Leaderless and Leader-Following Consensus With Communication and Input Delays Under a Directed Network Topology

Ziyang Meng, Wei Ren, *Member, IEEE*, Yongcan Cao, *Student Member, IEEE*, and Zheng You

Abstract—**In this paper, time-domain (Lyapunov theorems) and frequency-domain (the Nyquist stability criterion) approaches are used to study leaderless and leader-following consensus algorithms with communication and input delays under a directed network topology. We consider both the first-order and second-order cases and present stability or boundedness conditions. Several interesting phenomena are analyzed and explained. Simulation results are presented to support the theoretical results.**

Index Terms—**Communication and input delays, consensus tracking, directed network graph, leaderless consensus, multiagent system.**

I. Introduction

COOPERATIVE control of multiagent systems has received significant research attention in recent years. Compared with solo systems, additional benefits, such as high robustness and great efficiency, can be obtained by having a group of agents work cooperatively. Cooperative control has broad applications in formation control [1], flocking [2], and complex networks [3], [4]. A fundamental approach to achieve cooperative control is consensus [5]–[7]. Consensus means the agreement of a group of agents on their common states via local interaction. In a *leaderless consensus* problem, there does not exist a virtual leader, while in a *leader-following consensus* problem, there exists a virtual leader that specifies the objective for the whole group. More specifically, consensus with a static virtual leader is called a *consensus regulation* problem, and consensus with a dynamic virtual leader is called a *consensus tracking* problem. It is worthwhile to mention that the consensus tracking problem becomes much more complex if only a portion of the agents in the group has access to the virtual leader.

Since delays are inevitable in real systems, it is necessary and beneficial to study leaderless and leader-following consensus algorithms in the presence of the delays. Most existing references on consensus algorithms considered input delays. The authors in [6] first gave a leaderless consensus algorithm with input delays and then presented a frequency-domain approach to find the stability conditions. A similar leaderless consensus algorithm with uniform input delays was studied in [8], where a time-domain approach, i.e., the Lyapunov–Krasovaskii theorem, was used to obtain the stability conditions under strongly connected and balanced network topologies. Besides leaderless consensus algorithms, leader-following consensus algorithms with input delays were also studied. By combining the results in [8] and [9], the authors in [10] proposed a first-order consensus tracking algorithm with input delays, where an estimator was used to estimate the virtual leader's velocity. Due to the presence of the dynamic virtual leader and the input delays, the tracking errors were shown to be uniformly ultimately bounded instead of approaching zero. In the previous references, the network topology is assumed to be either undirected or strongly connected and balanced, which poses an obvious limitation. The extension to the case where the network topology has a directed spanning tree and the input delays are assumed to be nonuniform was provided in [11], where a frequency-domain method was used to find conditions to achieve leaderless consensus. Except for input delays, the influence of communication delays on consensus algorithms was also studied. The authors in [12] showed that communication delays will not jeopardize the stability of the first-order leaderless consensus algorithm under a directed network topology. A similar algorithm was discussed in [13], where the effect of initial conditions was highlighted. A second-order consensus regulation algorithm with nonuniform communication delays was studied in [14], but a damping term was used to regulate the velocities of all agents to zero, and the network topology was assumed to be undirected. The previous references considered either only the input delays or only the communication delays and, hence, lack completeness. The case with both the communication and input delays was studied in [15]. In particular, a first-order leaderless consensus algorithm with both the communication and input delays was studied in a discrete-time setting. However, a pure frequency-domain approach was used, thus leading the obtained stability conditions to be conservative.

This paper considers both leaderless and leader-following consensus algorithms with communication and input delays in, respectively, first-order kinematics and second-order dynamics under a directed network topology. The stability or boundedness conditions of four different cases, namely, leaderless consensus, consensus regulation, consensus tracking with full access to the virtual leader, and consensus tracking with partial

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Z. Meng and Z. You are with the Department of Precision Instruments and Mechanology, Tsinghua University, Beijing 100084, China.

W. Ren and Y. Cao are with the Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322 USA (e-mail: wei.ren@usu.edu).

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access to the virtual leader, are analyzed by using time-domain and frequency-domain approaches.

The contributions of this paper are fourfold. First, we assume a general network topology, i.e., a directed network topology with a directed spanning tree, instead of an undirected connected network topology or a directed strongly connected and balanced network topology [6], [8], [10], [16]. Second, both communication and input delays are considered in the cases of leaderless consensus, consensus regulation, and consensus tracking with full access to the virtual leader, which guarantees the completeness of the algorithms. Third, we show that the communication delay will not influence the stability of the first-order system in the case of consensus tracking with partial access to the virtual leader, which extends the results of [12] and [17]. Fourth, as a byproduct, we find that in the case of second-order leaderless consensus with both communication and input delays, the final group velocity is always dampened to zero rather than a possibly nonzero constant as compared with the standard second-order consensus algorithm studied in [18]. A preliminary version of the second-order case of the work is presented at the 2010 American Control Conference.

II. Preliminaries

A. Notations

$\mathbb{R}$ and $\mathbb{C}$ are the set of real numbers and the set of complex numbers, respectively.

$\mathbf{1}_n$ and $\mathbf{0}_n$ are the $n \times 1$ all-one vector and the $n \times 1$ all-zero vector, respectively.

I_n and $\mathbf{0}_{n\times n}$ are the $n \times n$ identity matrix and the $n \times n$ matrix with all zero entries, respectively.

$\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are, respectively, the minimal eigenvalue and the maximum eigenvalue of the matrix A.

$\|A\|$ is the norm of the matrix/vector A.

$\mathbf{j}$ is the imaginary unit.

$\Re\{\bullet\}$ and $\Im\{\bullet\}$ are, respectively, the real part and the imaginary part of a complex number.

$\rho(A)$ is the spectral radius of the matrix A.

$Q < 0$ means that the matrix Q is negative-definite.

B. Graph Theory Notions

Using graph theory, we can model the network topology in a multiagent system consisting of n agents. A directed graph $\mathbb{G}_n$ consists of a pair $(\mathbb{V}, \mathbb{E})$, where $\mathbb{V} = \{v_1, \ldots, v_n\}$ is a finite nonempty set of nodes, and $\mathbb{E} \in \mathbb{V} \times \mathbb{V}$ is a set of ordered pairs of nodes. An edge (v_i, v_j) denotes that node v_j can obtain information from node v_i, but not necessarily vice versa. All neighbors of node v_i are denoted as $N_i := \{v_j \mid (v_j, v_i) \in \mathbb{E}\}$.

A directed path is a sequence of edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots$. A directed graph has a directed spanning tree if there exists at least one node having a directed path to all other nodes.

For the leaderless consensus case, the adjacency matrix $\mathbb{A}_n = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathbb{G}_n$ is defined such that a_{ij} is positive if $(v_j, v_i) \in \mathbb{E}$, while $a_{ij} = 0$ otherwise. Here, we assume that $a_{ii} = 0$, $\forall\, i$. The (nonsymmetric) Laplacian matrix $\mathbb{L}_n = [\lessdot_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathbb{A}_n$ is defined as $\lessdot_{ii} = \sum_{j\neq i} a_{ij}$ and $\lessdot_{ij} = -a_{ij}$, where $i \neq j$.

For the leader-following case, we assume that besides agents 1 to n, there exists a virtual leader, labeled as agent $n+1$, in the system. We use $\mathbb{G}_{n+1}$ to model the network topology in this case. The adjacency matrix $\mathbb{A}_{n+1} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ associated with $\mathbb{G}_{n+1}$ is defined such that a_{ij} is positive if $(v_j, v_i) \in \mathbb{E}$, while $a_{ij} = 0$ otherwise, and $a_{(n+1)j} = 0$ for all $j = 1, \ldots, n+1$. Here, again, we assume that $a_{ii} = 0$, $\forall\, i$.

III. Definitions and Lemmas

Suppose that $f : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}^n$ is continuous and consider the retarded functional differential equation (RFDE)

$$\dot{x}(t) = f(t, x_t). \tag{1}$$

Let $\phi = x_t$ be defined as $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. Suppose that appropriate initial conditions are defined on the delay interval $[t_0 - \tau, t_0]$: $x_{t_0}(\theta) = \phi(\theta)$, $\forall\,\theta \in [-\tau, 0]$. Specifically, we assume that the initial condition satisfies $x(\theta) = 0$, $\forall\,\theta \in [t_0 - \tau, t_0]$, in this paper. Suppose that the solution $x(\sigma, \phi)(t)$ through (σ, ϕ) is continuous in (σ, ϕ, t) in the domain of definition of the function, where $\sigma \in \mathbb{R}$.

Definition 3.1 [19]: The solutions $x(\sigma, \phi)$ of the RFDE (1) are *uniformly ultimately bounded* if there is a $\beta > 0$ such that for any $\alpha > 0$, there is a constant $t_0(\alpha) > 0$ such that $|x(\sigma, \phi)(t)| \le \beta$ for $t \ge \sigma + t_0(\alpha)$ for all $\sigma \in \mathbb{R}$, $\phi \in \mathbb{C}$, $|\phi| \le \alpha$.

Suppose that $\mathcal{D} : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}^n$ is a linear operator on the second variable such that $\mathcal{D}(t, \phi) = A(t)\phi(0) - G(t, \phi)$, where $A(t)$ is a continuous nonsingular matrix, and $G(t, \phi) = \int_{-h}^{0} d\mu(t, \theta)\phi(\theta)$ satisfies $|\int_{-s^+}^{0} [d\mu(t, \theta)]\phi(\theta)| \le \gamma(s, t)|\phi|$ for $0 \le s \le h$, where μ is an $n \times n$ matrix function of bounded variation on θ, γ is continuous, and $\gamma(0, t) = 0$ for $t \ge 0$. If $g : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}^n$ is a continuous function, then the relation

$$\frac{d}{dt}\mathcal{D}(t, x_t) = g(t, x_t) \tag{2}$$

is a neutral functional differential equation (NFDE) [20].

Definition 3.2 [20]: Consider the NFDE (2). Suppose that operator $\mathcal{D}$ is stable. It defines a *uniform ultimately bounded* process if there is a $\beta > 0$ such that for any $\alpha > 0$, there is a constant $t_0(\alpha) > 0$ such that $|x(\sigma, \phi)(t)| \le \beta$ for $t \ge \sigma + t_0(\alpha)$ for all $\sigma \in \mathbb{R}$, $\phi \in \mathbb{C}$, $|\phi| \le \alpha$.

Lemma 3.1. (Degenerate Lyapunov–Krasovskii Stability Theorem) [21], [22]: Consider the NFDE (2). Suppose that operator $\mathcal{D}$ is stable, $g : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathbb{C}$) into bounded sets of $\mathbb{R}^n$, and $u(s)$, $v(s)$, and $w(s)$ are continuous, nonnegative, and nondecreasing functions with $u(s), v(s) > 0$ for $s \neq 0$ and $u(0) = v(0) = 0$. If there exists a continuous functional $V : \mathbb{R} \times \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{R}^n$, such that

1) $u(\|\mathcal{D}(t, \phi)\|) \le V(t, \mathcal{D}(t, \phi), \phi) \le v(\|\phi\|_c)$
2) $\dot{V}(t, \mathcal{D}(t, \phi), \phi) \le -w(\|\mathcal{D}(t, \phi)\|)$

then the trivial solution of (2) is *asymptotically stable*.

Lemma 3.1 will be used in the first-order and second-order leaderless consensus and consensus regulation problems.

Lemma 3.2. (Lyapunov–Razumikhin Uniformly Ultimately Bounded Theorem) [19]: Consider the RFDE (1). Suppose that $f : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathbb{C}$) into bounded sets of $\mathbb{R}^n$ and u, v, $w : \mathbb{R}^+ \mapsto \mathbb{R}^+$ are continuous nonincreasing functions, $u(s) \to \infty$ as $s \to \infty$. If there is a continuous function $V : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$, a continuous nondecreasing function $p : \mathbb{R}^+ \mapsto \mathbb{R}^+$, $p(s) > s$ for $s > 0$, and a constant $H \ge 0$ such that $u(|x|) \le V(x) \le v(|x|)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\dot{V}(t,\phi) \le -w(|\phi(0)|)$ if $|\phi(0)| \ge H$, $V(t+\theta,\phi(\theta)) < p(V(t,\phi(0)))$, $\theta \in [-\tau, 0]$, then the solutions of (1) are *uniformly ultimately bounded*.

Lemma 3.2 will be used in the first-order and second-order consensus tracking problems with full access to the virtual leader.

Lemma 3.3. (Lyapunov–Razumikhin Uniformly Ultimately Bounded Theorem for Neutral-Type Systems) [19], [20]: Consider the NFDE (2). Suppose that operator $\mathcal{D}$ is stable and $g : \mathbb{R} \times \mathbb{C} \mapsto \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathbb{C}$) into bounded sets of $\mathbb{R}^n$. If there is a continuous function $V : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$, a continuous nondecreasing function $p : \mathbb{R}^+ \mapsto \mathbb{R}^+$, $p(s) > s$ for $s > 0$ such that $u(|x|) \le V(x) \le v(|x|)$, $\forall x \in \mathbb{R}^n$, and $\dot{V}(\mathcal{D}(t,\phi)) \le -w(|\mathcal{D}(t,\phi)|)$ for all functions ϕ if $|\mathcal{D}(t,\phi)| \ge H$ and $V(\phi(\theta)) < p(V(\mathcal{D}(t,\phi)))$, $\theta \in [-\tau, 0]$, where $w(s)$ is a continuous positive function for $s \ge KH$, then the solution of (2) is *uniformly ultimately bounded*.

Lemma 3.3 will be used in the first-order and second-order consensus tracking problems with partial access to the virtual leader.

IV. First-Order Case With Communication and Input Delays Under a Directed Network Topology

Here, we model a group of agents with single-integrator kinematics as

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, 2, \ldots, n \tag{3}$$

where x_i and u_i are, respectively, the state and the control input of the ith agent.

A. First-Order Leaderless Consensus

Consider the following leaderless consensus algorithm with both communication and input delays:

$$u_i(t) = -\frac{1}{\sum_{j=1}^n a_{ij}} \sum_{j=1}^n a_{ij} \left[(x_i(t-\tau_1) - x_j(t-\tau_1-\tau_2)) \right], \quad i = 1, \ldots, n, \tag{4}$$

where τ_1 and τ_2 are the input and communication delays, respectively, and a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n$, is the (i, j) entry of the adjacency matrix $\mathbb{A}_n$. Here, we assume that every agent has a neighbor, which implies that $\sum_{j=1}^n a_{ij} \ne 0, \forall i$. To achieve consensus, that is, $x_i(t) \to x_j(t)$, as $t \to \infty$, the conditions on the input delay τ_1 and the communication delay τ_2 to guarantee the stability or the ultimately uniform boundedness of the closed-loop system should be addressed. Using (4), (3) can be written in the matrix form as

$$\dot{x}(t) = -x(t-\tau_1) + Ax(t-\tau_1-\tau_2) \tag{5}$$

where $x = [x_1, \ldots, x_n]^T$, and $A = [\widehat{a}_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $\widehat{a}_{ij} = a_{ij}/\sum_{j=1}^n a_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, n$. Define $L = I_n - A$. When $\mathbb{G}_n$ has a directed spanning tree, L has a simple zero eigenvalue, and all other eigenvalues are on the open right half-plane [7], [23]. The following singular vector decomposition is valid:

$$W^{-1}LW = \begin{bmatrix} \widetilde{L} & 0_{n-1} \\ 0_{n-1}^T & 0 \end{bmatrix}.$$

Here, among the infinite options of W, we choose the one that the last column of W is the vector $\mathbf{1}_n$. Note that, here, all the eigenvalues of $\widetilde{L}$ are on the open right half-plane. Before moving on, we need the following lemma.

Lemma 4.1 [10]: For any $a, b \in \mathbb{R}^n$ and any symmetric positive-definite matrix $\Phi \in \mathbb{R}^{n\times n}$, $2a^T b \le a^T \Phi^{-1} a + b^T \Phi b$.

Define $\widetilde{x} \triangleq W^{-1}x$. Denote $\widetilde{x}_{n-1}$ as the first $n-1$ rows of $\widetilde{x}$ and $\widetilde{x}_2$ as the last row of $\widetilde{x}$. Note that $A = I_n - L$. By multiplying W^{-1} on both sides of (5), it follows that (5) can be rewritten as

$$\begin{bmatrix} \dot{\widetilde{x}}_{n-1}(t) \\ \dot{\widetilde{x}}_2(t) \end{bmatrix} = -\begin{bmatrix} I_{n-1} & 0_{n-1} \\ 0_{n-1}^T & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_{n-1}(t-\tau_1) \\ \widetilde{x}_2(t-\tau_1) \end{bmatrix} + \begin{bmatrix} \widetilde{A} & 0_{n-1} \\ 0_{n-1}^T & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_{n-1}(t-\tau_1-\tau_2) \\ \widetilde{x}_2(t-\tau_1-\tau_2) \end{bmatrix}$$

where $\widetilde{A} = I_{n-1} - \widetilde{L}$. Equation (5) can be decoupled into the following two equations:

$$\dot{\widetilde{x}}_{n-1}(t) = -\widetilde{x}_{n-1}(t-\tau_1) + \widetilde{A}\widetilde{x}_{n-1}(t-\tau_1-\tau_2) \tag{6a}$$

$$\dot{\widetilde{x}}_2(t) = -\widetilde{x}_2(t-\tau_1) + \widetilde{x}_2(t-\tau_1-\tau_2). \tag{6b}$$

Theorem 4.1: If the fixed directed graph $\mathbb{G}_n$ has a directed spanning tree and every agent has a neighbor, there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that the following three conditions[1] are satisfied.

1) $2\overline{\tau_1} + \overline{\tau_2} < 1$.
2) $1 - ((1 - e^{-s\overline{\tau_1}})/s) + \lambda_i(\widetilde{A})((1 - e^{-s(\overline{\tau_1}+\overline{\tau_2})})/s) \ne 0$, for all $s \in C^+$.
3) $Q_{fc} = (-I_{n-1} + \widetilde{A})^T P_{fc} + P_{fc}(-I_{n-1} + \widetilde{A}) + \overline{\tau_1} S_{fc} + (\overline{\tau_1} + \overline{\tau_2}) H_{fc} + \overline{\tau_1}[(-I_{n-1} + \widetilde{A})^T P_{fc} S_{fc}^{-1} P_{fc}(-I_{n-1} + \widetilde{A})] + (\overline{\tau_1} + \overline{\tau_2})[(-I_{n-1} + \widetilde{A})^T P_{fc} \widetilde{A} H_{fc}^{-1} \widetilde{A}^T P_{fc}(-I_{n-1} + \widetilde{A})] < 0$, where P_{fc} is a symmetric positive-definite matrix chosen properly such that $(-I_{n-1} + \widetilde{A})^T P_{fc} + P_{fc}(-I_{n-1} + \widetilde{A}) < 0$, and S_{fc} and H_{fc} are arbitrary symmetric positive-definite matrices.

In addition, if the above conditions are satisfied, $\tau_1 \in [0, \overline{\tau_1}]$, and $\tau_2 \in [0, \overline{\tau_2}]$, system (3) using (4) reaches the consensus equilibrium $(p^T x(0)/(1 + \tau_2))\mathbf{1}_n$ asymptotically, where $p \in \mathbb{R}^n$ is a nonnegative left eigenvector of L associated with the zero eigenvalue satisfying $p^T \mathbf{1}_n = 1$.

Proof: We first prove that the stability of system (3) using (4) is guaranteed if the three conditions in Theorem 4.1 are satisfied. Then, we show that these three conditions are, indeed, satisfied if $\mathbb{G}_n$ has a directed spanning tree, and every agent has a neighbor. At last, the consensus equilibrium is explicitly presented by using the final value theorem.

We know that the stability of the following system:

$$\frac{d}{dt}\left(\widetilde{x}_{n-1}(t) - \int_{-\tau_1}^{0} \widetilde{x}_{n-1}(t+\theta)\, d\theta + \widetilde{A} \int_{-\tau_1-\tau_2}^{0} \widetilde{x}_{n-1}(t+\theta)\, d\theta \right) = -(I_{n-1} - \widetilde{A})\widetilde{x}_{n-1}(t) \tag{7}$$

[1]Note here that the three conditions are used to obtain the upper bounds $\overline{\tau_1}$ and $\overline{\tau_2}$ for allowable delays.

implies the stability of system (6a) if condition 2 in Theorem 4.1 is satisfied [22]. Consider a Lyapunov function candidate

$$\begin{aligned} V\left(\widetilde{x}_{(n-1)t}\right) = &\left[\widetilde{x}_{n-1}(t) - \int_{-\tau_1}^{0} \widetilde{x}_{n-1}(t+\theta)\,d\theta \right.\\ &\left. + \widetilde{A}\int_{-\tau_1-\tau_2}^{0} \widetilde{x}_{n-1}(t+\theta)\,d\theta\right]^T \\ &\times P_{fc}\left[\widetilde{x}_{n-1}(t) - \int_{-\tau_1}^{0} \widetilde{x}_{n-1}(t+\theta)\,d\theta\right.\\ &\left. + \widetilde{A}\int_{-\tau_1-\tau_2}^{0} \widetilde{x}_{n-1}(t+\theta)\,d\theta\right] \\ &+ \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \widetilde{x}_{n-1}(\xi)^T S_{fc}\widetilde{x}_{n-1}(\xi)\,d\xi\,d\theta \\ &+ \int_{-\tau_1-\tau_2}^{0}\int_{t+\theta}^{t} \widetilde{x}_{n-1}(\xi)^T H_{fc}\widetilde{x}_{n-1}(\xi)\,d\xi\,d\theta. \end{aligned}$$

Taking the derivative of V along (7) gives

$$\dot{V}\left(\widetilde{x}_{(n-1)t}\right) \le \widetilde{x}_{n-1}(t)^T Q_{fc}\widetilde{x}_{n-1}(t)$$

where Q_{fc} is defined as in Theorem 4.1, and we have used Lemma 4.1 to derive the inequality. Note that $Q_{fc} < 0$ satisfies condition 2 in Lemma 3.1. Also, note that $\alpha_1\|\mathcal{D}(\widetilde{x}_{(n-1)t})\| \le V(\widetilde{x}_{(n-1)t}) \le \alpha_2\|\widetilde{x}_{(n-1)t}\|_c$ [24], where $\mathcal{D}(\widetilde{x}_{(n-1)t}) = \widetilde{x}_{n-1}(t) - \int_{-\tau_1}^{0}\widetilde{x}_{n-1}(t+\theta)\,d\theta + \widetilde{A}\int_{-\tau_1-\tau_2}^{0}\widetilde{x}_{n-1}(t+\theta)\,d\theta$, $\|\widetilde{x}_{(n-1)t}\|_c = \sup_{\theta\in[-\tau_1-\tau_2,0]}\|\widetilde{x}_{(n-1)}(t+\theta)\|$, $\alpha_1 = \lambda_{\min}(P_{fc})$, and $\alpha_2 = \lambda_{\max}(P_{fc}) + \tau_1\lambda_{\max}(S_{fc}) + (\tau_1+\tau_2)\lambda_{\max}(H_{fc})$. This satisfies condition 1 in Lemma 3.1. Therefore, if conditions 2 and 3 in Theorem 5.1 are satisfied, the asymptotical stability of system (6a) is guaranteed by using Lemma 3.1.

For system (6b), we apply the Nyquist stability criterion to find its stability condition. After Laplace transformation, system (6b) can be written as

$$s\widetilde{x}_2(s) = -e^{-\tau_1 s}\widetilde{x}_2(s) + e^{-(\tau_1+\tau_2)s}\widetilde{x}_2(s).$$

Thus, the stability is determined by the roots' distribution of the following:

$$s = -e^{-\tau_1 s} + e^{-(\tau_1+\tau_2)s}. \tag{8}$$

Define $f(s) \triangleq (e^{-\tau_1 s} - e^{-(\tau_1+\tau_2)s})/s$. Based on the Nyquist stability criterion, if the trajectory of $f(j\omega)$, $\forall\,\omega \in (-\infty,\infty)$, does not enclose the point $(-1, \mathbf{j}0)$, then (8) is stable. One sufficient condition is that $\Re\{f(j\omega)\} > -1$, $\forall\,\omega\in(-\infty,\infty)$. Noting that $\Re\{f(j\omega)\} = (-\sin(\tau_1+\tau_2)\omega/\omega) + (\sin\tau_1\omega/\omega)$ and functions $(-\sin(\tau_1+\tau_2)\omega/\omega)$ and $(\sin\tau_1\omega/\omega)$ have minimum values, respectively, $-(\tau_1+\tau_2)$ and $-\tau_1$ with respect to $\forall\,\omega\in(-\infty,\infty)$, we have that $\Re\{f(j\omega)\} \ge -(2\tau_1+\tau_2)$. Therefore, it is easy to verify that the stability of system (6b) is guaranteed if condition 1 in Theorem 4.1 is satisfied.

Next, we show that these three conditions in Theorem 4.1 are, indeed, satisfied if $\mathbb{G}_n$ has a directed spanning tree, and every agent has a neighbor. It is straightforward to see that there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that conditions 1 and 2 are satisfied. For condition 3, we know that $\widetilde{L} = I_{n-1} - \widetilde{A}$ has all eigenvalues on the open right half-plane. Therefore, when $\tau_1 = \tau_2 = 0$, there always exists a P_{fc} to guarantee that $(-I_{n-1}+\widetilde{A})^T P_{fc} + P_{fc}(-I_{n-1}+\widetilde{A}) < 0$. Thus, based on the continuity, there must exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that $Q_{fc} < 0$ when $\tau_1 \in [0,\overline{\tau_1}]$ and $\tau_2\in[0,\overline{\tau_2}]$.

Finally, for the consensus equilibrium, we have that

$$\lim_{t\to\infty}\widetilde{x}_2(t) = \lim_{s\to 0}\frac{s\widetilde{x}_2(0)}{s + e^{-\tau_1 s} - e^{-(\tau_1+\tau_2)s}} = \frac{\widetilde{x}_2(0)}{1+\tau_2}$$

and $\widetilde{x}_{n-1}(t)\to 0$, as $t\to\infty$. It follows that the consensus equilibrium is given by $p^T x(0)/(1+\tau_2)\mathbf{1}_n$. ■

Remark 4.1: We know that the additional dynamics caused by the model transformation from (6a) to (7) can be characterized by the solutions of the following complex equation [25]:

$$\det\left(I_{n-1} - I_{n-1}\frac{1-e^{s\tau_1}}{s} + \widetilde{A}\frac{1-e^{s(\tau_1+\tau_2)}}{s}\right) = 0, \quad s\in\mathbb{C}.$$

Thus, if $\overline{\tau_1} + (\overline{\tau_1}+\overline{\tau_2})\|\widetilde{A}\| < 1$, there are no additional eigenvalues induced by the model transformation from (6a) to (7), which implies that the condition $\overline{\tau_1} + (\overline{\tau_1}+\overline{\tau_2})\|\widetilde{A}\| < 1$ can be used to replace condition 2 in Theorem 4.1.

Remark 4.2: If we let $S_{fc} = H_{fc} = I_{n-1}$, condition 3 in Theorem 4.1 can be written as

$$\overline{\tau_1}+\overline{\tau_2} < \frac{\lambda_{\min}\left[(I_{n-1}-\widetilde{A})^T P_{fc} + P_{fc}(I_{n-1}-\widetilde{A})\right]}{2 + \left\|(-I_{n-1}+\widetilde{A})^T P_{fc}\right\|^2 + \left\|(-I_{n-1}+\widetilde{A})^T P_{fc}\widetilde{A}\right\|^2}.$$

Remark 4.3: Note that, in Theorem 4.1, it is assumed that the fixed directed graph has a directed spanning tree, and every agent has a neighbor. Thus, the conclusion can be viewed as a generalization of [8], [10], and [16], where the directed graphs are assumed to be strongly connected and balanced.

Remark 4.4: For first-order leaderless consensus, the case of a general network topology that has a directed spanning tree was also considered in [11]. However, only input delays were considered. The extension to the case where there exist both communication and input delays was studied in [15]. A discrete-time setting was assumed, and a pure frequency-domain approach was used. In contrast, we here introduce both time-domain and frequency-domain approaches in a continuous-time setting.

B. First-Order Consensus Regulation

Here, we assume that there exists a virtual leader, labeled as agent $n+1$, whose state is a constant reference state x_d. The consensus regulation algorithm with both communication and input delays is proposed as

$$u_i = -\frac{1}{\sum_{j=1}^{n+1} a_{ij}}\sum_{j=1}^{n+1} a_{ij}\left[x_i(t-\tau_1) - x_j(t-\tau_1-\tau_2)\right], \qquad i = 1,\ldots,n \tag{9}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, a_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,n+1$, is the (i,j) entry of the adjacency matrix $\mathbb{A}_{n+1}$, and $x_{(n+1)} \equiv x_d$. Note that the condition that $\mathbb{G}_{n+1}$ has a directed spanning tree and the fact that all entries of the last row of $\mathbb{A}_{n+1}$ are zero imply that no other rows of $\mathbb{A}_{n+1}$ have all zero entries. It thus follows that $\sum_{j=1}^{n+1} a_{ij} \neq 0$, $i = 1, 2, \ldots, n$ [26]. The objective of (9) is to guarantee accurate regulation, i.e., $x_i(t) \to x_d$ as $t \to \infty$. Denote $\overline{x}_i = x_i - x_d$ and $\overline{x} = [\overline{x}_1, \ldots, \overline{x}_n]^T$. Define $\mathcal{A} = [\bar{a}_{ij}] \in \mathbb{R}^{n \times n}$ as $\bar{a}_{ij} = a_{ij} / \sum_{j=1}^{n+1} a_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, n$. Using (9), (3) can be written in the matrix form as

$$\dot{\overline{x}} = -\overline{x}(t - \tau_1) + \mathcal{A}\overline{x}(t - \tau_1 - \tau_2) \tag{10}$$

where we have used the fact that x_d is a constant. Before moving on, we need the following lemma regarding $(I_n - \mathcal{A})$.

Lemma 4.2 [27]: The real parts of all eigenvalues of $(I_n - \mathcal{A})$ are positive if the fixed directed graph $\mathbb{G}_{n+1}$ has a directed spanning tree.

Theorem 4.2: If the fixed directed graph $\mathbb{G}_{n+1}$ has a directed spanning tree, there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that the following two conditions are satisfied.

1) $1 - ((1 - e^{-s\overline{\tau_1}})/s) + \lambda_i(\mathcal{A})((1 - e^{-s(\overline{\tau_1}+\overline{\tau_2})})/s) \neq 0$, $\forall\, s \in C^+$.
2) $Q_{fr} = (-I_n + \mathcal{A})^T P_{fr} + P_{fr}(-I_n + \mathcal{A}) + \overline{\tau_1} S_{fr} + (\overline{\tau_1} + \overline{\tau_2}) H_{fr} + \overline{\tau_1}[(-I_n + \mathcal{A})^T P_{fr} S_{fr}^{-1} P_{fr}(-I_n + \mathcal{A})] + (\overline{\tau_1} + \overline{\tau_2})[(-I_n + \mathcal{A})^T P_{fr} \mathcal{A} H_{fr}^{-1} \mathcal{A}^T P_{fr}(-I_n + \mathcal{A})] < 0$, where P_{fr} is a symmetric positive-definite matrix chosen properly such that $(-I_n + \mathcal{A})^T P_{fr} + P_{fr}(-I_n + \mathcal{A}) < 0$, and S_{fr} and H_{fr} are arbitrary symmetric positive-definite matrices.

In addition, if the above conditions are satisfied, $\tau_1 \in [0, \overline{\tau_1}]$, and $\tau_2 \in [0, \overline{\tau_2}]$, system (3) using (9) guarantees $x_i(t) \to x_d$, $\forall\, i = 1, \ldots, n$, asymptotically as $t \to \infty$.

Proof: Similar to the analysis in Section IV-A, the stability of the following system:

$$\frac{d}{dt}\left(\overline{x}(t) - \int_{-\tau_1}^{0} \overline{x}(t+\theta)\, d\theta + \mathcal{A} \int_{-\tau_1-\tau_2}^{0} \overline{x}(t+\theta)\, d\theta\right) = -(I_n - \mathcal{A})\overline{x}(t) \tag{11}$$

implies the stability of system (10) if condition 1 in Theorem 4.2 is satisfied.

Consider a Lyapunov function candidate

$$\begin{aligned} V(\overline{x}_t) = &\left[\overline{x}(t) - \int_{-\tau_1}^{0} \overline{x}(t+\theta)\, d\theta + \mathcal{A} \int_{-\tau_1-\tau_2}^{0} \overline{x}(t+\theta)\, d\theta\right]^T \\ &\times P_{fr}\left[\overline{x}(t) - \int_{-\tau_1}^{0} \overline{x}(t+\theta)\, d\theta + \mathcal{A} \int_{-\tau_1-\tau_2}^{0} \overline{x}(t+\theta)\, d\theta\right] \\ &+ \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \overline{x}(\xi)^T S_{fr} \overline{x}(\xi)\, d\xi\, d\theta \\ &+ \int_{-\tau_1-\tau_2}^{0} \int_{t+\theta}^{t} \overline{x}(\xi)^T H_{fr} \overline{x}(\xi)\, d\xi\, d\theta. \end{aligned}$$

Taking the derivative of V along (11) gives

$$\dot{V}(\overline{x}_t) \leq \overline{x}(t)^T Q_{fr} \overline{x}(t)$$

where Q_{fr} is defined as in Theorem 4.2, and we have used Lemma 4.1 to derive the inequality. Thus, if the two conditions in Theorem 4.2 are satisfied, the stability of (10) can be guaranteed by using Lemma 3.1. In addition, it is straightforward to see that there exist $\overline{\tau}_1$ and $\overline{\tau}_2$ such that condition 1 is satisfied. For condition 2, we know that there also exist $\overline{\tau}_1$ and $\overline{\tau}_2$ such that $Q_{fr} < 0$ by following a similar analysis to that in Section IV-A since $I_n - \mathcal{A}$ has all eigenvalues with positive real parts if $\mathbb{G}_{n+1}$ has a directed spanning tree (Lemma 4.2). ■

Remark 4.5: Although the approaches used in the leaderless consensus case and the consensus regulation case are similar, the control goals of these two cases are different. For the leaderless consensus case, the final states of each agent are determined by the network topology, the control gains, and the time delays rather than being prespecified. However, for the consensus regulation case, there exists a virtual leader that determines the final state, and the control objective is to guarantee that the final states of all agents approach the state of the virtual leader. Plus, the result of the case of consensus regulation can be generalized to general weights, while the case of leaderless consensus requires special weights. Also, note that the remarks given in Remarks 4.1–4.3 are still valid in the consensus regulation case.

Remark 4.6: Using the similar model and analysis provided in [28], the results in this subsection can be extended to the case of multiple (nonuniform) delays.

C. First-Order Consensus Tracking With Full Access to the Virtual Leader

Here, we consider the case where the reference state x_d is time varying. Here, we assume that all agents have access to $\dot{x}_d$. The consensus tracking algorithm with both communication and input delays is proposed as

$$\begin{aligned} u_i = \dot{x}_d(t - \tau_1 - \tau_2) - \frac{1}{\sum_{j=1}^{n+1} a_{ij}} \sum_{j=1}^{n+1} a_{ij} \, [x_i(t - \tau_1) \\ - x_j(t - \tau_1 - \tau_2)], \qquad i = 1, \ldots, n \end{aligned} \tag{12}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$, is the (i, j) entry of the adjacency matrix $\mathbb{A}_{n+1}$, and $x_{n+1} \equiv x_d$. Using (12), (3) can be written in the matrix form as

$$\dot{\overline{x}} = -\overline{x}(t - \tau_1) + \mathcal{A}\overline{x}(t - \tau_1 - \tau_2) + R_{ft} \tag{13}$$

where $\mathcal{A}$ and $\overline{x}$ are defined as in Section IV-B, and $R_{ft} = \mathbf{1}_n[\dot{x}_d\,(t-\tau_1-\tau_2) - \dot{x}_d\,(t)] - \mathbf{1}_n[x_d\,(t-\tau_1) - x_d\,(t-\tau_1-\tau_2)] = -\mathbf{1}_n \int_{-\tau_1-\tau_2}^{0} \ddot{x}_d(t+\theta)\, d\theta - \mathbf{1}_n \int_{-\tau_2}^{0} \dot{x}_d(t+\theta)\, d\theta$ by using the Leibniz–Newton formula [19]. Here, we also assume that $|\dot{x}_d| < \delta_v$, $|\ddot{x}_d| < \delta_a$, where δ_v and δ_a are two positive constants.

Theorem 4.3: If the fixed directed graph $\mathbb{G}_{n+1}$ has a directed spanning tree, there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that $Q_{ft} = (-I_n + \mathcal{A})^T P_{fr} + P_{fr}(-I_n + \mathcal{A}) + \overline{\tau_1}(P_{fr} + P_{fr}\mathcal{A}P_{fr}^{-1}\mathcal{A}^T P_{fr} + 2q_f P_{fr}) + (\overline{\tau_1} + \overline{\tau_2})(P_{fr}\mathcal{A}P_{fr}^{-1}\mathcal{A}^T P_{fr} + P_{fr}\mathcal{A}\mathcal{A}P_{fr}^{-1}\mathcal{A}^T\mathcal{A}^T P_{fr} + 2q_f P_{fr}) < 0$, where P_{fr} is the same matrix given in Theorem 4.2 and $q_f > 1$. In addition, if $Q_{ft} < 0$, $\tau_1 \in [0, \overline{\tau_1}]$, and $\tau_2 \in [0, \overline{\tau_2}]$, system (3) using (12) guarantees that all $x_i - x_d$ are uniformly ultimately bounded. In particular, the ultimate bound of $\overline{x}$ is given by $\lambda_{\max}(P_{fr}) a_f / \lambda_{\min}(P_{fr}) \kappa_f \lambda_{\min}(-Q_{ft})$, where $a_f = 2[(\tau_1 + \tau_2)\delta_a + \tau_2\delta_v][\|P_{fr}\| + \tau_1\|P_{fr}\| + (\tau_1 + \tau_2)\|P_{fr}\mathcal{A}\|]$ and $0 < \kappa_f < 1$.

Proof: Using the Leibniz–Newton formula [19], we transform (13) to the following system:

$$\begin{aligned}\frac{d}{dt}\overline{x}(t) =& -(I_n-\mathcal{A})\overline{x}(t)+\int_{-\tau_1}^{0}\dot{\overline{x}}(t+\theta)\,d\theta\\ &-\mathcal{A}\int_{-\tau_1-\tau_2}^{0}\dot{\overline{x}}(t+\theta)\,d\theta+R_{ft}\\ =& -(I_n-\mathcal{A})\overline{x}(t)\\ &+\int_{-\tau_1}^{0}\left[-\overline{x}(t-\tau_1+\theta)+\mathcal{A}\overline{x}(t-\tau_1-\tau_2+\theta)\right]d\theta\\ &+\int_{-\tau_1}^{0}R_{ft}(t+\theta)\,d\theta\\ &-\mathcal{A}\int_{-\tau_1-\tau_2}^{0}\left[-\overline{x}(t-\tau_1+\theta)+\mathcal{A}\overline{x}(t-\tau_1-\tau_2+\theta)\right]d\theta\\ &-\mathcal{A}\int_{-\tau_1-\tau_2}^{0}R_{ft}(t+\theta)\,d\theta+R_{ft}\\ =& -(I_n-\mathcal{A})\overline{x}(t)-\int_{-2\tau_1}^{-\tau_1}\overline{x}(t+\theta)\,d\theta\\ &+\mathcal{A}\int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2}\overline{x}(t+\theta)\,d\theta+\int_{-\tau_1}^{0}R_{ft}(t+\theta)\,d\theta\\ &+\mathcal{A}\int_{-2\tau_1-\tau_2}^{-\tau_1}\overline{x}(t+\theta)\,d\theta-\mathcal{A}^2\int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2}\overline{x}(t+\theta)\,d\theta\\ &-\mathcal{A}\int_{-\tau_1-\tau_2}^{0}R_{ft}(t+\theta)\,d\theta+R_{ft}.\end{aligned}$$

Consider a Lyapunov function candidate $V(\overline{x})=\overline{x}^TP_{fr}\overline{x}$. Taking the derivative of $V(\overline{x})$ along (13) gives

$$\begin{aligned}\dot{V}(\overline{x})\le& \,\overline{x}^T\left[-(I_n-\mathcal{A})^TP_{fr}-P_{fr}(I_n-\mathcal{A})\right]\overline{x}\\ &+\tau_1\overline{x}^TP_{fr}P_{fr}^{-1}P_{fr}\overline{x}+\int_{-2\tau_1}^{-\tau_1}\overline{x}^T(t+\theta)P_{fr}\overline{x}(t+\theta)\,d\theta\\ &+\tau_1\overline{x}^TP_{fr}\mathcal{A}P_{fr}^{-1}\mathcal{A}^TP_{fr}\overline{x}\\ &+\int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2}\overline{x}^T(t+\theta)P_{fr}\overline{x}(t+\theta)\,d\theta\\ &+2\|\overline{x}\|\,\|P_{fr}\|\left[\tau_1(\tau_1+\tau_2)\delta_a+\tau_1\tau_2\delta_v\right]\\ &+(\tau_1+\tau_2)\overline{x}^TP_{fr}\mathcal{A}P_{fr}^{-1}\mathcal{A}^TP_{fr}\overline{x}\\ &+\int_{-2\tau_1-\tau_2}^{-\tau_1}\overline{x}^T(t+\theta)P_{fr}\overline{x}(t+\theta)\,d\theta\\ &+(\tau_1+\tau_2)\overline{x}^TP_{fr}\mathcal{A}\mathcal{A}P_{fr}^{-1}\mathcal{A}^T\mathcal{A}^TP_{fr}\overline{x}\\ &+\int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2}\overline{x}^T(t+\theta)P_{fr}\overline{x}(t+\theta)\,d\theta\\ &+2\|\overline{x}\|\,\|P_{fr}\mathcal{A}\|[(\tau_1+\tau_2)(\tau_1+\tau_2)\delta_a+(\tau_1+\tau_2)\tau_2\delta_v]\\ &+2\|\overline{x}\|\,\|P_{fr}\|[(\tau_1+\tau_2)\delta_a+\tau_2\delta_v]\end{aligned}$$

where we have used Lemma 4.1 and the facts that $|\dot{x}_d|<\delta_v$ and $|\ddot{x}_d|<\delta_a$ to derive the inequality. Take $p(s)=q_fs$ for some constant $q_f>1$. If $V(\overline{x}(t+\theta))<q_fV(\overline{x}(t))$, for $-2\tau_1-2\tau_2\le\theta\le 0$, we have

$$\begin{aligned}\dot{V}(\overline{x})\le& \,\overline{x}^T\left[-(I_n-\mathcal{A})^TP_{fr}-P_{fr}(I_n-\mathcal{A})\right]\overline{x}\\ &+\tau_1\overline{x}^T(P_{fr}+q_fP_{fr})\overline{x}\\ &+\tau_1\overline{x}^T\left(P_{fr}\mathcal{A}P_{fr}^{-1}\mathcal{A}^TP_{fr}+q_fP_{fr}\right)\overline{x}\\ &+(\tau_1+\tau_2)\overline{x}^T\left(P_{fr}\mathcal{A}P_{fr}^{-1}\mathcal{A}^TP_{fr}+q_fP_{fr}\right)\overline{x}\\ &+(\tau_1+\tau_2)\overline{x}^T\left(P_{fr}\mathcal{A}\mathcal{A}P_{fr}^{-1}\mathcal{A}^T\mathcal{A}^TP_{fr}+q_fP_{fr}\right)\overline{x}\\ &+2\|\overline{x}\|\,\|P_{fr}\|[\tau_1(\tau_1+\tau_2)\delta_a+\tau_1\tau_2\delta_v]\\ &+2\|\overline{x}\|\,\|P_{fr}\mathcal{A}\|[(\tau_1+\tau_2)(\tau_1+\tau_2)\delta_a+(\tau_1+\tau_2)\tau_2\delta_v]\\ &+2\|\overline{x}\|\,\|P_{fr}\|[(\tau_1+\tau_2)\delta_a+\tau_2\delta_v]\\ \le& \,\overline{x}(t)^TQ_{ft}\overline{x}(t)+a_f\|\overline{x}\|\end{aligned}$$

where Q_{ft} and a_f are defined as in Theorem 4.3. Because $I_n-\mathcal{A}$ has all eigenvalues on the open right half-plane (Lemma 4.2), there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that $Q_{ft}<0$ if P_{fr} is chosen such that $(-I_n+\mathcal{A})^TP_{fr}+P_{fr}(-I_n+\mathcal{A})<0$. Moreover, we have that $\lambda_{\min}(-Q_{ft})>0$. For $0<\kappa_f<1$, if $\|\overline{x}\|\ge(a_f/\kappa_f\lambda_{\min}(-Q_{ft}))$, we can obtain

$$\begin{aligned}\dot{V}(\overline{x})\le& -(1-\kappa_f)\lambda_{\min}(-Q_{ft})\|\overline{x}\|^2\\ &-\kappa_f\lambda_{\min}(-Q_{ft})\|\overline{x}\|^2+a_f\|\overline{x}\|\\ \le& -(1-\kappa_f)\lambda_{\min}(-Q_{ft})\|\overline{x}\|^2.\end{aligned}$$

Therefore, the uniformly ultimate boundedness of $\overline{x}$ follows from Lemma 3.2. Moreover, the ultimate bound is given by $\lambda_{\max}(P_{fr})a_f/\lambda_{\min}(P_{fr})\kappa_f\lambda_{\min}(-Q_{ft})$ by following a similar analysis to that in [29, pp. 172–174]. ■

Remark 4.7: Note that if $\tau_1=\tau_2=0$, $\lim_{t\to\infty}\|\overline{x}\|=0$. Also, note that when τ_1 and τ_2 are larger, the bound will be larger.

D. First-Order Consensus Tracking With Partial Access to the Virtual Leader

Here, we assume that the time-varying reference states x_d and $\dot{x}_d$ are available to only a portion of all agents and are bounded. We also assume that there exists only the communication delay. Enlightened by [17], we propose the following consensus tracking algorithm with the communication delay:

$$u_i=\frac{1}{\sum_{j=1}^{n+1}a_{ij}}\sum_{j=1}^{n+1}a_{ij}\left\{\dot{x}_j(t-\tau_2)-[x_i(t)-x_j(t-\tau_2)]\right\},\quad i=1,\ldots,n \tag{14}$$

where τ_2 is the communication delay, a_{ij}, $i=1,\ldots,n$, $j=1,\ldots,n+1$, is the (i,j) entry of the adjacency matrix $\mathbb{A}_{n+1}$, $x_{n+1}\equiv x_d$, and $\dot{x}_{n+1}\equiv\dot{x}_d$. Using (14), (3) can be written in the matrix form as

$$\dot{\overline{x}}=\mathcal{A}\dot{\overline{x}}(t-\tau_2)-\overline{x}(t)+\mathcal{A}\overline{x}(t-\tau_2)+R_{fft} \tag{15}$$

where $\mathcal{A}$ and $\overline{x}$ are defined as in Section IV-B, and $R_{fft}=[\dot{x}_d(t-\tau_2)-\dot{x}_d(t)]\mathbf{1}_n-[x_d(t)-x_d(t-\tau_2)]\mathbf{1}_n$.

Theorem 4.4: If the fixed directed graph $\mathbb{G}_{n+1}$ has a directed spanning tree, system (3) using (14) guarantees that all $x_i - x_d$ are uniformly ultimately bounded no matter how large the communication delay is.

Proof: The proof follows from Lemma 3.3. First, it is easy to verify that $\rho(\mathcal{A}) < 1$ based on the same analysis as that in [27], which means that the neutral operator $\mathcal{D}\overline{x}_t = \overline{x} - \mathcal{A}\overline{x}(t-\tau_2)$ is stable. Consider a Lyapunov function candidate $V(\overline{x}) = \overline{x}^T\overline{x}$. It is easy to show that $V(\overline{x})$ is positive definite. Taking the derivative of $V(\overline{x})$ along (15) gives

$$\begin{aligned}\dot{V}(\mathcal{D}\overline{x}_t) &= (\mathcal{D}\overline{x}_t)^T\left[-\overline{x}(t) + \mathcal{A}\overline{x}(t-\tau_2) + R_{fft}\right]\\ &= -(\mathcal{D}x_t)^T(\mathcal{D}x_t) + (\mathcal{D}x_t)R_{fft}.\end{aligned}$$

We then have that[2]

$$\dot{V}(\mathcal{D}\overline{x}_t) \le -\|\mathcal{D}\overline{x}_t\|\left(\|\mathcal{D}\overline{x}_t\| - \|R_{fft}\|\right).$$

If $\|\mathcal{D}\overline{x}_t\| > \|R_{fft}\|$ (x_d and $\dot{x}_d$ are assumed bounded), we have that $\dot{V}(\mathcal{D}\overline{x}_t) < 0$. Therefore, the uniformly ultimate boundedness of $\overline{x}$ is guaranteed according to Lemma 3.3. ■

Remark 4.8: From Theorem 4.4, it can be noted that the communication delay does not jeopardize the stability of the first-order system for the consensus tracking problem with partial access to the virtual leader. However, with the increase in the communication delay, the tracking errors will increase as well.

Remark 4.9: In real applications, the derivatives of the neighbors' information states $\dot{x}_j(t-\tau_2)$ can be calculated by using numerical differentiation. For example, $\dot{x}_j(t-\tau_2)$ can be approximated by $(x_j(kT-\tau_2) - x_j(kT-T-\tau_2))/T$, where T is the sampling period, and k is the discrete-time index.

V. Second-Order Case With Communication and Input Delays Under a Directed Network Topology

Here, we model a group of agents with double-integrator dynamics as

$$\dot{r}_i(t) = v_i(t) \quad \dot{v}_i(t) = u_i(t), \qquad i = 1,\ldots,n \tag{16}$$

where r_i, v_i, and u_i denote, respectively, the position, the velocity, and the control input of the ith agent.

A. Second-Order Leaderless Consensus

The proposed leaderless consensus algorithm with both communication and input delays is given as

$$\begin{aligned}u_i(t) = &-\frac{1}{\sum_{j=1}^n a_{ij}}\sum_{j=1}^n a_{ij}\left[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2)\right]\\ &-\frac{\gamma_c}{\sum_{j=1}^n a_{ij}}\sum_{j=1}^n a_{ij}\left[v_i(t-\tau_1) - v_j(t-\tau_1-\tau_2)\right],\\ &\qquad i = 1,\ldots,n\end{aligned} \tag{17}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, a_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,n$, is the (i,j) entry of the adjacency matrix $\mathbb{A}_n$, and γ_c is a positive gain. Here, we also assume that every agent has a neighbor, which implies that $\sum_{j=1}^n a_{ij} \neq 0$, $\forall i$. The control objective here is to guarantee that $r_i(t) \to r_j(t)$ and $v_i(t) \to v_j(t)$ as $t \to \infty$ when there exist both communication and input delays. Using (17), (16) can be written in the matrix form as

$$\begin{aligned}\begin{bmatrix}\dot{r}(t)\\ \dot{v}(t)\end{bmatrix} = &\begin{bmatrix}\mathbf{0}_{n\times n} & I_n\\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\end{bmatrix}\begin{bmatrix}r(t)\\ v(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ -I_n & -\gamma_c I_n\end{bmatrix}\\ &\times\begin{bmatrix}r(t-\tau_1)\\ v(t-\tau_1)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ A & \gamma_c A\end{bmatrix}\begin{bmatrix}r(t-\tau_1-\tau_2)\\ v(t-\tau_1-\tau_2)\end{bmatrix}\end{aligned}$$

where A is defined as in Section IV-A, $r = [r_1,\ldots,r_n]^T$, and $v = [v_1,\ldots,v_n]^T$. Define $\widetilde{r} \triangleq W^{-1}r$ and $\widetilde{v} \triangleq W^{-1}v$, where W is defined as in Section IV-A. Denote $\widetilde{r}_{n-1}$ and $\widetilde{v}_{n-1}$ as, respectively, the first $n-1$ rows of $\widetilde{r}$ and $\widetilde{v}$. Denote $\widetilde{r}_2$ and $\widetilde{v}_2$ as, respectively, the last row of $\widetilde{r}$ and $\widetilde{v}$. System (17) can be decoupled into the following:

$$\begin{aligned}\dot{\widetilde{x}}_{n-1}(t) = &A_0\widetilde{x}_{n-1}(t) + A_1\widetilde{x}_{n-1}(t-\tau_1)\\ &+ A_2\widetilde{x}_{n-1}(t-\tau_1-\tau_2)\end{aligned} \tag{18a}$$

$$\begin{aligned}\begin{bmatrix}\dot{\widetilde{r}}_2(t)\\ \dot{\widetilde{v}}_2(t)\end{bmatrix} = &\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\widetilde{r}_2(t)\\ \widetilde{v}_2(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ -1 & -\gamma_c\end{bmatrix}\begin{bmatrix}\widetilde{r}_2(t-\tau_1)\\ \widetilde{v}_2(t-\tau_1)\end{bmatrix}\\ &+\begin{bmatrix}0 & 0\\ 1 & \gamma_c\end{bmatrix}\begin{bmatrix}\widetilde{r}_2(t-\tau_1-\tau_2)\\ \widetilde{v}_2(t-\tau_1-\tau_2)\end{bmatrix}\end{aligned} \tag{18b}$$

where

$$\begin{aligned}\widetilde{x}_{n-1} &= \left[\widetilde{r}_{n-1}^T, \widetilde{v}_{n-1}^T\right]^T\\ A_0 &= \begin{bmatrix}\mathbf{0}_{(n-1)\times(n-1)} & I_{n-1}\\ \mathbf{0}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times(n-1)}\end{bmatrix}\\ A_1 &= \begin{bmatrix}\mathbf{0}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times(n-1)}\\ -I_{n-1} & -\gamma_c I_{n-1}\end{bmatrix}\\ A_2 &= \begin{bmatrix}\mathbf{0}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times(n-1)}\\ \widetilde{A} & \gamma_c\widetilde{A}\end{bmatrix}\end{aligned}$$

and $\widetilde{A}$ is defined as in Section IV-A.

Theorem 5.1: If the fixed directed graph $\mathbb{G}_n$ has a directed spanning tree, every agent has a neighbor, and $\gamma_c > \overline{\gamma}_c = \max_{\mu_i\neq 0}\{\sqrt{\Im(\mu_i)^2/\Re(\mu_i)|\mu_i|^2}\}$, where μ_i is the ith eigenvalue of $L = I_n - A$, there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that the following three conditions are satisfied.

1) $\gamma_c(2\overline{\tau_1} + \overline{\tau_2}) + ((2\overline{\tau_1} + \overline{\tau_2})\overline{\tau_2}/2) < 1$.
2) $1 + \lambda_i(A_1)((1 - e^{-s\overline{\tau_1}})/s) + \lambda_i(A_2)((1 - e^{-s(\overline{\tau_1}+\overline{\tau_2})})/s) \neq 0$, for all $s \in C^+$.
3) $Q_{sc} = (A_0 + A_1 + A_2)^T P_{sc} + P_{sc}(A_0 + A_1 + A_2) + \overline{\tau_1}S_{sc} + (\overline{\tau_1} + \overline{\tau_2})H_{sc} + \overline{\tau_1}[(A_0 + A_1 + A_2)^T P_{sc}A_1S_{sc}^{-1}A_1^T P_{sc}(A_0 + A_1 + A_2)] + (\overline{\tau_1} + \overline{\tau_2})[(A_0 + A_1 + A_2)^T P_{sc}A_2H_{sc}^{-1}A_2^T P_{sc}(A_0 + A_1 + A_2)] < 0$, where P_{sc} is a symmetric positive-definite matrix chosen properly such that $(A_0 + A_1 + A_2)^T P_{sc} + P_{sc}(A_0 + A_1 + A_2) < 0$, and S_{sc} and H_{sc} are arbitrary symmetric positive-definite matrices.

If the above conditions are satisfied, $\tau_1 \in [0, \overline{\tau_1}]$, and $\tau_2 \in [0, \overline{\tau_2}]$, system (16) using (17) reaches consensus asymptotically. Specifically, $r_i(t) \to (p^T v(0)/\tau_2)$ and $v_i(t) \to 0$, where p is defined as in Theorem 4.1.

Proof: Similar to the analysis given in Section IV-A, we first prove that the stability of system (16) using (17) is guaranteed if the three conditions in Theorem 5.1 are satisfied. Then,

[2] According to Lemma 3.3, if we let $p(s) = q_{ff}^2 s$ for some constant $q_{ff} > 1$, we then know that $p(V(\mathcal{D}\overline{x}_t)) > V(\overline{x}(\xi))$ for $t - \tau_2 \le \xi \le t$. However, this condition is not used in the proof because of the special expression of $\dot{V}$.

we show that these three conditions are, indeed, satisfied when $\mathbb{G}_n$ has a directed spanning tree, every agent has a neighbor, and $\gamma_c > \overline{\gamma}_c$. At last, the consensus equilibrium is explicitly presented by using the final value theorem.

For system (18a), consider a Lyapunov function candidate

$$V\left(\widetilde{x}_{(n-1)t}\right) = \left[\widetilde{x}_{n-1}(t) + A_1\int_{-\tau_1}^{0}\widetilde{x}_{n-1}(t+\theta)\,d\theta + A_2\int_{-\tau_1-\tau_2}^{0}\widetilde{x}_{n-1}(t+\theta)\,d\theta\right]^T \times P_{sc}\left[\widetilde{x}_{n-1}(t) + A_1\int_{-\tau_1}^{0}\widetilde{x}_{n-1}(t+\theta)\,d\theta + A_2\int_{-\tau_1-\tau_2}^{0}\widetilde{x}_{n-1}(t+\theta)\,d\theta\right] + \int_{-\tau_1}^{0}\int_{t+\theta}^{t}\widetilde{x}_{n-1}(\xi)^T S_{sc}\widetilde{x}_{n-1}(\xi)\,d\xi\,d\theta + \int_{-\tau_1-\tau_2}^{0}\int_{t+\theta}^{t}\widetilde{x}_{n-1}(\xi)^T H_{sc}\widetilde{x}_{n-1}(\xi)\,d\xi\,d\theta.$$

Taking the derivative of V gives

$$\dot{V}\left(\widetilde{x}_{(n-1)t}\right) \le \widetilde{x}_{n-1}(t)^T Q_{sc}\widetilde{x}_{n-1}(t)$$

where Q_{sc} is defined in Theorem 5.1. Thus, the stability of system (18a) is guaranteed if conditions 2 and 3 are satisfied by using Lemma 3.1.

For system (18b), define $g(s) \triangleq (\gamma_c s + 1)(e^{-\tau_1 s} - e^{-(\tau_1+\tau_2)s})/s^2$. By using the Nyquist stability criterion, we know that the stability of (18b) can be guaranteed if $\Re\{g(j\omega)\} > -1, \forall\,\omega \in (-\infty, \infty)$. Because

$$\begin{aligned}\Re\{g(j\omega)\} &= \frac{-\gamma_c\sin\tau_1\omega + \gamma_c\sin(\tau_1+\tau_2)\omega}{\omega} + \frac{-\cos\tau_1\omega + \cos(\tau_1+\tau_2)\omega}{\omega^2}\\ &= \frac{-\gamma_c\sin\tau_1\omega + \gamma_c\sin(\tau_1+\tau_2)\omega}{\omega} - \frac{2\sin\frac{(2\tau_1+\tau_2)}{2}\omega\sin\frac{\tau_2}{2}\omega}{\omega^2}\\ &\ge -\gamma_c\tau_1 - \gamma_c(\tau_1+\tau_2) - \frac{(2\tau_1+\tau_2)\tau_2}{2}\end{aligned}$$

condition 1 in Theorem 5.1 guarantees the stability of system (18b).

Next, we show that the three conditions in Theorem 5.1 are, indeed, satisfied if $\mathbb{G}_n$ has a directed spanning tree, every agent has a neighbor, and $\gamma_c > \overline{\gamma}_c$. It is straightforward to see that there exist $\overline{\tau}_1$ and $\overline{\tau}_2$ such that conditions 1 and 2 are satisfied. For condition 3, noting that

$$A_0 + A_1 + A_2 = \begin{bmatrix}\mathbf{0}_{(n-1)\times(n-1)} & I_{n-1}\\ -\widetilde{L} & -\gamma_c\widetilde{L}\end{bmatrix}$$

we know that the assumptions that $\mathbb{G}_n$ has a directed spanning tree, every agent has a neighbor, and $\gamma_c > \overline{\gamma}_c$ imply that all eigenvalues of $A_0 + A_1 + A_2$ are on the open left half-plane according to [30]. Thus, there always exists a P_{sc} to guarantee that $(A_0 + A_1 + A_2)^T P_{sc} + P_{sc}(A_0 + A_1 + A_2) < 0$, which implies that condition 3 is satisfied.

For the consensus equilibrium, we know that the asymptotical stability of (18b) implies that $\begin{bmatrix}\widetilde{r}_2(t)\\ \widetilde{v}_2(t)\end{bmatrix} \to \begin{bmatrix}\widetilde{v}_2(0)/\tau_2\\ 0\end{bmatrix}$ as $t \to \infty$, and the asymptotical stability of (18a) implies that $\widetilde{x}_{n-1} \to 0$ as $t \to \infty$. Thus, it follows that $\begin{bmatrix}r(t)\\ v(t)\end{bmatrix} \to \begin{bmatrix}(p^T v(0)/\tau_2)\mathbf{1}_n\\ \mathbf{0}_n\end{bmatrix}$ as $t \to \infty$. ■

Remark 5.1: Due to the existence of the communication delay, the final velocity is dampened to zero instead of a possible nonzero constant as compared with the standard second-order consensus algorithm studied in [18]. Also, note that if there exists only the input delay, the final velocity is a possibly nonzero constant, and the final position is a ramp signal, which are consistent with the results in [18].

Remark 5.2: Note that compared with the first-order case in Section IV-A, the second-order case requires more stringent conditions to guarantee stability, and the final consensus states are different.

B. Second-Order Consensus Regulation With a Constant Final Velocity

Here, we assume that there exists a virtual leader, labeled as agent $n+1$ with position r_d and velocity v_d. Here, we assume that v_d is constant. The control objective here is to guarantee that all agents can track the virtual leader under limited communication in the presence of delays. The proposed consensus regulation algorithm is given as

$$\begin{aligned}u_i = &-\frac{1}{\sum_{j=1}^{n+1}a_{ij}}\sum_{j=1}^{n+1}a_{ij}\left[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2)\right]\\ &-\frac{\gamma_r}{\sum_{j=1}^{n+1}a_{ij}}\sum_{j=1}^{n+1}a_{ij}\left[v_i(t-\tau_1) - v_j(t-\tau_1-\tau_2)\right],\\ &\qquad i = 1,\ldots,n\end{aligned}\tag{19}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, $a_{ij}, i = 1,\ldots,n, j = 1,\ldots,n+1$, is the (i,j) entry of the adjacency matrix $\mathbb{A}_{n+1}$, $r_{n+1} \equiv r_d$, $v_{n+1} \equiv v_d$, and γ_r is a positive gain. Note that if $\mathbb{G}_{n+1}$ has a directed spanning tree, then it follows that $\sum_{j=1}^{n+1}a_{ij} \ne 0$, $i = 1,\ldots,n$, [26]. Using (19), (16) can be written in the matrix form as

$$\dot{\overline{x}}(t) = \mathcal{A}_0\overline{x}(t) + \mathcal{A}_1\overline{x}(t-\tau_1) + \mathcal{A}_2\overline{x}(t-\tau_1-\tau_2) + R_{sr}\tag{20}$$

where

$$\overline{r} \triangleq [r_1 - r_d, \ldots, r_n - r_d]^T$$
$$\overline{v} \triangleq [v_1 - v_d, \ldots, v_n - v_d]^T \qquad \overline{x} = [\overline{r}^T, \overline{v}^T]^T$$
$$\mathcal{A}_0 = \begin{bmatrix}\mathbf{0}_{n\times n} & I_n\\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\end{bmatrix} \qquad \mathcal{A}_1 = \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ -I_n & -\gamma_r I_n\end{bmatrix}$$
$$\mathcal{A}_2 = \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ \mathcal{A} & \gamma_r\mathcal{A}\end{bmatrix} \qquad R_{sr} = \begin{bmatrix}\mathbf{0}_n\\ -\tau_2 v_d\mathbf{1}_n\end{bmatrix}.$$

Note that, here, we have used the fact that v_d is constant and $\mathcal{A}$ is defined as in Section IV-B.

By letting $M = (A_0 + A_1 + A_2)^{-1} R_{sr}$ and $\widehat{x} = \overline{x} - M$, we can transform (20) as

$$\dot{\widehat{x}} = \mathcal{A}_0 \widehat{x}(t) + \mathcal{A}_1 \widehat{x}(t - \tau_1) + \mathcal{A}_2 \widehat{x}(t - \tau_1 - \tau_2). \quad (21)$$

Theorem 5.2: If the fixed directed graph $\mathbb{G}_{n+1}$ has a directed spanning tree and $\gamma_r > \overline{\gamma}_r = \max_{\mu_i}\{\sqrt{\Im(\mu_i)^2/\Re(\mu_i)|\mu_i|^2}\}$, where μ_i is the ith eigenvalue of $I_n - \mathcal{A}$, there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that the following two conditions are satisfied.

1) $1+\lambda_i(\mathcal{A}_1)((1-e^{-s\overline{\tau_1}})/s)+\lambda_i(\mathcal{A}_2)((1-e^{-s(\overline{\tau_1}+\overline{\tau_2})})/s) \neq 0$, for all $s \in C^+$.
2) $Q_{sr} = (\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2)^T P_{sr} + P_{sr}(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2) + \overline{\tau_1} S_{sr} + (\overline{\tau_1} + \overline{\tau_2}) H_{sr} + \overline{\tau_1}[(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2)^T P_{sr} \mathcal{A}_1 S_{sr}^{-1} \mathcal{A}_1^T P_{sr}(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2)] + (\overline{\tau_1} + \overline{\tau_2})[(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2)^T P_{sr} \mathcal{A}_2 H_{sr}^{-1} \mathcal{A}_2^T P_{sr}(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2)] < 0$, where P_{sr} is a symmetric positive-definite matrix chosen properly such that $(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2)^T P_{sr} + P_{sr}(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2) < 0$, and S_{sr} and H_{sr} are arbitrary symmetric positive-definite matrices.

In addition, if the above conditions are satisfied, $\tau_1 \in [0, \overline{\tau_1}]$, and $\tau_2 \in [0, \overline{\tau_2}]$, system (16) using (19) guarantees that $\lim_{t\to\infty} \overline{r}(t) \to \tau_2 v_d (I_n - \mathcal{A})^{-1} \mathbf{1}_n$ and $\lim_{t\to\infty} \overline{v}(t) \to \mathbf{0}_n$ asymptotically as $t \to \infty$.

Proof: Consider a Lyapunov function candidate

$$\begin{aligned} V(\widehat{x}_t) = &\left[\widehat{x}(t) + \mathcal{A}_1 \int_{-\tau_1}^{0} \widehat{x}(t+\theta)\, d\theta + \mathcal{A}_2 \int_{-\tau_1-\tau_2}^{0} \widehat{x}(t+\theta)\, d\theta\right]^T \\ &\times P_{sr}\left[\widehat{x}(t) + \mathcal{A}_1 \int_{-\tau_1}^{0} \widehat{x}(t+\theta)\, d\theta + \mathcal{A}_2 \int_{-\tau_1-\tau_2}^{0} \widehat{x}(t+\theta)\, d\theta\right] \\ &+ \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \widehat{x}(\xi)^T S_{sr} \widehat{x}(\xi)\, d\xi\, d\theta \\ &+ \int_{-\tau_1-\tau_2}^{0}\int_{t+\theta}^{t} \widehat{x}(\xi)^T H_{sr} \widehat{x}(\xi)\, d\xi\, d\theta. \end{aligned}$$

Taking the derivative of V along (21) gives

$$\dot{V}(\overline{x}_t) \leq \overline{x}(t)^T Q_{sr} \overline{x}(t)$$

where Q_{sr} is defined as in Theorem 5.2.

By following a similar analysis to that in Section V-A, we can prove the stability of (21) and the existence of $\overline{\tau}_1$ and $\overline{\tau}_2$ such that the two conditions in Theorem 5.3 are satisfied. Since $\widehat{x}(t) \to \mathbf{0}_{2n}$, as $t \to \infty$, and $M = [\tau_2 v_d[(I_n - \mathcal{A})^{-1}\mathbf{1}_n]^T, \mathbf{0}_n^T]^T$, it follows that $\lim_{t\to\infty} \overline{r}(t) \to \tau_2 v_d (I_n - \mathcal{A})^{-1}\mathbf{1}_n$ and $\lim_{t\to\infty} \overline{v}(t) \to \mathbf{0}_n$ asymptotically as $t \to \infty$. ■

Corollary 5.1: If $v_d = 0$, we can get that $\lim_{t\to\infty} r_i(t) \to r_d$ and $\lim_{t\to\infty} v_i(t) \to 0$ as $t \to \infty$ given that the conditions in Theorem 5.2 are satisfied.

Remark 5.3: Note that different from the results in the first-order case in Section IV-B, the final positions of the followers might not be equal in the second-order case. The final relative positions of the followers are constant.

C. Second-Order Consensus Tracking With Full Access to the Virtual Leader

Here, the reference states r_d, v_d, and $\dot{v}_d$ are assumed to be time-varying, and $\dot{v}_d$ is assumed to be available to all agents. The following consensus tracking algorithm with both communication and input delays is proposed as

$$\begin{aligned} u_i = \dot{v}_d(t - \tau_1 - \tau_2) - &\frac{1}{\sum_{j=1}^{n+1} a_{ij}} \\ \times \sum_{j=1}^{n+1} &a_{ij}\left\{[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2)]\right. \\ &\left. + \gamma_t [v_i(t-\tau_1) - v_j(t-\tau_1-\tau_2)]\right\}, \\ &i = 1, 2 \ldots, n \end{aligned} \quad (22)$$

where τ_1 and τ_2 are the input and communication delays, respectively, a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$, is the (i, j) entry of the adjacency matrix $\mathbb{A}_{n+1}$, $r_{n+1} \equiv r_d(t)$, $v_{n+1} \equiv v_d(t)$, and γ_t is a positive gain. We also assume that $|v_d| < \delta_v$, $|\dot{v}_d| < \delta_a$, and $|\ddot{v}_d| < \delta_{\dot{a}}$, where δ_v, δ_a, and $\delta_{\dot{a}}$ are positive constants. Using (22), (16) can be written in the matrix form as

$$\dot{\overline{x}}(t) = \mathcal{A}_0 \overline{x}(t) + \mathcal{A}_1 \overline{x}(t-\tau_1) + \mathcal{A}_2 \overline{x}(t-\tau_1-\tau_2) + R_{st} \quad (23)$$

where $\overline{r}$, $\overline{v}$, $\overline{x}$, $\mathcal{A}$, $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ are defined as in Section V-B, $R_{st} = \begin{bmatrix}\mathbf{0}_n \\ R_1\end{bmatrix}$, and $R_1 = -\mathbf{1}_n \int_{-\tau_1-\tau_2}^{0} \ddot{v}_d(t+\theta)\, d\theta - \mathbf{1}_n \int_{-\tau_2}^{0} v_d(t+\theta)\, d\theta - \gamma_t \mathbf{1}_n \int_{-\tau_1-\tau_2}^{-\tau_1} \dot{v}_d(t+\theta)\, d\theta$ by using the Leibniz–Newton formula [19].

Theorem 5.3: If the fixed directed graph $\mathbb{G}_{n+1}$ has a directed spanning tree and $\gamma_t > \overline{\gamma}_r$, where $\overline{\gamma}_r$ is defined as in Theorem 5.2, there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that $Q_{st} = (\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2)^T P_{sr} + P_{sr}(\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2) + \overline{\tau_1}(P_{sr}\mathcal{A}_1\mathcal{A}_0 P_{sr}^{-1}\mathcal{A}_0^T\mathcal{A}_1^T P_{sr} + P_{sr}\mathcal{A}_1\mathcal{A}_1 P_{sr}^{-1}\mathcal{A}_1^T\mathcal{A}_1^T P_{sr} + P_{sr}\mathcal{A}_1\mathcal{A}_2 P_{sr}^{-1}\mathcal{A}_2^T\mathcal{A}_1^T P_{sr} + 3q_s P_{sr}) + (\overline{\tau_1} + \overline{\tau_2})(P_{sr}\mathcal{A}_2\mathcal{A}_0 P_{sr}^{-1}\mathcal{A}_0^T\mathcal{A}_2^T P_{sr} + P_{sr}\mathcal{A}_2\mathcal{A}_1 P_{sr}^{-1}\mathcal{A}_1^T\mathcal{A}_2^T P_{sr} + P_{sr}\mathcal{A}_2\mathcal{A}_2 P_{sr}^{-1}\mathcal{A}_2^T\mathcal{A}_2^T P_{sr} + 3q_s P_{sr}) < 0$, where P_{sr} is the same matrix given in Theorem 5.2, and $q_s > 1$. In addition, if $Q_{st} < 0$, $\tau_1 \in [0, \overline{\tau_1}]$, and $\tau_2 \in [0, \overline{\tau_2}]$, system (16) using (22) guarantees that all $r_i - r_d$ and $v_i - v_d$ are uniformly ultimately bounded. In particular, the ultimate bound of $\overline{x}$ is given by $\lambda_{\max}(P_{sr}) a_s / \lambda_{\min}(P_{sr}) \kappa_s \lambda_{\min}(-Q_{st})$, where $a_s = 2[\|P_{sr}\| + \|P_{sr}\mathcal{A}_1\|\tau_1 + \|P_{sr}\mathcal{A}_2\|(\tau_1+\tau_2)][(\tau_1+\tau_2)\delta_{\dot{a}} + \tau_2\delta_v + \gamma_t\tau_2\delta_a]$ and $0 < \kappa_s < 1$.

Proof: Similar to the analysis given in Section IV-C, by using the Leibniz–Newton formula [19], we transform (23) to the following system:

$$\begin{aligned}\frac{d}{dt}\overline{x}(t) = &(\mathcal{A}_0+\mathcal{A}_1+\mathcal{A}_2)\overline{x}(t) - \mathcal{A}_1\mathcal{A}_0\int_{-\tau_1}^{0}\overline{x}(t+\theta)\,d\theta\\ &- \mathcal{A}_1^2\int_{-2\tau_1}^{-\tau_1}\overline{x}(t+\theta)\,d\theta - \mathcal{A}_1\mathcal{A}_2\int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2}\overline{x}(t+\theta)\,d\theta\\ &- \mathcal{A}_2\mathcal{A}_0\int_{-\tau_1-\tau_2}^{0}\overline{x}(t+\theta)\,d\theta - \mathcal{A}_2\mathcal{A}_1\int_{-2\tau_1-\tau_2}^{-\tau_1}\overline{x}(t+\theta)\,d\theta\\ &- \mathcal{A}_2^2\int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2}\overline{x}(t+\theta)\,d\theta + R_{st}\\ &- \mathcal{A}_1\int_{-\tau_1}^{0}R_{st}(t+\theta)\,d\theta - \mathcal{A}_2\int_{-\tau_1-\tau_2}^{0}R_{st}(t+\theta)\,d\theta.\end{aligned}$$

Consider a Lyapunov function candidate $V(\overline{x}) = \overline{x}^T P_{sr}\overline{x}$. Taking the derivative of $V(\overline{x})$ along (23) gives

$$\begin{aligned}\dot{V}(\overline{x}) \le\, &\overline{x}^T\left[(\mathcal{A}_0+\mathcal{A}_1+\mathcal{A}_2)^T P_{sr} + P_{sr}(\mathcal{A}_0+\mathcal{A}_1+\mathcal{A}_2)\right]\overline{x}\\ &+ \tau_1\overline{x}^T P_{sr}\mathcal{A}_1\mathcal{A}_0 P_{sr}^{-1}\mathcal{A}_0^T\mathcal{A}_1^T P_{sr}\overline{x}\\ &+ \int_{-\tau_1}^{0}\overline{x}^T(t+\theta)P_{sr}\overline{x}(t+\theta)\,d\theta\\ &+ \tau_1\overline{x}^T P_{sr}\mathcal{A}_1\mathcal{A}_1 P_{sr}^{-1}\mathcal{A}_1^T\mathcal{A}_1^T P_{sr}\overline{x}\\ &+ \int_{-2\tau_1}^{-\tau_1}\overline{x}^T(t+\theta)P_{sr}\overline{x}(t+\theta)\,d\theta\\ &+ \tau_1\overline{x}^T P_{sr}\mathcal{A}_1\mathcal{A}_2 P_{sr}^{-1}\mathcal{A}_2^T\mathcal{A}_1^T P_{sr}\overline{x}\\ &+ \int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2}\overline{x}^T(t+\theta)P_{sr}\overline{x}(t+\theta)\,d\theta\\ &+ (\tau_1+\tau_2)\overline{x}^T P_{sr}\mathcal{A}_2\mathcal{A}_0 P_{sr}^{-1}\mathcal{A}_0^T\mathcal{A}_2^T P_{sr}\overline{x}\\ &+ \int_{-\tau_1-\tau_2}^{0}\overline{x}^T(t+\theta)P_{sr}\overline{x}(t+\theta)\,d\theta\\ &+ (\tau_1+\tau_2)\overline{x}^T P_{sr}\mathcal{A}_2\mathcal{A}_1 P_{sr}^{-1}\mathcal{A}_1^T\mathcal{A}_2^T P_{sr}\overline{x}\\ &+ \int_{-2\tau_1-\tau_2}^{-\tau_1}\overline{x}^T(t+\theta)P_{sr}\overline{x}(t+\theta)\,d\theta\\ &+ (\tau_1+\tau_2)\overline{x}^T P\mathcal{A}_2\mathcal{A}_2 P_{sr}^{-1}\mathcal{A}_2^T\mathcal{A}_2^T P_{sr}\overline{x}\\ &+ \int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2}\overline{x}^T(t+\theta)P_{sr}\overline{x}(t+\theta)\,d\theta\\ &+ 2\|\overline{x}\|\,\|P_{sr}\|\left[(\tau_1+\tau_2)\delta_{\dot{a}} + \tau_2\delta_v + \gamma_t\tau_2\delta_a\right]\\ &+ 2\|\overline{x}\|\,\|P_{sr}\mathcal{A}_1\|\tau_1\left[(\tau_1+\tau_2)\delta_{\dot{a}} + \tau_2\delta_v + \gamma_t\tau_2\delta_a\right]\\ &+ 2\|\overline{x}\|\,\|P_{sr}\mathcal{A}_2\|(\tau_1+\tau_2)\\ &\times\left[(\tau_1+\tau_2)\delta_{\dot{a}} + \tau_2\delta_v + \gamma_t\tau_2\delta_a\right]\end{aligned}$$

where we have used Lemma 4.1 and the facts that $|v_d| < \delta_v$, $|\dot{v}_d| < \delta_a$, and $|\ddot{v}_d| < \delta_{\dot{a}}$ to derive the inequality. Take $p(s) = q_s s$ for some constant $q_s > 1$. If $V(\overline{x}(t+\theta)) \le q_s V(\overline{x}(t))$ for $-2\tau_1 - 2\tau_2 \le \theta \le 0$, by following a similar analysis to that in Section IV-C, we have that

$$\dot{V}(\overline{x}) \le \overline{x}(t)^T Q_{st}\overline{x}(t) + a_s\|\overline{x}\|$$

where Q_{st} and a_s are defined as in Theorem 5.3. It is easy to verify that there exist $\overline{\tau_1}$ and $\overline{\tau_2}$ such that $Q_{st} < 0$ by following a similar analysis to that in Section IV-C. Moreover, we have that $\lambda_{\min}(-Q_{st}) > 0$. For $0 < \kappa_s < 1$, if $\|\overline{x}\| \ge a_s/\kappa_s\lambda_{\min}(-Q_{st})$, we can obtain that

$$\begin{aligned}\dot{V}(\overline{x}) \le &-(1-\kappa_s)\lambda_{\min}(-Q_{st})\|\overline{x}\|^2\\ &-\kappa_s\lambda_{\min}(-Q_{st})\|\overline{x}\|^2 + a_s\|\overline{x}\|\\ \le &-(1-\kappa_s)\lambda_{\min}(-Q_{st})\|\overline{x}\|^2.\end{aligned}$$

The uniformly ultimate boundedness of $\overline{x}$ then follows from Lemma 3.2. Moreover, we can obtain that $\lambda_{\max}(P_{sr})a_s/\lambda_{\min}(P_{sr})\kappa_s\lambda_{\min}(-Q_{st})$ is the ultimate bound of $\overline{x}$ by following a similar analysis to that in [29, pp. 172–174]. ■

D. Second-Order Consensus Tracking With Partial Access to the Virtual Leader

Here, we assume that the reference states r_d, v_d, and $\dot{v}_d$ are time-varying and available to only a portion of all agents. We also assume that the system is only influenced by the communication delay. The proposed consensus tracking algorithm is given as

$$\begin{aligned}u_i = &\frac{1}{\sum_{j=1}^{n+1} a_{ij}}\sum_{j=1}^{n+1} a_{ij}\left\{\dot{v}_j(t-\tau_2) - [r_i(t) - r_j(t-\tau_2)]\,,\right.\\ &\left.-\gamma_{ft}\left[v_i(t) - v_j(t-\tau_2)\right]\right\}, \qquad i = 1,2,\ldots,n \quad (24)\end{aligned}$$

where τ_2 is the communication delay, a_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,n+1$, is the (i,j) entry of the adjacency matrix $\mathbb{A}_{n+1}$, $r_{n+1} \equiv r_d$, $v_{n+1} \equiv v_d$, $\dot{v}_{n+1} \equiv \dot{v}_d$, and γ_{ft} is a positive gain. Using (24), (16) can be written in the matrix form as

$$\dot{\overline{x}}(t) = D_f\dot{\overline{x}}(t-\tau_2) + \mathcal{A}_{f0}\overline{x} + \mathcal{A}_{f1}\overline{x}(t-\tau_2) + R_{sft} \quad (25)$$

where

$$D_f = \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathcal{A}\end{bmatrix} \qquad \mathcal{A}_{f0} = \begin{bmatrix}\mathbf{0}_{n\times n} & I_n\\ -I_n & -\gamma_{ft}I_n\end{bmatrix}$$

$$\mathcal{A}_{f1} = \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ \mathcal{A} & \gamma_{ft}\mathcal{A}\end{bmatrix} \qquad R_{sft} = \begin{bmatrix}\mathbf{0}_n\\ R_2\end{bmatrix}$$

$$\begin{aligned}R_2 = &\left[\dot{v}_d(t-\tau_2) - \dot{v}_d(t)\right]\mathbf{1}_n - \left[r_d(t) - r_d(t-\tau_2)\right]\mathbf{1}_n\\ &-\gamma_{ft}\left[v_d(t) - v_d(t-\tau_2)\right]\mathbf{1}_n\end{aligned}$$

and $\overline{r}$, $\overline{v}$, $\mathcal{A}$, and $\overline{x}$ are defined as in Section V-B.

Theorem 5.4: If the fixed directed graph $\mathbb{G}_{n+1}$ has a directed spanning tree, and $\gamma_{ft} > \overline{\gamma}_r$, where $\overline{\gamma}_r$ is defined as in Theorem 5.2, system (16) using (24) guarantees that all $r_i - r_d$ and $v_i - v_d$ are uniformly ultimately bounded if

$$\lambda > 2q_{sf}\left\|P_{sr}(\mathcal{A}_{f0}D_f + \mathcal{A}_{f1})\right\| + 2\|P_{sr}\mathcal{A}_{f1}\| \quad (26)$$

where $\lambda = \lambda_{\min}[-(\mathcal{A}_{f0} + \mathcal{A}_{f1})^T P_{sr} - P_{sr}(\mathcal{A}_{f0} + \mathcal{A}_{f1})]$, P_{sr} is the same matrix given in Theorem 5.2, and $q_{sf} > 1$.

Proof: First, it is easy to verify that $\rho(D_f) < 1$ based on the same analysis as in [27], which means that the neutral operator $\mathcal{D}\overline{x}_t = \overline{x} - D_f \overline{x}(t - \tau_2)$ is stable. Consider a Lyapunov function candidate $V(\overline{x}) = \overline{x}^T P_{sr} \overline{x}$. Taking the derivative of V along (25) gives

$$\begin{aligned}
\dot{V}(\mathcal{D}\overline{x}_t) &= 2(\mathcal{D}\overline{x}_t)^T P_{sr} \left[\mathcal{A}_{f0}\overline{x} + \mathcal{A}_{f1}\overline{x}(t-\tau_2) + R_{sft}\right] \\
&= 2(\mathcal{D}\overline{x}_t)^T P_{sr} \left[\mathcal{A}_{f0}\mathcal{D}\overline{x}_t + \mathcal{A}_{f0} D_f \overline{x}(t-\tau_2) \right. \\
&\qquad \left. + \mathcal{A}_{f1}\overline{x}(t-\tau_2) + R_{sft}\right] \\
&= (\mathcal{D}\overline{x}_t)^T \left[(\mathcal{A}_{f0} + \mathcal{A}_{f1})^T P_{sr} + P_{sr}(\mathcal{A}_{f0} + \mathcal{A}_{f1})\right] \\
&\quad \times \mathcal{D}\overline{x}_t + 2(\mathcal{D}\overline{x}_t)^T P_{sr}(\mathcal{A}_{f0} D_f + \mathcal{A}_{f1})\overline{x}(t-\tau_2) \\
&\quad - 2(\mathcal{D}\overline{x}_t)^T P_{sr}\mathcal{A}_{f1}(\mathcal{D}\overline{x}_t) + 2(\mathcal{D}\overline{x}_t)^T P_{sr} R_{sft}.
\end{aligned}$$

Letting $f(s) = q_{sf}^2 s$ for some constant $q_{sf} > 1$, $f(V(\mathcal{D}\overline{x}_t)) > V(\overline{x}(\xi))$ for $t - \tau_2 \le \xi \le t$ implies that $q_{sf}^2 (\mathcal{D}\overline{x}_t)^T (\mathcal{D}\overline{x}_t) > \overline{x}(\xi)^T \overline{x}(\xi)$. It follows that $\overline{x}(t - \tau_2) < q_{sf}(\mathcal{D}\overline{x}_t)$. Thus, it follows that

$$\begin{aligned}
\dot{V}(\mathcal{D}\overline{x}_t) \le -\lambda \|\mathcal{D}\overline{x}_t\|^2 &+ 2q_{sf}\|P_{sr}(\mathcal{A}_{f0} D_f + \mathcal{A}_{f1})\| \, \|\mathcal{D}\overline{x}_t\|^2 \\
&+ 2\|P_{sr}\mathcal{A}_{f1}\| \, \|\mathcal{D}\overline{x}_t\|^2 + 2\|P_{sr} R_{sft}\| \, \|\mathcal{D}\overline{x}_t\|
\end{aligned}$$

where λ is defined in Theorem 5.4. Note here that the assumptions that $\mathbb{G}_{n+1}$ has a directed spanning tree and $\gamma_{ft} > \overline{\gamma}_{fr}$ guarantee that there exists $\mathcal{A}$ such that (26) is satisfied. Therefore, if $\lambda > 2q_{sf}\|P_{sr}(\mathcal{A}_{f0} D_f + \mathcal{A}_{f1})\| + 2\|P_{sr}\mathcal{A}_{f1}\|$, the uniformly ultimate boundedness of $\overline{x}$ can be achieved according to Lemma 3.3. ■

Remark 5.4: Note that different from the first-order case where uniformly ultimate boundedness is guaranteed no matter how large the communication delay is, a certain delay-independent condition has to be satisfied beforehand to ensure the possibility of uniformly ultimate boundedness in the second-order case.

VI. Simulation

Here, we present simulation results to validate the theoretical results in Sections IV and V. We consider a group of six agents. For the leaderless consensus problem, the adjacency matrix $\mathbb{A}_n$ is chosen as

$$\mathbb{A}_n = \begin{bmatrix} 0 & 5 & 0 & 2.5 & 0 & 2.5 \\ 8 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 3 & 3 \\ 1 & 0 & 1 & 0 & 8 & 0 \\ 0 & 1.2 & 0 & 1.8 & 0 & 7 \\ 5 & 1 & 0 & 2 & 2 & 0 \end{bmatrix}.$$

For the leader-following cases, the adjacency matrix $\mathbb{A}_{n+1}$ is defined as

$$\mathbb{A}_{n+1} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 8 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 3 & 4 \\ 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 1.2 & 0 & 1.8 & 0 & 7 & 0 \\ 5 & 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

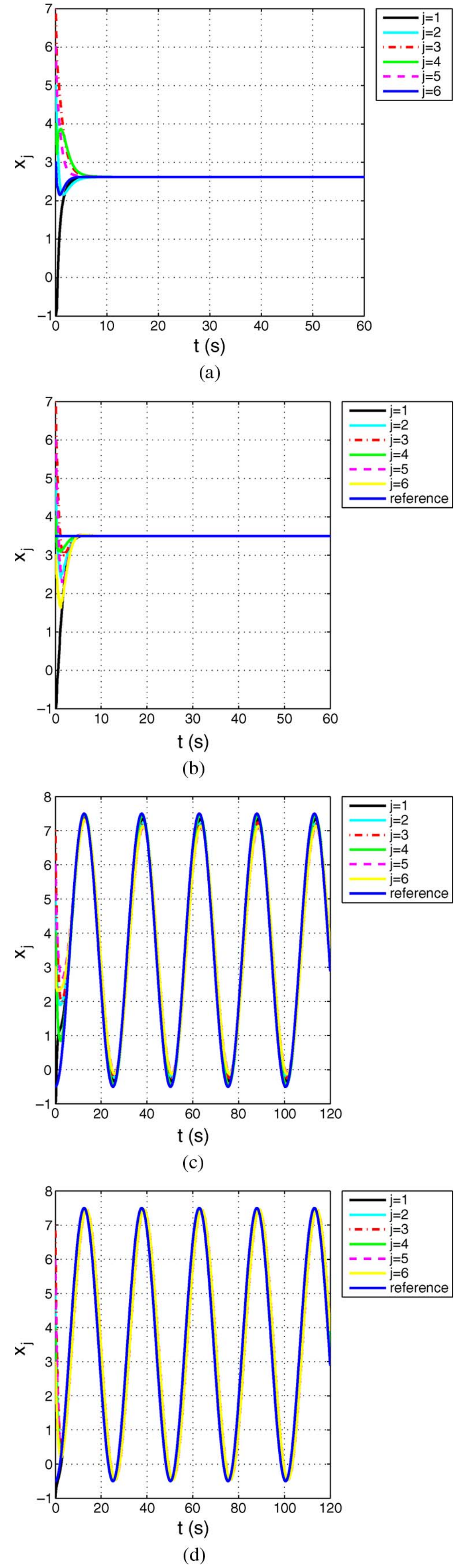

Fig. 1. First-order cases. (a) Simulation results using (4). (b) Simulation results using (9). (c) Simulation results using (12). (d) Simulation results using (14).

For the first-order cases, the initial states are chosen as $x(0) = [-1, 5, 7, 4, 6, 3]^T$. The input delay and the communication delay are chosen, respectively, as $\tau_1 = 0.1$ s and $\tau_2 = 0.2$ s. In the case of the first-order consensus regulation, we let the reference state be $x_d = 3.5$. In the case of the first-order consensus tracking with full access to the virtual leader, we let the reference state be $x_d(t) = 3.5 - 4\cos(t/4)$. In the case of the first-order consensus tracking with partial access to the virtual leader, we let the reference state be $x_d(t) = 3.5 - 4\cos(t/4)$ and the communication delay be $\tau_2 = 0.2$ s.

Fig. 1(a)–(d) shows the states of the agents for system (3) using, respectively, (4), (9), (12), and (14). It can be seen that for the leaderless consensus and consensus regulation problems, there are no final tracking errors between the agents and the virtual leader, while for the consensus tracking problem, there exist bounded tracking errors between the agents and the virtual leader due to the existence of the delays and the fact that the virtual leader is dynamic.

For the second-order cases, we choose $r(0) = [-0.4, 0.5, 0.7, 0.4, 1.2, 0.3]^T$ and $v(0) = [-0.1, 0.2, 0.7, 0.4, -0.1, 0.3]^T$ as the initial states. The input delay and the communication delay are chosen, respectively, as $\tau_1 = 0.3$ s and $\tau_2 = 0.1$ s. In the case of the second-order consensus regulation with a zero final velocity, we let the reference states be $r_d = -0.2$ and $v_d = 0$. In the case of the second-order consensus regulation with a nonzero constant final velocity, we let the reference states be $r_d(t) = -0.2 + 0.1t$ and $v_d(t) = 0.1$. In the case of the second-order consensus tracking with full access to the virtual leader, we let the reference states be $r_d(t) = -0.2 + 0.3t - 1.6\sin(t/4)$ and $v_d(t) = 0.3 - 0.4\cos(t/4)$. In the case of the second-order tracking with partial access to the virtual leader, we let the reference states be $r_d(t) = -0.2 + 0.3t - 1.6\sin(t/4)$ and $v_d(t) = 0.3 - 0.4\cos(t/4)$, and the communication delay be $\tau_2 = 0.1$ s.

Fig. 2(a) shows the states r_i and v_i of system (16) using (17). It is interesting to notice that unlike the standard second-order consensus algorithm in [18], the final velocities are always dampened to zero rather than a possibly nonzero constant. Fig. 2(b) and (c) shows, respectively, the states r_i and v_i of system (16) using (19) when $v_d = 0$ and $v_d = 0.1$. It is worth noticing that when v_d is a nonzero constant, the final tracking errors of all $r_i - r_d$ approach constant (not necessary identical) values.

Fig. 2(d) and (e) shows the states r_i and v_i of system (16) using, respectively, (22) and (24). There exist bounded tracking errors between the agents and the virtual leader due to the existence of the delays and the fact that the virtual leader is dynamic.

VII. Conclusion

Leaderless consensus, consensus regulation, and consensus tracking problems for both first-order and second-order integrators have been discussed under a directed network topology with communication and input delays. By using decoupling techniques, we have presented the stability conditions for the leaderless consensus problems. The consensus regulation problems can be viewed as a direct extension of the leaderless

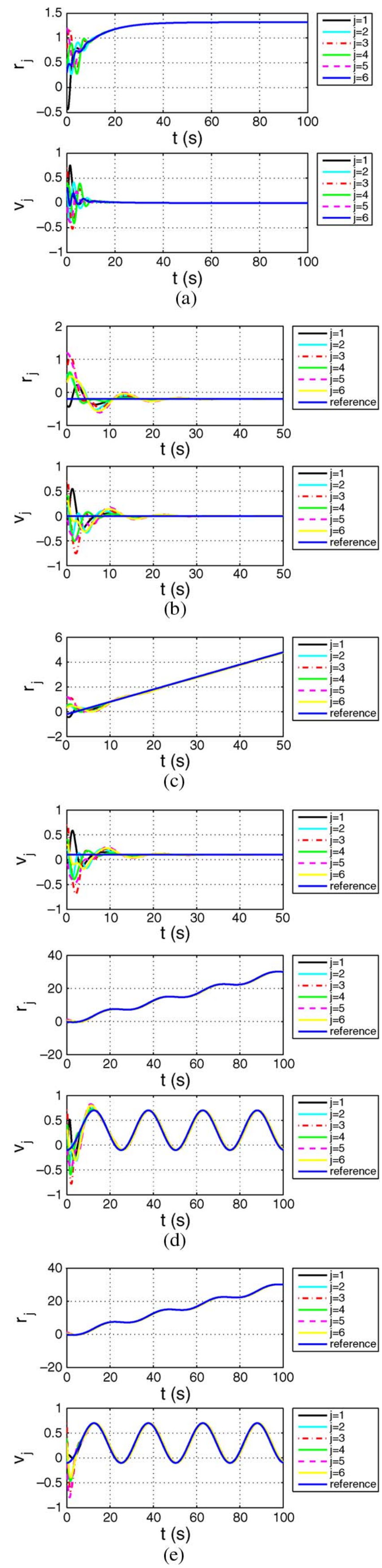

Fig. 2. Second-order cases. (a) Simulation results using (17). (b) Simulation results using (19). (c) Simulation results using (19). (d) Simulation results using (22). (e) Simulation results using (24).

consensus problems. In particular, the final velocities of the agents have been shown to be dampened to zero for the second-order leaderless consensus problem when there exists a communication delay. For the consensus tracking problems, the conditions to guarantee the uniformly ultimate boundedness of the tracking errors with full/partial access to the virtual leader have been presented. Finally, simulation results have been given to validate the theoretical results. Future works will include the design of zero-error consensus tracking algorithms in the presence of delays, the study on the case of consensus tracking algorithms with partial access to the virtual leader when there exist both communication and input delays, and the discussion on the influence of multiple time-varying delays.

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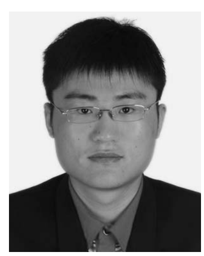

Ziyang Meng received the B.S. degree from Huazhong University of Science and Technology, Wuhan, China, in 2006. He is currently working toward the Ph.D. degree in the Department of Precision Instruments and Mechanology, Tsinghua University, Beijing, China.

From 2008 to 2009, he was an exchange Ph.D. student supported by China Scholarship Council with the Department of Electrical and Computer Engineering, Utah State University, Logan. His research interest focuses on cooperative control of distributed multiagent systems and spacecraft attitude determination and control.

Wei Ren (S'01–M'04) received the B.S. degree in electrical engineering from Hohai University, Nanjing, China, in 1997, the M.S. degree in mechatronics from Tongji University, Shanghai, China, in 2000, and the Ph.D. degree in electrical engineering from Brigham Young University, Provo, UT, in 2004.

From October 2004 to July 2005, he was a Research Associate with the Department of Aerospace Engineering, University of Maryland, College Park. Since August 2005, he has been with the Department of Electrical and Computer Engineering, Utah State University, Logan, where he is currently an Associate Professor. He is an author of the book *Distributed Consensus in Multi-Vehicle Cooperative Control* (Springer-Verlag, 2008). His research focuses on cooperative control of multivehicle systems, networked cyber-physical systems, and autonomous control of unmanned vehicles.

Dr. Ren was a recipient of the National Science Foundation CAREER Award in 2008. He is currently an Associate Editor for *Systems and Control Letters* and an Associate Editor on the IEEE Control Systems Society Conference Editorial Board.

Yongcan Cao (S'07) received the B.S. degree in electrical engineering from Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 2003 and the M.S. degree in electrical engineering from Shanghai Jiao Tong University, Shanghai, China, in 2006. He is currently working toward the Ph.D. degree in the Department of Electrical and Computer Engineering, Utah State University, Logan.

His research interest focuses on cooperative control and information consensus of multiagent systems.

Zheng You received the B.S., M.S., and Ph.D. degrees from Huazhong University of Science and Technology, Wuhan, China, in 1985, 1987, and 1990, respectively.

In 1990, he joined the faculty of the Department of Precision Instruments and Mechanology, Tsinghua University, Beijing, China, as an Assistant Professor, where he became an Associate Professor in 1992, a Full Professor in 1994, and was awarded specially appointed Professor for Chang Jiang Scholar by the Ministry of Education in 2001. He has published more than 300 papers and 32 research reports. He is the holder of 12 Chinese invention patents. His main research interests include micro–nano technology and micro–nano satellite technology.