GORENSTEIN HOMOLOGICAL DIMENSIONS AND AUSLANDER CATEGORIES

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Abstract. In this paper, we study Gorenstein injective, projective, and flat modules over a local Noetherian ring $(R, \mathfrak{m})$. We correspond to a dualizing complex $\mathbf{D}$ of $\hat{R}$, the classes $A'(R)$ and $B'(R)$ of R-modules. For an R-module M, we show that $M \in A'(R)$ if and only if Gorenstein projective dimension of M is finite and if and only if Gorenstein flat dimension of M is finite. In dual situation by using the class $B'(R)$, we provide a characterization for modules of finite Gorenstein injective dimension.

1. Introduction

Throughout this paper, R will denote a commutative ring with nonzero identity and $\hat{R}$ will denote the completion of a local ring $(R, \mathfrak{m})$. When discussing the completion of a local ring $(R, \mathfrak{m})$, we will mean the $\mathfrak{m}$-adic completion.

Auslander and Bridger [**3**] introduced the G-dimension, $\mathrm{G}-\dim_R M$, for every finitely generated R-module M (see also [**2**]). They proved the inequality $\mathrm{G}-\dim_R M \leq \mathrm{pd}_R M$, with equality $\mathrm{G}-\dim_R M = \mathrm{pd}_R M$ when $\mathrm{pd}_R M$ is finite. The G-dimension has strong parallels to the projective dimension. For instance, over a local Noetherian ring $(R, \mathfrak{m})$, the following conditions are equivalent:

(i) R is Gorenstein.

(ii) $\mathrm{G}-\dim_R R/\mathfrak{m} < \infty$.

(iii) All finitely generated R-modules have finite G-dimension.

This characterization of Gorenstein rings is parallel to Auslander-Buchsbaum-Serre characterization of regular rings. G-dimension also differs from projective dimension in that it is defined only for finitely generated modules. Enochs and Jenda defined in [**9**] Gorenstein projective modules (i.e. modules of G-dimension 0) whether the modules are finitely generated or not. Also, they defined a homological dimension, namely the Gorenstein projective dimension, $\mathrm{Gpd}_R(-)$, for arbitrary (non-finitely generated) modules. It is known that for finitely generated modules, the Gorenstein projective dimension agrees with the G-dimension. Along the same lines, Gorenstein flat and Gorenstein injective modules were introduced in [**9**,**10**].

Let R be a Cohen-Macaulay local ring admitting a dualizing module D. Foxby [**12**] defined the class $\mathcal{G}_0(R)$ to be those R-modules M such that $\mathrm{Tor}_i^R(D, M) = \mathrm{Ext}_R^i(D, D \otimes_R M) = 0$ for all $i \geq 1$ and such that the natural map $M \longrightarrow \mathrm{Hom}_R(D, D \otimes_R M)$ is an isomorphism, and $\mathcal{I}_0(R)$ to

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be those R-modules N such that $\operatorname{Ext}^i_R(D,N) = \operatorname{Tor}^R_i(D, \operatorname{Hom}_R(D,N)) = 0$ for all $i \geq 1$ and such that the natural map $D \otimes_R \operatorname{Hom}_R(D,N) \longrightarrow N$ is an isomorphism. In [**11**] Enochs, Jenda and Xu characterize Gorenstein injective, projective and flat dimensions in terms of $\mathcal{G}_0(R)$ and $\mathcal{I}_0(R)$. The main aim of this paper is to extend these characterizations for Gorenstein injective, projective and flat dimensions to arbitrary local Noetherian rings.

Let R be a Noetherian ring with dualizing complex $\mathbf{D}$. The Auslander categories $A(R)$ and $B(R)$ with respect to $\mathbf{D}$ are defined in [**4**, 3.1]. In [**5**], it is shown that the modules in $A(R)$ are precisely those of finite Gorenstein projective dimension (Gorenstein flat dimension), see [**5**, Theorem 4.1], and the modules in $B(R)$ are those of finite Gorenstein injective dimension, see [**5**, Theorem 4.4]. This may be viewed as an extension of the results of [**11**]. Note that, by [**4**, Proposition 3.4], if R is a Cohen-Macaulay local ring with a dualizing module, then an R-module M is in $A(R)$ if and only if $M \in \mathcal{G}_0(R)$ (resp. an R-module M is in $B(R)$ if and only if $M \in \mathcal{I}_0(R)$).

Let R be a local Noetherian ring probably without dualizing complex, and let $\mathbf{D}$ denote the dualizing complex of $\hat{R}$. We define $A'(R)$ to be those R-modules M such that $\hat{R} \otimes_R M \in A(\hat{R})$ and $B'(R)$ to be those R-modules N such that $\operatorname{Hom}_R(\hat{R}, N) \in B(\hat{R})$. In sections 2, 3, and 4, we characterize Gorenstein injective, projective, and flat modules in terms of the classes $A'(R)$ and $B'(R)$. To be more precise, we show the following results.

Theorem 1.1. *Let R be a local Noetherian ring and M an R-module.*
(i) (See Theorem 2.5) M is Gorenstein flat if and only if M belongs to $A'(R)$ and $\operatorname{Tor}^R_i(L,C) = 0$ for all injective R-modules L and all $i > 0$.
(ii) (See Theorem 3.2) M is Gorenstein projective if and only if M belongs to $A'(R)$ and $\operatorname{Ext}^i_R(M,P) = 0$ for all projective R-modules P and all $i > 0$.
(iii) (See Theorem 4.8) M is Gorenstein injective if and only if M belongs to $B'(R)$, M is cotorsion and $\operatorname{Ext}^i_R(E,M) = 0$ for all injective R-modules E and all $i > 0$.

Even more generally, by using the classes $A'(R)$ and $B'(R)$, we characterize modules of finite Gorenstein injective, projective and flat dimensions. Namely, we prove the following two results.

Theorem 1.2. *(See Theorem 3.4) Let R be a local Noetherian ring and M an R-module. Then the following conditions are equivalent:*
(i) $\operatorname{Gfd}_R M < \infty$.
(ii) $\operatorname{Gpd}_R M < \infty$.
(iii) $M \in A'(R)$.

Theorem 1.3. *(See Theorem 4.10) Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d and $\operatorname{Ext}^i_R(\hat{R}, M) = 0$ for all $i > 0$. Then Gorenstein injective dimension of M is finite if and only if M belongs to $B'(R)$. In particular, if $M \in B'(R)$ then $\operatorname{Gid}_R(M) \leq d$.*

Setup and notation If M is any R-module, we use $\operatorname{pd}_R M$, $\operatorname{fd}_R M$ and $\operatorname{id}_R M$ to denote the usual projective, flat and injective dimension of M, respectively. Furthermore, we write $\operatorname{Gpd}_R M$, $\operatorname{Gfd}_R M$ and $\operatorname{Gid}_R M$ for the Gorenstein projective, Gorenstein flat and Gorenstein injective dimension of M, respectively. Let $\mathcal{X}$ be any class of R-modules and let M be an R-module. An

$\mathcal{X}$-precover of M is an R-homomorphism $\varphi : X \longrightarrow M$, where $X \in \mathcal{X}$ and such that the sequence,

$$\operatorname{Hom}_R(X', X) \stackrel{\operatorname{Hom}_R(X',\varphi)}{\longrightarrow} \operatorname{Hom}_R(X', M) \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. If, moreover, $f\varphi = \varphi$ for $f \in \operatorname{Hom}_R(X, M)$ implies f is an automorphism of M, then φ is called an $\mathcal{X}$-cover of M. Also, an $\mathcal{X}$-preenvelope and $\mathcal{X}$-envelope of M are defined "dually". By $P(R)$, $F(R)$ and $I(R)$ we denote the classes of all projective, flat and injective R-modules, respectively. Furthermore, we let $\overline{P(R)}$, $\overline{F(R)}$ and $\overline{I(R)}$ denote the classes of all R-modules with finite projective, flat and injective dimension, respectively.

We may use the following facts without comment. If R is Noetherian of finite Krull dimension, then $\overline{P(R)} = \overline{F(R)}$ (see [**16**, Theorem 4.2.8]). Also, if R is Noetherian then for any $M \in \overline{P(R)}$, we have $\operatorname{pd}_R(M) \leq \dim R$ (see [**15**, p. 84]).

2. Gorensein flat dimension

Let R be a local Noetherian ring an let $\mathbf{D}$ denote the dualizing complex of $\hat{R}$. Let $A(\hat{R})$ denote the full subcategory of $\mathbf{D}_b(\hat{R})$, consisting of those complexes X for which $\mathbf{D} \otimes^{\mathbf{L}}_{\hat{R}} X \in \mathbf{D}_b(\hat{R})$ and the canonical morphism

$$\gamma_X : X \longrightarrow \mathbf{R}\operatorname{Hom}_{\hat{R}}(\mathbf{D}, \mathbf{D} \otimes^{\mathbf{L}}_{\hat{R}} X),$$

is an isomorphism. $\mathbf{D}_b(\hat{R})$ denote the full subcategory of $\mathbf{D}(\hat{R})$ (the derived category of $\hat{R}$-modules) consisting of complexes X with $H_n(X) = 0$ for $| n | >> 0$, see [**4**].
Now, we define $A'(R)$ to be the class of all R-modules M such that $\hat{R} \otimes_R M \in A(\hat{R})$.

Lemma 2.1. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of modules over a local Noetherian ring R. Then if any two of M', M, M'' are in $A'(R)$, so is the third.*

Proof. The exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ yields, the exact sequence $0 \longrightarrow \hat{R} \otimes_R M' \longrightarrow \hat{R} \otimes_R M \longrightarrow \hat{R} \otimes_R M'' \longrightarrow 0$. Now, the conclusion follows by using [**5**, Theorem 4.1] and [**13**, Theorem 2.24]. $\square$

Proposition 2.2. *Let R be a local Noetherian ring and let M be an R-module. If $\operatorname{Gfd}_R M < \infty$, then $M \in A'(R)$.*

Proof. By [**13**, Proposition 3.10], we have $\operatorname{Gfd}_{\hat{R}}(\hat{R} \otimes_R M) < \infty$. Using [**5**, Theorem 4.1], we conclude that $\hat{R} \otimes_R M$ belongs to $A(\hat{R})$. So, the assertion follows by the definition. $\square$

In the proof of the following lemma we use the method of the proof of [**11**, Lemma 3.1].

Lemma 2.3. *Suppose K is cotorsion of finite flat dimension and suppose M is an R-module. If $\operatorname{Tor}^R_i(E, M) = 0$ for all $i > 0$ and all injective R-modules E, then $\operatorname{Ext}^i_R(M, K) = 0$ for all $i > 0$.*

Proof. We prove by induction on $\operatorname{fd}_R K$. First, let K be flat and cotorsion. Then K is a summand of a module of the form $\operatorname{Hom}_R(E, E')$ where E and E' are injective ([**8**, Lemma 2.3]).

It is enough to show that $\mathrm{Ext}_R^i(M, \mathrm{Hom}_R(E, E')) = 0$ for all $i > 0$. We have

$$\mathrm{Ext}_R^i(M, \mathrm{Hom}_R(E, E')) \cong \mathrm{Hom}_R(\mathrm{Tor}_i^R(M, E), E')$$

for all $i \geq 0$. Thus $\mathrm{Ext}_R^i(M, K) = 0$ for all $i > 0$. Now, let K be cotorsion and of finite flat dimension. Let $F_0 \longrightarrow K$ be a flat cover of K with kernel L. Then L is cotorsion, see [**8**, Lemma 2.2]. Also, we have the exact sequence

$$\mathrm{Ext}_R^i(M, F_0) \longrightarrow \mathrm{Ext}_R^i(M, K) \longrightarrow \mathrm{Ext}_R^{i+1}(M, L).$$

Since K and L are cotorsion, then so is F_0. Hence, by inductive hypothesis $\mathrm{Ext}_R^i(M, K) = 0$ for all $i > 0$. □

Lemma 2.4. *Let R be a Noetherian ring and M an R-module.*
(i) If R be a local ring and $M \in A'(R)$, then there exists a monomorphism $M \longrightarrow L$ with $\mathrm{fd}_R L < \infty$.
(ii) Assume $\psi : M \longrightarrow L$ is a monomorphism such that $\mathrm{fd}_R L < \infty$ *and that* $\mathrm{Tor}_i^R(N, M) = 0$ *for all injective R-modules N and all $i > 0$. Then M possesses a monic $\overline{F(R)}$-preenvelope $M \longrightarrow F$, in which F is flat.*
(iii) Let R be a local. Assume $\varphi : M \longrightarrow L$ is a monomorphism such that $\mathrm{pd}_R L < \infty$ *and that* $\mathrm{Ext}_R^i(M, N) = 0$ *for all projective R-modules N and all $i > 0$. Then there exists a monic $\overline{P(R)}$-preenvelope $M \longrightarrow P$, in which P is projective.*

Proof. (i) Since M belongs to $A'(R)$, $\mathrm{Gfd}_{\hat{R}}(M \otimes_R \hat{R})$ is finite by the definition and [**5**, Theorem 4.1]. Therefore, by [**5**, lemma 2.19], we have an exact sequence of $\hat{R}$-modules and $\hat{R}$-homomorphisms $0 \longrightarrow M \otimes_R \hat{R} \longrightarrow L$, where flat dimension of L is finite as an $\hat{R}$-module. So, we obtain an exact sequence $0 \longrightarrow M \longrightarrow L$ of R-modules and R-homomorphism, where flat dimension of L is finite as an R-module. Not that every flat $\hat{R}$-module is also flat as an R-module.

(ii) Using [**7**, Proposition 5.1], there exists a flat preenvelope $f : M \longrightarrow F$. We show that f is $\overline{F(R)}$-preenvelope. To this end, let $\psi' : M \longrightarrow L'$ be an R-homomorphism such that $\mathrm{fd}_R L' < \infty$ and let $0 \longrightarrow K \longrightarrow F' \xrightarrow{\pi} L' \longrightarrow 0$ be an exact sequence such that $\pi : F' \longrightarrow L'$ is a flat cover. Then K is of finite flat dimension and also by [**8**, lemma 2.2], it is cotorsion. Lemma 2.3 implies that $\mathrm{Ext}_R^i(M, K) = 0$ for all $i > 0$. So, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, K) \longrightarrow \mathrm{Hom}_R(M, F') \longrightarrow \mathrm{Hom}_R(M, L') \longrightarrow \mathrm{Ext}_R^1(M, K) = 0.$$

Therefore, there exists an R-homomorphism $h : M \longrightarrow F'$ such that $\pi h = \psi'$. Since $f : M \longrightarrow F$ is flat preenvelope, there exists an R-homomorphism $g : F \longrightarrow F'$ such that $h = gf$. Hence, there exists the R-homomorphism $\pi g : F \longrightarrow L'$ such that $\pi g f = \psi'$. Thus f is $\overline{F(R)}$-preenvelope. Consequently, f is monic, because ψ is monic.

(iii) Using [**11**, Proposition 1.1], there exists $f : M \longrightarrow L'$ which is a $\overline{P(R)}$-preenvelope. Since $\varphi : M \longrightarrow L$ is monic, it turns out that $f : M \longrightarrow L'$ is also monic. Now, let $0 \longrightarrow K \longrightarrow P \xrightarrow{\pi} L' \longrightarrow 0$ be an exact sequence such that P is projective R-module. It is easy to see that $K \in \overline{P(R)}$. On the other hand, by hypothesis and induction on projective dimension, $\mathrm{Ext}_R^i(M, Q) = 0$ for all

$i > 0$ and for all $Q \in \overline{P(R)}$. Therefore, $\mathrm{Ext}^i_R(M,K) = 0$ for all $i > 0$. Hence $f : M \longrightarrow L'$ has a lifting $M \longrightarrow P$ which is monic and still an $\overline{P(R)}$-preenvelope. □

Theorem 2.5. *Let $(R,\mathfrak{m})$ be a local Noetherian ring and C an R-module. Then the following conditions are equivalent:*
(i) C is Gorenstein flat.
(ii) C belongs to $A'(R)$ and $\mathrm{Tor}^R_i(L,C) = 0$ for all injective R-modules L and all $i > 0$.

Proof. $(i) \Rightarrow (ii)$ By Proposition 2.2, C belongs to $A'(R)$. Also, [**13**, Theorem 3.6], implies the last assertion in (ii).
$(ii) \Rightarrow (i)$ By [**13**, Theorem 3.6], it is enough to show that C admits a right flat resolution

$$\mathbf{X} = 0 \longrightarrow C \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \dots$$

such that $\mathrm{Hom}_R(\mathbf{X}, Y)$ is exact for all flat R-modules Y (i.e. C admits a co-proper right flat resolution). Lemma 2.4 (i) implies that there exists an exact sequence $0 \longrightarrow C \longrightarrow L$ of R-modules and R-homomorphisms such that $\mathrm{fd}_R L < \infty$. Using Lemma 2.4 (ii), there exists a monomorphism $f : C \longrightarrow K$ which is a flat preenvelope. We obtain the short exact sequence $0 \longrightarrow C \xrightarrow{f} K \longrightarrow B \longrightarrow 0$ and so for every flat R-module F' we have the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(B,F') \longrightarrow \mathrm{Hom}_R(K,F') \longrightarrow \mathrm{Hom}_R(C,F') \longrightarrow 0.$$

Let E be an injective R-module. Since $\mathrm{Hom}_R(E, E_R(R/\mathfrak{m}))$ is a flat R-module, we conclude that

$$0 \longrightarrow C \otimes_R E \longrightarrow K \otimes_R E \longrightarrow B \otimes_R E \longrightarrow 0$$

is an exact sequence. So, $\mathrm{Tor}^R_i(E,B) = 0$ for all $i > 0$ and all injective R-modules E, because K is a flat R-module. Also, by Lemma 2.1 and Proposition 2.2, we obtain $B \in A'(R)$. Then proceeding in this manner, we get the desired co-proper right flat resolution of C. □

Corollary 2.6. *Let $(R,\mathfrak{m})$ be a local Noetherian ring of dimension d and let $M \in A'(R)$. Then* $\mathrm{Gfd}_R(M) = \mathrm{Gfd}_{\hat{R}}(\hat{R} \otimes_R M)$. *In particular, if $M \in A'(R)$ then* $\mathrm{Gfd}_R M \leq \dim R$.

Proof. By [**13**, Proposition 3.10], $\mathrm{Gfd}_{\hat{R}}(\hat{R} \otimes_R M) \leq \mathrm{Gfd}_R(M)$. We show that $\mathrm{Gfd}_R(M) \leq \mathrm{Gfd}_{\hat{R}}(\hat{R} \otimes_R M)$ and so [**13**, Theorem 3.24] completes the proof. As M belongs to $A'(R)$, we get that $\hat{R} \otimes_R M$ belongs to $A(\hat{R})$. So, by [**5**, Theorem 4.1] $\mathrm{Gfd}_{\hat{R}}(\hat{R} \otimes_R M)$ is finite. Set $\mathrm{Gfd}_{\hat{R}}(\hat{R} \otimes_R M) = t$ and let

$$0 \longrightarrow C \longrightarrow P_{t-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

be an exact sequence of R-modules and R-homomorphisms such that P_i's are projective. We obtain the exact sequence

$$0 \longrightarrow \hat{R} \otimes_R C \longrightarrow \hat{R} \otimes_R P_{t-1} \longrightarrow \dots \longrightarrow \hat{R} \otimes_R P_1 \longrightarrow \hat{R} \otimes_R P_0 \longrightarrow \hat{R} \otimes_R M \longrightarrow 0.$$

By [**13**, Theorems 3.14], $\hat{R} \otimes_R C$ is a Gorenstein flat $\hat{R}$-module. Also, Lemma 2.1 and the above exact sequence, imply that C belongs to $A'(R)$. In view of Theorem 2.5, it is enough to show that $\mathrm{Tor}^R_i(C,E) = 0$ for all injective R-modules E and all $i > 0$. Let E be an injective R-module and let $\mathrm{Hom}_R(-, E_R(R/\mathfrak{m}))$ denote by $(-)^\vee$. From the natural monomorphism $E \longrightarrow (E^\vee)^\vee$, we conclude that E is a direct summand of $(E^\vee)^\vee$. So, it is enough to show that $\mathrm{Tor}_i^R((E^\vee)^\vee, C) = 0$ for all $i > 0$. By the next result, $\mathrm{id}_{\hat{R}}\,(E^\vee)^\vee$ is finite. It therefore follows from [**13**, Theorem 3.14] that $\mathrm{Tor}_i^{\hat{R}}(C \otimes_R \hat{R}, (E^\vee)^\vee) = 0$ for all $i > 0$. Suppose $\mathbf{F}_\bullet \longrightarrow C$ is a flat resolution of C. For every $i > 0$, we have

$$\begin{aligned}\mathrm{Tor}_i^R(C, (E^\vee)^\vee) &\cong H_i(\mathbf{F}_\bullet \otimes_R (E^\vee)^\vee)\\ &\cong H_i((\mathbf{F}_\bullet \otimes_R \hat{R}) \otimes_{\hat{R}} (E^\vee)^\vee)\\ &\cong \mathrm{Tor}_i^{\hat{R}}(C \otimes_R \hat{R}, (E^\vee)^\vee)\end{aligned}$$

The last isomorphism comes from the fact that $F_\bullet \otimes_R \hat{R}$ is a flat resolution of $C \otimes_R \hat{R}$, considered as an $\hat{R}$-module. Thus, $\mathrm{Tor}_i^R(C, (E^\vee)^\vee) = 0$ for all $i > 0$. □

Lemma 2.7. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and let K be an R-module such that $\mathrm{id}_R(K)$ is finite. Let $\mathrm{Hom}_R(-, E_R(R/\mathfrak{m}))$ denote by $(-)^\vee$. The R-module $(K^\vee)^\vee$ considered with the $\hat{R}$-module structure coming from $E_R(R/\mathfrak{m})$, that is, $(\hat{r}f)(x) = \hat{r}(f(x))$, for all $\hat{r} \in \hat{R}$, $f \in \mathrm{Hom}_R(K^\vee, E_R(R/\mathfrak{m}))$ and $x \in K^\vee$. Then $\mathrm{id}_{\hat{R}}\,(K^\vee)^\vee$ is finite.*

Proof. We deduce that $\mathrm{fd}_R(K^\vee)$ is finite. It is easy to see that $\mathrm{fd}_{\hat{R}}(K^\vee \otimes_R \hat{R})$ is finite. By the adjoint isomorphism, we have the following isomorphism

$$\mathrm{Hom}_{\hat{R}}(K^\vee \otimes_R \hat{R}, E_R(R/\mathfrak{m})) \cong \mathrm{Hom}_R(K^\vee, E_R(R/\mathfrak{m})),$$

as an $\hat{R}$-modules. This ends the proof, because the injective dimension of $\mathrm{Hom}_{\hat{R}}(K^\vee \otimes_R \hat{R}, E_R(R/\mathfrak{m}))$ is finite as an $\hat{R}$-module. □

3. Gorensein projective dimension

In this section, we show that Gorenstein projective dimension of an R-module is finite if and only if its Gorenstein flat dimension is finite.

Proposition 3.1. *Let R be a Noetherian ring with finite Krull dimension and C be an R-module. Then $\mathrm{Gfd}_R(C) \leq \mathrm{Gpd}_R(C)$.*

Proof. See [**13**, Remark 3.3 and Proposition 3.4]. □

Theorem 3.2. *Let R be a local Noetherian ring and M an R-module. Then the following conditions are equivalent:*
(i) M is Gorenstein projective.
(ii) $M \in A'(R)$ and $\mathrm{Ext}^i_R(M, P) = 0$ for all projective R-modules P and all $i > 0$.

Proof. Assume that M is Gorenstein projective. Then M belongs to $A'(R)$, by Proposition 2.2 and Proposition 3.1. Also, [**13**, Proposition 2.3], implies that $\mathrm{Ext}^i_R(M, P) = 0$ for all projective R-modules P and all $i > 0$.
Now, we show that $(ii) \Rightarrow (i)$. By [**13**, Proposition 2.3], it is enough to show that M admits a right projective resolution

$$\mathbf{X} = 0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \dots$$

such that $\mathrm{Hom}_R(\mathbf{X}, Y)$ is exact for every projective R-module Y (i.e. M admits a co-proper right projective resolution). Using parts (i) and (iii) of Lemma 2.4, there exists a monomorphism $\psi : M \longrightarrow Q$ which is a projective preenvelope. We consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\psi} Q \longrightarrow B \longrightarrow 0.$$

Let P be a projective R-module. Applying the functor $\mathrm{Hom}_R(-, P)$ to the above exact sequence. Since $\psi : M \longrightarrow Q$ is a projective preenvelope, $\mathrm{Ext}^i_R(B, P) = 0$ for all $i > 0$. Also, $B \in A'(R)$ by Lemma 2.1 and Proposition 2.2. Then proceeding in this manner, we get the desired co-proper right projective resolution for M and so we obtain the result. □

Theorem 3.3. *Let R be a local Noetherian ring of dimension d and M be an R-module. Then*

$$\mathrm{Gpd}_R\, M \leq \dim R + \mathrm{Gfd}_R\, M.$$

Proof. We can assume that $\mathrm{Gfd}_R\, M$ is finite. We prove the inequality by induction on $\mathrm{Gfd}_R\, M$. First, let M be a Gorenstein flat R-module. Let F be a flat R-module. Consider the minimal pure injective resolution

$$0 \longrightarrow F \longrightarrow PE^0(F) \longrightarrow PE^1(F) \longrightarrow \dots$$

(see [**16**, pages 39 and 92]). Note that, by [**16**, Lemma 3.1.6], $PE^n(F)$ is flat for all $n \geq 0$ and also, by [**16**, Corollary 4.2.7], $PE^n(F) = 0$ for all $n > d$. Since, every pure injective module is cotorsion, by [**13**, Proposition 3.22], $\mathrm{Ext}^j_R(M, PE^i(F)) = 0$ for all $i \geq 0$ and all $j \geq 1$. Therefore, $\mathrm{Ext}^{d+i}_R(M, F) \cong \mathrm{Ext}^i_R(M, PE^d(F))$ for all $i \geq 1$, and so $\mathrm{Ext}^{d+i}_R(M, F) = 0$ for all $i \geq 1$. Next, let

$$0 \longrightarrow C \longrightarrow P_{d-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence such that P_i's are projective. We have $\mathrm{Ext}^{d+i}_R(M, F) \cong \mathrm{Ext}^i_R(C, F)$ for all $i \geq 1$, and so $\mathrm{Ext}^i_R(C, F) = 0$ for all $i \geq 1$. On the other hand, using Lemma 2.1 and Proposition 2.2, we conclude that C belongs to $A'(R)$. Therefore, by Theorem 3.2, C is Gorenstein projective, and hence $\mathrm{Gpd}_R\, M \leq \dim R$.

Now, let $\mathrm{Gfd}_R\, M = t > 0$ and let $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ be an exact sequence such that P is projective. By [**13**, Proposition 3.12], $\mathrm{Gfd}_R\, K = t - 1$. Now, the induction hypothesis implies that

$$\mathrm{Gpd}_R\, K \leq \dim R + t - 1.$$

Therefore, $\mathrm{Gpd}_R\, M \leq \dim R + t - 1 + 1 = \dim R + t = \dim R + \mathrm{Gfd}_R\, M$. □

Now, we are ready to deduce the main result of this section by using Propositions 2.2 and 3.1, Corollary 2.6 and Theorem 3.3.

Theorem 3.4. *Let R be a local Noetherian ring and M an R-module. Then the following conditions are equivalent:*

(i) $\mathrm{Gfd}_R\, M < \infty$.

(ii) $\mathrm{Gpd}_R\, M < \infty$.

(iii) $M \in A'(R)$.

4. Gorenstein injective dimension

Let R be a local Noetherian ring an let $\mathbf{D}$ denote the dualizing complex of $\hat{R}$. Let $B(\hat{R})$ denote the full subcategory of $\mathbf{D}_b(\hat{R})$, consisting of those complexes X for which $\mathbf{R}\operatorname{Hom}_{\hat{R}}(\mathbf{D}, X) \in \mathbf{D}_b(\hat{R})$ and the canonical morphism

$$\tau_X : \mathbf{D} \otimes^{\mathbf{L}}_{\hat{R}} \mathbf{R}\operatorname{Hom}_{\hat{R}}(\mathbf{D}, X) \longrightarrow X,$$

is an isomorphism, see [**4**, 3.1].

Now, we define $B'(R)$ to be the class of all R-modules M such that $\operatorname{Hom}_R(\hat{R}, M) \in B(\hat{R})$.

In the Theorem 4.8, we want to characterize Gorenstein injective modules in terms of the class $B'(R)$. To prove Theorem 4.8, we need the following results.

Definition 4.1. (See [**6**, Definition 5.1]) For every R-module M, we show the small restricted injective dimension by $\operatorname{Ed}_R M$ and define

$$\operatorname{Ed}_R M = \sup\{i \in \mathbb{N}_0 \mid \exists L \in \overline{F(R)} \mid \operatorname{Ext}^i_R(L, M) \neq 0\}.$$

Theorem 4.2. (**Dimension inequality**) *Let R be a Noetherian ring of finite Krull dimension. For every R-module M, we have the following inequality:*

$$\operatorname{Ed}_R M \leq \operatorname{Gid}_R M \leq \operatorname{id}_R M.$$

Proof. Every injective module is Gorenstein injective, and so $\operatorname{Gid}_R M \leq \operatorname{id}_R M$. We can assume that $\operatorname{Gid}_R M$ is finite. We show that $\operatorname{Ed}_R M \leq \operatorname{Gid}_R M$ by induction on $\operatorname{Gid}_R M = n$. First assume that $n = 0$. It is enough to show that for every R-module L with $\operatorname{pd}_R L = l$ and $i > 0$, $\operatorname{Ext}^i_R(L, M) = 0$. Since M is Gorenstein injective, we have an exact sequence

$$0 \longrightarrow H \longrightarrow E_{l-1} \longrightarrow \ldots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

such that E_i is injective module for all $0 \leq i \leq l-1$. For any $i > 0$, we have

$$\operatorname{Ext}^i_R(L, M) \cong \operatorname{Ext}^{i+l}_R(L, H).$$

For $i > 0$, since $i + l > \operatorname{pd}_R L$, we get $\operatorname{Ext}^i_R(L, M) = 0$. This means that $\operatorname{Ed}_R M \leq 0$, and so that the result holds. Now, let $n > 0$. Using the Gorenstein injective version of [**13**, Proposition 2.18], there exists exact sequence $0 \longrightarrow M \longrightarrow T \longrightarrow K \longrightarrow 0$ such that T is injective R-module and $\operatorname{Gid}_R K = n - 1$. By induction, we have $\operatorname{Ed}_R K \leq \operatorname{Gid}_R K = n - 1$, and so $\operatorname{Ext}^j_R(L, K) = 0$ for all $L \in \overline{F(R)}$ and all $j > n - 1$. For each $i > n$ and each $L \in \overline{F(R)}$, we have the following exact sequence

$$0 = \operatorname{Ext}^{i-1}_R(L, K) \longrightarrow \operatorname{Ext}^i_R(L, M) \longrightarrow \operatorname{Ext}^i_R(L, T) = 0.$$

So $\operatorname{Ed}_R M \leq n = \operatorname{Gid}_R M$. This ends the proof. □

By Theorem 4.2, every Gorenstein injective R-module over a Noetherian ring of finite Krull dimension is strongly cotorsion (see [**16**, Definition 5.4.1]). The following example shows that there exists an R-module with finite Gorenstein injective dimension over a regular local ring which is not cotorsion.

Example 4.3. Let R be a regular local ring of Krull dimension one which is not complete. By [**1**, Lemma 3.3], $\mathrm{Hom}_R(\hat{R}, R) = 0$. So, $\hat{R}$ is not a projective R-module. Therefore, $\mathrm{pd}_R(\hat{R}) = 1$ and consequently there exists an R-module M such that $\mathrm{Ext}^1_R(\hat{R}, M) \neq 0$. On the other hand, $\mathrm{id}_R M \leq 1$. So, M is an R-module with finite Gorenstien injective dimension which is not cotorsion.

Proposition 4.4. *Let R be a local Noetherian ring and M an R-module.*
(i) If M is a Gorenstein injective R-module, then $\mathrm{Hom}_R(\hat{R}, M)$ is Gorenstein injective as an $\hat{R}$-module.
(ii) If M is a Gorenstein injective R-module, then $M \in B'(R)$.

Proof. (i) Let

$$\mathbf{X} = \ldots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \xrightarrow{\rho^0} G^0 \longrightarrow G^1 \longrightarrow \ldots$$

be an exact sequence of injective R-modules such that $\mathrm{Hom}_R(I, \mathbf{X})$ is exact for every injective R-modules I with $\ker \rho^0 = M$. If

$$0 \longrightarrow G'' \longrightarrow E \longrightarrow G' \longrightarrow 0$$

is an exact sequence such that G', G'' are Gorenstein injective and E is injective, then Theorem 4.2 yields the short exact sequence,

$$0 \longrightarrow \mathrm{Hom}_R(\hat{R}, G'') \longrightarrow \mathrm{Hom}_R(\hat{R}, E) \longrightarrow \mathrm{Hom}_R(\hat{R}, G') \longrightarrow 0.$$

Hence, we obtain the exact sequence

$$\mathbf{Y} = \ldots \longrightarrow \mathrm{Hom}_R(\hat{R}, E_1) \longrightarrow \mathrm{Hom}_R(\hat{R}, E_0) \xrightarrow{\mathrm{Hom}_R(\hat{R},\rho^0)} \mathrm{Hom}_R(\hat{R}, G^0) \longrightarrow \ldots$$

of $\hat{R}$-modules and $\hat{R}$-homomorphisms in which $\ker(\mathrm{Hom}_R(\hat{R}, \rho^0)) \cong \mathrm{Hom}_R(\hat{R}, M)$. On the other hand, if E is an injective R-module, we can conclude that $\mathrm{Hom}_R(\hat{R}, E)$ is injective as an $\hat{R}$-module, because $\mathrm{Hom}_{\hat{R}}(-, \mathrm{Hom}_R(\hat{R}, E)) \cong \mathrm{Hom}_R(- \otimes_{\hat{R}} \hat{R}, E)$. It is enough to show that $\mathrm{Hom}_{\hat{R}}(E', \mathbf{Y})$ is exact, for all injective $\hat{R}$-modules E'. This follows from the following isomorphisms of complexes

$$\mathrm{Hom}_{\hat{R}}(E', \mathbf{Y}) \cong \mathrm{Hom}_{\hat{R}}(E', \mathrm{Hom}_R(\hat{R}, \mathbf{X})) \cong \mathrm{Hom}_R(E', \mathbf{X})$$

and the fact that every injective $\hat{R}$-module is also injective as an R-module.

(ii) Let M be Gorenstein injective. By (i), $\mathrm{Hom}_R(\hat{R}, M)$ is Gorenstein injective $\hat{R}$-module. Hence, by [**5**, Theorem 4.4], $\mathrm{Hom}_R(\hat{R}, M) \in B(\hat{R})$, and so $M \in B'(R)$, by the definition. □

Proposition 4.5. *An R-module M is Gorenstein injective if and only if $\mathrm{Ext}^i_R(E, M) = 0$ for all injective R-modules E and for all $i > 0$ and there exists an exact sequence*

$$\mathbf{X} = \ldots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

of R-modules and R-homomorphisms with E_i is injective R-module for all $i \geq 0$, such that $\mathrm{Hom}_R(E, \mathbf{X})$ is exact for all injective R-modules E (i.e. M admits a proper left injective resolution).

Proof. It is the dual version of [**13**, Proposition 2.3] and we leave the proof to the reader. □

Lemma 4.6. *(i) Let R be a local Noetherian ring and M a cotorsion R-module such that M belongs to $B'(R)$. Then there exists an epimorphism $L \longrightarrow M$ with $\mathrm{id}_R(L) < \infty$.*
(ii) Let R be a Noetherian ring and $\varphi : L \longrightarrow M$ an R-epimorphism with $\mathrm{id}_R(L) < \infty$ and $\mathrm{Ext}^i_R(N, M) = 0$ for all injective R-modules N and all $i > 0$. Then there exists an epic $\overline{I(R)}$-precover $E \longrightarrow M$, in which E is injective.

Proof. (i) Since M belongs to $B'(R)$, then $\mathrm{Hom}_R(\hat{R}, M))$ belongs to $B(\hat{R})$. So, $\mathrm{Hom}_R(\hat{R}, M))$ has finite Gorenstein injective dimension as an $\hat{R}$-module by [**5**, Theorem 4.4]. By [**5**, Lemma 2.18], There are an $\hat{R}$-module L and an $\hat{R}$-epimorphism $L \longrightarrow \mathrm{Hom}_R(\hat{R}, M)$ such that injective dimension of L as an $\hat{R}$-module is finite. Since every injective $\hat{R}$-module is injective as an R-module, injective dimension of L as an R-module is finite. Consider the following exact sequence

$$0 \longrightarrow R \longrightarrow \hat{R} \longrightarrow \hat{R}/R \longrightarrow 0,$$

that yields the following exact sequence

$$\mathrm{Hom}_R(\hat{R}, M)) \longrightarrow \mathrm{Hom}_R(R, M)) \longrightarrow \mathrm{Ext}^1_R(\hat{R}/R, M).$$

On the other hand, since $\hat{R}/R$ is a flat R-module and M is a cotorsion R-module, $\mathrm{Ext}^1_R(\hat{R}/R, M) = 0$. So, the natural R-homomorphism $\mathrm{Hom}_R(\hat{R}, M) \longrightarrow M$ is epic. The result follows.

(ii) By [**16**, Theorem 2.4.3], there exists an $I(R)$-precover $f : E \longrightarrow M$. We claim that f is an $\overline{I(R)}$-precover. Let $\varphi' : L' \longrightarrow M$ be an R-homomorphism such that $\mathrm{id}_R(L') < \infty$. Consider an exact sequence

$$0 \longrightarrow L' \xrightarrow{g} E' \longrightarrow K \longrightarrow 0$$

such that E' is an injective R-module. It is clear that injective dimension of K is finite. By induction on injective dimension, we can deduce from assumption that $\mathrm{Ext}^1_R(K, M)$ is zero. We obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(K, M) \longrightarrow \mathrm{Hom}_R(E', M) \longrightarrow \mathrm{Hom}_R(L', M) \longrightarrow \mathrm{Ext}^1_R(K, M) = 0.$$

Hence, we conclude that there exists an R-homomorphism $\psi : E' \longrightarrow M$ such that $\varphi' = \psi g$. On the other hand, since f is an $I(R)$-precover, there exists an R-homomorphism $h : E' \longrightarrow E$ such that $\psi = fh$. Hence, there exists an R-homomorphism $hg : L' \longrightarrow E$ such that $f(hg) = \varphi'$. It therefore follows that f is an $\overline{I(R)}$-precover. Consequently f is epic, because φ is epic. □

Lemma 4.7. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, M a cotorsion R-module, and K a cotorsion $\hat{R}$-module. Then*
(i) $\mathrm{Ext}^i_R(F, M) = 0$ for all flat R-modules F and all $i > 0$.
(ii) K is cotorsion as an R-module.
(iii) For all $j > 0$, $\mathrm{Ext}^j_R(E, M) = 0$ for all injective R-modules E if and only if $\mathrm{Ext}^j_{\hat{R}}(I, \mathrm{Hom}_R(\hat{R}, M)) = 0$ for all injective $\hat{R}$-modules I.

Proof. (i) See the proof of [**16**, Proposition 3.1.2].
(ii) Suppose F is a flat R-module and $\mathbf{P}_\bullet \longrightarrow F$ a projective resolution of F. For all $i > 0$, we

have

$$\begin{aligned}\operatorname{Ext}^i_R(F,K) &\cong H^i(\operatorname{Hom}_R(\mathbf{P}_\bullet,K))\\ &\cong H^i(\operatorname{Hom}_{\hat{R}}(\mathbf{P}_\bullet\otimes_R\hat{R},K))\\ &\cong \operatorname{Ext}^i_{\hat{R}}(F\otimes_R\hat{R},K).\end{aligned}$$

The last isomorphism comes from the fact that K is a cotorsion $\hat{R}$-module and $F\otimes_R\hat{R}$ is flat as an $\hat{R}$-module for all flat R-modules F. This ends the proof of (ii).
(iii) Suppose L is an $\hat{R}$-module and $\mathbf{F}_\bullet\longrightarrow L$ is a free resolution of L, considered as an $\hat{R}$-module. For every $j>0$, we have

$$\begin{aligned}\operatorname{Ext}^j_{\hat{R}}(L,\operatorname{Hom}_R(\hat{R},M)) &\cong H^j(\operatorname{Hom}_{\hat{R}}(\mathbf{F}_\bullet,\operatorname{Hom}_R(\hat{R},M))\\ &\cong H^j(\operatorname{Hom}_R(\mathbf{F}_\bullet\otimes_{\hat{R}}\hat{R},M))\\ &\cong H^i(\operatorname{Hom}_R(\mathbf{F}_\bullet,M))\\ &\cong \operatorname{Ext}^j_R(L,M)\end{aligned}$$

The last isomorphism follows from the fact that M is cotorsion and every flat $\hat{R}$-module is flat as an R-module.
$\Rightarrow$) We know that every injective $\hat{R}$-module is injective as an R-module. So, the result follows from the above isomorphism.
$\Leftarrow$) By assumption, it is easy to see that

$$\operatorname{Ext}^i_{\hat{R}}(N,\operatorname{Hom}_R(\hat{R},M))=0,$$

for all $\hat{R}$-modules N of finite injective dimension and all $i>0$. Let E be an injective R-module and let $\operatorname{Hom}_R(-,E_R(R/\mathfrak{m}))$ denote by $(-)^\vee$. From the natural monomorphism $E\longrightarrow(E^\vee)^\vee$, we conclude that E is a direct summand of $(E^\vee)^\vee$. So, it is enough to show that $\operatorname{Ext}^i_R((E^\vee)^\vee,M)=0$ for all $i>0$. Since, by Lemma 2.7, $\operatorname{id}_{\hat{R}}((E^\vee)^\vee)<\infty$, the result follows from the above isomorphism.
□

Theorem 4.8. *Let R be a local Noetherian ring and M an R-module. Then the following conditions are equivalent:*
(i) M is Gorenstein injective.
(ii) M is cotorsion and $\operatorname{Hom}_R(\hat{R},M)$ is Gorenstein injective as an $\hat{R}$-module.
(iii) $M\in B'(R)$, M is cotorsion and $\operatorname{Ext}^i_R(E,M)=0$ for all injective R-modules E and all $i>0$.

Proof. *(i)* $\Rightarrow$ *(ii)* This follows from Theorem 4.2 and Proposition 4.4.

(ii) $\Rightarrow$ *(iii)* By [**5**, Theorem 4.4], $\operatorname{Hom}_R(\hat{R},M)$ belongs to $B(\hat{R})$, and so M belongs to $B'(R)$. Also, Proposition 4.5 implies that

$$\operatorname{Ext}^i_{\hat{R}}((I,\operatorname{Hom}_R(\hat{R},M))=0$$

for all injective $\hat{R}$-modules I and all $i>0$. The result follows from Lemma 4.7 (iii).

$(iii) \Rightarrow (i)$ In view of Proposition 4.5, it is enough to show that M admits a proper left injective resolution. It follows from Lemma 4.6 (i) and (ii) that there exists an exact sequence

$$0 \longrightarrow B \longrightarrow E \overset{f}{\longrightarrow} M \longrightarrow 0$$

such that f is an $\overline{I(R)}$-precover and E an injective R-module. It is enough to show that B satisfies the given assumptions on M.

Let I be an injective R-module. It is easy to deduce from the above exact sequence that $\mathrm{Ext}^i_R(I,B)=0$ for all $i \geq 2$. Also, we have the following exact sequence

$$\mathrm{Hom}_R(I,E) \longrightarrow \mathrm{Hom}_R(I,M) \longrightarrow \mathrm{Ext}^1_R(I,B) \longrightarrow \mathrm{Ext}^1_R(I,E)=0.$$

On the other hand, $\mathrm{Hom}_R(I,f)$ is epimorphism. So $\mathrm{Ext}^1_R(I,B)=0$.

Now, we prove that B is a cotorsion R-module. In view of assumption and Lemma 4.7, we conclude that

$$\mathrm{Ext}^i_{\hat{R}}(I, \mathrm{Hom}_R(\hat{R}, M)) = 0$$

for all injective $\hat{R}$-modules I and all $i > 0$. On the other hand, $M \in B'(R)$ implies that $\mathrm{Hom}_R(\hat{R}, M) \in B(\hat{R})$. Therefore, by [**5**, Lemma 4.7], $\mathrm{Hom}_R(\hat{R}, M)$ is Gorenstein injective as an $\hat{R}$-module. Hence, we have an exact sequence

$$0 \longrightarrow K \longrightarrow E' \longrightarrow \mathrm{Hom}_R(\hat{R}, M) \longrightarrow 0,$$

of $\hat{R}$-modules and $\hat{R}$-homomorphism such that E' is an injective and K is a Gorenstein injective $\hat{R}$-module. By Theorem 4.2, K is a cotorsion $\hat{R}$-module. Lemma 4.7 implies that K is cotorsion as an R-module. Now, let $\varphi : \mathrm{Hom}_R(\hat{R}, M) \longrightarrow M$ be the natural R-homomorphism. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & \mathrm{Hom}_R(\hat{R}, M) & \longrightarrow & 0 \\ & & & & & & \downarrow \varphi & & \\ 0 & \longrightarrow & B & \longrightarrow & E & \overset{f}{\longrightarrow} & M & \longrightarrow & 0. \end{array}$$

Since E' is an injective R-module and $f : E \longrightarrow M$ is an $\overline{I(R)}$-precover, there exists an R-homomorphism $\psi : E' \longrightarrow E$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & \mathrm{Hom}_R(\hat{R}, M) & \longrightarrow & 0 \\ & & & & \downarrow \psi & & \downarrow \varphi & & \\ 0 & \longrightarrow & B & \longrightarrow & E & \overset{f}{\longrightarrow} & M & \longrightarrow & 0. \end{array}$$

It is easy to see that there exists an R-homomorphism $\theta : K \longrightarrow B$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & \mathrm{Hom}_R(\hat{R}, M) & \longrightarrow & 0 \\ & & \downarrow \theta & & \downarrow \psi & & \downarrow \varphi & & \\ 0 & \longrightarrow & B & \longrightarrow & E & \overset{f}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

Suppose F is a flat R-module. Then we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_R(F,\mathrm{Hom}_R(\hat{R},M)) & \xrightarrow{\beta} & \mathrm{Ext}^1_R(F,K) & \longrightarrow 0 & \\ \downarrow{\scriptstyle \mathrm{Hom}_R(F,\varphi)} & & \downarrow{\scriptstyle \theta_1} & & (*) \\ \mathrm{Hom}_R(F,M) & \xrightarrow{\delta} & \mathrm{Ext}^1_R(F,B) & \longrightarrow 0. & \end{array}$$

The natural exact sequence

$$0 \longrightarrow R \longrightarrow \hat{R} \longrightarrow \hat{R}/R \longrightarrow 0,$$

yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\hat{R}/R, M) \longrightarrow \mathrm{Hom}_R(\hat{R}, M) \xrightarrow{\varphi} M \longrightarrow 0,$$

because M is a cotorsion R-module and $\hat{R}/R$ is a flat R-module. Thus, we obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(F,\mathrm{Hom}_R(\hat{R}/R,M)) \longrightarrow \mathrm{Hom}_R(F,\mathrm{Hom}_R(\hat{R},M)) \xrightarrow{\mathrm{Hom}_R(F,\varphi)} \mathrm{Hom}_R(F,M) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_R(F,\mathrm{Hom}_R(\hat{R}/R,M)).$$

Since M is a cotorsion and $\hat{R}/R$ is a flat R-module,

$$\mathrm{Ext}^1_R(F,\mathrm{Hom}_R(\hat{R}/R,M)) \cong \mathrm{Ext}^1_R(F \otimes_R \hat{R}/R, M)).$$

On the other hand, $F \otimes_R \hat{R}/R$ is a flat R-module, so $\mathrm{Ext}^1_R(F \otimes_R \hat{R}/R, M))$ is zero R-module. Therefore $\mathrm{Hom}_R(F,\varphi)$ is an epimorphism. By (*), $\theta_1\beta$ is epic and so θ_1 is epic. Thus, since K is a cotorsion R-module, $\mathrm{Ext}^1_R(F,B)$ is the zero module. This means that B is cotorsion.

Now, we apply the functor $\mathrm{Hom}_R(\hat{R},-)$ on the following exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow M \longrightarrow 0,$$

and obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\hat{R},B) \longrightarrow \mathrm{Hom}_R(\hat{R},E) \longrightarrow \mathrm{Hom}_R(\hat{R},M) \longrightarrow 0.$$

It is easy to see that $\mathrm{Hom}_R(\hat{R},E)$ is an injective $\hat{R}$-module. Since $\mathrm{Hom}_R(\hat{R},M)$ is Gorenstein injective as an $\hat{R}$-module, by [**13**, theorem 2.25], $\mathrm{Hom}_R(\hat{R},B)$ has finite Gorenstein injective dimension. So, it follows from [**5**, Theorem 4.4] that $B \in B'(R)$. This ends the proof. □

The following example shows that the dual version of Theorem 3.4 is not true.

Example 4.9. Let R be a non-complete local Noetherian domain which is not Gorenstein. By [**14**, Theorem 2.1], $\mathrm{Gid}_R(R) = \infty$. On the other hand, by [**1**, Lemma 3.3], $\mathrm{Hom}_R(\hat{R},R) = 0$. So R has infinite Gorenstein injective dimension as an R-module but $R \in B'(R)$.

Theorem 4.10. *Let $(R,\mathfrak{m})$ be a local Noetherian ring of dimension d and $\mathrm{Ext}^i_R(\hat{R},M) = 0$ for all $i > 0$. Then the Gorenstein injective dimension of M is finite if and only if M belongs to $B'(R)$. In particular, if $M \in B'(R)$ then $\mathrm{Gid}_R(M) \leq d$.*

Proof. $\Rightarrow$) Let $\mathrm{Gid}_R\, M = t$ and

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \ldots \longrightarrow G^t \longrightarrow 0$$

be an exact sequence such that G^i is Gorenstein injective for all $0 \leq i \leq t$. Using hypothesis, we obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\hat{R}, M) \longrightarrow \mathrm{Hom}_R(\hat{R}, G^0) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_R(\hat{R}, G^t) \longrightarrow 0.$$

By Proposition 4.4 (i), $\mathrm{Gid}_{\hat{R}}(\mathrm{Hom}_R(\hat{R}, M))$ is finite as an $\hat{R}$-module and so by [**5**, Theorem 4.4], $\mathrm{Hom}_R(\hat{R}, M)$ belongs to $B(\hat{R})$. The assertion follows from the definition.

$\Leftarrow$) Since M belongs to $B'(R)$, $\mathrm{Hom}_R(\hat{R}, M)$ belongs to $B(\hat{R})$. Now, by using [**5**, Theorem 4.4], the Gorenstein injective dimension of $\mathrm{Hom}_R(\hat{R}, M)$ is finite as an $\hat{R}$-module. By [**13**, Theorem 2.29], $\mathrm{Gid}_{\hat{R}}(\mathrm{Hom}_R(\hat{R}, M)) \leq \mathrm{FID}(R)$, where $\mathrm{FID}(R) = \sup\{\ \mathrm{id}_R(M) | M\ is\ an\ R-module\ of\ finite\ injective\ dimension\}$. It is known that $\mathrm{id}_R(N) = \mathrm{fd}_R(\mathrm{Hom}_R(N, E_R(R/\mathfrak{m})))$, for all R-modules N. So, we have $\mathrm{Gid}_{\hat{R}}(\mathrm{Hom}_R(\hat{R}, M)) \leq d$.

Consider the following exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots \longrightarrow E^{d-1} \longrightarrow L \longrightarrow 0,$$

of R-modules and R-homomorphisms such that E^i is injective R-module for all $0 \leq i \leq d-1$. We have the following exact sequence,

$$0 \longrightarrow \mathrm{Hom}_R(\hat{R}, M) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_R(\hat{R}, E^{d-1}) \longrightarrow \mathrm{Hom}_R(\hat{R}, L) \longrightarrow 0.$$

So, by [**13**, Theorem 2.22], $\mathrm{Hom}_R(\hat{R}, L)$ is a Gorenstein injective $\hat{R}$-module. On the other hand, for any flat R-module F and any $i > 0$, we have

$$\mathrm{Ext}^i_R(F, L) \cong \mathrm{Ext}^{i+d}_R(F, M).$$

Therefore, $\mathrm{Ext}^i_R(F, L)$ is zero for all $i > 0$, because the projective dimension of F is less than $d+1$. So, L is cotorsion. It therefore follows from Theorem 4.8 that L is a Gorenstein injective R-module. Thus, $\mathrm{Gid}_R(M) \leq d$. $\square$

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