{X}{\cdot}C_S(\mathfrak{X})$ is both centric in S and weakly $\mathcal{F}$-closed. So by Proposition 2.4, there is a functor

$$\overline{F}\colon \mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(\mathfrak{X}^*)) \longrightarrow \mathsf{Ab},$$

where

$$\overline{F}(P/\mathfrak{X}^*) = \frac{Z(P)\cap T}{Z(P)\cap T_0} \cong {M_1}^P/{M_0}^P$$

for all

$$P/\mathfrak{X}^* \cong \mathrm{Out}_P(\mathfrak{X}^*) \le \mathrm{Out}_S(\mathfrak{X}^*) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(\mathfrak{X}^*));$$

and such that

$$\varprojlim_{\mathcal{O}_p(\mathrm{Out}_{\mathcal{F}}(\mathfrak{X}^*))}^{*} (\overline{F}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^*)}^{*}(F). \tag{3}$$

Finally, $\varprojlim^i(\overline{F}) = 0$ for $i > 0$ by Proposition 1.3 ($\overline{F} \cong H^0M_1/H^0M_0$). Together with (1), (2), and (3), this finishes the proof of the proposition. □

It now remains to determine, for each saturated fusion system $\mathcal{F}$ over a p-group S (p odd), whether there always exists a sequence of strongly $\mathcal{F}$-closed subgroups for which Proposition 3.5 applies to each successive pair. For convenience, we define a subgroup $Q \le S$ to be *universally weakly closed* in S if for every saturated fusion system $\mathcal{F}$ over a p-group $S' \ge S$ such that S is strongly $\mathcal{F}$-closed, Q is weakly $\mathcal{F}$-closed in S'.

Lemma 3.6. *Fix an odd prime p and a p-group S. Then a subgroup $Q \le S$ is universally weakly closed if for all $P \le S$ containing Q, Q is a characteristic subgroup of P.*

Proof. Assume that $Q \le S$ is not universally weakly closed. Then there exist a saturated fusion system $\mathcal{F}$ over a p-group $S' \ge S$ such that S is strongly $\mathcal{F}$-closed, and such that Q is not weakly $\mathcal{F}$-closed in S'. By Alperin's fusion theorem for saturated fusion systems [BLO2, Theorem A.10], there is a subgroup $P' \le S'$ containing Q, and an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P')$ such that $\alpha(Q) \ne Q$. Set $P = T \cap P'$. Then $\alpha(P) = P$ since T is strongly $\mathcal{F}$-closed, and hence α induces an automorphism of $P \le S$ which does not send Q to itself. Thus Q is not a characteristic subgroup of P. □

The next lemma gives some simple conditions on the p-group for being able to apply Proposition 3.5. For any p-group S, let $J(S)$ denote Thompson's subgroup: the subgroup generated by all elementary abelian subgroups of S of maximal rank.

Proposition 3.7. *Fix an odd prime p, and a p-group S which satisfies any of the following conditions.*

(a) $\mathfrak{X}(S) \ge J(S)$.

(b) *S contains a unique elementary abelian p-subgroup E of maximal rank.*

(c) *$S/\mathfrak{X}(S)$ is abelian.*

Then there is a subgroup $P \le \mathfrak{X}(S)$ which is centric and universally weakly closed in S.

Proof. Write $\mathfrak{X} = \mathfrak{X}(S)$ for short.

(a) Assume $\mathfrak{X} \ge J(S)$. Clearly, $J(S)$ is universally weakly closed in S; however, it need not be centric. So instead, consider the subgroup $Q = J(S){\cdot}C_S(J(S)) \le S$. This is clearly normal and centric in S, and is characteristic in any subgroup of S which contains it since $J(S)$ is. Thus Q is universally weakly closed in S by Lemma 3.6.

It remains to check that $Q \le \mathfrak{X}$. Since $J(S) \le \mathfrak{X}$, every elementary abelian subgroup of S of maximal rank commutes with $Z(\mathfrak{X})$, and thus contains $\Omega_1(Z(\mathfrak{X}))$ since otherwise it would not be maximal. Thus $\Omega_1(Z(\mathfrak{X})) \le Z(J(S))$, so

$$[\Omega_1(Z(\mathfrak{X})), Q] \le [Z(J(S)), J(S){\cdot}C_S(J(S))] = 1.$$

Hence $Q \le \mathfrak{X}$ by Lemma 3.3.

(b) If $E \le S$ is the unique elementary abelian subgroup of maximal rank, then $J(S) = E$, and $E \le \mathfrak{X}$ by Lemma 3.2. The result thus follows from (a).

(c) Assume that $S/\mathfrak{X}$ is abelian, and that $\mathfrak{X}$ is not universally weakly closed in S. By Lemma 3.6, there is a subgroup $P \le S$ containing $\mathfrak{X}$, and an automorphism $\alpha \in \mathrm{Aut}(P)$ such that $\alpha(\mathfrak{X}) \ne \mathfrak{X}$. We claim that this is impossible.

Assume first that $\alpha(Z(\mathfrak{X})) \not\le \mathfrak{X}$, and fix an element $g \in \alpha(Z(\mathfrak{X}))\smallsetminus\mathfrak{X}$. Then $[\Omega_1(Z(\mathfrak{X})), g] \le \alpha(\mathfrak{X})$, since $\alpha(\mathfrak{X}) \lhd P$, and hence

$$[[\Omega_1(Z(\mathfrak{X})), g], g] \le [\alpha(\mathfrak{X}), g] = 1$$

since $g \in Z(\alpha(\mathfrak{X}))$. Set $Q = \langle g, \mathfrak{X}\rangle$; then $[\Omega_1(Z(\mathfrak{X})), Q; 2] = 1$, and $Q \lhd S$ since $S/\mathfrak{X}$ is abelian. Then $Q \in \mathfrak{X}$ by Lemma 3.3, and this contradicts the original assumption on g.

Now assume that $\alpha(Z(\mathfrak{X})) \le \mathfrak{X}$, and thus that $Z(\mathfrak{X}) \le \alpha^{-1}(\mathfrak{X})$ (and $\alpha^{-1}(\mathfrak{X}) \ne \mathfrak{X}$). Fix a chain of subgroups

$$1 = Q_0 \le Q_1 \le \cdots \le Q_n = \mathfrak{X},$$

all normal in S (hence in P), which satisfy condition (1) in Definition 3.1. Let $i \le n$ be such that $Q_i \not\le \alpha(\mathfrak{X})$ but $Q_{i-1} \le \alpha(\mathfrak{X})$. Then

$$\alpha^{-1}\big(\Omega_1(C_P(Q_{i-1}))\big) = \Omega_1(C_P(\alpha^{-1}Q_{i-1})) \ge \Omega_1(C_P(\mathfrak{X})) = \Omega_1(Z(\mathfrak{X})),$$

and hence

$$[\Omega_1(Z(\mathfrak{X})), \alpha^{-1}Q_i; p{-}1] \le \alpha^{-1}\big([\Omega_1(C_P(Q_{i-1})), Q_i; p{-}1]\big) = 1$$

by the assumption on the Q_i. Hence by Lemma 3.3 again, $\langle \mathfrak{X}, Q_i\rangle \le \mathfrak{X}$, which contradicts the original assumption on Q_i. □

We note the following immediate corollary to Propositions 3.7(a) and 3.5.

Corollary 3.8. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and let $1 = T_0 \le T_1 \le \cdots \le T_k = S$ be any sequence of subgroups which are all strongly $\mathcal{F}$-closed in S. Assume, for all $1 \le i \le k$, that $\mathfrak{X}(T_i/T_{i-1}) \ge J(T_i/T_{i-1})$. Then $\varprojlim^i(\mathcal{Z}_{\mathcal{F}}) = 0$ for all $i > 0$.* □

Corollary 3.8 motivates the following

Conjecture 3.9. *For any odd prime p and any p-group P, $\mathfrak{X}(P) \ge J(P)$.*

By Corollary 3.8 (together with Lemma 2.5), in order to prove that $\mathcal{Z}_{\mathcal{F}}$ is acyclic for all saturated fusion systems $\mathcal{F}$, it suffices to prove Conjecture 3.9 for all p-groups P which can occur as minimal strongly closed subgroups in saturated fusion systems. However, it seems to be very difficult to prove or find a counterexample to this conjecture, even in this restricted form. This also indicates that it will be very difficult to find an example of a saturated fusion system $\mathcal{F}$ for which $\mathcal{Z}_{\mathcal{F}}$ is not acyclic, if there are any.

We finish this section with one other elementary result about the groups $\mathfrak{X}(S)$, a result which will be useful in the next section.

Proposition 3.10. *Fix an odd prime p and a p-group S. Then either* $\mathrm{rk}(Z(\mathfrak{X}(S))) \geq p$, *or* $\mathfrak{X}(S) = S$. *In particular,* $\mathfrak{X}(S) = S$ *if* $\mathrm{rk}(S) \leq p-1$.

Proof. Set $\mathfrak{X} = \mathfrak{X}(S)$ for short. Assume $\mathrm{rk}(Z(\mathfrak{X})) \leq p-1$, and set $E \overset{\text{def}}{=} \Omega_1(Z(\mathfrak{X})) \lhd S$. For each $i \geq 0$, either

$$[E,S;i{+}1] = [[E,S;i],S] \lneqq [E,S;i],$$

or $[E,S;i] = 1$. Since $|E| \cong (C_p)^k$ for $k \leq p-1$, this shows that $[E,S;p{-}1] = 1$, and hence that $\mathfrak{X}(S) = S$ by Lemma 3.3. $\square$

4. The acyclicity of $\mathcal{Z}_G$ at odd primes

We are now ready to show, for any finite group G and any odd prime p, that all higher limits of $\mathcal{Z}_G$ vanish when p is odd. This will be based on the following proposition, which gives for any finite group G a sufficient condition for the acyclicity of $\mathcal{Z}_G$ in terms of its simple composition factors. When G is a finite group and $S \in \mathrm{Syl}_p(G)$, set

$$\mathfrak{X}_G(S) = \bigcap \bigl\{ P \leq S \bigm| P \geq \mathfrak{X}(S),\ O_p(\mathrm{Out}_G(P)) = 1, N_S(P) \in \mathrm{Syl}_p(N_G(P)) \bigr\} \ :$$

the intersection of all subgroups of S which contain $\mathfrak{X}(S)$, and are fully normalized and $\mathcal{F}_p(G)$-radical.

Proposition 4.1. *For any prime p and any finite group G, $\mathcal{Z}_G$ is acyclic if for each nonabelian simple group L which occurs in the decomposition series for G, and any $S \in \mathrm{Syl}_p(L)$, there is a subgroup $Q \leq \mathfrak{X}_L(S)$ which is centric and weakly* $\mathrm{Aut}(L)$*-closed in S. In particular, $\mathcal{Z}_G$ is acyclic for each finite solvable group G.*

Proof. Fix a sequence of normal subgroups

$$1 = K_0 \lneqq K_1 \lneqq \cdots \lneqq K_n = G$$

such that each subquotient K_{i+1}/K_i is a minimal normal subgroup of G/K_i. We show that $\mathcal{Z}_G^{K_{i+1}}/\mathcal{Z}_G^{K_i}$ is acyclic for each i. Choose $S \in \mathrm{Syl}_p(G)$, and set $S_i = S \cap K_i \in \mathrm{Syl}_p(K_i)$ and $\mathcal{F} = \mathcal{F}_S(G)$.

Assume that $Q = \widetilde{Q}/S_i \leq S_{i+1}/S_i$ is centric in S_{i+1}/S_i and that $\widetilde{Q}$ is fully $\mathcal{F}$-normalized (i.e., $N_S(\widetilde{Q}) \in \mathrm{Syl}_p(N_G(\widetilde{Q}))$). If $\widetilde{Q}$ is $\mathcal{F}|_{S_{i+1}}$-radical, then

$$1 = O_p(\mathrm{Out}_G(\widetilde{Q})) \cap \mathrm{Out}_{K_{i+1}}(\widetilde{Q}) = O_p(\mathrm{Out}_{K_{i+1}}(\widetilde{Q})) \xrightarrow{\text{proj}} O_p(\mathrm{Out}_{K_{i+1}/K_i}(Q)),$$

and hence Q is a radical p-subgroup of K_{i+1}/K_i. This proves that

$$\mathfrak{X}_{\mathcal{F}}(S_{i+1}/S_i) \leq \mathfrak{X}_{K_{i+1}/K_i}(S_{i+1}/S_i).$$

So by Proposition 3.5 (and Lemma 2.1), to prove that $\mathcal{Z}_G^{K_{i+1}}/\mathcal{Z}_G^{K_i}$ is acyclic, it suffices to show

$\mathfrak{X}_{K_{i+1}/K_i}(S_{i+1}/S_i)$ contains a subgroup Q which is centric and weakly G/K_i-closed in S_{i+1}/S_i. (1)

To simplify notation, we replace G by G/K_i (so $K_i = 1$), and set $K = K_{i+1}$ and $P = S_{i+1} \in \mathrm{Syl}_p(K)$. Thus, K is a minimal normal subgroup of G, and we must find

$Q \le \mathfrak{X}_K(P)$ which is centric and weakly G-closed in P. This is clear if K has order prime to p (i.e., $Q = P = 1$).

Since K is a minimal normal subgroup, it is a product of finite simple groups isomorphic to each other (cf. [Go, Theorem 2.1.5]). If K is an elementary abelian p-group, then $\mathfrak{X}(K) = K$, and is centric and weakly closed in K. So assume $K \cong L^n$ where L is simple and nonabelian and $n \ge 1$. We can choose this identification in a way such that $P = (P')^n$ for some fixed $P' \in \mathrm{Syl}_p(L)$. Then $\mathfrak{X}(P) = \mathfrak{X}(P')^n$ (see Definition 3.1), and $\mathfrak{X}_K(P) = \mathfrak{X}_L(P')^n$ since each radical p-subgroup of K splits as a product of n radical p-subgroups of L [JMO, Proposition 1.6(ii)]. By assumption, there is a subgroup $Q' \le \mathfrak{X}_L(P')$ which is centric and weakly $\mathrm{Aut}(L)$-closed in P'. Then $Q \stackrel{\mathrm{def}}{=} (Q')^n$ is centric in P, and $Q \le \mathfrak{X}_K(P)$. It remains to show that Q is weakly $\mathrm{Aut}(K)$-closed in P, and hence weakly G-closed in P.

Assume otherwise: assume there is $\alpha \in \mathrm{Aut}(L^n)$ such that $Q \ne \alpha(Q) \le P$. The n factors L are the unique minimal normal subgroups of L^n, so each automorphism of L^n permutes these factors, and hence $\mathrm{Aut}(L^n) \cong \mathrm{Aut}(L) \wr \Sigma_n$. Thus $\alpha = \sigma \circ (\alpha_1, \ldots, \alpha_n)$ for some $\alpha_i \in \mathrm{Aut}(L)$ and some $\sigma \in \Sigma_n$ (regarded as an automorphism of L^n); and $Q' \ne \alpha_i(Q') \le P'$ for some i. Which contradicts the assumption that Q' is weakly $\mathrm{Aut}(L)$-closed in P'. □

We now prove that all finite nonabelian simple groups L satisfy the condition in Proposition 4.1: for any odd prime $p\big||L|$ and any $S \in \mathrm{Syl}_p(L)$, there is a subgroup $Q \le \mathfrak{X}_L(S)$ (or $\mathbb{Q} \le \mathfrak{X}(S)$) which is centric and weakly $\mathrm{Aut}(L)$-closed in S. We first consider some cases where this can be shown using Proposition 3.7(b).

Proposition 4.2. *Assume p is odd, and let L be a simple group which is either an alternating group, or a group of Lie type in characteristic different from p. Then for $S \in \mathrm{Syl}_p(L)$, $J(S) \le \mathfrak{X}(S)$, and hence there is a subgroup $Q \le \mathfrak{X}(S)$ which is centric and weakly* $\mathrm{Aut}(L)$*-closed in S.*

Proof. If $L \cong A_n$, then S contains a unique elementary abelian p-subgroup E of maximal rank, generated by a product of $[n/p]$ disjoint p-cycles (cf. [GL, 10-5]). Hence $J(S) = E \le \mathfrak{X}(S)$ by Lemma 3.2, and the result follows from Proposition 3.7(a) or (b).

Now assume that L is a simple group of Lie type in characteristic $\ne p$. If $\mathrm{rk}_p(L) \le 2$, then $\mathfrak{X}(S) = S$ by Proposition 3.10. So assume $\mathrm{rk}_p(L) > 2$. Then by [GL, 10-2(1)], each Sylow p-subgroup of L contains a unique elementary abelian p-subgroup of maximal rank; and the result follows from Lemma 3.2 and Proposition 3.7(a,b) again. (Note that all of the exceptional cases listed in [GL] — the simple groups $A_2(q)$, ${}^2A_2(q)$, $G_2(q)$, ${}^3D_4(q)$, and ${}^2F_4(q)$ when $p = 3$ — have 3-rank at most 2 by [GL, 10-2(2)] and Tables 10:1 and 10:2.) □

We next consider simple groups of Lie type in characteristic p. We first summarize the structures in these groups which will be needed, referring to [Ca] as a general reference.

Assume first that L is a Chevalley group: $L = \mathbb{G}(q)$, where $\mathbb{G}$ is one of the groups A_n, B_n, etc., defined over the finite field $\mathbb{F}_q$ ($q = p^a$). For example, $A_n(q) \cong PSL_{n+1}(\mathbb{F}_q)$. Let $\Phi \subseteq \mathbb{V}$ denote the root system of $\mathbb{G}$, where $\mathbb{V}$ is a real vector space. Let Φ_+ be the set of positive roots; thus $\Phi = \{\pm r \mid r \in \Phi_+\}$. Let I denote the set of primitive roots, an $\mathbb{R}$-basis of $\mathbb{V}$.

To each root $r \in \Phi$ corresponds a root subgroup $X_r \cong \mathbb{F}_q$ in $L = \mathbb{G}(q)$. Then $U \stackrel{\text{def}}{=} \prod_{r\in\Phi_+} X_r$ is a Sylow p-subgroup of L. Also, $B \stackrel{\text{def}}{=} N_L(U) = U \rtimes H$ (the Borel subgroup), where H is the subgroup of diagonal elements (and has order prime to p). Set $N = N_L(H)$; then $W \cong N/H$ is the Weyl group of $\mathbb{G}$ (and of the root system Φ).

For example, when $L = A_n(q) = PSL_{n+1}(q)$, then we can take

$$\mathbb{V} = \{x \in \mathbb{R}^{n+1} \mid \textstyle\sum x_i = 0\}, \qquad \Phi = \{e_i - e_j \mid i \neq j\}$$

(where $\{e_1, \ldots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$),

$$\Phi_+ = \{e_i - e_j \mid i < j\}, \qquad \text{and} \qquad I = \{e_i - e_{i+1}\}.$$

Then $X_{e_i-e_j}$ is the subgroup of matrices which have 1's along the diagonal and are zero elsewhere except at entry (i, j), U is the group of upper triangular matrices with 1's along the diagonal, and B is the group of all upper triangular matrices. Diagonal elements are represented by diagonal matrices, N is the image of the subgroup of monomial matrices, and $W \cong \Sigma_{n+1}$.

We next fix the notation for the twisted groups ${}^t\mathbb{G}(q)$. Let $\tau \in \mathrm{Aut}(\mathbb{V}, \Phi, \Phi_+)$ be an automorphism of the root system of $\mathbb{G}$ of order t. Set $\sigma = \widehat{\tau}\circ\widehat{\varphi} \in \mathrm{Aut}(\mathbb{G}(q'))$, where $\widehat{\tau}$ is induced by τ and $\widehat{\varphi}$ is induced by $\varphi \in \mathrm{Aut}(\mathbb{F}_{q'})$ of order t. (In most cases, $q' = q^t$, and so $\mathbb{F}_q$ is the fixed subfield of the automorphism φ.) In all cases, $\tau \in \mathrm{Aut}(\Phi_+)$ can be seen as an automorphism of the Dynkin diagram (and $t = 2, 3$). Also, $\sigma(X_r) = X_{\tau(r)}$ for each $r \in \Phi$, and thus σ leaves invariant the subgroups U, H, and N. The twisted group $L = {}^t\mathbb{G}(q)$ is defined to be the commutator subgroup of $\mathbb{G}(q')^\sigma$, or alternatively as the subgroup of $\mathbb{G}(q')$ generated by U^σ and the analogous subgroup for the root groups of negative roots. Its Borel subgroup is defined to be $B = N_L(U^\sigma) = U^\sigma \rtimes (H^\sigma \cap L)$.

Proposition 4.3. *Assume p is odd, let L be a simple group of Lie type in characteristic p, and fix $S \in \mathrm{Syl}_p(L)$. Then $\mathfrak{X}_L(S)$ is weakly $\mathrm{Aut}(L)$-closed in S.*

Proof. Write $L = {}^t\mathbb{G}(q)$, where $q = p^a$ (possibly $t = 1$). We use the above notation. In particular, Φ_+ denotes the set of positive roots, and $U \in \mathrm{Syl}_p(L)$ is the product of the root subgroups X_r for $r \in \Phi_+$.

For each τ-invariant subset $J \subseteq I$, consider the subgroups

$$U_J = \prod_{r\in\Phi_+\smallsetminus\langle J\rangle} X_r,$$

In particular, $U_\emptyset = U$, and $U_I = 1$. We claim that the following statement holds:

> The subgroups U_J^σ, for τ-invariant subsets $J \subseteq I$, are the only subgroups of $S = U^\sigma$ which are radical p-subgroups of L; and they are all weakly L-closed in U^σ. (1)

By a theorem of Borel and Tits (see the corollary in [BW]), every radical p-subgroup of L is conjugate to one of the subgroups U_J^σ, and by [Gr, Lemma 4.2], if U_J^σ is L-conjugate to $U_{J'}^\sigma$ for $J, J' \leq I$ then $J = J'$. But since we need to know that each radical p-subgroup of L contained in S is actually *equal* to one of the U_J^σ, we modify Grodal's proof to show this.

Assume that $P \le S = U^\sigma$ is L-conjugate to U_J^σ. We show that $P = U_J^\sigma$; this proves that U_J^σ is weakly L-closed in S, and hence (using [BW]) proves (1). Since $L \le U^\sigma{\cdot}N^\sigma{\cdot}U^\sigma$ [Ca, Proposition 8.2.2], and since $U_J \lhd U$, we have $P = ux(U_J^\sigma)x^{-1}u^{-1}$ for some $u \in U^\sigma$ and $x \in N^\sigma$, and $P = U_J^\sigma$ if and only if $x(U_J^\sigma)x^{-1} = U_J^\sigma$. So we can assume $u = 1$ and $P = x(U_J^\sigma)x^{-1}$. Now, x permutes the root subgroups via the action of $w = xH \in W^\tau$ on Φ, and so $w(\Phi_+ \smallsetminus \langle J\rangle) \subseteq \Phi_+$. Write $\Delta = \Phi_+ \smallsetminus \langle J\rangle$ for short; this is closed in the sense that any $r \in \Phi$ which is a positive linear combination of elements of Δ also lies in Δ. So $w(\Delta)$ has the same property. This implies that all primitive roots for the system $w(\pm\langle J\rangle) \cap \Phi_+$ are primitive roots in Φ_+, and thus that $w(\Delta) = \Phi_+ \smallsetminus \langle J'\rangle$ for some $J' \subseteq I$. After replacing w by its product with some element in the Weyl group of $\pm\langle J'\rangle$, we can assume that it sends positive roots to positive roots, and hence must be the identity. So $J' = J$, and $P = U_J^\sigma$.

Thus, by (1), if $Q \le S$ is a radical p-subgroup of L, then $Q = U_J^\sigma$ for some J. Hence $\mathfrak{X}_L(S)$ is the intersection of the subgroups U_J^σ which contain $\mathfrak{X}(S)$. Also, $U_J \cap U_{J'} = U_{J\cup J'}$, since each element of U has a unique decomposition as product of elements of the root groups taken in an appropriate order [Ca, Theorem 5.3.3(ii)]. So any intersection of subgroups U_J^σ is again of the same form, and thus $\mathfrak{X}_L(S) = U_J^\sigma$ for some τ-invariant subset $J \subseteq I$. Hence $\mathfrak{X}_L(S)$ is weakly L-closed in S by (1) again.

Each automorphism of L is congruent mod $\mathrm{Inn}(L)$ to some $\alpha \in \mathrm{Aut}(L)$ which sends S to itself, and α permutes the radical p-subgroups of L contained in S and sends $\mathfrak{X}(L)$ to itself. Thus each coset in $\mathrm{Out}(L)$ contains an automorphism which sends $\mathfrak{X}_L(S)$ to itself, and $\mathfrak{X}_L(S)$ is weakly $\mathrm{Aut}(L)$-closed in S since it is weakly L-closed. □

In fact, when L is simple of Lie type in characteristic p (and p is odd, as usual), then for $S \in \mathrm{Syl}_p(L)$, $\mathfrak{X}(S) = S$ except when $p = 3$ and and $L \cong C_n(q) \cong PSp_{2n}(q)$ $(n \ge 2)$ or $L \cong {}^2A_n(q) \cong PSU_{n+1}(q^2)$ $(n \ge 3)$.

We are now ready to consider the sporadic groups.

Proposition 4.4. *Assume p is odd, let L be a sporadic simple group, and fix $S \in \mathrm{Syl}_p(L)$. Then there is a subgroup $Q \le \mathfrak{X}(S)$ which is centric and weakly* $\mathrm{Aut}(L)$*-closed in S.*

Proof. If $p \ge 5$, then $\mathrm{rk}_p(L) < p$ by [GLS, §5.6], and so $\mathfrak{X}(S) = S$ by Proposition 3.10. So assume $p = 3$. We consider several different cases.

(a) If L is one of the groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_4, *HS*, *He*, or *Ru*, then $\mathrm{rk}_3(L) \le 2$ by [GL, p.123], and so $\mathfrak{X}(S) = S$ by Proposition 3.10.

(b) Assume L is one of the groups J_3, Co_3, Co_2, *McL*, *Suz*, *Ly*, *O'N*, or F_5. In all of these cases, S contains a normal elementary abelian 3-subgroup E of index ≤ 9, $\mathfrak{X}(S) \ge E$ by Lemma 3.2, so $S/\mathfrak{X}(S)$ is abelian, and $\mathfrak{X}(S)$ is weakly $\mathrm{Aut}(L)$-closed by Proposition 3.7(c). More precisely, there are the following inclusions of index prime to 3:

L	J_3	Co_3	Co_2	McL	Suz	Ly	$O'N$	F_5
E	C_3^3	C_3^5	C_3^4	C_3^4	C_3^5	C_3^5	C_3^4	C_3^4
$N_L(E)/E$	$C_3^2 \rtimes C_8$	$2 \times M_{11}$	M_{10}	M_{10}	M_{11}	$2 \times M_{11}$	order 320	order 1152

See [GL, §5] for references. (In fact, in all of the above cases, E is the unique elementary abelian subgroup of S of maximal rank.)

(c) Assume $L \cong Co_1$. By [Cu, p.424], S is contained in a semidirect product $C_3^6 \rtimes 2M_{12}$, and the elementary abelian subgroup C_3^6 is generated by all elements of order 3 in S which lie in the conjugacy class (3A). Thus S contains a unique elementary abelian 3-subgroup E of maximal rank, and hence $C_S(E) \leq \mathfrak{X}(S)$ is centric and weakly $\mathrm{Aut}(L)$-closed in S by Proposition 3.7(b).

(e) Assume $L = F_3$. By [Ho] or [Pa] (see also [As, 14.2]), there are subgroups

$$D \leq K \leq M \leq S,$$

all normal in S, such that $K \cong C_3^5$ is abelian, $C_S(K) = K$, and $[M, K] = D = Z(M) \cong (C_3)^2$. (Also, $M/K \cong C_3^4$ and $N_L(D)/M \cong GL_2(3)$.) Thus $M \leq \mathfrak{X}(S)$, so $\mathrm{rk}(Z(\mathfrak{X}(S))) \leq \mathrm{rk}(Z(M)) = 2$, and hence $\mathfrak{X}(S) = S$ by Proposition 3.10.

In the remaining cases, for a p-group R, we use the notation $Z_n(R) \lhd R$: $Z_1(R) = Z(R)$, and $Z_n(R)/Z_{n-1}(R) = Z(R/Z_{n-1}(R))$. The group R is of class n if $R = Z_n(R) \gneqq Z_{n-1}(R)$. Also, following the notation of [As], we say that a subgroup $H \leq L$ is of type $H'/m_t/m_{t-1}/\cdots/m_1$ if upon setting $R = O_p(H)$, then $H/R \cong H'$, $Z(R) \cong C_p^{m_1}$, and $Z_i(R)/Z_{i-1}(R) \cong C_p^{m_i}$ for all i. (We restrict, for simplicity, to the case where $Z_i(R)/Z_{i-1}(R)$ is elementary abelian for all i.)

(d) Assume $L \cong Fi_{22}$. Then L contains a subgroup $L_0 \cong \Omega_7(3)$ with index prime to 3 (cf. [As, p.26]). Regard L_0 as acting on $V = \mathbb{F}_3^7$, let $W \subseteq V$ be a maximal isotropic subspace ($\dim(W) = 3$), and let $H \leq L_0$ be the subgroup of elements which leave W invariant. Then $[L_0{:}H]$ is prime to 3, and hence we can assume that $S \leq H$. One easily checks that H is of type $SL_3(3)/3/3$, where $R \stackrel{\text{def}}{=} O_3(H)$ is the subgroup of elements whose restriction to W (and to $V/W^\perp$) is the identity, and $Z(R)$ is the subgroup of elements whose restriction to $W^\perp$ and to V/W are the identity. Also, $Z(R) = [R, R]$, and $H/R \cong SL_3(3)$ acts on $Z(R)$ as the group of 3×3 antisymmetric matrices. From this, one quickly sees that $R \leq \mathfrak{X}(S)$. If we set $S_0/R = Z(S/R) \cong C_3$, then $[[Z(R), S_0], S_0] = 1$, so $S_0 \leq \mathfrak{X}(S)$. Hence $\mathrm{rk}(Z(\mathfrak{X}(S))) \leq \mathrm{rk}(Z(S_0) = 2$, so $\mathfrak{X}(S) = S$ by Proposition 3.10. (Alternatively, one can show that S contains a unique elementary abelian subgroup of maximal rank 5, and then apply Proposition 3.7(b).)

(f) Assume $L = Fi_{23}$ or F_2. By [As, p. 33], there is an inclusion $Fi_{23} \leq F_2$ with index prime to 3, so these groups have isomorphic Sylow 3-subgroups. By [As, p. 27 & 208–209], there is a subgroup $H \leq Fi_{23}$ of index prime to 3 and of type $SL_3(3)/3/3/1/3$. We can thus assume $R \stackrel{\text{def}}{=} O_3(H) \leq S \leq H$. Also, $Z_2(R) \cong C_3^4$, $Z_3(R) = C_H(Z_2(R)) \cong Q \times C_3$ where $Z(Q) = [Q, Q] = Z(R)$ and $Q/Z(Q) \cong Z_3(R)/Z_2(R) \cong C_3^3$, and $R/Z_3(R)$ acts on $Z_2(R)$ as the group of all automorphisms which are the identity on $Z(R)$ and on $Z_2(R)/Z(R)$. Since $[[Z_3(R), Z_3(R)], Z_3(R)] = 1$ and $[[Z_2(R), R], R] = 1$, we see that $R \leq \mathfrak{X}(S)$; and hence $\mathfrak{X}(S) = S$ by the same argument as was used for $L \cong Fi_{22}$.

(g) Assume $L = Fi'_{24}$. By [As, pp. 29 & 210–211], there is a subgroup $H \le Fi'_{24}$ of index prime to 3 and of type $(A_5 \times SL_2(3))/8/4/2$, and we can assume $R \overset{\text{def}}{=} O_3(H) \le S \le H$. Also, $R/Z_2(R)$ acts on $Z_2(R) \cong C_3^8$ as the group of all automorphisms which are the identity on $Z(R)$ and on $Z_2(R)/Z(R)$; and the actions of $SL_2(3) \le H/R$ on $Z(R)$ and of $A_5 \le H/R$ on $Z_2(R)/Z(R) \cong C_3^4$ are faithful. Thus $Z_2(R)$ is centric in H and $[[Z_2(R), R], R] = 1$. It follows that $R \le \mathfrak{X}(S)$, and hence (since $\mathrm{rk}(Z(R)) = 2$) that $\mathfrak{X}(S) = S$ by Proposition 3.10.

(h) Assume $L = F_1$. By [As, pp. 35 & 211–212], there is a subgroup $H \le F_1$ of index prime to 3 of type $(GL_2(3) \times M_{11})/10/5/2$. We can thus assume that $R \overset{\text{def}}{=} O_3(H) \le S \le H$. Also, $R/Z_2(R)$ acts on $Z_2(R) \cong C_3^7$ as the group of automorphisms which are the identity on $Z(R)$ and on $Z_2(R)/Z(R)$, and the actions of $GL_2(3) \le H/R$ on $Z(R)$ and of $M_{11} \le H/R$ on $Z_2(R)/Z(R)$ are faithful. Thus $Z_2(R)$ is centric in H and $[[Z_2(R), R], R] = 1$. It follows that $R \le \mathfrak{X}(S)$, and hence by Proposition 3.10 (since $\mathrm{rk}(Z(R)) = 2$) that $\mathfrak{X}(S) = S$. □

We are now ready to prove Theorem A.

Theorem 4.5. *For any odd prime p and any finite group G, $\mathcal{Z}_G$ is acyclic.*

Proof. Let L be a finite simple group, and fix $S \in \mathrm{Syl}_p(L)$. If L is an alternating group, or of Lie type in characteristic $\neq p$, then by Proposition 4.2, there is a subgroup $Q \le \mathfrak{X}(S)$ which is centric and weakly $\mathrm{Aut}(L)$-closed in S. If L is of Lie type in characteristic p, then $\mathfrak{X}_L(S)$ itself is centric and weakly $\mathrm{Aut}(L)$-closed in S by Proposition 4.3. If L is a sporadic group, then there is a subgroup $Q \le \mathfrak{X}(S)$ which is centric and weakly $\mathrm{Aut}(L)$-closed in S by Proposition 4.4. The theorem now follows from Proposition 4.1, together with the classification theorem for finite simple groups. □

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