THE COXETER QUOTIENT OF THE FUNDAMENTAL GROUP OF A GALOIS COVER OF $\mathbb{T}\times\mathbb{T}$

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Abstract. Let X be the surface $\mathbb{T}\times\mathbb{T}$ where $\mathbb{T}$ is the complex torus. This paper is the third in a series, studying the fundamental group of the Galois cover of X with respect to a generic projection onto $\mathbb{CP}^2$.

Van Kampen Theorem gives a presentation of the fundamental group of the complement of the branch curve, with 54 generators and more than 2000 relations. Here we introduce a certain natural quotient (obtained by identifying pairs of generators), prove it is a quotient of a Coxeter group related to the degeneration of X, and show that this quotient is virtually nilpotent.

1. Overview

For an algebraic surface X embedded in a projective space $\mathbb{CP}^N$, let X_{Gal} be the Galois cover of X with respect to the full symmetric group. The fundamental group of the Galois cover is a deformation invariant of surfaces. The first computation of this invariant can be found in [MT1], where an algorithm is outlined for the computation of the fundamental group in terms of generators and relations. Techniques to get a compact presentation and identify the group are also presented in this paper, and yet, in the general case it is very difficult to obtain concrete information

[1]Partially supported by the DAAD fellowship (Germany), Eager (Eu-network, HPRN-CT-2009-00099) and the LDFT postdoctoral fellowship (the Einstein mathematics institute, Hebrew university, Jerusalem). The Emmy Noether Research Institute for Mathematics (center of the Minerva Foundation of Germany), the Excellency Center "Group Theoretic Methods in the Study of Algebraic Varieties of the Israel Science Foundation.

Date: March 24, 2009.

on such groups from their presentation, for example whether the group is virtually solvable.

The group $\pi_1(X_{\mathrm{Gal}})$ was computed and identified for embeddings of the surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ [MT1], the Hirzebruch surfaces ([FRT] and [MRT]), and, recently, $\mathbb{CP}^1 \times \mathbb{T}$ [AGTV] where $\mathbb{T}$ is the complex torus. (See the references therein for more cases).

The computation is done along the following lines. Take a generic projection $X \to \mathbb{CP}^2$ (of degree n) with a branch curve S. Let $X_0 \to \mathbb{CP}^2$ be the degeneration of $X\rangle\mathbb{CP}^2$ to a union of planes, and S_0 its branch curve in $\mathbb{C}^2$. The first step is to compute the braid monodromy corresponding to S_0 and use 'regeneration rules' [MT2] to get the braid monodromy factorization of S (see [AT1] for the braid monodromy notion). Then one applies the van Kampen Theorem [vK] to get a presentation of $\pi_1(\mathbb{C}^2 - S)$ on a standard set of generators $\Gamma_1, \Gamma_{1'}, \dots, \Gamma_m, \Gamma_{m'}$, where $2m$ is the degree of S (see [AT2]). Let $\tilde{\pi}_1 = \pi_1(\mathbb{C}^2 - S)/\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle$. There is a natural homomorphism $\psi : \tilde{\pi}_1 \to S_n$, derived from the natural monodromy $\pi_1(\mathbb{C}^2 - S)\rangle S_n$. It is shown in [MT1], that the kernel of this map is isomorphic to $\pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})$, the fundamental group of the affine part of the Galois cover. Thus we have a short exact sequence

$$1 \longrightarrow \pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}}) \longrightarrow \tilde{\pi}_1 \longrightarrow S_n \longrightarrow 1.$$

The group $\pi_1(X_{\mathrm{Gal}})$ is then obtained by adding the 'projective relation'

$$\Gamma_1 \Gamma_{1'} \dots \Gamma_m \Gamma_{m'} = 1. \tag{1}$$

Under the above mentioned monodromy, each pair of generators Γ_j and $\Gamma_{j'}$ is mapped to the same transposition in S_n. Let C denote the quotient of $\tilde{\pi}_1$ under the identification $\Gamma_{j'} = \Gamma_j$ (for all j). Taking the previous sequence modulo the new relation, we get the short exact sequence

$$1 \longrightarrow \pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})/\langle \Gamma_j = \Gamma_{j'} \rangle \longrightarrow C \longrightarrow S_n \longrightarrow 1.$$

It is easy to see that ψ splits through C, and so we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}}) \arrow[r] \arrow[d] & \tilde{\pi}_1 \arrow[r, "\psi"] \arrow[d] & S_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(X_{\mathrm{Gal}}^{\mathrm{Aff}})/\langle \Gamma_j = \Gamma_{j'} \rangle \arrow[r] & C \arrow[r, "\psi_C"] & S_n \arrow[r] & 1
\end{tikzcd}$$

The kernel $K_C = \mathrm{Ker}(\psi_C)$ is then a quotient of $\pi_1(X_{\mathrm{Gal}})$, since the projective relation vanishes when we identify $\Gamma_j = \Gamma_{j'}$.

For $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the Hirzebruch surfaces, the group C is isomorphic to S_n. On the other hand, for $X = \mathbb{CP}^1 \times \mathbb{T}$, C was identified to be the Coxeter group of type $\tilde{A}_5$ (namely isomorphic to $S_6 \ltimes \mathbb{Z}^5$) [AGTV].

In this paper we obtain a presentation for the group C associated to the surface $X = \mathbb{T} \times \mathbb{T}$, and show that C is a quotient of a certain Coxeter group (which belongs to the family that was studied in [RTV], which are Coxeter groups with a natural projection onto a symmetric group, sending the generators to transpositions). The computation of $\pi_1(X_{\mathrm{Gal}})$ for this surface started in [A], and continued in [AT1] and [AT2]. We briefly review in the next section.

Eventually we prove that K_C is abelian by cyclic:

Theorem 1.1. *The group C is a semidirect product $C = S_{18} \ltimes K_C$, where K_C is a central extension of $\mathbb{Z}^{34}$ by $\mathbb{Z}$.*

More precisely, let H be the group generated by $x_1, \ldots, x_{18}$, $y_1, \ldots, y_{18}$ and z, with the relations

$$\begin{aligned}
[x_i, x_j] &= 1, \\
[y_i, y_j] &= 1, \\
[x_i, y_j] &= 1, \\
[x_i, y_i] &= z
\end{aligned}$$

for all $i \neq j$, and z central. Then K_C is isomorphic to the kernel of the map $H \to \mathbb{Z}^2 = \langle x, y \rangle$ defined by $x_i \mapsto x$ and $y_i \mapsto y$ (and $z \mapsto 1$). The *action of S_{18} on K_C is via indices, and in particular $\langle z \rangle$ is the center of C.*

2. The Coxeter quotient associated to $\mathbb{T} \times \mathbb{T}$

From now on let X denote the surface $\mathbb{T} \times \mathbb{T}$. Let us recall what has been done in [AT1] and [AT2]. The torus $\mathbb{T}$ embeds in $\mathbb{CP}^2$, so by the Segre map, X embeds in $\mathbb{CP}^{(2+1)(2+1)-1} = \mathbb{CP}^8$. The torus degenerates as a union of three lines (in general position), which we depict in Figure

$$1 \text{ ——— } 2 \text{ ——— } 3 \text{ ——— } 1$$

Figure 1. Degeneration of $\mathbb{T}$

1 (with the repeating index indicating the two points being identified). Multiplying two such degenerations, we obtain a degeneration of X as a union of nine copies of $\mathbb{CP}^1 \times \mathbb{CP}^1$, which are each further degenerated into two copies of $\mathbb{CP}^2$, as seen in Figure 2. This surface, composed of 18 planes with 27 intersection lines and 9 intersection points, is called X_0.

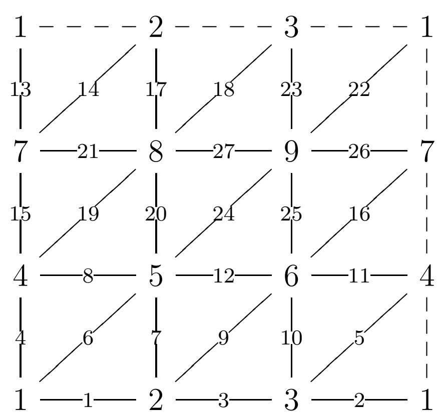

Figure 2. Degeneration of $X = \mathbb{T} \times \mathbb{T}$

Projecting X_0 onto $\mathbb{CP}^2$, we get a line arrangement S_0, which is the 1-skeleton of X_0, composed of 27 lines. Regenerating X_0, we get an induced regeneration downstairs from S_0 to the branch curve S of X. In [AT1] the degeneration process was described in details, and the braid monodromy factorization of S was obtained. This was used in [AT2] to compute a presentation $\pi_1(\mathbb{C}^2 - S)$, with the generators $\Gamma_1, \Gamma_{1'}, \ldots, \Gamma_{27}, \Gamma_{27'}$, and about 2000 relations. Therefore we have a presentation of $\tilde{\pi}_1 = \pi_1(\mathbb{C}^2 - S)/\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle$. The generators correspond (in pairs) to the 27 lines, and the map $\psi : \tilde{\pi}_1 \rightarrow S_{18}$ is defined by sending Γ_j and $\Gamma_{j'}$ to the transposition $(\alpha\beta)$ where α and β are the planes intersecting in line j of X_0.

As described above for the general case, the group C is the quotient of $\tilde{\pi}_1$ obtained by adding the relations $\Gamma_{j'} = \Gamma_j$. We will denote this pair of generators by u_j, so $C = \langle u_1, \ldots, u_{27} \rangle$ with the relations induced from $\tilde{\pi}_1$ under the projection $\theta : \Gamma_j, \Gamma_{j'} \mapsto u_j$. Since $\psi(\Gamma_{j'}) = \psi(\Gamma_j)$, ψ splits as the composition $\psi_C \circ \theta$ where ψ_C is defined by

$$\psi_C(u_j) = \psi(\Gamma_j). \tag{2}$$

We remark that the map $u_j \mapsto \Gamma_j$ does not define a homomorphism from C back to $\tilde{\pi}_1$. The issue of splitting the short exact sequence

$$1 \longrightarrow \operatorname{Ker}(\theta) \longrightarrow \tilde{\pi}_1 \longrightarrow C \longrightarrow 1$$

is important, but will not be discussed further in this paper.

Our main result was stated as Theorem 1.1. For the proof, we first present C as a quotient of a certain Coxeter group (which is discussed in the next section), and then apply the general results of [RTV].

3. Presentations of S_n via transpositions

Let T be an arbitrary graph on n points. To every edge $u \in T$ we attach the transposition $(\alpha\,\beta)$ where u connects the vertices α and β. This set of transpositions generates S_n if and only if T is connected. It is known that S_n has a presentation with the edges of T as generators, and the following five sets of relations:

$$u^2 = 1 \quad \text{for all } u \in T, \tag{3}$$

$$uv = vu \quad \text{if } u, v \text{ are disjoint}, \tag{4}$$

$$uvu = vuv \quad \text{if } u, v \text{ share a common vertex}, \tag{5}$$

$$[u, wvw] = 1 \quad \text{if } u, v, w \text{ meet in a common vertex}, \tag{6}$$

and, for every cycle $u_1, \dots, u_m$ in T, the relation

$$u_1 \dots u_{m-1} = u_2 \dots u_m, \tag{7}$$

which we say is the relation associated to the cycle (see [RTV] for details). It is easy to see that (assuming (3)–(6) hold) any ordered numeration of the edges along a cycle gives the same relation (7).

Let $\mathrm{C}(T)$ denote the Coxeter group generated by $T = \{u\}$ (one generator for every edge of T), with the relations (3)–(5); this is obviously a Coxeter group. Let $\mathrm{C}_{\mathrm{Y}}(T)$ denote the quotient obtained by adding the relations (6). As we assume T to be connected, the map sending u to the associated transposition is obviously a surjection

$$\psi_T : \mathrm{C}_{\mathrm{Y}}(T) \to S_n. \tag{8}$$

We will later show that the group C is a quotient of $\mathrm{C}_{\mathrm{Y}}(T)$ for the graph T of Figure 3. In fact C is obtained by adding some (but not all) of the cyclic relations (7) to $\mathrm{C}_{\mathrm{Y}}(T)$, so we get a chain of surjections

$$\mathrm{C}(T) \longrightarrow \mathrm{C}_{\mathrm{Y}}(T) \longrightarrow C \longrightarrow S_{18}.$$

4. A presentation of C

The group C was defined above as the image of $\tilde{\pi}_1 = \pi_1(\mathbb{C}^2 - S)/\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle$ under the map θ, sending Γ_j and $\Gamma_{j'}$ to an abstract generator u_j. A presentation for $\tilde{\pi}_1$ was described in [AT2] (with the complete list of relations given in [A]).

Let T denote the graph obtained by connecting the centers of every two neighboring triangles in Figure 2. The resulting graph is given in Figure 3. Therefore, the edges of T correspond to lines in X_0, and the

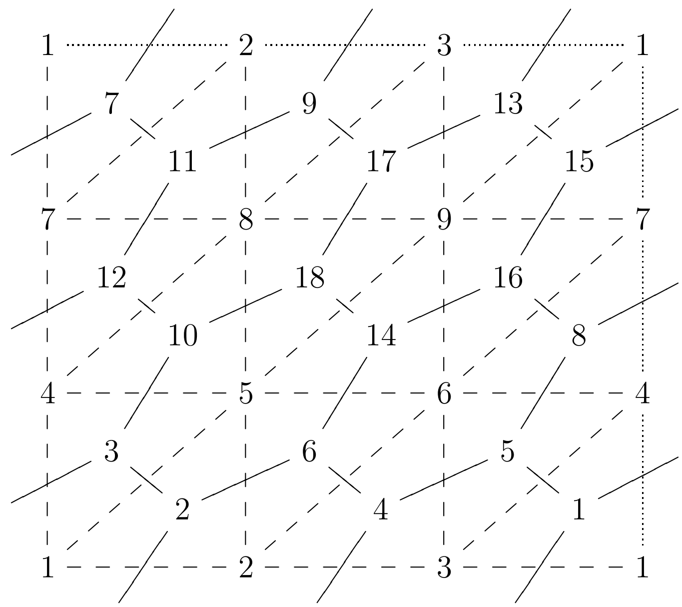

FIGURE 3. The graph T

vertices correspond to planes. Two edges of T have a joint vertex if and only if the corresponding lines (in X_0) belong to the same plane (depicted as a triangle in Figure 2).

Theorem 4.1. *The group C is generated by $\{u_j\}_{j=1,\dots,27}$, with the relations (3)–(5) arising from the graph T, and the cyclic relations (7) associated to the nine hexagons in T (those centered in the intersection points V_1–V_9 of the diagram).*

Proof. A presentation for C is obtained by substituting u_j for Γ_j and $\Gamma_{j'}$ in the presentation of $\tilde{\pi}_1$ ([A] and [AT2]), which has around 2000 relations. Fortunately, most of these relations fall into easy to describe families.

Start with the obvious relations: since $\Gamma_j^2 = 1$ in $\tilde{\pi}_1$, we have $u_j^2 = 1$ in C, and so (3) is proved. Moreover some relations of C have the form $u_iu_ju_iu_j$, namely u_i commutes with u_j. The whole list of relations was 'cleaned' by removing every subword of the form u_j^2, and by replacing every $u_i u_j u_i$ by u_j, if u_i and u_j are known to commute. These redundant words were also removed if they appear in a rotated version of a relation (namely $rs = 1$ where $sr = 1$ is given). During this process new commutation relations were 'discovered', and they too were used to further clean the list. The result of this procedure is a presentation with 333 relations: 264 commutation relations (of the form $u_i u_j = u_j u_i$), 44 'triple' relations (of the form $u_i u_j u_i = u_j u_i u_j$), and 25 miscellaneous, which are listed as Equations (9)–(33) below (sorted by length). In order to save space, we write the index j instead of u_j.

$$1 \cdot 13 \cdot 22 \cdot 6 \cdot 4 \cdot 2 \cdot 4 \cdot 6 \cdot 22 \cdot 13 = e \tag{9}$$

$$24 \cdot 25 \cdot 26 \cdot 25 \cdot 24 \cdot 22 \cdot 23 \cdot 27 \cdot 23 \cdot 22 = e \tag{10}$$

$$5 \cdot 4 \cdot 8 \cdot 4 \cdot 5 \cdot 19 \cdot 15 \cdot 11 \cdot 15 \cdot 19 = e \tag{11}$$

$$9 \cdot 10 \cdot 11 \cdot 10 \cdot 9 \cdot 16 \cdot 25 \cdot 12 \cdot 25 \cdot 16 = e \tag{12}$$

$$12 \cdot 24 \cdot 20 \cdot 24 \cdot 12 \cdot 24 \cdot 20 \cdot 24 \cdot 12 \cdot 24 \cdot 20 \cdot 24 = e \tag{13}$$

$$15 \cdot 21 \cdot 15 \cdot 14 \cdot 13 \cdot 14 \cdot 16 \cdot 26 \cdot 16 \cdot 14 \cdot 13 \cdot 14 = e \tag{14}$$

$$19 \cdot 20 \cdot 19 \cdot 21 \cdot 19 \cdot 20 \cdot 19 \cdot 21 \cdot 19 \cdot 20 \cdot 19 \cdot 21 = e \tag{15}$$

$$20 \cdot 19 \cdot 21 \cdot 19 \cdot 20 \cdot 18 \cdot 17 \cdot 18 \cdot 27 \cdot 18 \cdot 17 \cdot 18 = e \tag{16}$$

$$24 \cdot 25 \cdot 24 \cdot 22 \cdot 23 \cdot 22 \cdot 24 \cdot 25 \cdot 24 \cdot 22 \cdot 23 \cdot 22 = e \tag{17}$$

$$3 \cdot 9 \cdot 7 \cdot 9 \cdot 3 \cdot 9 \cdot 7 \cdot 9 \cdot 3 \cdot 9 \cdot 7 \cdot 9 = e \tag{18}$$

$$5 \cdot 2 \cdot 5 \cdot 18 \cdot 23 \cdot 18 \cdot 10 \cdot 3 \cdot 10 \cdot 18 \cdot 23 \cdot 18 = e \tag{19}$$

$$5 \cdot 4 \cdot 5 \cdot 19 \cdot 15 \cdot 19 \cdot 5 \cdot 4 \cdot 5 \cdot 19 \cdot 15 \cdot 19 = e \tag{20}$$

$$6 \cdot 4 \cdot 6 \cdot 22 \cdot 13 \cdot 22 \cdot 6 \cdot 4 \cdot 6 \cdot 22 \cdot 13 \cdot 22 = e \tag{21}$$

$$6 \cdot 7 \cdot 6 \cdot 24 \cdot 20 \cdot 24 \cdot 6 \cdot 7 \cdot 6 \cdot 24 \cdot 20 \cdot 24 = e \tag{22}$$

$$6 \cdot 7 \cdot 6 \cdot 8 \cdot 6 \cdot 7 \cdot 6 \cdot 8 \cdot 6 \cdot 7 \cdot 6 \cdot 8 = e \tag{23}$$

$$9 \cdot 10 \cdot 9 \cdot 16 \cdot 25 \cdot 16 \cdot 9 \cdot 10 \cdot 9 \cdot 16 \cdot 25 \cdot 16 = e \tag{24}$$

$$9 \cdot 7 \cdot 9 \cdot 14 \cdot 17 \cdot 14 \cdot 9 \cdot 7 \cdot 9 \cdot 14 \cdot 17 \cdot 14 = e \tag{25}$$

$$1 \cdot 14 \cdot 17 \cdot 14 \cdot 9 \cdot 7 \cdot 9 \cdot 3 \cdot 9 \cdot 7 \cdot 9 \cdot 14 \cdot 17 \cdot 14 = e \tag{26}$$

$$6 \cdot 7 \cdot 6 \cdot 8 \cdot 6 \cdot 7 \cdot 6 \cdot 24 \cdot 20 \cdot 24 \cdot 12 \cdot 24 \cdot 20 \cdot 24 = e \tag{27}$$

$$\begin{aligned} 10 \cdot 3 \cdot 10 \cdot 18 \cdot 23 \cdot 18 \cdot 10 \cdot 3 \cdot 10 \cdot 18 \cdot 23 \cdot 18 \cdot 10 \cdot \\ \cdot 3 \cdot 10 \cdot 18 \cdot 23 \cdot 18 = e \end{aligned} \tag{28}$$

$$\begin{aligned} 15 \cdot 14 \cdot 13 \cdot 14 \cdot 15 \cdot 21 \cdot 15 \cdot 14 \cdot 13 \cdot 14 \cdot 15 \cdot 21 \cdot 15 \cdot \\ \cdot 14 \cdot 13 \cdot 14 \cdot 15 \cdot 21 = e \end{aligned} \tag{29}$$

$$\begin{aligned} 19 \cdot 20 \cdot 18 \cdot 17 \cdot 18 \cdot 20 \cdot 19 \cdot 21 \cdot 19 \cdot 20 \cdot 18 \cdot 17 \cdot 18 \cdot \\ \cdot 20 \cdot 19 \cdot 21 \cdot 19 \cdot 20 \cdot 18 \cdot 17 \cdot 18 \cdot 20 \cdot 19 \cdot 21 = e \end{aligned} \tag{30}$$

$$\begin{aligned} 25 \cdot 24 \cdot 22 \cdot 23 \cdot 22 \cdot 24 \cdot 25 \cdot 26 \cdot 25 \cdot 24 \cdot 22 \cdot 23 \cdot 22 \cdot \\ \cdot 24 \cdot 25 \cdot 26 \cdot 25 \cdot 24 \cdot 22 \cdot 23 \cdot 22 \cdot 24 \cdot 25 \cdot 26 = e \end{aligned} \tag{31}$$

1 2	1 3	1 4	1 7	1 13
1 17	2 3	2 10	2 13	2 23
3 7	3 17	3 23	4 11	4 13
4 15	7 8	7 12	7 17	7 20
8 11	8 12	8 15	8 20	10 12
10 25	11 12	11 25	12 20	13 21
13 26	15 26	17 21	17 27	20 21
20 27	21 26	21 27	23 25	23 26
23 27	25 27	26 27		

FIGURE 4. Pairs i, j for which the order of $u_i u_j$ is *not given* as a relation

$$6 \cdot 4 \cdot 2 \cdot 4 \cdot 6 \cdot 22 \cdot 13 \cdot 22 \cdot 6 \cdot 4 \cdot 2 \cdot 4 \cdot 6 \cdot 22 \cdot 13 \cdot \\ \cdot 22 \cdot 6 \cdot 4 \cdot 2 \cdot 4 \cdot 6 \cdot 22 \cdot 13 \cdot 22 \; = \; e \tag{32}$$

$$9 \cdot 7 \cdot 9 \cdot 3 \cdot 9 \cdot 7 \cdot 9 \cdot 14 \cdot 17 \cdot 14 \cdot 9 \cdot 7 \cdot 9 \cdot 3 \cdot 9 \cdot \\ 7 \cdot 9 \cdot 14 \cdot 17 \cdot 14 \cdot 9 \cdot 7 \cdot 9 \cdot 3 \cdot 9 \cdot 7 \cdot 9 \cdot 14 \cdot 17 \cdot 14 \; = \; e \tag{33}$$

All the 264+44=308 relations of the first two types, which give the order of products $u_i u_j$ (as 2 or 3), match our expectations: in all the relations $(u_i u_j)^2 = 1$, the edges $i, j \in T$ do not have a joint vertex, and in all the relations $(u_i u_j)^3 = 1$, i and j do share a joint vertex. The first two sets of relations are perhaps best described by listing what is missing: the $\binom{27}{2} - 308 = 43$ pairs i, j for which the order is not given in the relations. These 43 'non-relations' are listed in Table 4.

It remains to compute the orders of $u_i u_j$ for the pairs of Table 4 (thus completing the proof that relations (4) and (5) hold), and show that Equations (9)–(33) transform into the nine cyclic relations promised in (7). Of course these are not all the the cycles in the graph T.

Notice that each of the relations (9)–(33) involves generators which correspond to lines of X_0 with one common point (namely the edges in T belong to one of the nine hexagons). Of these, there are nine relations

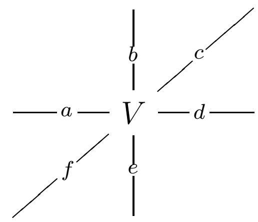

FIGURE 5. Generic names for the lines in X_0

which involve all the six lines around a point: (9)–(12),(14),(16),(19) and (26)–(27). We start by transforming these nine relations into the required cyclic relations. Some caution is in order here: we consider every equality of the form $u_m = u_{m-1}\dots u_2u_1u_2\dots u_{m-1}$ to be a 'version' of the cyclic relation; once the orders of the u_iu_j are known to be 2 or 3 (according to whether or not i and j intersect), all these versions are equivalent. At this time, however, we do not have all the order relations, and we will only establish one version of the cyclic relation around every point.

To simplify reading, we will use the notation given in Figure 5 for the lines around a point: If the point is understood from the context, a–f refer to the appropriate lines around it. In general, we will use a_r–f_r for the lines around the point V_r. For example, $a_6 = 12$, $b_6 = 25$, and $d_5 = 12$ (see Figure 2). The advantage of this cumbersome notation is that we now see that all the missing relations in Table 4 involve pairs of the generators a_r, b_r, d_r and e_r. In fact, in all the pairs i, j of this table, the lines i, j have a common point; so lines which do no intersect are known to commute. Likewise diagonals cannot be found in the table, so every order relation in which c_r or f_r is involved, is known to hold.

Table 6 lists the relations which are not given in advance.

Moreover, many relations in (9)–(33) are instances of the same 'generic' relation in the letters a–f, as we shall now see.

	a_rb_r	d_re_r	b_rd_r	e_ra_r	a_rd_r	b_re_r
V_1		x	x	x	x	x
V_2	x	x	x	x	x	x
V_3		x	x	x	x	
V_4			x	x	x	x
V_5	x	x	x	x	x	x
V_6			x	x	x	x
V_7	x		x	x	x	
V_8	x	x	x	x	x	
V_9	x		x	x	x	x

FIGURE 6. Order relations which are not given in advance

Remark 4.2. *If x, y, z are elements of order 2 in a group, and satisfy the relations $(xy)^3 = (yz)^3 = (xz)^2 = 1$, then $\langle x, y, z\rangle$ is a homomorphic image of S_4 (which is the Coxeter group of type A_3). In particular $xyzyx = zyxyz$.*

We start with the relations around V_r for $r = 1, 4, 6, 9$. Notice that Equations (9),(11),(12) and (10) are the relation

$$f_re_rd_re_rf_r = c_rb_ra_rb_rc_r$$

around these four points, respectively. For $r = 4, 6, 9$, the generators d_r, e_r, f_r satisfy the relations of Remark 4.2, so we have that $f_re_rd_re_rf_r = d_re_rf_re_rd_r$, resulting in the cyclic relation

$$d_re_rf_re_rd_r = c_rb_ra_rb_rc_r \qquad \text{for } r = 4, 6, 9. \tag{34}$$

The situation is slightly different for $r = 1$, since the order of d_1e_1 is not known yet (this is the exception $1, 13$ from Table 4). However, the orders of a_1b_1, a_1c_1, b_1c_1 are known, and applying Remark 4.2 for a_1, b_1, c_1 we have

$$f_re_rd_re_rf_r = a_rb_rc_rb_ra_r \qquad \text{for } r = 1. \tag{35}$$

Next, consider the relation (19), which can be written as $c_rd_rc_r = f_re_rf_rb_ra_rb_rf_re_rf_r$ around V_3. Since $f_3 = 18$ commutes with $c_3 = 5$ and with $d_3 = 2$ (proof: the pairs $2, 18$ and $5, 18$ cannot be found in Figure 4), we have $cdc = efbabfe$. Since $cdc = dcd$ (for every j) and $bab = aba$ (for $r = 3$), we have that

$$e_rd_rc_rd_re_r = f_ra_rb_ra_rf_r \qquad \text{for } r = 3. \tag{36}$$

The case of (14) is similar: it is the relation $faf = cbcedecbc$ around V_7. Since c_7 commutes with a_7 and f_7, $faf = afa$ (for every r), and $ede = ded$ (for $r = 7$), we have that

$$b_ra_rf_ra_rb_r = c_rd_re_rd_rc_r \qquad \text{for } r = 7. \tag{37}$$

The relations (26) and (27) both have the form

$$a = fefcbcdcbcfef$$

around V_2 and V_5, respectively. We also note that (18) and (13) are the relation $cbcdcbc = dcbcd$ around these points. Combining this with the fact that f_r commutes with b_r, c_r and d_r (for every j), we have

$$e_rf_ra_rf_re_r = d_rc_rb_rc_rd_r \qquad \text{for } r = 2, 5. \tag{38}$$

Finally, we derive the cyclic relation around V_8. Let $x = e_8f_8a_8f_8e_8$ and $y = c_8b_8c_8 = b_8c_8b_8$. The relation (30) is $yxy = xyx$, so Equation (16), which is the relation $x = yd_8y$, transforms into $d = xyx = efafebcbefafe$. But e_r, f_r commute with b_r, c_r (for every r, except for the relation $b_re_r = e_rb_r$ which does hold for $r = 8$), so we obtain

$$f_re_rd_re_rf_r = a_rb_rc_rb_ra_r \qquad \text{for } r = 8, \tag{39}$$

and this is the last of our nine cyclic relations.

We are now ready to prove relations (3) and (5) for the pairs i, j of Figure 4. As seen from Table 6, the 43 pairs i, j for which the order of u_iu_j is not given, fall into four categories: $a_rd_r = d_ra_r$ (9 pairs, around all the points), $b_re_r = e_rb_r$ (6 pairs), $a_rb_ra_r = b_ra_rb_r$ and $d_re_rd_r = e_rd_re_r$ (10 pairs), and finally $b_rd_r = d_rb_r$ and $a_re_r =$

$e_r a_r$ (18 pairs, two around every point). The orders of all the other pairs ($a_r c_r, a_r f_r, b_r c_r, b_r f_r, c_r d_r, c_r e_r, c_r f_r, d_r f_r$ and $e_r f_r$) are known as relations for every r.

Equations (17), (20), (21), (22), (24) and (25) all have the same form, $c_r b_r c_r f_r e_r f_r = f_r e_r f_r c_r b_r c_r$, around the points V_r with $r = 9, 4, 1, 5, 6$ and 2, respectively. For the other points (V_3, V_7 and V_8), we already know that $b_r e_r = e_r b_r$. Since f_r commutes with b_r and with c_r, and c_r commutes with e_r, these relations translate to

$$b_r e_r = e_r b_r \qquad \text{for every } r. \tag{40}$$

Next, we prove the ten relations of the third kind: that $u_i u_j u_j = u_j u_i u_j$ if i, j are horizontal and vertical lines which share a common triangle. The idea is, in each case, to express u_i (or u_j) as a conjugate of another generator using a cyclic relation, and then show that the conjugate satisfy the triple relation with u_j (or u_i).

As an illustration for this method, consider the pair $d_8 = 27$ and $e_8 = 20$ (the pair 20, 27 does appear in Figure 4, so the order of $u_{20}u_{27}$ is not yet known). Notice that $e_8 = b_5$. The cyclic relation (38) around V_5 provides the equality $b_5 = c_5 \alpha c_5$ where $\alpha = d_5 e_5 f_5 a_5 f_5 e_5 d_5$ commutes with d_8 (since a_5, f_5, d_5, e_5 have no point in common with d_8). As $d_8 = a_9$ and $c_5 = f_9$, these two generators satisfy $d_8 c_5 d_8 = c_5 d_8 c_5$. Finally, $d_8 e_8 = d_8 b_5 = d_8 c_5 \alpha c_5 \sim c_5 d_8 c_5 \alpha = d_8 c_5 d_8 \alpha = c_5 d_8 \alpha d_8 = c_5 \alpha = b_5 c_5$ (where $\sim$ denotes conjugate in the group), and we are done since $(b_5 c_5)^3 = 1$.

The following proposition will be used to prove the relations $a_r b_r a_r = b_r a_r b_r$, which we need to show for $r = 2, 5, 7, 8, 9$.

Proposition 4.3. *Let V_s be a point to the left of V_r in X_0 (so that $a_r = d_s$). If $c_s d_s c_s$ commutes with b_r, then $(a_r b_r)^3 = 1$.*

Proof. Let $\alpha = c_s d_s c_s$. Let t be the index of the point above r, so that V_r, V_s, V_t form a clockwise triangle. The edges of this triangle are $a_r = d_s$, $c_s = f_t$ and $e_t = b_r$. Since $e_t f_t e_t = f_t e_t f_t$, we have

$a_r b_r = d_s b_r = c_s \alpha c_s b_r \sim \alpha c_s b_r c_s = \alpha b_r c_s b_r \sim b_r \alpha b_r c_s = \alpha c_s = c_s d_s$, and $c_s d_s$ is known to have order 3. $\square$

The point to the left of V_r for $r = 9$ is V_s for $s = 8$, and by (39), around V_8 we have $d = efabcbafe = efacbcafe = cefabafec$, so that $c_8 d_8 c_8$ is a word in a_8, b_8, e_8, f_8 which all commute with b_9. By the proposition, $(a_9 b_9)^3 = 1$.

For $r = 5, 7$ we have $s = 4, 9$, respectively. In both cases (34) apply, and we have $f_s e_s d_s e_s f_s = d_s e_s f_s e_s d_s = c_s b_s a_s b_s c_s$. But c_s commutes with e_s, f_s, so again $c_s d_s c_s$ is a word in the other letters around V_s, which all commute with b_r. The proposition thus gives $(a_r b_r)^3 = 1$.

The case $r = 2$ (where $s = 1$) is similar. Around V_1 we have $d = feabcbaef = feacbcaef$, but c commutes with a, e, f, and the same argument applies.

For $r = 8$ the point to the left is V_s for $s = 7$, and the cyclic relation (37) is $b_7 a_7 f_7 a_7 b_7 = c_7 d_7 e_7 d_7 c_7$. But since $d_7 e_7 d_7 = e_7 d_7 e_7$ and e_7, c_7 commute, we find as before that $c_7 d_7 c_7$ commutes with b_8 and the result $(a_8 b_8)^3 = 1$ follows.

Together with the cases $r = 1, 3, 4, 6$ which are given as relations, we conclude that

$$a_r b_r a_r = b_r a_r b_r \qquad \text{for every } r. \tag{41}$$

Next, we prove the other half of the third set of relations, i.e. the relations of the form $d_r e_r d_r = e_r d_r e_r$, which we need to show for $r = 1, 2, 3, 5, 8$. The case $j = 8$ was settled above.

Proposition 4.4. *Let V_s be a point below V_r (so that $e_r = b_s$). If $c_s b_s c_s$ commutes with d_r, then $(d_r e_r)^3 = 1$.*

Proof. As in Proposition 4.3. Let $\alpha = c_s b_s c_s$. Let t be the index of the point to the right of r, so that r, s, t form a counterclockwise triangle. The edges of this triangle are $e_r = b_s$, $c_s = f_t$ and $a_t = d_r$. Since $a_t f_t a_t = f_t a_t f_t$, we have $d_r e_r = d_r b_s = d_r c_s \alpha c_s \sim c_s d_r c_s \alpha = d_r c_s d_r \alpha \sim$

$c_s d_r \alpha d_r = c_s \alpha = b_s c_s$, and $b_s c_s$ is known to be of order 3 for every s. $\square$

This proposition immediately applies for $r = 2$ and $r = 5$: in the first case $s = 8$ and we have from (39) that $c_s b_s c_s = b_s c_s b_s = a_s f_s e_s d_s e_s f_s a_s$, and a_8, d_8, e_8, f_8 all commute with d_2. In the second case $s = 2$ and by (38) $c_s b_s c_s = d_s e_s f_s a_s f_s e_s d_s$ and we are done by the same argument.

The point below $r = 3$ is $s = 9$. The cyclic relation (34) provides $d_s e_s f_s e_s d_s = c_s b_s a_s b_s c_s$. However, using (41) we have $c_s b_s c_s = a_s d_s e_s f_s e_s d_s a_s$, and the usual argument applies.

The final case is $r = 1$, where $s = 7$. Then the cyclic relation (37) gives $b_s a_s f_s a_s b_s = c_s d_s e_s d_s c_s$. Again by (41) we can apply Remark 4.2, so that $b_s a_s f_s a_s b_s = f_s a_s b_s a_s f_s$. But c_s commutes with a_s, f_s, so $c_s b_s c_s = a_s f_s d_s e_s d_s f_s a_s$ and Proposition 4.4 applies (as these generators all commute with d_r). With this, we proved

$$d_r e_r d_r = e_r d_r e_r \qquad \text{for every } r, \tag{42}$$

and we are done with the third set.

There are three kinds of relations we still need to prove, namely $(a_r d_r)^2 = 1$, $(b_r d_r)^2 = 1$ and $(a_r e_r)^2 = 1$, for every r.

In order to prove that b_r, d_r commute for every r, let s be the point to the right of r, so that $d_r = a_s$. All we need is to write a_s in terms of the other generators around V_s (since they have no common point with d_r). For $r = 1, 4, 5, 6, 8$, the relations (34) and (38) provide the needed expressions directly. In the other cases the cyclic relations express $a_7 f_7 a_7$, $a_3 b_3 a_3$ or $a_s b_s c_s b_s a_s$ ($s = 2, 8$) in terms of the other generators, but using (41) (and Remark 4.2) we can write these too as conjugates of the appropriate a_s. Thus we have proved

$$b_r d_r = d_r b_r \qquad \text{for every } r. \tag{43}$$

In a similar manner we can prove

$$a_r e_r = e_r a_r \qquad \text{for every } r. \tag{44}$$

Indeed, writing $a_r = d_s$ for an appropriate s (V_s to the left of V_r), all we need is to express d_s in terms of the other generators around V_s. The cyclic relations (34)–(39) express $d_s, d_sc_sd_s, d_se_sd_s, d_se_sf_se_sd_s$ and $d_sc_sb_sc_sd_s$ in terms of the other relations for every s; but all these are conjugate to d_s using the relations we know so far, and we are done.

By now we know the orders of all the products of two generators around a point, except for a_rd_r. In particular, when we consider $\langle b_r, c_r, d_r, e_r, f_r\rangle$ or $\langle a_r, b_r, c_r, e_r, f_r\rangle$ for a fixed r, we have a homomorphic image of the Coxeter group of type A_5, namely the symmetric group S_6. The cyclic relations now present a_r as an element in $\langle b_r, c_r, d_r, e_r, f_r\rangle$, which is a transposition disjoint from d_r. For example, around V_7 we have (by (37)) that $bfafb = bafab = cdedc = edcde$, so that $dad = dfbedcdebfd = fbdedcdedbf = fbedecedebf = fbedcdebf = a$. This shows that

$$a_rd_r = d_ra_r \qquad \text{for every } r, \tag{45}$$

which completes the proof of relations (4) and (5). In particular the remark made after (7) applies, and the cyclic relations we proved become the relations required in (7).

To finish the proof of the theorem, we need to check that Equations (9)–(33) do not introduce more relations. Since every relation involves only generators around one point V_r, they can easily be evaluated in $\langle a_r, b_r, c_r, d_r, e_r, f_r\rangle$, which is a homomorphic image of the Coxeter group of type $\tilde{A}_5$, which is isomorphic to $S_6 \ltimes \mathbb{Z}^5$ (in fact the cyclic relations express f_r, say, in terms of the other generators, so we are computing in homomorphic images of the Coxeter group of type A_5, namely S_6). For example, Relation (33) involves generators around V_2, and translates to $(c_2b_2c_2d_2c_2b_2c_2 \cdot f_2e_2f_2)^3 = 1$. This can easily be verified in $\langle b_2, c_2, d_2, e_2, f_2\rangle$ which by now is known to be a homomorphic image of S_6. $\square$

Corollary 4.5. *Relation (6) is also satisfied by the generators of C. In particular C is a quotient of* $\mathrm{C_Y}(T)$*.*

Proof. Suppose that u_i, u_j, u_k are edges of T which meet in a point. Then u_j and u_k belong to the same hexagon in T. Use the cyclic relation associated to this cycle to rewrite $u_j u_k u_j$ as a product of generators from the other edges of the cycle, which in particular to not intersect u_i and therefore commute with u_i. □

5. The structure of C

The fundamental group of the graph T is freely generated by 10 generators. To see this, choose a spanning subtree T_0 (which will contain $18 - 1 = 17$ edges since T connects 18 vertices); then there are $27 - 17 = 10$ basic cycles, since T has 27 edges. We label the complement of T_0 in T by $x^{(1)}, \dots, x^{(10)}$, as in Figure 7, where the edges of the spanning subtree are denoted by double lines. The generator corresponding to $x^{(\tau)}$ is of course the loop resulting from connecting the end point of $x^{(\tau)}$ to the starting point with the (unique) path on T_0.

It is proven in [RTV] that the natural map from the abstract group $\mathrm{C_Y}(T_0)$ to $\mathrm{C_Y}(T)$ (sending a generator to itself) is in fact an embedding. Since T_0 is a tree, this group is isomorphic to the symmetric group on 18 letters. Moreover, the cyclic relations defining C as a quotient of $\mathrm{C_Y}(T)$ can be 'solved' in S_{18} (by assigning transpositions to the generators outside of T_0; this will also be evident from the computations below), and so the subgroup $\langle u_j : j \in T_0 \rangle$ of C is isomorphic to S_{18}. This constitutes a splitting of the map $\psi_C : C \to S_{18}$.

Fix $n = 18$ and $t = 10$. Let $F^{\star}_{t,n}$ be the group generated by the $t \cdot n$ generators $\{x_i^{(\tau)}\}$ ($\tau = 1, \dots, t$, $i = 1, \dots, n$), with the relations

$$[x_i^{(\tau)}, x_j^{(\tau')}] = 1 \qquad \text{for every } \tau, \tau' \text{ and } i \neq j$$

(therefore $F^{\star}_{t,n}$ is a direct product of n copies of $\pi_1(T)$ which is the free group on t generators). Let $e^1, \dots, e^t$ denote a set of generators of $\mathbb{Z}^t$, and let $\mathrm{ab} : F^{\star}_{t,n} \to \mathbb{Z}^t$ be the map defined by $\mathrm{ab}(x_i^{(\tau)}) = e^{\tau}$ (for all i). Let $F_{t,n}$ denote the kernel of this map (note that this is the kernel of

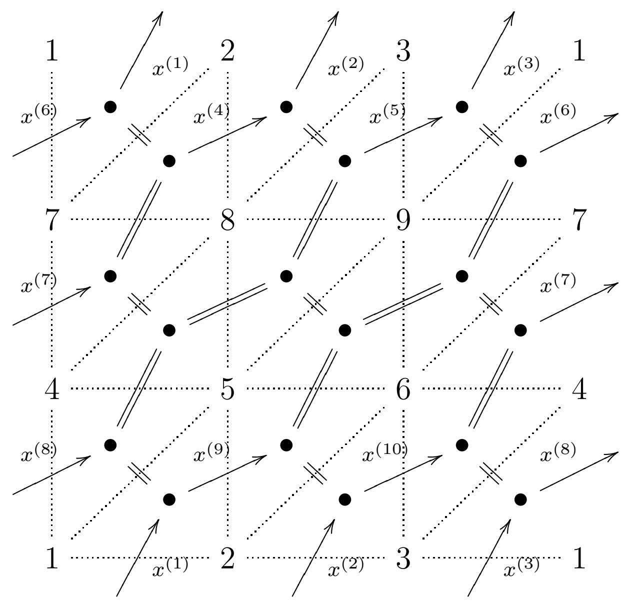

FIGURE 7. Spanning subtree of T

the natural diagonal projection $\pi_1(T)^n \to \mathrm{H}_1(T)$ since the homology group H_1 is the abelianization of π_1).

Recall the definition of $\mathrm{C_Y}(T)$ from Section 3, where T is the graph of Figure 7. Let N denote the normal subgroup of $\mathrm{C_Y}(T)$ generated by the nine cyclic relations, associated to the hexagons around the points $V_1,\dots,V_9$. By Theorem 4.1 and Corollary 4.5, the group C is isomorphic to the quotient $\mathrm{C_Y}(T)/N$. The cyclic relations trivially hold in S_n, so the map $\psi_C\colon C\to S_n$ of Equation (2) is induced from the natural surjection $\psi_T\colon \mathrm{C_Y}(T)\to S_n$ of Equation (8).

In [RTV] (Theorems 5.7 and 6.1) it is shown that $\mathrm{C_Y}(T)\cong S_n\ltimes F_{t,n}$, where S_n acts on $F_{t,n}$ by permuting the lower indices. To specify an isomorphism, one chooses a spanning subtree T_0 of T (we take the one given in Figure 7). Then, let $u\in T$ be a (directed) edge, pointing from α to β. The isomorphism $\Phi\colon \mathrm{C_Y}(T)\to S_n\ltimes F_{t,n}$ is defined by taking u to the transposition $\Phi(u)=(\alpha\,\beta)$ if $u\in T_0$ (i.e. u is on the spanning subtree), and $\Phi(u) = (\alpha\,\beta)(x_\beta^{(\tau)})^{-1}x_\alpha^{(\tau)}$ if $u = x^{(\tau)}$ is an edge outside of T_0. Note that $\Phi(u)$ is an element of order 2 in $S_n \ltimes F_{t,n}$. The edges of $T - T_0$ are ordered for the sake of this definition (since $(x_\beta^{(\tau)})^{-1}x_\alpha^{(\tau)} \neq (x_\alpha^{(\tau)})^{-1}x_\beta^{(\tau)}$ in $F_{t,n}$), but of course $x^{(\tau)}$ and $(x^{(\tau)})^{-1}$ is the same element in C (or even in $\mathrm{C_Y}(T)$).

Since the cyclic relations hold in the symmetric group, $\psi_T(N) = 1$ and so $\Phi(N)$ is contained in the kernel of the natural projection $S_n \ltimes F_{t,n} \to S_n$, namely $\Phi(N) \subseteq F_{t,n}$. Moreover $\Phi(N)$ is normal in $F^\star_{t,n}$ so

$$C \cong \mathrm{C_Y}(T)/N \cong (S_n \ltimes F_{t,n})/\Phi(N) = S_n \ltimes (F_{t,n}/\Phi(N))$$

is the kernel of the induced map $\mathrm{ab}\colon S_n \ltimes (F^\star_{t,n}/\Phi(N)) \to \mathbb{Z}^t$. We will the quotient $F^\star_{t,n}/\Phi(N)$, and then apply the map ab and compute its kernel. See Figure 8 for some of the groups involved.

$$\begin{tikzcd}
\mathrm{C_Y}(T) \arrow[r, "\Phi"] \arrow[d] & S_n \ltimes F_{t,n} \arrow[r, hook] \arrow[d] & S_n \ltimes F^\star_{t,n} \arrow[d] \\
C \arrow[r] & S_n \ltimes F_{t,n}/\Phi(N) \arrow[r, hook] & S_n \ltimes F^\star_{t,n}/\Phi(N)
\end{tikzcd}$$

FIGURE 8.

We first compute the image of the cyclic relation associated to V_8. Let α and β denote the vertices of $x^{(4)}$ (these were planes 11 and 9 in Figure 3). Let $x^{(4)}, u_1, \dots, u_5$ denote the edges of the hexagon around V_8 (in that order), then the cyclic relation is that $x^{(4)} = u_1u_2u_3u_4u_5u_4u_3u_2u_1$ in C (note that $x^{(4)}$ and the u_j have order 2). Applying Φ, the right hand side is mapped to the transposition $(\alpha\,\beta)$ while $x^{(4)}$ is mapped to $(\alpha\,\beta)(x_\beta^{(4)})^{-1}x_\alpha^{(4)}$. The equality then becomes $x_\beta^{(4)} = x_\alpha^{(4)}$, which under the action of S_n becomes $x_j^{(4)} = x_i^{(4)}$ for every i and j. Thus $y^4 = x_i^{(4)}$ is independent of i, and therefore central (as it commutes with every generator). The same computation, around V_5, V_6 and V_9, proves that (in $S_n \ltimes (F^\star_{t,n}/\Phi(N))$) $y^5 = x_i^{(5)}$, $y^9 = x_i^{(9)}$ and $y^{10} = x_i^{(10)}$ are all independent of i and thus central.

Let us now evaluate the cyclic relation around V_4. Let $x^{(7)}, u_1, u_2, x^{(8)}, u_3$ and u_4 denote the edges of the hexagon around V_4. Moreover let α and β denote the end points of $x^{(7)}$, and γ, δ denote the end points of $x^{(8)}$ ($x^{(8)}$ points from γ to δ). The relation in C is

$$x^{(7)}u_1u_2x^{(8)}u_3 = u_1u_2x^{(8)}u_3u_4.$$

Applying Φ, we obtain

$$(\alpha\,\delta)(x_\delta^{(7)})^{-1}x_\alpha^{(7)}(\gamma\,\delta)(x_\delta^{(8)})^{-1}x_\gamma^{(8)} = (\gamma\,\delta)(x_\delta^{(8)})^{-1}x_\gamma^{(8)}(\alpha\,\gamma),$$

which is equivalent to

$$(x_\delta^{(7)})^{-1}x_\alpha^{(7)} = (x_\gamma^{(8)})^{-1}x_\alpha^{(8)}(x_\delta^{(8)})^{-1}x_\gamma^{(8)},$$

but since $x_\gamma^{(8)}$, $x_\alpha^{(8)}$ and $x_\delta^{(8)}$ commute, we obtain $x_\alpha^{(7)}(x_\alpha^{(8)})^{-1} = x_\delta^{(7)}(x_\delta^{(8)})^{-1}$. Acting with S_n, we obtain

$$x_i^{(7)}(x_i^{(8)})^{-1} = x_j^{(7)}(x_j^{(8)})^{-1}$$

for every i, j. In particular $y^{7,8} = x_i^{(7)}(x_i^{(8)})^{-1}$ is independent of i, and therefore central.

In a similar manner (working around V_7, V_2 and V_3), we prove that $y^{6,8} = x_i^{(6)}(x_i^{(8)})^{-1}$ is independent of i and central, and likewise for $y^{2,1} = x_i^{(2)}(x_i^{(1)})^{-1}$ and $y^{3,1} = x_i^{(3)}(x_i^{(1)})^{-1}$.

It remains to evaluate the cyclic relation around V_1. The surrounding hexagon is given in Figure 9, where the triangles $3, 2, 7, 15, 13$ and 1 (see Figure 3) were relabelled $\alpha, \beta, \gamma, \delta, \epsilon$ and ϕ.

The cyclic relation around V_1 in C is

$$u_1x^{(1)}x^{(6)}u_2x^{(3)} = x^{(1)}x^{(6)}u_2x^{(3)}x^{(8)}.$$

Applying Φ, we obtain

$$\begin{aligned} & (\alpha\,\beta)(\beta\,\gamma)(x_\beta^{(1)})^{-1}x_\gamma^{(1)}(\gamma\,\delta)(x_\gamma^{(6)})^{-1}x_\delta^{(6)}(\delta\,\epsilon)(\epsilon\,\phi)(x_\phi^{(3)})^{-1}x_\epsilon^{(3)} \\ = \; & (\beta\,\gamma)(x_\beta^{(1)})^{-1}x_\gamma^{(1)}(\gamma\,\delta)(x_\gamma^{(6)})^{-1}x_\delta^{(6)}(\delta\,\epsilon)(\epsilon\,\phi)(x_\phi^{(3)})^{-1}x_\epsilon^{(3)}(\phi\,\alpha)(x_\alpha^{(8)})^{-1}x_\phi^{(8)}, \end{aligned}$$

which translates to

$$(x_\gamma^{(1)})^{-1}x_\beta^{(1)}(x_\delta^{(6)})^{-1}x_\beta^{(6)}(x_\beta^{(3)})^{-1}x_\phi^{(3)} = (x_\gamma^{(1)})^{-1}x_\alpha^{(1)}(x_\delta^{(6)})^{-1}x_\alpha^{(6)}(x_\alpha^{(3)})^{-1}x_\phi^{(3)}(x_\alpha^{(8)})^{-1}x_\beta^{(8)},$$

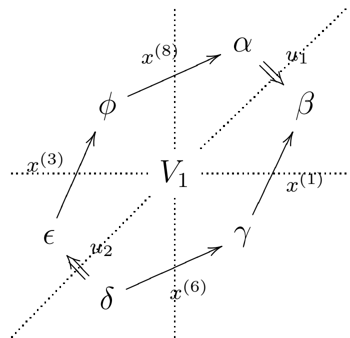

FIGURE 9. The hexagon around V_1

and then (using commutation) to

$$x_\beta^{(1)}x_\beta^{(6)}(x_\beta^{(3)})^{-1}(x_\beta^{(8)})^{-1} = x_\alpha^{(1)}x_\alpha^{(6)}(x_\alpha^{(3)})^{-1}(x_\alpha^{(8)})^{-1}.$$

But $x_i^{(6)}(x_i^{(8)})^{-1} = y^{6,8}$ and $x_i^{(3)}(x_i^{(1)})^{-1} = y^{3,1}$, are central, so acting with S_n, we obtain

$$x_j^{(1)}x_j^{(8)}(x_j^{(1)})^{-1}(x_j^{(8)})^{-1} = x_i^{(1)}x_i^{(8)}(x_i^{(1)})^{-1}(x_i^{(8)})^{-1}$$

for every i, j. It follows that $z = [x_i^{(1)}, x_i^{(8)}]$ is independent of i (and therefore central).

Summarizing, $F_{t,n}^\star/\Phi(N)$ is generated by $y^4, y^5, y^9, y^{10}, y^{7,8}, y^{6,8}, y^{3,1}, y^{2,1}$ and z which are all central, and by $\{x_i^{(1)}, x_i^{(8)}\}_{i=1,\dots,18}$, subject to the relations

$$[x_i^{(1)}, x_j^{(1)}] = 1, \tag{46}$$

$$[x_i^{(8)}, x_j^{(8)}] = 1, \tag{47}$$

$$[x_i^{(1)}, x_j^{(8)}] = 1, \tag{48}$$

(for all $i \neq j$) and

$$[x_i^{(1)}, x_i^{(8)}] = z, \tag{49}$$

for all i. Chasing back the definition of the various y generators, we see that the map $\mathrm{ab}\colon F_{t,n}^\star \to \mathbb{Z}^t = \{e^1, \dots, e^t\}$ is defined by $\mathrm{ab}(y^\tau) = e^\tau$ for $\tau = 4, 5, 9, 10$, $\mathrm{ab}(y^{\tau,\tau'}) = e^\tau(e^{\tau'})^{-1}$ for $(\tau, \tau') = (6,8), (7,8), (3,1), (2,1)$,

$\mathrm{ab}(x_i^{(\tau)}) = e^\tau$ for $\tau = 1, 8$ and $\mathrm{ab}(z) = 1$. Let $g \in F^\star_{t,n}$ be an arbitrary element. For every $\tau \neq 1, 8$, the exponent of e^τ in $\mathrm{ab}(g)$ is equal to the exponent of y^τ (or $y^{\tau,1}$, or $y^{\tau,8}$) in g. Therefore, the kernel $F_{t,n}$ is generated by $x_i^{(1)}$, $x_i^{(8)}$ and z.

Recall that $n = 18$.

Corollary 5.1. *Let H denote the group generated by $\{z, x_i^{(1)}, x_i^{(8)}\}_{i=1,\dots,n}$, with the relations (46)–(49) and z central. Define a map $\mathrm{ab}\colon H \to \mathbb{Z}^2 = \mathbb{Z}e^1 \oplus \mathbb{Z}e^8$ by $\mathrm{ab}(z) = 0$, $\mathrm{ab}(x_i^{(1)}) = e^1$ and $\mathrm{ab}(x_i^{(8)}) = e^8$. The symmetric group is acting on H by indices, and the action is compatible with* ab.

Then $K_C = \mathrm{Ker}(\psi_C\colon C \to S_n)$ is isomorphic to $\mathrm{Ker}(\mathrm{ab})$*, and C is the semidirect product $S_n \ltimes \mathrm{Ker}(\mathrm{ab})$ (action on the indices).*

Note that H is an extension of $\mathbb{Z}^{2n} = \mathbb{Z}^{36}$ by $\mathbb{Z} = \langle z \rangle$, and $\mathrm{Ker}(\mathrm{ab})$ is an extension of $\mathbb{Z}^{2(n-1)} = \mathbb{Z}^{34}$ by $\mathbb{Z}$. This proves Theorem 1.1.

Since z is invariant under the action of S_n, it generates the center of $S_n \ltimes H$ for the group H just defined. Therefore $\Phi^{-1}(z)$ (or more precisely its image in C) generates the center of C. The computations above allow us to identify this element. Recall that $C = \langle u_1, \dots, u_{27} \rangle$ corresponding to the intersecting pairs of planes (with the generators numbered as in Figure 2); here too we write j for u_j.

Proposition 5.2. *Let $\sigma_1 = 21 \cdot 19 \cdot 8 \cdot 6$, $\tau_1 = \sigma_1^{-1} \cdot 14 \cdot \sigma_1$, $\sigma_2 = 20 \cdot 24 \cdot 25 \cdot 16 \cdot 11 \cdot 5$, $\tau_2 = 19 \cdot 21 \cdot 14 \cdot 21 \cdot 19$, $\tau_3 = \sigma_2^{-1} \tau_2 \sigma_2$ and $\tau_4 = \sigma_2^{-1} \cdot 8 \cdot \sigma_2$.*

Then the center of C is the infinite cyclic group generated by $[\tau_1 \cdot 1, \tau_3^{-1}(\tau_4 \cdot 4)\tau_3]$.

Proof. Consider the above as elements of $\mathrm{C_Y}(T)$. The only generators used which are not in T_0, are 1 and 4. Recall from [RTV] that $\mathrm{C_Y}(T_0) \cong S_n$ if T_0 is a spanning subtree. We thus compute in the group $\mathrm{C_Y}(T_0) \cong S_{18}$ (numbering as in Figure 3): $\tau_1 = (2\,7)$, $\tau_2 = (7\,10)$, $\tau_3 =$

$(1\,7)$ and $\tau_4 = (1\,3)$. Now, $\tau_1 \cdot 1$ and $\tau_4 \cdot 4$ are in the kernel of ψ_T (see Section 3 for the definition of this map), and Φ maps them to $(x_2^{(1)})^{-1}x_7^{(1)}$ and $(x_3^{(8)})^{-1}x_1^{(8)}$ respectively. Moreover $\Phi(\tau_3(\tau_4 \cdot 4)\tau_3) = (x_3^{(8)})^{-1}x_7^{(8)}$.

Now $\Phi([\tau_1 \cdot 1, \tau_3(\tau_4 \cdot 4)\tau_3]) = [(x_2^{(1)})^{-1}x_7^{(1)}, (x_3^{(8)})^{-1}x_7^{(8)}] = [x_7^{(1)}, x_7^{(8)}]$ since $x_3^{(8)}$ and $x_2^{(1)}$ commute with $x_7^{(i)}$ and with each other; and the last commutator is z of Equation (49), which generates the center of $S_n \ltimes F_{t,n}$ modulo the cyclic relations. $\square$

We conclude with a general remark, motivated by a topological interpretation of the computation done in this section. Originally, $\mathrm{C_Y}(T)$ is isomorphic to S_n acting on a certain subgroup of $\pi_1(T)^n$. Adding a cyclic relation to $\mathrm{C_Y}(T)$ trivializes one generator, and this can be achieved by patching a 2-cell (homeomorphic to D^2) on this cycle of T. The degenerated object X_0 can be viewed as a triangulation of a torus; then T is the dual graph of its 1-skeleton S_0. Adding all the patches to T results with a surface homeomorphic to X_0, namely to the torus $\mathbb{T}$. The fundamental group is now $\pi_1(\mathbb{T}) = \mathbb{Z}^2$, and indeed the kernel of the map $\pi_1(\mathbb{T})^n \to \mathrm{H}_1(\mathbb{T})$ is $\mathbb{Z}^{2(n-1)} = \mathbb{Z}^{34}$, which is the abelianization of K_C.

Let X be a surface of general type of degree n, with a degeneration to a union X_0 of planes where no three planes meet in a line. In all cases computed so far (including the Hirzebruch and Veronese surfaces, embeddings of $\mathbb{CP}^1 \times \mathbb{CP}^1$ with respect to the full linear system $|aL_1 + bL_2|$, as well as $\mathbb{CP}^1 \times \mathbb{T}$ and $\mathbb{T} \times \mathbb{T}$ which is dealt with here), the kernel $K_C = \mathrm{Ker}(\psi_C : C \to S_n)$ has the same abelianization as the kernel of $\pi_1(X_0)^n \to \mathrm{H}_1(X_0)$. It would be interesting to know how far this observation goes.

References

[A] Amram, M., *The Galois Covers of Algebraic Surfaces*, doctoral dissertation, Bar Ilan, Israel, 2001.

[AGTV] Amram, M., Goldberg, D., Teicher, M., Vishne U., *The fundamental group of a Galois cover of* $\mathbb{CP}^1 \times \mathbb{T}$, Algebraic and Geometric Topology Volume 2, 2002, paper no. 20, 403-432.

[AT1] Amram, M., Teicher, M., *On the degeneration, regeneration and braid monodromy of* $\mathbb{T} \times \mathbb{T}$, Acta Applicandae mathematicae, 75(1), 2003, 195-270.

[AT2] Amram, M., Teicher, M., *The fundamental group of the complement of the branch curve of the surface* $\mathbb{T} \times \mathbb{T}$ *in* $\mathbb{C}^2$, Osaka, Japan, 40(4), 2003, 1-37.

[FRT] Freitag, P., Robb, A., Teicher, M., *The fundamental group of Galois cover of Hirzebruch surfaces*, in preparation.

[MRT] Moishezon, B., Robb, A., Teicher, M., *On Galois covers of Hirzebruch surface*, Math. Ann. **305**, 1996, 493-539.

[MT1] Moishezon, B., Teicher, M., *Simply connected algebraic surfaces of positive index*, Invent. Math. **89**, 1987, 601-643.

[MT2] Moishezon, B., Teicher, M., *Braid group techniques in complex geometry. IV. Braid monodromy of the branch curve* S_3 *of* $V_3 \to \mathbb{CP}^2$ *and application to* $\pi_1(\mathbb{CP}^2 - S_3, *)$, Classification of algebraic varieties (L'Aquila, 1992), 333–358, Contemp. Math., 162, Amer. Math. Soc., Providence, RI, 1994.

[RTV] Rowen, L.H., Teicher, M. and Vishne, U., *Coxeter Covers of the Symmetric Groups*, J. Group Theory, to appear, (2004).

[vK] van Kampen, E.R., *On the fundamental group of an algebraic curve*, Amer. J. Math. **55**, 1933, 255-260.

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