implementation based on micro-controllers; 2) the development of a FDI method addressing sensor faults, actuator faults, and process faults under one unified framework [15].

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Sampled-Data Based Consensus of Continuous-Time Multi-Agent Systems With Time-Varying Topology

Yanping Gao and Long Wang

Abstract—This technical note studies consensus problems of multiple agents with continuous-time second-order dynamics, where each agent can obtain its positions and velocities relative to its neighbors only at sampling instants. It is assumed that the sampling period of each agent is independent of the others' and the interaction topology among agents is time-varying, where the associated direct graphs may not have spanning trees. If the union graph of all direct graphs has a spanning tree, then there exist controller gains and sampling periods such that consensus is reached. Moreover, two approaches are presented to design such controller gains and sampling periods. Simulations are performed to validate the theoretical results.

Index Terms—Consensus, multi-agent systems, sampled-data control, second-order agents, time-varying topology.

I. INTRODUCTION

There has been much work on consensus problems of first-order agents, and many research topics, such as consensus under time-varying topology [1]–[4], finite-time consensus [5], consensus over random networks [6], asynchronous consensus [7], [8], and consensus with predictive mechanisms [9], have been studied thoroughly. In some practical situations, agents such as unmanned aerial vehicles and mobile robots can be controlled directly by their accelerations rather than by their velocities. Hence, it is also necessary to investigate consensus problems of second-order agents. In [10] and [11], two typical protocols were proposed for continuous-time second-order agents. In [12], a relaxed sufficient condition was obtained for consensus of continuous-time second-order agents with switching topology. Formation control problems of continuous-time second-order agents, which can be transformed into consensus problems, were considered in [13] and [14]. In [15], motion coordination problems were discussed for continuous-time second-order agents with switching topology, variation of link gain, and unmodeled dynamics. In [16], consensus problems were investigated for discrete-time second-order agents with stochastic switching topology. For details, see the survey papers [17], [18] and the references therein.

In most of the work on continuous-time multi-agent systems, it is assumed that all information is transmitted continuously. However, information transmission may be interrupted due to the unreliability of communication channels and the limitations of sensing ability of agents. Hence, it is more practical to take account of intermittent information transmission. In [19]–[22], consensus problems were addressed for continuous-time second-order agents in a sampled-data setting, where the sampling periods of all agents are the same. Moreover, all agents update their control inputs at the same discrete times. On the

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other hand, it is difficult to guarantee synchrony because of technology limitations and environment disturbances, and thus, the asynchronous case, namely, each agent acts on its own pace, also deserves to be studied. Asynchronous consensus problems of first-order agents have been investigated extensively (see, e.g., [7], [8]). To the authors' best knowledge, there are few research results on asynchronous consensus problems of second-order agents. Based on the above considerations, we investigate asynchronous consensus problems of continuous-time second-order agents in a sampled-data setting, where the sampling periods of all agents may be different. The main contribution of this work is to provide some sufficient conditions for consensus under intermittent information transmission, asynchronous update, and time-varying topology which may not have spanning trees.

Notations: Let $I_n \in \mathbb{R}^{n\times n}$ be an identity matrix and $\mathbf{1}_n = [1 \ \cdots \ 1]^T \in \mathbb{R}^n$; Let $\mathbb{Z}^+$ denote the set of all positive integers; for any symmetric matrices A, $A < 0$ (respectively, $A > 0$) means that A is a negative (respectively, positive) definite matrix; for any square matrix H, $H(i,:)$ represents the i-th row of H and $\Lambda(H)$ denotes the set of all eigenvalues of H; Ψ is called the transformation matrix from x to y if $y = \Psi x$, where $y, x \in \mathbb{R}^n$, $\Psi \in \mathbb{R}^{n\times n}$.

II. Preliminaries

A. Graph Theory

We introduce some basic definitions in graph theory [23].

A directed graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G})$ and an edge set $\mathcal{E}(\mathcal{G})$, where $\mathcal{V}(\mathcal{G}) = \{v_1,\ldots,v_n\}$ and $\mathcal{E}(\mathcal{G}) \subset \{(v_j,v_i) : v_j, v_i \in \mathcal{V}(\mathcal{G})\}$. For edge (v_j,v_i), v_j is called the parent vertex of v_i and v_i is called the child vertex of v_j. The set of neighbors of vertex v_i is defined by $N(\mathcal{G},v_i) = \{v_j : (v_j,v_i) \in \mathcal{E}(\mathcal{G}) \text{ and } j \neq i\}$, and the associated index set is denoted by $N(\mathcal{G},i) = \{j : v_j \in N(\mathcal{G},v_i)\}$. A (directed) path from v_{i_1} to v_{i_k} is a sequence, $v_{i_1},\ldots,v_{i_k}$, of distinct vertices such that $(v_{i_j},v_{i_{j+1}}) \in \mathcal{E}(\mathcal{G})$, $j = 1,\ldots,k-1$. A directed graph $\mathcal{G}$ is strongly connected if there is a path from every vertex to every other vertex. A directed tree is a directed graph, where every vertex except one special vertex has exactly one parent vertex, and the special vertex, called root vertex, has no parent vertices and can be connected to any other vertices via paths. A subgraph $\mathcal{G}_s$ of $\mathcal{G}$ is a graph such that $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$. $\mathcal{G}_s$ is said to be a spanning subgraph if $\mathcal{V}(\mathcal{G}_s) = \mathcal{V}(\mathcal{G})$. For any $v_i, v_j \in \mathcal{V}(\mathcal{G}_s)$, if $(v_i,v_j) \in \mathcal{E}(\mathcal{G}_s) \Leftrightarrow (v_i,v_j) \in \mathcal{E}(\mathcal{G})$, then $\mathcal{G}_s$ is said to be an induced subgraph of $\mathcal{G}$, and $\mathcal{G}_s$ is also said to be induced by $\mathcal{V}(\mathcal{G}_s)$. A spanning tree of $\mathcal{G}$ is a directed tree which is a spanning subgraph of $\mathcal{G}$. $\mathcal{G}$ is said to have a spanning tree if some edges form a spanning tree of $\mathcal{G}$. The union graph of a collection of graphs $\{\mathcal{G}_1,\ldots,\mathcal{G}_Z\}$, where $\mathcal{G}_1,\ldots,\mathcal{G}_Z$ have the same vertex set $\mathcal{V}$, is a graph with vertex set $\mathcal{V}$ and edge set equaling the union of edge sets of $\mathcal{G}_1,\ldots,\mathcal{G}_Z$.

A matrix is called nonnegative if each of its elements is nonnegative. A weighted directed graph $\mathcal{G}(A)$ is a directed graph $\mathcal{G}$ plus a nonnegative matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} > 0 \Leftrightarrow (v_j,v_i) \in \mathcal{E}(\mathcal{G})$, and a_{ij} is called the weight of edge (v_j,v_i). The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ of

$$\mathcal{G}(A) \text{ is defined as } l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum\limits_{s=1,s\neq i}^{n} a_{is}, & i = j \end{cases}.$$

B. Model

Consider a group of agents with the following second-order dynamics:

$$\dot{x}_i(t) = v_i(t),\ \dot{v}_i(t) = u_i(t),\ i = 1,\ldots,n \tag{1}$$

where $x_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ are the position and velocity vectors of agent i, respectively, and u_i is the control input or the protocol. Although we only consider the case of second-order dynamics, similar analysis can also be done for the case of high-order dynamics.

Given u_i, $i = 1,\ldots,n$, we say that u_i or multi-agent system (1) solves a consensus problem asymptotically if $\lim_{t\to\infty}(x_p(t) - x_s(t)) = 0$ and $\lim_{t\to\infty}(v_p(t) - v_s(t)) = 0$, $p, s = 1,\ldots,n$, for any initial states. Such a consensus problem can find application in formation control of multiple vehicles/robots (see, e.g., [13], [14]).

We consider the following protocol:

$$\begin{aligned} u_i(t) = &- k_1 \sum_{j\in N_i\left(t_s^{(i)}\right)} a_{ij}\left(t_s^{(i)}\right)\left(x_i\left(t_s^{(i)}\right) - x_j\left(t_s^{(i)}\right)\right) \\ &- k_2 \sum_{j\in N_i\left(t_s^{(i)}\right)} a_{ij}\left(t_s^{(i)}\right)\left(v_i\left(t_s^{(i)}\right) - v_j\left(t_s^{(i)}\right)\right), \\ &t_s^{(i)} \le t < t_{s+1}^{(i)}, s = 0,1,\ldots, i = 1,\ldots,n \end{aligned} \tag{2}$$

where $k_1, k_2 > 0$, and $t_s^{(i)} = t_0 + sh_i$. Moreover, it is assumed that the sampling period of each agent is independent of the others' and $h_i = l_i h$, $i = 1,\ldots,n$, where $h > 0$ and $l_i \in \mathbb{Z}^+$.

III. Main Results

In this section, first we show that there exist controller gains k_1, k_2 and sampling periods $h_1,\ldots,h_n$ such that protocol (2) solves a consensus problem if the time-varying topology satisfies some conditions, and then we provide some methods to design such controller gains and sampling periods. For this purpose, we need some preparations.

To facilitate the following analysis, introduce a new topology, denoted by $\mathcal{G}(\widehat{A}(t))$, which is different from the actual topology $\mathcal{G}(A(t))$.

Definition 1: $\mathcal{G}(\widehat{A}(t))$ is a weighted directed graph with the same vertex set as $\mathcal{G}(A(t))$, where $\widehat{A}(t) = [\widehat{a}_{ij}(t)] \in \mathbb{R}^{n\times n}$. For any $i \in \{1,\ldots,n\}$, $s \in \{0,1,\ldots\}$, and $t \in \left[t_s^{(i)}, t_{s+1}^{(i)}\right)$, if the information of relative state between agent i and agent j, $\forall\, j \neq i$, is available for agent i at time $t_s^{(i)}$, then $(v_j,v_i) \in \mathcal{E}(\mathcal{G}(\widehat{A}(t)))$, or else $(v_j,v_i) \notin \mathcal{E}(\mathcal{G}(\widehat{A}(t)))$; if $(v_j,v_i) \in \mathcal{E}(\mathcal{G}(\widehat{A}(t)))$, then $\widehat{a}_{ij}(t) = a_{ij}\left(t_s^{(i)}\right)$. It is assumed that $\widehat{a}_{ii}(t) = 0$, $i = 1,\ldots,n$, $\forall\, t \ge t_0$.

Each agent interacts with other agents only at discrete times, and we use $\mathcal{G}(A(t))$ to denote the actual interaction among all agents. Under protocol (2), the effect of interaction at a discrete time will last to next discrete time, and we use $\mathcal{G}(\widehat{A}(t))$ to represent such effect. Hence, $\mathcal{G}(\widehat{A}(t))$ can be viewed as the extension of actual interaction among all agents on entire time. Furthermore, we make the following assumptions:

(A1) $\widehat{A}(t+T) = \widehat{A}(t)$, $T > 0$, $\forall\, t \ge t_0$, i.e., $\mathcal{G}(\widehat{A}(t))$ is periodically time-varying.

(A2) The union graph of $\{\mathcal{G}(\widehat{A}(t)) : \forall\, t \ge t_0\}$ has a spanning tree.

Remark 1: Periodic processes exist extensively in nature and engineering [24]. In some cases, the communication among agents exhibits periodic phenomena, which implies that the topology among agents is periodically time-varying. Hence, we focus on the case of periodically time-varying topology. Note that $\mathcal{G}(\widehat{A}(t))$ is periodic if and only if $\mathcal{G}(A(t))$ is periodic. Actually, (A1) and (A2) can be replaced with a more general topology case, which will be discussed in Section IV.

Let $\left\{t_s^{(i)} : i = 1,\ldots,n, s = 0,1,\ldots\right\} = \{t_0, t_1, \ldots\}$, where $t_0 < t_1 < \cdots$. Obviously, $\mathcal{G}(\widehat{A}(t))$ is time-invariant during each time interval $[t_r, t_{r+1})$, and each $t_{r+1} - t_r$ is an integer multiple of h. For convenience, let $\{\mathcal{G}(\widehat{A}(t)) : \forall\, t \ge t_0\} = \{\mathcal{G}(\widehat{A}_1),\ldots,\mathcal{G}(\widehat{A}_M)\}$ and introduce a switching signal $\sigma : \{0,1,\ldots\} \to \{1,\ldots,M\}$.

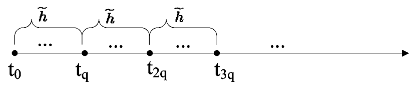

Fig. 1. Distribution of discrete times during all time intervals $[t_{jq}, t_{(j+1)q})$, $j = 0, 1, \ldots$, is identical.

From (2), we see that the control input of agent i during $\left[t_s^{(i)}, t_{s+1}^{(i)}\right)$ is time-invariant. By solving (1), we have $x_i(t) = x_i\left(t_s^{(i)}\right) + \left(t - t_s^{(i)}\right) v_i\left(t_s^{(i)}\right) + (\left(t - t_s^{(i)}\right)^2/2) u_i\left(t_s^{(i)}\right)$, $v_i(t) = v_i\left(t_s^{(i)}\right) + \left(t - t_s^{(i)}\right) u_i\left(t_s^{(i)}\right)$, $s = 0, 1, \ldots, i = 1, \ldots, n$, $\forall\, t \in \left(t_s^{(i)}, t_{s+1}^{(i)}\right]$. For any $r \in \{0, 1, \ldots\}$ and $i \in \{1, \ldots, n\}$, there exists $d(r, i) \in \{0, 1, \ldots\}$ such that $t_r \in \left[t_{d(r,i)}^{(i)}, t_{d(r,i)+1}^{(i)}\right)$. Then

$$
\begin{aligned}
x_i(t) &= x_i(t_r) + (t - t_r) v_i(t_r) + \frac{(t - t_r)^2}{2} u_i\left(t_{d(r,i)}^{(i)}\right), \\
v_i(t) &= v_i(t_r) + (t - t_r) u_i\left(t_{d(r,i)}^{(i)}\right), \forall\, t \in (t_r, t_{r+1}] \qquad (3)
\end{aligned}
$$

and

$$
\begin{aligned}
x_i(t_{r+1}) &= x_i(t_r) + (t_{r+1} - t_r) v_i(t_r) \\
&\quad + \frac{(t_{r+1} - t_r)^2}{2} u_i\left(t_{d(r,i)}^{(i)}\right), \\
v_i(t_{r+1}) &= v_i(t_r) + (t_{r+1} - t_r) u_i\left(t_{d(r,i)}^{(i)}\right). \qquad (4)
\end{aligned}
$$

Let l denote the least common multiple of $l_1, \ldots, l_n$, then the least common multiple of $h_1, \ldots, h_n$ is lh. Let $\widetilde{h} = lh$. By $t_s^{(i)} = t_0 + sh_i$, $i = 1, \ldots, n$, $s = 0, 1, \ldots$, $\{t_r : t_r \in [t_0 + j\widetilde{h}, t_0 + (j+1)\widetilde{h}), r \in \{0, 1, \ldots\}\}$, $j = 0, 1, \ldots$, have the same number of elements, denoted by q, where $1 \leq q \leq \sum_{i=1}^{n} (l/l_i - 1) + 1$. Hence, $\{t_{jq}, t_{jq+1}, \ldots, t_{(j+1)q-1}\} \subset [t_0 + j\widetilde{h}, t_0 + (j+1)\widetilde{h})$, where $t_{jq} = t_0 + j\widetilde{h}$, $j = 0, 1, \ldots$, and the distribution of discrete times during each two time intervals $[t_{kq}, t_{(k+1)q})$ and $[t_{jq}, t_{(j+1)q})$ is identical, i.e.

$$
\begin{aligned}
t_{kq+1} - t_{kq} &= t_{jq+1} - t_{jq}, \ldots, \\
t_{(k+1)q} - t_{(k+1)q-1} &= t_{(j+1)q} - t_{(j+1)q-1}, \\
k, j &= 0, 1, \ldots \qquad (5)
\end{aligned}
$$

as shown in Fig. 1. By the definition of $t_{d(r+jq,i)}^{(i)}$, $t_r - t_{d(r,i)}^{(i)} = t_{r+jq} - t_{d(r+jq,i)}^{(i)}$. Thus

$$
t_{d(r+jq,i)}^{(i)} = t_{d(r,i)}^{(i)} + t_{r+jq} - t_r = t_{d(r,i)}^{(i)} + j\widetilde{h}. \qquad (6)
$$

Let $f(r) = \lfloor r/q \rfloor$, $r = 0, 1, \ldots$, where $\lfloor r/q \rfloor$ is the maximum integer not larger than r/q, then $f(r)q \leq r < (f(r)+1)q$, and

$$
t_{f(r)q} \leq t_{d(r,i)}^{(i)} \leq t_r, i = 1, \ldots, n. \qquad (7)
$$

Let $\widehat{L}_{\sigma(r)}$ denote the Laplacian matrix of $\mathcal{G}(\widehat{A}(t_r))$, and let $\theta(r) = [\,\theta_1^T(r) \ \ldots \ \theta_n^T(r)\,]^T$, $\theta_i(r) = [\,\theta_{i1}(r) \ \ \theta_{i2}(r)\,]^T = [\,x_i(t_r) \ \ v_i(t_r)\,]^T$. By (7), system (4) can be rewritten as

$$
\begin{aligned}
\theta(r+1) &= (I_n \otimes B_1(r))\theta(r) \\
&\quad - \sum_{j=0}^{r-f(r)q} \left(\widehat{L}_{\sigma(r)}^{(j)} \otimes B_2(r)\right)\theta(r-j), \\
r &= 0, 1, \ldots \qquad (8)
\end{aligned}
$$

where $B_1(r) = \begin{bmatrix} 1 & \tau_r \\ 0 & 1 \end{bmatrix}$, $B_2(r) = \begin{bmatrix} k_1\tau_r^2/2 & k_2\tau_r^2/2 \\ k_1\tau_r & k_2\tau_r \end{bmatrix}$, $\tau_r = t_{r+1} - t_r$, $\widehat{L}_{\sigma(r)}^{(j)} \in \mathbb{R}^{n \times n}$, and

$$
\widehat{L}_{\sigma(r)}^{(j)}(i,:) = \begin{cases} \widehat{L}_{\sigma(r)}(i,:), & t_{d(r,i)}^{(i)} = t_{r-j}, i = 1, \ldots, n \\ 0, & t_{d(r,i)}^{(i)} \neq t_{r-j}. \end{cases}
$$

Clearly, $\sum_{j=0}^{r-f(r)q} \widehat{L}_{\sigma(r)}^{(j)} = \widehat{L}_{\sigma(r)}$, $\widehat{L}_{\sigma(r)}^{(j)}\mathbf{1}_n = 0$, and the delays of system (8) are not larger than $q-1$. Note that system (8) is different from the discrete-time second-order multi-agent system in [16] and the discrete-time systems obtained by discretization in [19]–[22]. Obviously, asynchrony induces more time-delays.

We say that system (8) solves a consensus problem if $\lim_{r\to\infty}(\theta_i(r) - \theta_j(r)) = 0$, $i, j = 1, \ldots, n$, for any initial value $\theta(0)$. By (3) and the definition of θ, system (1) with protocol (2) solves a consensus problem if and only if system (8) solves a consensus problem. Nonnegative matrix theory is generally applied to deal with consensus problems of discrete-time systems (see, e.g., [2], [4], [8]). However, some coefficient matrices of system (8) may not be nonnegative. Hence, we resort to the Lyapunov's direct method to treat the consensus problem of system (8) [25].

First we make a state transformation for system (8). Let $U = [\,\mathbf{1}_n \ \ U_1\,]$ be an invertible matrix, then $U^{-1}\widehat{L}_{\sigma(r)}U = \begin{bmatrix} 0 & \alpha_{\sigma(r)} \\ 0 & \widehat{H}_{\sigma(r)} \end{bmatrix}$, $U^{-1}\widehat{L}_{\sigma(r)}^{(j)}U = \begin{bmatrix} 0 & \alpha_{\sigma(r)}^{(j)} \\ 0 & \widehat{H}_{\sigma(r)}^{(j)} \end{bmatrix}$ where $\widehat{H}_{\sigma(r)}$, $\widehat{H}_{\sigma(r)}^{(j)} \in \mathbb{R}^{(n-1)\times(n-1)}$. Let $\delta(r) = (U^{-1} \otimes I_2)\theta(r)$, where $\delta(r) = [\,\delta_1^T(r) \ \ \widehat{\delta}^T(r)\,]^T$, $\widehat{\delta}(r) = [\,\delta_2^T(r) \ \ldots \ \delta_n^T(r)\,]^T$, and $\delta_i(r) = [\,\delta_{i1}(r) \ \ \delta_{i2}(r)\,]^T$, then $\widehat{\delta}(r+1) = (I_{n-1} \otimes B_1(r))\widehat{\delta}(r) - \sum_{j=0}^{r-f(r)q}\left(\widehat{H}_{\sigma(r)}^{(j)} \otimes B_2(r)\right)\widehat{\delta}(r-j)$, $r = 0, 1, \ldots$. Let $\widetilde{\delta}(r) = [\,\delta_{21}(r) \ \ldots \ \delta_{n1}(r) \ \ \delta_{22}(r) \ \ldots \ \delta_{n2}(r)\,]^T$, then

$$
\begin{aligned}
\widetilde{\delta}(1) &= \Phi_0^{(0)}\widetilde{\delta}(0), \\
\widetilde{\delta}(r+1) &= \Phi_r^{(0)}\widetilde{\delta}(r) + \sum_{j=1}^{r-f(r)q} \Phi_r^{(j)}\widetilde{\delta}(r-j), \\
r &= 1, 2, \ldots \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_r^{(0)} &= I_{2(n-1)} + \tau_r \begin{bmatrix} 0 & I_{n-1} \\ -k_1\widehat{H}_{\sigma(r)}^{(0)} & -k_2\widehat{H}_{\sigma(r)}^{(0)} \end{bmatrix} \\
&\quad + \tau_r^2 \begin{bmatrix} -\frac{k_1}{2}\widehat{H}_{\sigma(r)}^{(0)} & -\frac{k_2}{2}\widehat{H}_{\sigma(r)}^{(0)} \\ 0 & 0 \end{bmatrix}, \\
r &= 0, 1, \ldots, \\
\Phi_r^{(j)} &= \tau_r \begin{bmatrix} 0 & 0 \\ -k_1\widehat{H}_{\sigma(r)}^{(j)} & -k_2\widehat{H}_{\sigma(r)}^{(j)} \end{bmatrix} \\
&\quad + \tau_r^2 \begin{bmatrix} -\frac{k_1}{2}\widehat{H}_{\sigma(r)}^{(j)} & -\frac{k_2}{2}\widehat{H}_{\sigma(r)}^{(j)} \\ 0 & 0 \end{bmatrix}, \\
r &= 1, 2, \ldots.
\end{aligned}
$$

With the above preparations, we obtain the following result.

Lemma 1: System (1) with protocol (2) solves a consensus problem if and only if system (9) is globally asymptotically stable.

Hence, we can establish the following main result by analyzing the stability of system (9).

Theorem 1: Assume (A1) and (A2) hold. Then there exist controller gains k_1, k_2 and sampling periods $h_1, \ldots, h_n$ such that system (1) with protocol (2) solves a consensus problem.

Proof: First we show that the global asymptotic stability of system (9) is equivalent to the asymptotic stability of a discrete-time time-invariant system without delays.

The least common multiple of $\widetilde{h}$ and T is denoted by $\widetilde{N}h$, Where $\widetilde{N} \in \mathbb{Z}^+$. obviously, there exists $m \in \mathbb{Z}^+$ such that $t_{mq} = \widetilde{N}h$. By (A1) and (A2), $\mathcal{G}(\widehat{A}(\cdot))$ during $[t_{rmq}, t_{(r+1)mq})$ is the same as $\mathcal{G}(\widehat{A}(\cdot))$ during $[t_{smq}, t_{(s+1)mq})$, $\forall\, r,\, s \in \{0, 1, \ldots\}$, and the union graph of $\{\mathcal{G}(\widehat{A}(t)) : t \in [t_{rmq}, t_{(r+1)mq})\}$, $\forall\, r \in \{0, 1, \ldots\}$, has a spanning tree.

By (5), $\tau_{jq} = \tau_0, \ldots, \tau_{jq+q-1} = \tau_{q-1}$, $j = 0, 1, \ldots$. Clearly, τ_r, $r = 0, 1, \ldots$, are all integer multiples of h. Thus, let $\tau_{jq} = m_0 h, \ldots, \tau_{jq+q-1} = m_{q-1}h$, $j = 0, 1, \ldots$, where $m_0, \ldots, m_{q-1} \in \mathbb{Z}^+$. Note that q, $m_0, \ldots, m_{q-1}$ are determined only by $l_1, \ldots, l_n$. Obviously, each coefficient matrix of system (9) can be written as a polynomial matrix of h, namely

$$\begin{aligned}
\Phi_r^{(0)} =& I_{2(n-1)} \\
&+ h \begin{bmatrix} 0 & m_{r-f(r)q} I_{n-1} \\ -k_1 m_{r-f(r)q} \widehat{H}_{\sigma(r)}^{(0)} & -k_2 m_{r-f(r)q} \widehat{H}_{\sigma(r)}^{(0)} \end{bmatrix} \\
&+ h^2 \begin{bmatrix} -\frac{k_1 m_{r-f(r)q}^2}{2} \widehat{H}_{\sigma(r)}^{(0)} & -\frac{k_2 m_{r-f(r)q}^2}{2} \widehat{H}_{\sigma(r)}^{(0)} \\ 0 & 0 \end{bmatrix}, \\
\Phi_r^{(j)} =& h \begin{bmatrix} 0 & 0 \\ -k_1 m_{r-f(r)q} \widehat{H}_{\sigma(r)}^{(j)} & -k_2 m_{r-f(r)q} \widehat{H}_{\sigma(r)}^{(j)} \end{bmatrix} \\
&+ h^2 \begin{bmatrix} -\frac{k_1 m_{r-f(r)q}^2}{2} \widehat{H}_{\sigma(r)}^{(j)} & -\frac{k_2 m_{r-f(r)q}^2}{2} \widehat{H}_{\sigma(r)}^{(j)} \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{10}$$

The following aim is to obtain the transformation matrix from $\widetilde{\delta}(rmq)$ to $\widetilde{\delta}((r+1)mq)$, $\forall\, r \in \{0, 1, \ldots\}$, which will be finished by three steps.

Let $\mathcal{H}_{\sigma(r)} = [\widehat{H}_{\sigma(r)}^{(0)} \ \ldots \ \widehat{H}_{\sigma(r)}^{(r-f(r)q)}]$, $r = 0, 1, \ldots$. The transformation matrices from $\widetilde{\delta}(0)$ to $\widetilde{\delta}(z)$, $z = 1, \ldots, q$, can be calculated by (9). Moreover, by (10), the transformation matrix from $\widetilde{\delta}(0)$ to $\widetilde{\delta}(z)$, $\forall\, z \in \{1, \ldots, q\}$, can be written as a polynomial matrix of h with degree $2z$; let $E_{z-1}^{(i)}$ denote the constant coefficient matrix associated with h^i, $\forall\, i \in \{0, 1, \ldots, 2z\}$. Clearly, $E_{z-1}^{(0)} = I_{2(n-1)}$, $z = 1, \ldots, q$. Then $\widetilde{\delta}(1) = \Phi_0^{(0)}\widetilde{\delta}(0) = \left(I_{2(n-1)} + hE_0^{(1)} + h^2 E_0^{(2)}\right)\widetilde{\delta}(0)$, $\widetilde{\delta}(2) = \Phi_1^{(0)}\widetilde{\delta}(1) + \Phi_1^{(1)}\widetilde{\delta}(0) = \left(I_{2(n-1)} + \sum_{i=1}^{4} h^i E_1^{(i)}\right)\widetilde{\delta}(0), \ldots, \widetilde{\delta}(q) = \left(I_{2(n-1)} + \sum_{i=1}^{2q} h^i E_{q-1}^{(i)}\right)\widetilde{\delta}(0)$, where for any $z \in \{1, \ldots, q\}$

$$E_{z-1}^{(1)} = \begin{bmatrix} 0 & \sum_{j=0}^{z-1} m_j I_{n-1} \\ -k_1 \sum_{j=0}^{z-1} m_j \widehat{H}_{\sigma(j)} & -k_2 \sum_{j=0}^{i} m_j \widehat{H}_{\sigma(j)} \end{bmatrix}$$

and $[\, E_{z-1}^{(1)} \ \ldots \ E_{z-1}^{(2z)}\,]$ is determined only by k_1, k_2, $m_0, \ldots, m_{z-1}, \mathcal{H}_{\sigma(0)}, \mathcal{H}_{\sigma(1)}, \ldots, \mathcal{H}_{\sigma(z-1)}$.

By similar manipulation, for any $r \in \{1, 2, \ldots\}$, the transformation matrix from $\widetilde{\delta}(rq)$ to $\widetilde{\delta}((r+1)q)$ can also be written as a polynomial matrix of h with degree $2q$; let $E_{(r+1)q-1}^{(i)}$ denote the constant coefficient matrix associated with h^i, $\forall\, i \in \{0, 1, \ldots, 2q\}$. Then

$$\widetilde{\delta}((r+1)q) = \left(I_{2(n-1)} + \sum_{i=1}^{2q} h^i E_{(r+1)q-1}^{(i)}\right)\widetilde{\delta}(rq), \quad r = 0, 1, \ldots \tag{11}$$

where

$$E_{(r+1)q-1}^{(1)} = \begin{bmatrix} 0 & \sum_{j=0}^{q-1} m_j I_{n-1} \\ -k_1 \sum_{j=rq}^{(r+1)q-1} m_{j-rq} \widehat{H}_{\sigma(j)} & -k_2 \sum_{j=rq}^{(r+1)q-1} m_{j-rq} \widehat{H}_{\sigma(j)} \end{bmatrix}$$

and $[\, E_{(r+1)q-1}^{(1)} \ \ldots \ E_{(r+1)q-1}^{(2q)}\,]$ is determined only by k_1, k_2, $m_0, \ldots, m_{q-1}, \mathcal{H}_{\sigma(rq)}, \ldots, \mathcal{H}_{\sigma((r+1)q-1)}$.

By (11), for any $r \in \{0, 1, \ldots\}$, the transformation matrix from $\widetilde{\delta}(rmq)$ to $\widetilde{\delta}((r+1)mq)$ can also be written as a polynomial matrix of h with degree $2mq$; let $\Gamma_r^{(i)}$ denote the constant coefficient matrix associated with h^i, $\forall\, i \in \{0, 1, \ldots, 2mq\}$. Hence, for any $r \in \{0, 1, \ldots\}$, $\widetilde{\delta}((r+1)mq) = \left(I_{2(n-1)} + \sum_{i=1}^{2q} h^i E_{(r+1)mq-1}^{(i)}\right) \times \cdots \times \left(I_{2(n-1)} + \sum_{i=1}^{2q} h^i E_{(rm+1)q-1}^{(i)}\right)\widetilde{\delta}(rmq) = \left(I_{2(n-1)} + \sum_{i=1}^{2mq} h^i \Gamma_r^{(i)}\right)\widetilde{\delta}(rmq)$. clearly

$$\begin{aligned}
\Gamma_r^{(1)} &= \sum_{k=(rm+1)q-1}^{(r+1)mq-1} E_k^{(1)} \\
&= \begin{bmatrix} 0 & m\sum_{j=0}^{q-1} m_j I_{n-1} \\ -k_1 \sum_{j=rmq}^{(r+1)mq-1} \Theta_j & -k_2 \sum_{j=rmq}^{(r+1)mq-1} \Theta_j \end{bmatrix}
\end{aligned}$$

where $\Theta_j = m_{j-f(j)q}\widehat{H}_{\sigma(j)}$, and $[\,\Gamma_r^{(1)} \ \cdots \ \Gamma_r^{(2mq)}\,]$ Is Determined Only By k_1, k_2, $m_0, \ldots, m_{q-1}$, $\mathcal{H}_{\sigma(rmq)}, \ldots, \mathcal{H}_{\sigma((r+1)mq-1)}$.

By (5), (6) and (A1), for any r, $s \in \{0, 1, \ldots\}$, $\mathcal{G}(\widehat{A}(\cdot))$ during $[t_{rmq}, t_{(r+1)mq})$ is the same as $\mathcal{G}(\widehat{A}(\cdot))$ during $[t_{smq}, t_{(s+1)mq})$, and thus, $\mathcal{H}_{\sigma(rmq)} = \mathcal{H}_{\sigma(smq)}, \ldots, \mathcal{H}_{\sigma((r+1)mq-1)} = \mathcal{H}_{\sigma((s+1)mq-1)}$, which means $[\,\Gamma_r^{(1)} \ \cdots \ \Gamma_r^{(2mq)}\,] = [\,\Gamma_s^{(1)} \ \cdots \ \Gamma_s^{(2mq)}\,]$, i.e., $I_{2(n-1)} + \sum_{i=1}^{2mq} h^i \Gamma_r^{(i)} = I_{2(n-1)} + \sum_{i=1}^{2mq} h^i \Gamma_s^{(i)}$. Let $\Gamma_i = \Gamma_r^{(i)}$, $r = 0, 1, \ldots$, $i = 1, \ldots, 2mq$, $\Gamma = i_{2(n-1)} + \sum_{i=1}^{2mq} h^i \Gamma_i$, and $\xi(r) = \widetilde{\delta}(rmq)$, Then

$$\xi(r+1) = \Gamma\xi(r), r = 0, 1, \ldots. \tag{12}$$

Obviously, the global asymptotic stability of system (9) is equivalent to the asymptotic stability of system (12). By Lemma 1, system (1) with protocol (2) solves a consensus problem if and only if system (12) is asymptotically stable or Γ is Schur stable. Note that Γ is a polynomial matrix of h with degree $2mq$ and is determined only by controller gains k_1, k_2, sampling periods $h_1, \ldots, h_n$ and the interaction topology.

Next we prove that there exist controller gains k_1, k_2 and sampling periods $h_1, \ldots, h_n$ such that system (12) is asymptotically stable. Consider any given positive integers $l_1, \ldots, l_n$. Since the union graph of $\{\mathcal{G}(\widehat{A}(t)) : \forall\, t \in [t_0, t_{mq})\}$ has a spanning tree, $\sum_{j=0}^{mq-1} m_{j-f(j)q}\widehat{L}_{\sigma(j)}$ can be viewed as the Laplacian matrix of a directed graph with a spanning tree.

Clearly

$$U^{-1}\left(\sum_{j=0}^{mq-1} m_{j-f(j)q}\widehat{L}_{\sigma(j)}\right)U = \begin{bmatrix} 0 & \sum_{j=0}^{mq-1} m_{j-f(j)q}\alpha_{\sigma(j)} \\ 0 & \sum_{j=0}^{mq-1} m_{j-f(j)q}\widehat{H}_{\sigma(j)} \end{bmatrix}.$$

Let $S = \sum_{j=0}^{mq-1} m_{j-f(j)q}\widehat{H}_{\sigma(j)}$ and $\alpha = m\sum_{j=0}^{q-1} m_j$, then $\Gamma_1 = \begin{bmatrix} 0 & \alpha I_{n-1} \\ -k_1 S & -k_2 S \end{bmatrix}$. By Lemma 3.3 in [4], $-S$ is Hurwitz stable. By analyzing the eigenvalues of Γ_1 and by Liénard-Chipart criterion, for any $k_1 > 0$, $k_2 > \max_{\lambda \in \lambda(s)} \sqrt{K_1 \alpha \mathrm{Im}(\lambda)^2 / \mathrm{Re}(\lambda) \mid \lambda \mid^2}$, there exists $Q > 0$ such that $\Gamma_1{}^T Q + Q\Gamma_1$. By calculation, $\Gamma^T Q\Gamma - Q = h\left(\Gamma_1{}^T Q + Q\Gamma_1 + hv_0(h)\right)$, where $v_0(h)$ is a polynomial matrix of h with degree $4mq - 2$ and $\Gamma_1{}^T Q + Q\Gamma_1$ is independent of h. Hence, for any given $l_i \in \mathbb{Z}^+$, $i = 1, \ldots, n$, $k_1 > 0$, $k_2 > \max_{\lambda \in \Lambda(S)} \sqrt{k_1 \alpha \mathrm{Im}(\lambda)^2 / \mathrm{Re}(\lambda) \mid \lambda \mid^2}$, if h is small enough, then $\Gamma^T Q\Gamma - Q < 0$, namely, system (12) is asymptotically stable. Hence, there exist k_1, k_2, $h_1, \ldots, h_n$ such that system (1) with protocol (2) solves a consensus problem. ■

Theorem 1 shows the existence of controller gains and sampling periods which ensure consensus. In the following, we provide methods to design such controller gains and sampling periods. By the proof of Theorem 1, it is natural to have the following result.

Corollary 1: Assume (A1) and (A2) hold. For any given controller gains k_1, k_2 and sampling periods $h_1, \ldots, h_n$, system (1) with protocol (2) solves a consensus problem if and only if there exists $P > 0$ such that $\Gamma^T P \Gamma - P < 0$.

Hence, we can find $k_1, k_2, h_1, \ldots, h_n$, which ensure consensus, by verifying the feasibility of a group of LMIs, i.e., $\Gamma^T P\Gamma - P < 0$, $P > 0$. By the proof of Theorem 1, for any given $l_i \in \mathbb{Z}^+$, $i = 1, \ldots, n$, $k_1 > 0$ and $k_2 > \max_{\lambda \in \Lambda(S)} \sqrt{k_1 \alpha \mathrm{Im}(\lambda)^2 / \mathrm{Re}(\lambda) \mid \lambda \mid^2}$, the LMIs hold if h is small enough. Furthermore, we can obtain an allowable upper bound of h by applying the linear matrix inequality technique.

Corollary 2: Assume (A1) and (A2) hold. Let $k_1 > 0$, $k_2 > \max_{\lambda \in \Lambda(S)} \sqrt{k_1 \alpha \mathrm{Im}(\lambda)^2 / \mathrm{Re}(\lambda) \mid \lambda \mid^2}$, $l_i \in \mathbb{Z}^+$, $i = 1, \ldots, n$, and $\varepsilon > 0$. System (1) with protocol (2) solves a consensus problem if $h < h^*$, where

$$h^* = \max y, s.t.\ P\Gamma_1 + \Gamma_1^T P + \sum_{k=2}^{4mq} y^{k-1} D_k < 0, y > 0, P > 0 \tag{13}$$

where $y \in \mathbb{R}$, $P \in \mathbb{R}^{2(n-1)\times 2(n-1)}$, $D_2 = (1/\varepsilon)P + \Gamma_1^T P \Gamma_1 + \varepsilon \Gamma_2^T P \Gamma_2$, $D_{4mq} = \Gamma_{2mq}^T P \Gamma_{2mq}$, and

$$D_k = \begin{cases} \frac{1}{\varepsilon}P + \frac{1}{\varepsilon}\sum_{j=1}^{k-1/2} \Gamma_j^T P \Gamma_j + \varepsilon \sum_{j=k+1/2}^{k} \Gamma_j^T P \Gamma_j, \\ k \in \{3, \ldots, 2mq\} \text{ and } k \text{ is odd} \\ \frac{1}{\varepsilon}P + \frac{1}{\varepsilon}\sum_{j=1}^{k/2-1} \Gamma_j^T P \Gamma_j + \Gamma_{k/2}^T P \Gamma_{k/2} \\ +\varepsilon \sum_{j=k/2+1}^{k} \Gamma_j^T P \Gamma_j, \\ k \in \{3, \ldots, 2mq\} \text{ and } k \text{ is even} \\ \frac{1}{\varepsilon}\sum_{j=k-2mq}^{k-1/2} \Gamma_j^T P \Gamma_j + \varepsilon \sum_{j=k+1/2}^{2mq} \Gamma_j^T P \Gamma_j, \\ k \in \{2mq+1, \ldots, 4mq-1\} \text{ and } k \text{ is odd} \\ \frac{1}{\varepsilon}\sum_{j=k-2mq}^{k/2-1} \Gamma_j^T P \Gamma_j + \Gamma_{k/2}^T P \Gamma_{k/2} \\ +\varepsilon \sum_{j=k/2+1}^{2mq} \Gamma_j^T P \Gamma_j, \\ k \in \{2mq+1, \ldots, 4mq-1\} \text{ and } k \text{ is even} \end{cases}.$$

Proof: We first show that optimization problem (13) is solvable. By the proof of Theorem 1, there exists $Q > 0$ such that $Q\Gamma_1 + \Gamma_1^T Q < 0$. Thus, the LMIs in (13) are feasible if $y > 0$ is small enough. Moreover, If the LMIs in (13) are feasible for $y = y_1 > 0$, then they are also feasible for any $y \in (0, y_1)$. Clearly, the set of all y satisfying the LMIs in (13), i.e., $\{y : y > 0 \text{ and there exists } P > 0 \text{ such that the LMIs in (13) are satisfied}\}$, is a bounded open interval in $(0, +\infty)$. Hence, (13) is solvable.

Let $\widetilde{m} = 2mq$. Consider any $h < h^*$. By the above analysis, there exists $\widetilde{P} > 0$ such that the LMIs in (13) hold for $P = \widetilde{P}$ and $y = h$. Let $\widetilde{D}_k$ denote D_k in the Case of $P = \widetilde{P}$, then

$$\widetilde{P}\Gamma_1 + \Gamma_1^T \widetilde{P} + \sum_{k=2}^{2\widetilde{m}} h^{k-1} \widetilde{D}_k < 0. \tag{14}$$

Choose the following Lyapunov function for system (12): $V(r) = \xi(r)^T \widetilde{P} \xi(r)$, $r = 0, 1, \ldots$. Then $V(r+1) - V(r) = \xi(r)^t (\Gamma^T \widetilde{P} \Gamma - \widetilde{P})\xi(r) = h\xi(r)^T \left(\widetilde{P}\Gamma_1 + \Gamma_1^T \widetilde{P} + \sum_{k=2}^{2\widetilde{m}} h^{k-1} J_k\right) \xi(r)$, where

$$J_k = \begin{cases} \widetilde{P}\Gamma_k + \Gamma_k{}^T \widetilde{P} + \sum_{i+j=k, i,j \in \{1,\ldots,\widetilde{m}\}} \Gamma_i{}^T \widetilde{P} \Gamma_j, \\ k \in \{2, \ldots, \widetilde{m}\} \\ \sum_{i+j=k, i,j \in \{1,\ldots,\widetilde{m}\}} \Gamma_i{}^T \widetilde{P} \Gamma_j, \\ k \in \{\widetilde{m}+1, \ldots, 2\widetilde{m}\} \end{cases}.$$

Clearly, $\widetilde{P}\Gamma_k + \Gamma_k^T \widetilde{P} \le (1/\varepsilon)\widetilde{P} + \varepsilon \Gamma_k^T \widetilde{P} \Gamma_k$ and $\Gamma_i^T \widetilde{P} \Gamma_j + \Gamma_j^T \widetilde{P} \Gamma_i \le (1/\varepsilon)\Gamma_j^T \widetilde{P} \Gamma_j + \varepsilon \Gamma_i^T \widetilde{P} \Gamma_i$ for $j < i$. Hence, $J_k \le \widetilde{D}_k$, $k = 2, \ldots, 2\widetilde{m}$, which mean $V(r+1) - V(r) \le h\xi(r)^T \left(\widetilde{P}\Gamma_1 + \Gamma_1^T \widetilde{P} + \sum_{k=2}^{2\widetilde{m}} h^{k-1} \widetilde{D}_k\right) \xi(r)$. By (14), $V(r+1) - V(r) < 0$, $r = 0, 1, \ldots$. Thus, system (12) is asymptotically stable, namely, system (1) with protocol (2) solves a consensus problem. ■

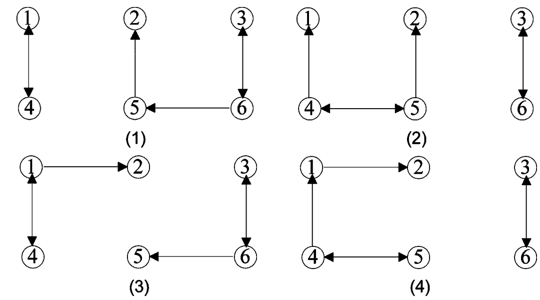

Fig. 2. Topologies.

Remark 2: By the proof of Corollary 2, h^* can be obtained by the following steps: (1) Let $\epsilon > 0$ be a tolerable error and $y_0 > 0$ be small enough such that the LMIs in (13) are feasible, where the feasibility of the LMIs can be verified by the feasp solver in Matlab's LMI Toolbox; (2) Let $y = y_0 + \epsilon$ and verify the feasibility of the LMIs in (13), if they are feasible, then replace y_0 with $y_0 + \epsilon$ and repeat step (2), or else $h^* \approx y_0$.

IV. Discussion

Although only the case of periodically time-varying topology is considered, the more general topology case, i.e., there exist nonempty, bounded, and contiguous time intervals such that the union graph of all graphs across each time interval has a spanning tree, can be analyzed similarly. In the case of periodically time-varying topology, the consensus problem is transformed into the asymptotic stability problem of a discrete-time time-invariant system; however, in the case of general topology mentioned above, the consensus problem can be transformed into the asymptotic stability problem of a discrete-time switched system. There have been many research results on the stability of switched systems (see [26]), and thus, the analysis in the general topology case is omitted due to space limitation.

V. Simulations

Consider a system with six agents. The sampling periods of agents 1, 2, 3 are $2h$, and the sampling periods of agents 4, 5, and 6 are h. The communication structures among six agents are shown as follows: (1) at sampling instants $4rh$, $r = 0, 1, \ldots$, agent 1 (respectively, 2,3) can obtain the information of relative state between it and agent 4 (respectively, 5,6); (2) at sampling instants $(4r+2)h$, $r = 0, 1, \ldots$, agent 1 (respectively, 2,3) can obtain the information of relative state between it and agent 4 (respectively, 1,6); (3) at sampling instants $2rh$, $r = 0, 1, \ldots$, agent 4 (respectively,5,6) can obtain the information of relative state between it and agent 1 (respectively, 6,3); (4) at sampling instants $(2r+1)h$, $r = 0, 1, \ldots$, agent 4 (respectively, 5,6) can obtain the information of relative state between it and agent 5 (respectively, 4,3). Assume that each edge weight is 1. Hence, for any $r \in \{0, 1, \ldots\}$, the topologies, defined by Definition 1, during time intervals $[4rh, (4r+1)h)$, $[(4r+1)h, (4r+2)h)$, $[(4r+2)h, (4r+3)h)$, and $[(4r+3)h, (4r+4)h)$ are $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$, respectively, where $\mathcal{G}_i$, $i = 1, 2, 3, 4$, are shown in Fig. 2. Obviously, none of $\mathcal{G}_i$, $i = 1, 2, 3, 4$, have spanning trees, while their union graph has spanning trees. By Theorem 1, there exist k_1, k_2, h such that consensus is reached. By calculation, for $k_1 = 1$, $k_2 = 4$, $h = 0.1$, Γ is Schur stable, which means that consensus is reached by Corollary 1. The state trajectories of six agents in the case of $k_1 = 1$, $k_2 = 4$, $h = 0.1$ are shown in Fig. 3, which validates our results.

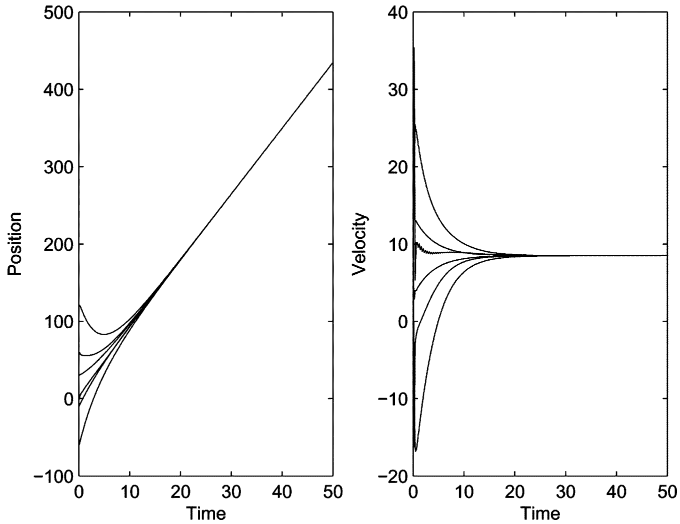

Fig. 3. State trajectories of six agents.

VI. Conclusion

This technical note has studied consensus problems of continuous second-order agents in a sampled-data setting, where the sampling period of each agent is independent of the others' and the interaction topology among agents is time-varying. In virtue of some state transformations, the consensus is shown to be equivalent to the asymptotic stability of a discrete-time time-invariant system without delays. Under the condition that the union graph of all graphs has a spanning tree, it has been proved that consensus can be reached for some controller gains and sampling periods. Furthermore, two methods have been presented to design such controller gains and sampling periods. Simulations have been provided to illustrate the effectiveness of the theoretical results.

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