Higher-Dimensional Categories:

an illustrated guide book

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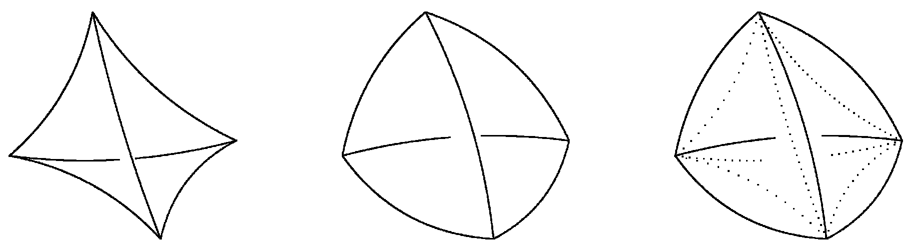

Draft Version

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Higher-dimensional categories are like a vast mountain that many people are trying to conquer. Some intrepid explorers have made the ascent, each taking a different route and each encountering different hazards. Each has made a map of his route, but do we know how all these maps fit together? Do we know that they fit together at all? In fact, are we even climbing the same mountain?

This work is an illustrated guide book to the world of higher-dimensional categories. A map would be more detailed and precise. An encyclopedia would be more comprehensive. Our aim is neither rigour nor completeness. Our aim is to provide would-be visitors with a sense of what they might find on arrival; to give them an idea of what landmarks to look out for; to warn them of the hazards of the territory; to introduce them to the language of the place; to whet their appetite for exploring by themselves.

To this end, we adopt an informal and friendly tone. We include enough detail to give visitors their bearings but not so much that they need a magnifying glass to find what they're looking for. And, most importantly, we provide copious pictures to illustrate our descriptions.

Inasmuch as this is a guide book, we will assume that the reader is already interested in visiting. This is not an advertising campaign. However, we hope that, with the appearance of a guide book, more people will consider visiting...

Contents

Preface

One person's clear mental image is another person's intimidation.
W. Thurston

We can regard the many definitions of n-category as an obstacle or a treasure-trove. In fact, it should always be helpful to have many different views of a structure — each viewpoint gives us greater scope for understanding. However, it also gives us a greater scope for being baffled.

The aim of this work is to promote understanding of the various definitions. We too have been thoroughly baffled, but have maintained a belief that each definition can look natural if viewed from the appropriate angle. Our general mission, then, is *demystification.* We will take each definition in turn and find a way of making everything seem as intuitive as possible. Of course, everyone arrives with different intuitions and we also arrived with our own, but we have tried to find out *what the appropriate intuitions are* that make each idea seem natural. This has involved shifting our mind-set around as we move between different theories, a sort of mental gymnastics that we feel is an essential warm-up for the exercise of understanding.

This work grew out of a very real situation: the second named author asked the first for an introduction to the various definitions of n-category, in advance of the IMA n-categories workshop in June 2004. A short discussion would not suffice, and we embarked on a journey of discovery in a series of about twenty afternoons. We made detailed notes of everything explained and, motivated by the impending workshop, expanded them into the present form. Our aim here is just as it was when we first sat down with our coffees: to shed as much light on the definitions as we can. These notes are by-and-large a record of our discussions and as such we lay no claims to completeness; we have simply included everything we found helpful along the way. We have chosen intuition over rigour wherever a choice seems to be required. This often means "waffle" over "concision" as we have aimed towards readable prose rather than elegant mathematical exposition that demands hours (or days) of reading and re-reading, sentence by sentence.

Prerequisites

We have tried to assume very little about our readers. Of course, we can't claim there are *no* prerequisites, but we can unequivocally state that we are only assuming the knowledge of a first-year graduate student: one of the authors! That is, since this is essentially a record of our discussions, the background we assume is essentially whatever we had in common at the start. This can be quickly summed up as: categories, functors, natural transformations and a little about limits and adjunctions. Where further background theory comes into a definition we have given some kind of account of it, even if only an impressionistic one; or else we have decided that a reasonable understanding of the definition can be reached without it.

In our discussions we took Leinster's invaluable survey paper [69] as a starting point, and the present work might be treated as a companion to it. We often make reference to its notation and terminology to help the reader who is proceeding in this fashion. However, the present work is also intended to stand alone.

Structure

We have included every definition to be presented at the IMA workshop. We have tried to keep each chapter self-contained and as such the ordering of the chapters is somewhat arbitrary. The idea is that it should not be necessary to read this guide book from start to finish. However we do include (often informal) comments relating or comparing notions in different chapters, but have tried to make them symmetric — we might ask the reader to "recall" something from a chapter that happens to occur later in our arbitrary ordering. The exception is of course the introduction; also, Chapters 2 (Penon) and 3 (Batanin and Leinster) are closely linked.

Acknowledgements

We would like to thank John Baez, Martin Hyland and Tom Leinster for their ongoing support and inspiration. We would also like to thank John Baez and Peter May for organising the IMA workshop, and for creating it so deliberately as a forum for interactive demystification. The extent to which this has motivated this work cannot be understated.

Chapter 1

Introduction

Weak higher-dimensional categories are difficult to define because of the words "weak" and "higher-dimensional". One or the other would be fine — weak low-dimensional categories, strict higher-dimensional categories — but together they produce more expressive power than we can easily describe.

The difficulty is in getting a structure that is weak enough to be "expressive", but coherent enough to be "sensible". There are many ways of interpreting this ideal and yet more ways of realising it. In this introduction we outline ways in which definitions of n-category can differ. We broadly think of these differences as falling into two classes:

- ideological: different ways of *interpreting* the ideal
- technical: different ways of *realising* the ideal.

Since this whole subject involves generalising something well-understood, we will begin with a discussion about the different possible starting points for generalisation. We then highlight more specifically some of the ways in which definitions of n-category can differ; we will use these as points of reference when we move onto discussing each definition in detail. The definitions and their various characteristics are summed up in the form of several tables in the Appendix.

1.1 Starting points for generalisation

> *If someone has stepped out into the unknown, a good way to start looking for them is to go to the place where they were last seen.*

For any mathematical structure there are likely to be many equivalent ways of describing it. We can each have our personal favourite, which is fine until we come to *g*eneralise. Generalisations of a structure can look very different depending on which particular description of the structure we started with.

The generalisation of *categories* to *weak higher-dimensional categories* is a prime example of this phenomenon. Even the (relatively) simple structure of an ordinary category can be thought of in very different ways. For the 1-dimensional case it might just be a matter of personal preference, but if we generalise into n dimensions those innocuous differences are likely be magnified n times. So it's important to know where we started, in order better to understand where we end up.

In this section we will present ordinary categories in three different ways. These lead to three different general directions for defining n-categories, and we will broadly classify the definitions along these lines.

1.1.1 Categories I: graphs with structure

Definition 1 *A* category *is given by*

i) DATA*: a diagram* $C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$ *in* **Set**

ii) STRUCTURE*: composition and identities*

iii) PROPERTIES*: unit and associativity axioms.*

The data $C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$ is also known by the (over-used) term “”. We can interpret it as a set C_1 of arrows with source and target in C_0 given by s, t. This viewpoint leads to the following generalisation

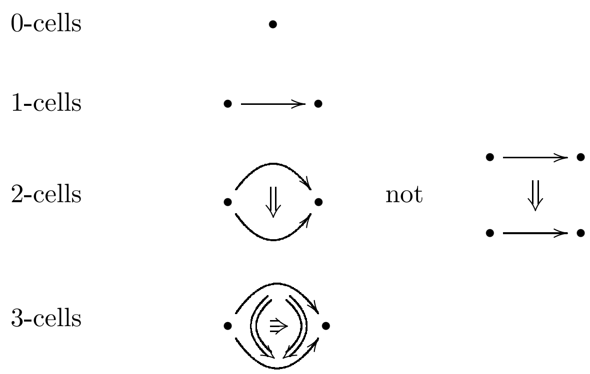
Definition 1-n *A* strict n-category *is given by*

i) DATA: *a diagram* $C_n \overset{s_n}{\underset{t_n}{\rightrightarrows}} C_{n-1} \overset{s_{n-1}}{\underset{t_{n-1}}{\rightrightarrows}} \cdots \overset{s_2}{\underset{t_2}{\rightrightarrows}} C_1 \overset{s_1}{\underset{t_1}{\rightrightarrows}} C_0$ *in* **Set**

ii) STRUCTURE: *composition and identities*

iii) PROPERTIES: *strict associativity and interchange axioms.*

We have to be a bit more careful about our generalised data: we would like our cells to look like this



so we need to impose conditions on the source and target maps to ensure that for any cell its source and target match up properly, unlike this one

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] \arrow[phantom, ur, "\Downarrow"] & \bullet \end{tikzcd}$$

So formally, we need

$$\begin{array}{rcl} s_{k-1}s_k &=& s_{k-1}t_k \\ t_{k-1}s_k &=& t_{k-1}t_k \end{array}$$

for all k. These are called the "globularity" conditions (ensuring that the cells look like "globs") and the resulting data is called a *globular set.*

To generalise this approach for *weak* n-categories the issue is then to work out how to *weaken* the coherence demands of the strict n-category structure. This sort of approach is taken by Penon, Batanin and Leinster.

Remark Definition 1 above yields an abstract account of categories as algebras for the "free category" monad on the category **Grph** of graphs (see Section 2.4 for details). This leads to the idea of defining n-categories as algebras for an appropriate monad on globular sets, and this is what the above authors do.

1.1.2 Categories II: objects and morphisms

Definition 2 *A* category *is given by*

i) objects: a set C_0

ii) hom-sets: for all $a, b \in C_0$, *a set* $C(a,b)$

iii) composition: for all $a, b, c \in C_0$ *a function*

$$\begin{array}{ccc} C(a,b) \times C(b,c) & \longrightarrow & C(a,c) \\ (f,g) & \mapsto & g \circ f \end{array}$$

iv) identities: for all $a \in C_0$ *a function*

$$\begin{array}{ccc} \{*\} & \longrightarrow & C(a,a) \\ * & \mapsto & 1_a \end{array}$$

satisfying associativity and unit axioms.

Here we interpret the sets $C(a,b)$ as sets of morphisms of the form $a \longrightarrow b$.

This viewpoint leads to the following generalisation:

Definition 2-n *A* strict n-category *is given by*

i) objects: a set C_0

ii) hom-$(n-1)$*-categories: for all* $a, b \in C_0$*, an* $(n-1)$*-category* $C(a,b)$

iii) composition: for all $a, b, c \in C_0$ *an* $(n-1)$*-functor*

$$C(a,b) \times C(b,c) \longrightarrow C(a,c)$$

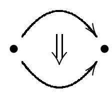
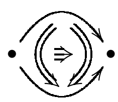
iv) identities: for all $a \in C_0$ *an* $(n-1)$*-functor*

$$1 \longrightarrow C(a,a)$$

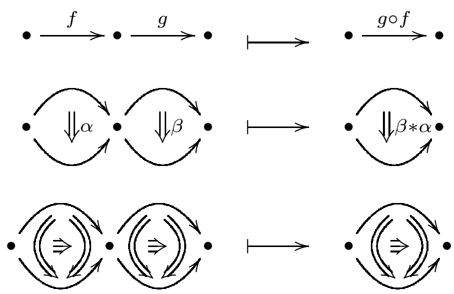
where 1 is the terminal $(n-1)$*-category*

satisfying associativity, unit and interchange axioms.

This gives a strict n-category as a "category enriched in $(n-1)$-categories". This means, for a start, that for any $a, b \in C_0$, the morphisms $a \longrightarrow b$ form an $(n-1)$-category $C(a,b)$ which we can interpret as having

0-cells		i.e. 1-cells of C
1-cells		i.e. 2-cells of C
2-cells		i.e. 3-cells of C

So the composition functor gives:



There is a general definition of "categories enriched in $\mathcal{V}$" for suitable $\mathcal{V}$ (see Section 8.1.2) and a well-developed theory of such enriched categories [55] but everything happens strictly. For generalisation to *w*eak n-categories, the issue in this approach is then to work out how to weaken the coherence demands in the definition of enrichment. This sort of approach is taken by Trimble and May explicitly, and by Tamsamani and Simpson less explicitly.

1.1.3 Categories III: nerves

Every category has an underlying simplicial set called its nerve. Conversely, a simplicial set arises as the nerve of a category if and only if it satisfies the "nerve condition". In fact we can use this characterisation as a definition:

Definition 3 *A c*ategory *is a simplicial set satisfying the "nerve condition".*

Essentially, the nerve condition asserts that the m-cells give m-fold composition in an associative way. This leads to the following generalisation due to Street:

Definition 3-n *A s*trict n-category *is a simplicial set satisfying the "n-nerve condition"*

This n-nerve condition involves some extremely intricate ("magical") combinatorial arguments about sets of indices. Once this is done, the move to the weak case is child's play (it is done by deleting the word "unique" from an existence condition). This is Street's approach; he has expressed the strict case in a form that makes generalisation easy.

We might wonder if we can take a "nerve-like" approach but avoid the intricate combinatorics. One way of doing this is to build in some more information to the underlying data, by using different "shapes of cell". This sort of approach is taken by the Joyal, Tamsamani, Simpson and the Opetopic. We will discuss the issue of cell shape in the next section.

1.2 Key points of difference

In this section we will highlight some of the key points of difference we encounter in the various definitions.

1.2.1 The Data-Structure-Properties (DSP) trichotomy

We begin with one of the more difficult issues to articulate, and one that causes the most difficulty when it comes to proving comparison theorems. We can think of any mathematical definition in terms of

i) underlying *data*, equipped with

ii) certain extra *structure*, satisfying

iii) some *properties*.

For example a group is given by

i) an underlying set, equipped with

ii) a binary operation, identity and inverses, satisfying

iii) some axioms.

We will pompously call this the *Data-Structure-Properties (DSP) trichotomy.*

Now, some differences in definitions arise from the question of where the various components of n-category are given, with respect to this trichotomy. We already see a difference in the above definitions of category: the "graph" definition specifies composition as a piece of *additional structure* where the "nerve" definition uses the *property* of "being able to extract coherent composition from the given data". It doesn't make a serious difference for ordinary categories, but becomes more serious as we generalise/weaken.

We can illustrate this by considering "weakening" an associativity axiom:

$$(hg)f = h(gf) \quad \text{becomes} \quad (hg)f \cong h(gf).$$

Now "$(hg)f$ is equal to $h(gf)$" is reasonably thought of as a *property*, and "$(hg)f$ is isomorphic to $h(gf)$" is also a *property*, but as soon as we demand a *specified isomorphism*

$$(hg)f \xrightarrow{\sim} h(gf)$$

it has become a piece of structure. Where trichotomies don't match up, comparisons between theories become much technically harder.

The DSP question is closely related to questions of "algebraic vs non-algebraic" and cell shape.

1.2.2 Algebraic vs non-algebraic

Some definitions specify composition and constraints uniquely, where others just assert that they "exist". This can be thought of as the difference between being *algebraic* (specifying things uniquely) and *non-algebraic*. If it seems strange to consider composition that is *not* uniquely specified, consider the example of 1-cell composition in a bicategory.

In a bicategory, 1-cell composition is not strictly associative, so there is *not* a unique way composing a string of three composable 1-cells:

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

although there *is* a unique composite of any composable pair. Some definitions of n-category don't specify unique composites *at all*; instead, "exhibiting a composite" is a *property* of certain cells in the theory, and we just demand that enough of those structure-giving cells exist.

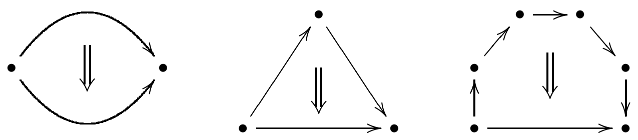
Algebraic structures might be thought of as being easier to calculate with, as they actually *give* the crucial pieces of information we need rather than just asserting their existence. However, one curiously convenient feature of non-algebraic approaches is that the whole issue of coherence constraints pretty much vanishes. If we don't have unique composites it doesn't even make sense to write down "$(hg)f$" and "$h(gf)$", so we can't ask if they are equal, isomorphic or anything else. We see that, in a way, this coherence issue only arises once we start choosing specified composites.

An algebraic approach is taken by Penon, Batanin, Leinster, Trimble and May; non-algebraic by Street, Simpson, Tamsamani and the Opetopic.

Remark on bias There is also the related issue of *bias*. Composition is called biased when some composites are specified and not others; typically nullary (i.e. identities) and binary composites might be used to generate the others, as with the case of classical bicategories. Unbiased composition specifies composites for *all* arities. This might be seen as the opposite extreme of specifying none at all.

1.2.3 Cell shape

The underlying shapes of cells is one of the most basic and evident differences between definitions. However, although it appears to occur at the *data* level it causes (or is caused by) much more far-reaching differences. The point is that different cell shapes are invoked in order that cells might play a wider role than just "being cells". That is, differences in cell shape often belies differences in the *role that cells play* in the structure.



The most basic cell shape is the globular shape. The rule of thumb is that globular cells play no role except "being cells"; other shapes of cells are needed if they are to be used for, say, giving composition. This is a feature of what we call the "nerve-like" approaches (Street, Opetopic, Joyal, Simpson/Tamsamani), and we will see that these are the ones taking non-globular cell shapes.

Remark *In the nerve-like definitions, the data for an n-category is a presheaf*

$$A : \Sigma^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

where Σ is the category of "shapes" for the theory in question. The functor A is then thought of as giving us, for each shape, a set of cells of that shape. The morphisms in the category Σ give us the clues as to how we might interpret the "shapes" as actual geometrical shapes.

1.2.4 Weak n-categories or n-weak categories?

As we said at the beginning, the difficulty is in making something both weak *and* higher-dimensional. In generalising from ordinary categories to weak n-categories, we could:

- weaken the structure first, and then increase dimensions, or

- increase dimensions strictly, and then weaken the structure.

We have the following schematic diagram:

$$
\begin{tikzcd}
\begin{array}{c}\text{strict}\\\text{low-dimensional}\\\text{categories}\end{array} \arrow[r, "\mathsf{weaken}"] \arrow[d, "\begin{array}{c}\mathsf{build}\\\mathsf{dimensions}\end{array}"'] & \begin{array}{c}\text{weak}\\\text{low-dimensional}\\\text{categories}\end{array} \arrow[d, "\begin{array}{c}\mathsf{build}\\\mathsf{dimensions}\end{array}"] \\
\begin{array}{c}\text{strict}\\\text{higher-dimensional}\\\text{categories}\end{array} \arrow[r, "\mathsf{weaken}"'] & \begin{array}{c}\text{weak}\\\text{higher-dimensional}\\\text{categories}\end{array}
\end{tikzcd}
$$

The bottom left route is taken by Street, Penon, Batanin, Leinster and Joyal. The top right is taken by Trimble, May, Simpson, Tamsamani and the Opetopic. A technical way of thinking about it is that the top right route is inductive; one consequence of it is that we can't reach ω-categories in this way, only n-categories for finite n. We will not dwell much on this difference, and will sometimes say "n-category" even when the possibility of ω is there.

1.2.5 Beheading vs headshrinking

Some definitions include in their data k-cells for *all* k, not just $k \leq n$. The idea to assert that, for dimensions greater than n, the structure has "shrunk" to being trivial, in some suitable sense. Other definitions actually chop off the data at n dimensions. Those definitions whose cells play a wider role than just "being cells" are likely to need cells of higher dimensions than n, since in those cases

- structure on k-cells is given by $(k+1)$-cells, with
- properties given by $(k+2)$-cells.

So in order to give structure (such as composition) on n-cells in such a theory, $(n+1)$-cells will be required, and beheading the structure is therefore too violent.

1.2.6 Conclusion

The definitions and their features as discussed above are collected in table form in the Appendix. We must stress that in highlighting the above differences we are not trying to suggest that the differences between definitions are irreconcilable. On the contrary, it is our view that understanding differences is an important part of understanding similarities.

Chapter 2

Penon

2.1 Introduction

We have chosen to start with the definition of Penon as we find it the most direct, invoking little complicated machinery. Readers who are unfamiliar with monads may disagree, but we include a crash course for their benefit at the end of this chapter. The definition was given in [86] and takes the following form:

> *An n-category is an algebra for a certain monad on the category* **RefGSet** *of reflexive globular sets.*

The starting point is:

> *An n-category should have*
>
> - *cells of each dimension, with a suitable source and target of the dimension below*
> - *binary composition at all dimensions*
> - *identities*
>
> *satisfying some* weak *associativity and unit "conditions".*

These weak conditions actually take the form of "mediating cells" giving higher morphisms between certain composites. The composites that need to be related in this way are those that *would have been equal in a strict setting.*

In a *strict* n-category, we can take a pasting diagram of cells and compose them in any order we like—the result will be the same. To make a *weak* n-category, we are going to "stretch out" a strict n-category a bit, so that there is a bit of "distance" in between these composites done in different orders—but not *too* much. And it shouldn't be empty space in between; they must be connected by a suitable mediating cell.

It must be said that this definition can look very unintuitive if presented in its most concise form, for example as in [69]. In particular the so-called "mysterious category $\mathcal{Q}$" does not look so mysterious in Penon's original paper, where it is reached in two stages rather than plucked suddenly from thin air.

In fact, approached from the right direction, this definition can be one of the most intuitively straightforward of the definitions — provided one is comfortable with the idea of algebras for a monad. Of course, as with all the definitions, it is important to start from the right place in order to see how a construction is "natural". We end this chapter with an introduction to monads and their algebras, in Section 2.4.

Finally we note that there is a natural "non-reflexive" variant of Penon's definition which we discuss in Section 2.3.4; it seems that the original "reflexive" version gives a notion of ω-category that is a little stricter than we would aim for in general, but this can be fixed by using the non-reflexive version instead.

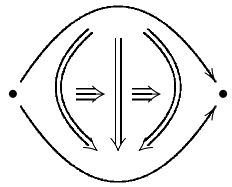
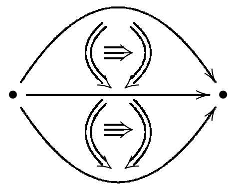
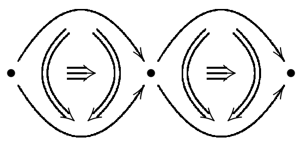
2.2 Intuition

We begin by thinking very naively about what a weak ω-category should look like, in the spirit of the "graph with structure" type of definition (Section 1.1.1). We might start with

i) globular cells of each dimension,

ii) identity cells: for every k-cell α a $(k+1)$-cell $i_\alpha\colon \alpha \longrightarrow \alpha$ even if we don't know yet how these cells are going to behave,

iii) binary composition: we can compose k-cells along boundary p-cells, for $0 \leq p < k$. We call this "p-composition" and write $\alpha \circ_p \beta$. The case $k = 3$ is depicted below.

k=3	
	2-composition or composition along a bounding 2-cell
	1-composition or composition along a bounding 1-cell
	0-composition or composition along a bounding 0-cell

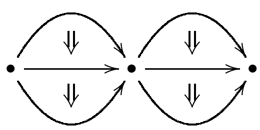
To make a *strict* ω-category we would now give *strict* associativity and interchange axioms — or we could require some universal property like:

> *Given any pasting diagram there is precisely one way of composing it.*

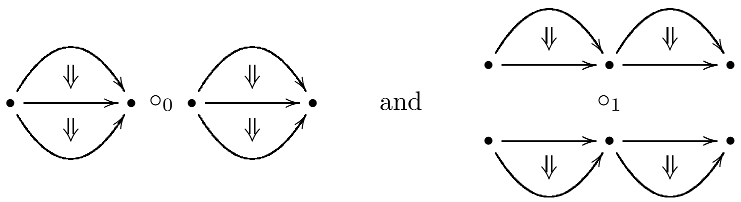
For a *weak* ω-category, we want to say something like

> *Given any pasting diagram there is a whole bunch of ways of composing it in different orders, but they should all be suitably related.*

"Suitably related" means related by mediating cells , perhaps pseudo-invertible ones, satisfying some coherence . For example here is a pasting diagram

 .

In a strict ω-category, the interchange law tells us that all of the ways of composing this diagram are equal. So it doesn't matter in what order we perform this composition. However, in a weak ω-category it does matter — we need to distinguish between



This is a bit like putting brackets into a "strict pasting diagram" to make it into a "weak pasting diagram". We can always then forget those brackets and get back to the underlying strict pasting diagram.

So we have some sort of map:

$$\begin{array}{c}\text{“Weak pasting diagrams”}\\ \text{(Order is remembered)}\\ \Big\downarrow \\ \text{“Strict pasting diagrams”}\\ \text{(Order doesn'tmatter)}\end{array}$$

Coherence is imposed by requiring that any two weak pasting diagrams with the same underlying strict pasting diagram have a mediating cell between them (one dimension up).

This is where *contractions* come in. The idea will be something like:

> *The space of weak pasting diagrams lying over any given strict pasting diagram should be contractible.*

So the components of Penon's definition are:

i) globular sets (with putative identities)

ii) binary composition

iii) contractions

2.2.1 Magmas

A *magma* is a structure given by the components (i) and (ii) above. That is, we have:

i) a reflexive globular set = a globular set with putative identities picked out

ii) binary p-composition of k-cells (see diagrams on page 11)

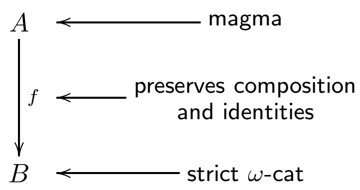
A map of magmas is a map of underlying globular sets preserving the putative identities and the composition.

A magma can be thought of as a primitive sort of ω-structure, like an ω-category with absolutely no coherence imposed at all. We will then use a contraction to impose order on the magma and force the composition to be something sensible.

2.2.2 Contractions

> *Where does a contraction live? What are we requiring to be contractible?*

The idea is to relate a magma to a strict ω-category and to use the coherence of the strict composition to keep control over the composition in the magma. So we use a diagram of the following form:



This is where our contraction is going to "live" (although actually all we need in order to state the definition of contraction is that A be a globular set and B a reflexive globular set).

The definition of contraction *almost* says:

> *Given any two cells with the same image under f, there must be a given cell in between them that maps to the identity.*

However, if we try to formalise this we see that it doesn't make sense to ask for a cell in between two cells unless they have the same source and target as one another. We say two k-cells are *parallel* if $k > 0$ and they have the same source and target as one another; we call all 0-cells parallel.

Then the definition of contraction *actually* says:

> *Given any two parallel k-cells α and β with the same image under f, there must be a given $(k+1)$-cell $\alpha \longrightarrow \beta$ that maps to the identity under f.*

Stated formally we have

Definition *A* contraction[1] $[\ ,\ ]$ *on a map f gives: for all α, $\beta \in A(k)$ such that*

[1] The definition given in [69] is much more abstractly formal. There, $V_\phi(m)$ is "the set of all pairs of m-cells that need a contraction" and a contraction is then defined as a map

$$V_\phi(m) \longrightarrow A(m+1)$$

for each m, which picks out a suitable contraction cell for each pair that needs one.

i) α and β are parallel, and

ii) $f(\alpha) = f(\beta)$,

a "contraction cell" $[\alpha,\beta]\colon \alpha \longrightarrow \beta$ such that

$$f([\alpha,\beta]) = 1_{f\alpha} = 1_{f\beta} \in B.$$

Further, for all α we must have $[\alpha,\alpha] = 1_\alpha \in A$.

Pictorially:

given
$$\begin{tikzcd}[row sep=large]
\bullet \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[d, phantom, "\big\downarrow"] & \bullet \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$
the contraction gives a
$$\begin{tikzcd}[row sep=large]
\bullet \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "{[\alpha,\beta]}"] \arrow[d, phantom, "\big\downarrow"] & \bullet \\
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\mathrm{id}\Downarrow"] & \bullet
\end{tikzcd}$$

Note on contractibility

Note that a contraction actually specifies the cells $[\alpha,\beta]$; "contractibility" is about *existence* of such cells, without actually specifying them.

Remark on pseudo-invertibility

There is some symmetry in the definition of contraction — if the pair (α,β) needs a contraction, then so does the pair (β,α). So contraction cells come in pairs

$$\begin{tikzcd}
\alpha \arrow[r, bend left, "{[\alpha,\beta]}"] & \beta \arrow[l, bend left, "{[\beta,\alpha]}"]
\end{tikzcd}$$

and the idea is for these to be pseudo-inverse to one another. If we compose them (in A) we get something that lies over the identity in B since

$$f([\alpha,\beta]) = \mathrm{id}$$

$$f([\beta,\alpha]) = \mathrm{id}.$$

Now f preserves composition and identities, so

$$f([\beta,\alpha] \circ [\alpha,\beta]) = \mathrm{id} = f(1_\alpha)$$

which means we must have a contraction cell

$$\begin{tikzcd}
{[\beta,\alpha]\circ[\alpha,\beta]} \arrow[r, bend left] & 1_\alpha \arrow[l, bend left]
\end{tikzcd} .$$

We can keep going, and the contraction will keep explicitly giving cells witnessing the fact that $[\alpha,\beta]$ and $[\beta,\alpha]$ really are pseudo-inverse to one another.

Remark on flavours of contraction

An issue of terminology: Penon and Batanin both use this flavour of contraction where Leinster uses a "richer" one. However, Batanin and Leinster both use the term "contraction" whereas Penon uses the French "étirement" which means "stretching". It is not so strange that these words seem to be opposites — saying that A contracts to B amounts to the same as saying B stretches to A, just seen from the other direction. The emphasis in Penon's case is perhaps the idea of taking a strict structure and "stretching" it out to allow weakness; the emphasis for Leinster and Batanin is the idea of ensuring a structure is not too weak by making sure it can contract down to a strict one.

2.2.3 The all-important category $\mathcal{Q}$

Given the above data for a contraction, we can immediately make a category out of it: the *no-longer-mysterious category* $\mathcal{Q}$ is the category whose objects are $\left(\begin{array}{c} A \\ \downarrow f \\ B \end{array}\right)$ as above (page 13), equipped with a specified contraction. Morphisms preserve everything possible.

2.3 The actual definition

We will now launch in and give the definition directly. The final step is technically simple (to *state* — harder to prove) but not necessarily intuitively clear. We will therefore immediately unravel the definition to see what it actually looks like.

2.3.1 Formalities

The idea is to define n-categories as algebras for a certain monad: we have a forgetful functor

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{G} & \mathbf{RefGSet} \\ \\ \begin{array}{c} A \\ f\downarrow \\ B \end{array} & \longmapsto & A \end{array}$$

which sends $\left(\begin{array}{c} A \\ \downarrow f \\ B \end{array}\right)$ to the underlying reflexive globular set of A. The interesting and crucial result is that G has a left adjoint F.

Definition *A weak ω-category is an algebra for the monad $P = GF$.*

Note that $F \dashv G$ is not monadic so the left adjoint F is not actually a "free ω-category" functor, and so $\mathcal{Q}$ is not equivalent to **Weak-ω-Cat** (see Section 2.4). That is, the objects of $\mathcal{Q}$ are not themselves ω-categories but they do play an instrumental role in defining them.

2.3.2 Explanations

So, what does an algebra for this monad "look like"? Given a reflexive globular set A, PA is supposed to be the "free weak ω-category on A" (or rather, the underlying reflexive globular set of the free weak ω-category) obtained by constructing a $\mathcal{Q}$-object "freely" from A. This free $\mathcal{Q}$-object FA will look like

$$\begin{tikzcd} PA \arrow[d, "f"] \\ TA \end{tikzcd}$$

where TA is the free *strict* ω-category on A.

Remark *The fact that TA is the right thing to have at the bottom is perhaps more evident in Penon's original version, but here it should at least seem the obvious thing to try. What other strict ω-category could we possibly use, if we are only given A to start with?*

We can then construct PA from A in quite a hands-on way, dimension by dimension, freely adding in precisely what is necessary to ensure that

i) PA is a magma, and

ii) f has a contraction.

The first few steps proceed as follows:

0-cells These stay the same and f acts as the identity.

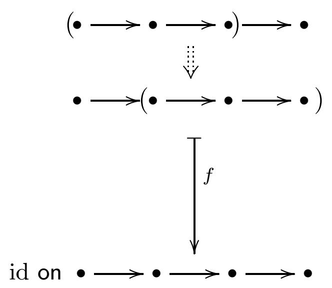
1-cells

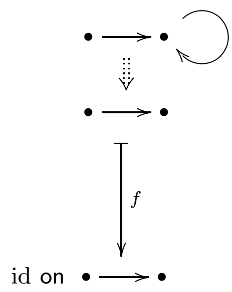
i) New 1-cells required for contraction on 0-cells:
but f is the identity on 0-cells so we only need contraction cells $[\alpha, \alpha]$, i.e. 1_α, but these already exist in the reflexive globular structure of A.

ii) New 1-cells required for magma structure:
we already have identities, but we need to add binary composites freely, and then binary composites of binary composites, and so on. We do it by induction over the 'depth' of the composite. f sends these to the underlying strict composite, i.e. removes the brackets.

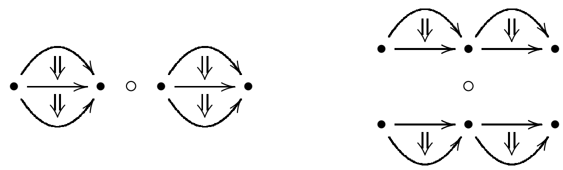
2-cells

i) New 2-cells required for contraction on 1-cells:
we need to add in a contraction cell mediating between any pair of weak composites that have the same underlying strict composite. For example,

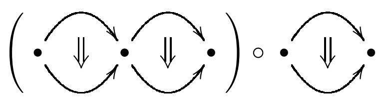


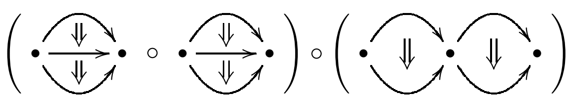


ii) New 2-cells required for magma structure:
identities 1_α are all given by contraction cells $[\alpha, \alpha]$ so we just need to add in binary composites, and binary composites of binary composites and so on. For example,



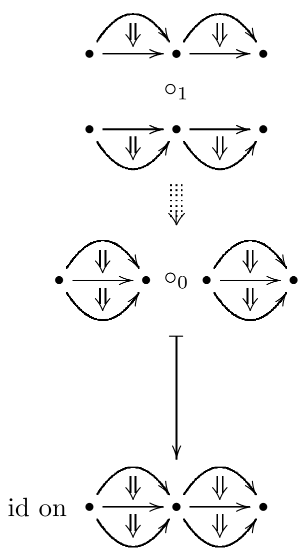
and





and we start to see how difficult these weak composites are to draw.

3-cells Finally we give an example of a 3-cell from contraction.



giving the interchange constraint.

This gives an idea of what PA looks like.

Finally, a P-algebra gives us an action $\begin{pmatrix} PA \\ \downarrow \\ A \end{pmatrix}$ so it gives

i) composition, i.e. it evaluates the formal weak composites as actual cells of A

ii) mediating cells i.e. it evaluates the formal contraction cells as actual cells of A

Remark *In the next chapter we will use a similar "dimension-by-dimension" method to construct the monads for the Batanin/Leinster definitions.*

2.3.3 Things to tinker with

In the next chapter we will discuss the definitions of Leinster and Batanin, and we will informally "compare and contrast" these definitions as we go along. We pause for a moment now to emphasise the places in this definition where an alternative choice of approach could have been taken; the remainder of this chapter will be an introduction to monads.

i) **Reflexivity:** we could do this whole definition starting with plain (non-reflexive) globular sets rather than reflexive ones. That is, identities could be given along with composites as part of the algebra action $PA \longrightarrow A$, rather than as *a priori* data. It might seem that this should give the same notion of ω-category, but in fact this is not the case; see Section 2.3.4.

ii) **Bias:** Again, we could repeat the whole definition but using *unbiased* composition for the magma structure (see Section 1.2.2). It would, however, make the freely generated composites technically harder to describe. We expect that the end result should be *in some sense* equivalent, but making this precise is difficult.

iii) **Flavour of contraction:** as discussed in Section 2.2.2.

iv) **Use of operads to describe weak composition:** we could also use other algebraic structures like operads to describe the weak composition. This approach will be taken in the next chapter for the definitions of Batanin and Leinster.

2.3.4 Reflexivity vs non-reflexivity

We include some brief remarks here about the difference between the reflexive and non-reflexive approaches. A full treatment would take more time than is appropriate in this guidebook, but we include a few comments so that the clued-up reader might be able to work out the details for himself; the less clued-up reader will at least get an idea of where the issues are.

The problem can be seen in the following fact:

> *In the reflexive Penon set-up, braided monoidal categories are forced to be symmetric.*

Here we are constructing braided monoidal categories as degenerate 3-categories with only one 0-cell and one 1-cell. The idea is that the braiding should come from the "interchanger" of 2-cells, but in the reflexive set-up this interchange mediator is forced to be the identity.

The general slogan then is that *interchange is too strict.* More precisely, the problem arises for any cells (of any dimension) whose domain and codomain are identity cells. Since a braided monoidal category is (here) a 3-category with only one 1-cell, that 1-cell must be the identity. So *all* 2-cells have an identity 1-cell as domain and codomain, and hence the problem arises for all of them. Essentially, the problem comes down to the fact that cells with an identity as domain and codomain can be "interchanged" past one another using an Eckmann-Hilton type argument.

The problem does not arise in the non-reflexive case because in the underlying globular set we don't *know* which cells are identities. At the moment of constructing interchangers, this information about identities is still a "secret"; in the reflexive case the secret is made public too soon, and this results in the interchangers becoming too strict.

2.4 A crash course on monads and their algebras

In this section we will briefly and informally review monads and their algebras. We will not aim to be comprehensive or completely rigorous; rather, we aim to give a sort of crash course so that the reader who is unfamiliar with these ideas might at least get a feel for how a monad is used to get a handle on an algebraic structures. For a full treatment we refer the reader to any standard text such as [75].

We will explain how the algebras for a certain monad give small categories. This construction can be generalised to give strict n-categories as algebras for the "strict n-category monad". Some of the definitions of *weak* n-category discussed in this work arise as algebras for a certain monad; others (the "non-algebraic") do not. We will start with a simpler example but first we must get the definition out of the way.

2.4.1 Monads

The slogan is:

> *A monad is an algebraic theory and an algebra for a monad is a model of that theory.*

The idea is that a monad gives us a way of describing a theory (such as "the theory of groups", "the theory of categories", "the theory of compact Hausdorff spaces") by encapsulating all the information about how structures in that theory are required to behave. This works provided the theory is well enough behaved.

Monads are intimately related to adjunctions. Any adjunction gives rise to a monad. We will use this to motivate the definition but the definition can just as well be made directly.

Suppose we have an adjunction $F \dashv G \colon \mathcal{C} \longrightarrow \mathcal{D}$ with unit $\eta \colon 1_{\mathcal{C}} \longrightarrow GF$ and counit $\varepsilon \colon FG \longrightarrow 1_{\mathcal{D}}$. If we write $T = GF \colon \mathcal{C} \longrightarrow \mathcal{C}$ we have the following two natural transformations:

- $\eta \colon 1_{\mathcal{C}} \Rightarrow GF = T$ with components
$$\eta_X \colon X \longrightarrow TX$$
- $G\varepsilon F \colon GFGF \Rightarrow GF$ which we write as $\mu \colon T^2 \Rightarrow T$ with components
$$\mu_X \colon T^2X \longrightarrow TX$$

where η and ε are the unit and counit of the adjunction. From the axioms of an adjunction the following diagrams commute:

$$\begin{tikzcd} TX \arrow[r, "T\eta_X"] \arrow[dr, "1_{TX}"'] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "\eta_{TX}"'] \arrow[dl, "1_{TX}"] \\ & TX & \end{tikzcd} \qquad \begin{tikzcd} T^3X \arrow[r, "\mu_X T"] \arrow[d, "T\mu_X"'] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

If we now use the above facts as axioms we arrive at the definition of a monad:

Definition *A* monad *on a category $\mathcal{C}$ consists of a functor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ equipped with natural transformations*

- $\eta\colon 1_{\mathcal{C}} \Rightarrow T$ *the "unit", and*
- $\mu\colon T^2 \Rightarrow T$ *the "multiplication",*

satisfying the above axioms.

Example 1: groups

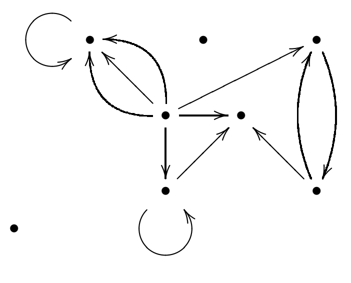
A standard example is the monad for groups. There is a forgetful functor $U\colon \mathbf{Grp} \longrightarrow \mathbf{Set}$ from the category of groups to the category of sets which simply forgets about the multiplication, identities and inverses. The interesting thing is that this functor has a left adjoint $F\colon \mathbf{Set} \longrightarrow \mathbf{Grp}$ which sends every set to the free group on that set. The composite UF is a monad on **Set**, the "free group" monad, and we will later see that algebras for this monad are precisely groups. This trick works for an important class of structures which we think of as "well-behaved" by virtue of this highly desirable property.

Example 2: categories

As another example, consider the small category monad. A small category is completely specified by a set of objects C_0, a set of morphisms C_1 and functions $s, t\colon C_1 \rightrightarrows C_0$, equipped with identities and composition satisfying some axioms. If we forget the extra structure (i.e. identities and composition) we get a forgetful functor $U\colon \mathbf{Cat} \longrightarrow \mathbf{Grph}$ from the category of small categories to the category of "graphs". Here, a graph (or "directed graph" or "1-globular set") is a diagram of sets

$$C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0\,.$$

We can think of a graph as a set of vertices with some arrows between them; a typical graph looks something like



where C_0 is the set of vertices and C_1 is the set of edges. The maps $s, t\colon C_1 \longrightarrow C_0$ assign a source and target to each edge, so we can add arrowheads accordingly. A small category is a graph equipped with suitable composition and identities.

Again, as with the case of groups, the forgetful functor U has a left adjoint F which sends any graph $C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$ to the category freely generated by it. This category will have the same set of objects but a larger set of morphisms because we have "thrown in" extra morphisms to give composites and identities. Specifically, we must throw in one morphism for each *composable string of k arrows*, for all $k \geq 0$. Composition is given by concatenation of strings. Note that a "composable string of 0 arrows" is interpreted as an "empty string" on an object, giving the identity.

As before, this adjunction gives a monad

$$T = FU\colon \mathbf{Grph} \longrightarrow \mathbf{Grph}.$$

We will consider this monad in more detail as it is an important starting point for the generalisation to n-categories.

The unit map $\eta_C : C \longrightarrow TC$ embeds our original graph in our bigger one by sending the original arrows to "strings of length 1" in our bigger graph. The multiplication $\mu_C : T^2C \longrightarrow TC$ turns a "composable string of composable strings of arrows"

$$\overbrace{\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet}\,\overbrace{\longrightarrow \bullet \longrightarrow \bullet}\,\overbrace{\longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet}$$

into a "composable string of arrows" in the obvious way, by forgetting the subdivisions.

The axioms confirm two obvious facts:

i) The triangular ("unit") axioms say that if we take a string of arrows, put a subdivision around each arrow and then forget that we did it, we get back to where we started; likewise if we forget we put brackets around the whole thing and then forget them.

ii) The square ("associativity") axiom says that if we have two layers of subdivision to forget about, it doesn't matter which one we forget first.

As with the previous example, we will see that the algebras for this monad are precisely categories.

2.4.2 Algebras for a monad

Recall our slogan:

> *A monad is an algebraic theory and an algebra for a monad is a model of that theory.*

We define an algebra for a monad as follows:

Definition *Let (T, η, μ) be a monad for $\mathcal{C}$. An* algebra *for T consists of an object $A \in \mathcal{C}$ together with a morphism $TA \xrightarrow{\theta} A$ such that the following diagrams commute:*

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\theta"] \\
& A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T^2A \arrow[r, "\mu_A"] \arrow[d, "T\theta"'] & TA \arrow[d, "\theta"] \\
TA \arrow[r, "\theta"'] & A
\end{tikzcd}$$

We refer to A as the "underlying object" of the algebra, and θ as the "algebra action". Many of our leading examples are monads on **Set**, so that an algebra is a "set equipped with extra structure" or a "set with some operations on it". We're quite used to thinking about the underlying set of a group, the underlying set of a topological space, and so on. The "algebra action" then tells us where the required extra structure is to be found, or how the operations are to be evaluated.

Example: categories

We will discuss the example of the small category monad $T\colon \mathbf{Grph} \longrightarrow \mathbf{Grph}$ from the previous section. What does an algebra for this monad look like?

The underlying object is a graph $C \in \mathbf{Grph}$, and the algebra action θ is a map from the set of "composable strings of arrows" back to the original set of arrows. This is a sort of "evaluation map" and tells us *how each possible composite is to be evaluated.*

We now examine the algebra axioms. The first axiom tells us that θ acts as the identity on the set of objects $TC_0 = C_0$ and that every string of length one evaluates to the actual arrow it came from in C_1. The second axiom tells us about "composable strings of composable strings". The lower-left side of the diagram says "compose the sub-strings and then compose the results" and the upper-right side says "forget the subdivisions and perform the whole composition all at once". This is a generalised form of associativity as illustrated in the following example:

$$\begin{tikzcd}
(\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2})\bullet(\xrightarrow{f_3} \bullet \xrightarrow{f_4} \bullet \xrightarrow{f_5} \bullet) \arrow[r, mapsto] \arrow[dd, mapsto] & \bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \xrightarrow{f_3} \bullet \xrightarrow{f_4} \bullet \xrightarrow{f_5} \bullet \arrow[d, mapsto] \\
& \bullet \xrightarrow{f_5\circ f_4\circ f_3\circ f_2\circ f_1} \bullet \arrow[d, equal] \\
(\bullet \xrightarrow{f_2\circ f_1})\bullet(\xrightarrow{f_5\circ f_4\circ f_3} \bullet) \arrow[r, mapsto] & \bullet \xrightarrow{(f_5\circ f_4\circ f_3)(f_2\circ f_1)} \bullet
\end{tikzcd}$$

Note that the small category we construct as an T-algebra does not have a preferred binary composition. That is, a k-ary composite is given *for each* $k \geq 0$. A category with k-ary composites defined for all k rather than just a binary and nullary composite can be thought of as an *unbiased* category. However, it is a happy consequence of strict associativity that we can, equivalently, specify just binary composites and identities, and check some simple special cases of the general associativity axiom.

Remark on terminology

Note that monads were originally called by another name which we prefer to forget. We agree with Mac Lane [75] that that terminology has achieved "a maximum of needless confusion".

Chapter 3

Batanin and Leinster

Introduction

The definitions of Batanin [9] and Leinster [72] look a bit similar to that of Penon in that each gives n-categories as the algebra for a monad, and the underlying data is a globular set of cells. Furthermore, in each case the idea is to use the well-understood strict theory and weaken it using a notion of "contraction".

The striking difference is the use of an *operad* to produce a monad in the Batanin/Leinster approach. The definitions look like this:

> ***Batanin:*** *An ω-category is an algebra for an initial "operad with contraction and system of compositions."*
>
> ***Leinster:*** *An ω-category is an algebra for an initial "operad with (a slightly different kind of) contraction."*

This statement of the definitions is rather crude, but highlights the idea of Leinster's slightly different contraction, which is to handle Batanin's notions of *contraction* and *system of composition* all at once.

Points of difference

The use of operads can be thought of as a technical difference, but the definitions also look quite different because of some "ideological" differences. There are essentially four points of difference:

	Penon	Batanin	Leinster
Reflexivity	✓	-	-
Biased composition	✓	✓	-
Biased contraction	✓	✓	-
Use of operad	-	✓	✓

We think of reflexivity and the use of an operad as "technical" differences whereas biased composition and biased contraction can be thought of as "ideological" differences. We will now discuss each of these differences a little.

1. Reflexivity

This is about whether identities are given as part of the underlying data ("reflexive") or as additional structure afterwards ("non-reflexive"). This turns out to make an important difference to the resulting structure; see Section 2.3.4.

2. Biased composition

Penon and Batanin specify only *binary* and *nullary* composites ("biased"), whereas Leinster specifies *all arities* ("unbiased"). The "general feeling" is that the biased and the unbiased should be in some sense equivalent, but the hard part is finding the right framework for expressing this equivalence.

3. Biased contraction

Considering contraction as a process of lifting cells, Penon and Batanin lift only identity cells ("biased") where Leinster lifts all cells ("unbiased"). This can be thought of as a question of "bias" as in one case we are specifying only a particular type of contraction cell, and (implicitly) using these to generate all the others. For an example of this in more details, see Section 3.3.4.

4. Use of operad

There are (at least) two ways to think of the use of operads:

i) Often, one of the most technically hard parts of any definition is how to express all the weak composites (i.e. weakly associative composites). Operads give a "slick" way of expressing weak composites.

ii) In Penon's definition we start with actual labelled cells and make composites of them; alternatively we can use an operad to make composites of unlabelled cells, and then put the labels on afterwards.

We will elaborate on this in the next section.

We note that while the use of an operad can be thought of as a technical device for expressing weak composition, in the end the composition turns out to be subtly different in some particular situations. We will discuss this further in Section 3.5.1.

3.1 Intuitions

We begin by giving an intuitive feel for the ideas of this definition. Probably the hardest idea to picture is the particular kind of operad being used here: by "operad" in this chapter we really mean "globular operad".

The actual definition of a globular operad involves various pieces of background material that are not essential for an intuitive understanding; in any case the rigorous definition is so elegantly abstract and compact that it takes a great deal of unpacking to work out what it "looks" like. So we begin here by giving an intuitive account of why operads might arise in a definition of n-category, deferring a fuller definition to Section 3.2.

3.1.1 Why operads I

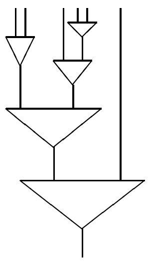
In this section we answer the question: Why operads? We can answer this question from two points of view; we will take the opposite direction in the next section. In both cases the starting point is going to be the following question:

How can we make weak composites of globular cells?

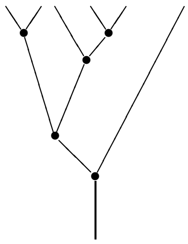
First, let's think about how we can represent a weak composite of 1-cells e.g.

$$\Big((\bullet \longrightarrow \bullet \longrightarrow \bullet)(\longrightarrow (\bullet \longrightarrow \bullet \longrightarrow \bullet)) \longrightarrow \bullet\Big)$$

So what we really mean by "weak composite" is a composite in which we have to remember the order of composition, since associativity is not strict. We can 'remember' in what order we did the composition by using the following diagram

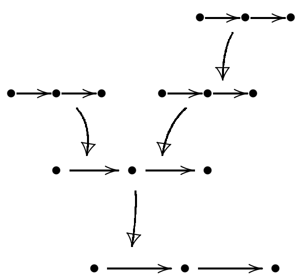


or



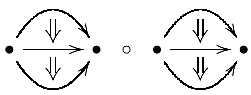
Here,  is supposed to be like an arrow pointing downwards but with two inputs and one output. It is telling us that we have fed in two composable arrows and composed them to make one single arrow.

In fact, these diagrams illustrate an operadic type of composition. We can think of it as "sticking a configuration of cells into the place where one cell used to be" or building up a "compound" composite stage by stage. The following diagram illustrates this idea for the example above:

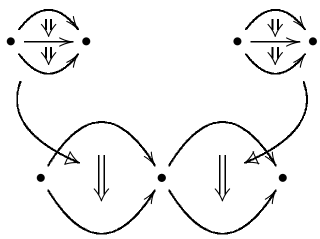


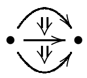
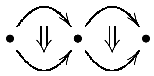
Some more examples in higher dimensions

The weak composite

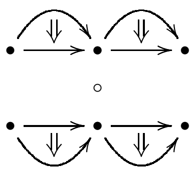


can be expressed as

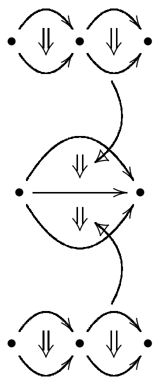


which says: first do each vertical composite , and then compose the results together by .

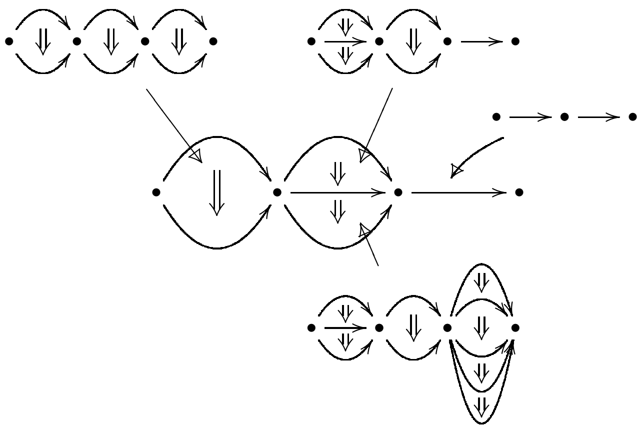
Similarly, the weak composite



can be expressed as



A more general operadic composite of 2-cells might look like



Now that we're in two dimensions the difference is that "the things we're sticking in aren't linear." This is where *globular* operads come in:

> *The things we are sticking in have* globular *pasting diagrams as their underlying shape.*

So we can now give the first answer to our question.

> **Question:** *Why operads?*
>
> **Answer I:** *Because we can use operadic composition to express compound composites of globular cells.*

3.1.2 Why operads II

Why operads? We will now answer this question from the opposite direction. The following table sums up the ideas we will now consider.

	Penon	**Batanin/Leinster**
technique	non-operadic	operadic
idea	label the cells and then build weak composites	build weak composites and then label the cells

The starting point is still going to be the issue of how to make weak composites of globular cells. We illustrate this by thinking about the following three composable 1-cells:

$$\boxed{a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d}$$

- In the Penon definition we take the cells and build the possible weak composites directly

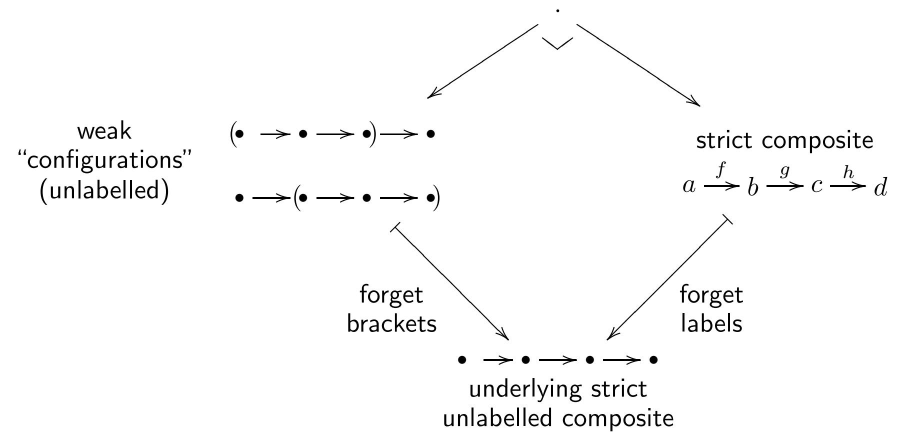
$$(a \xrightarrow{f} b \xrightarrow{g} c) \xrightarrow{h} d$$

$$a \xrightarrow{f} (b \xrightarrow{g} c \xrightarrow{h} d)$$

(ignoring possibilities with identities).

- Alternatively, we could "compose-then-label", that is, find the weak composites for *unlabelled* cells first, and put the labels on afterwards.

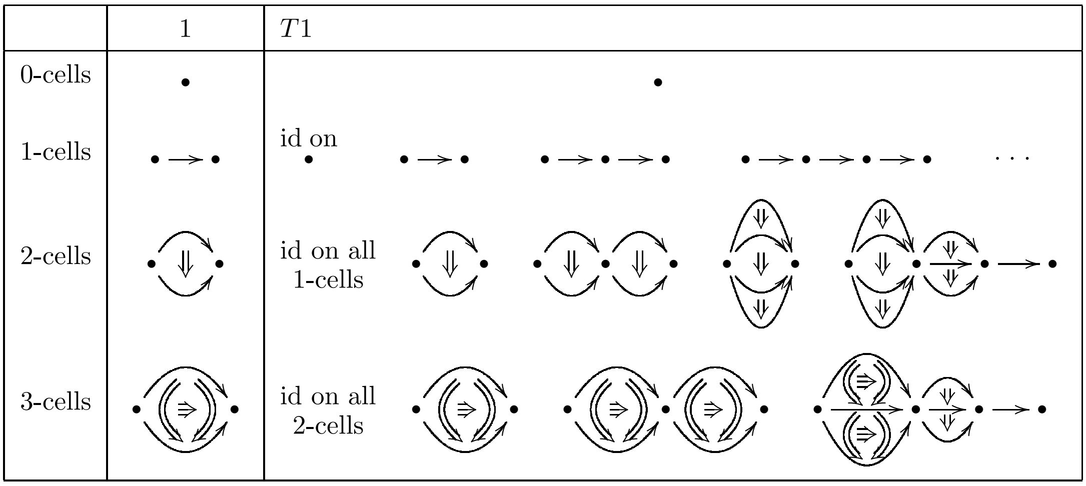
We then get the following pullback



Taking the pullback here "puts the labels on the weak configurations". That is, an element of the pullback is

- a *strict* formal composite, together with
- a *weak* configuration with the same underlying strict unlabelled composite.

A weak configuration can be thought of as an "order of composition" for a strict pasting diagram. Strict composites are things we can handle: we have the free strict ω-category monad T on the category **GSet** of globular sets, which takes a globular set A and gives us all strict formal composites (including identities) of cells in A. We can also do this to unlabelled cells by applying T to the terminal globular set 1, which has precisely one cell of each dimension. Some examples of the unlabelled case are illustrated below:



So to make labelled composites as above, we want a pullback of the following form in **GSet**:

$$
\begin{tikzcd}
 & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
W \arrow[dr] & & TA \arrow[dl] \\
 & T1 &
\end{tikzcd}
$$

A priori, W could be any globular set of "weak configurations" that we choose. W gives us immediate and direct control over what we want our theory to "look like". We could put biased or unbiased composites in W, or strange combinations of things (or indeed strict composites), as long as we specify a strict composite in $T1$ that each element of W lies over.

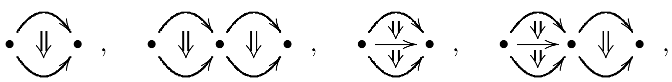
For example, for classical (biased) bicategories, W must at least have:

0-cells $\bullet$

1-cells

$\bullet \to \bullet$, $\bullet \to \bullet \to \bullet$, $(\bullet \to \bullet \to \bullet) \to \bullet$, $(\bullet \to \bullet \to \bullet)(\to \bullet \to \bullet)$, ...etc

2-cells



... and all 2-cell composites.

We also need an associator 2-cell

$$\begin{array}{c} \bullet \longrightarrow (\bullet \longrightarrow \bullet \longrightarrow \bullet\,) \\ \Downarrow \\ (\bullet \longrightarrow \bullet \longrightarrow \bullet) \longrightarrow \bullet \end{array}$$

lying over the identity 2-cell on $\bullet \to \bullet \to \bullet \to \bullet \ \in T1$, and unlabelled unit constraints.

Let's see how the "unlabelled associator" gives us all the associators we need in the pullback.

- An element of the pullback is a pair

$$(\theta \in W,\ \alpha \in TA)$$

such that θ and α lie over the same element of $T1$.

Now suppose θ is the unlabelled associator given above. We get an element of the pullback since:

i) $\theta \in W$ lies over the identity 2-cell on $\bullet \to \bullet \to \bullet \to \bullet \ \in T1$, and

ii) the identity on $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ in TA also lies over the same element of $T1$.

So we get an element of the pullback, giving the "labelled associator"

$$a_{hgf} \colon h(gf) \Rightarrow (hg)f.$$

What is a sensible choice of W?

So far we have only said that W is a globular set of weak configurations for our theory. We now need to ask the following question:

What choice of W would make a sensible theory of ω-categories?

The first issue is that we would like this construction to produce a *monad* on **GSet**, as this will at least give us some kind of well-behaved algebraic theory. The monad would take a globular set of cells A and produce the "weak composites labelled by A", given by the pullback. There is certainly one known way of ensuring that this is really a monad:

The pullback construction produces a monad if $W \longrightarrow T1$ is an operad.

So we can now give our second answer to our motivating question.

Question: *Why operads?*

> **Answer II:** *Because this will ensure that the construction gives us a monad.*

Of course, just having a "sensible" algebraic theory is not enough: there are plenty of sensible algebraic theories which are nothing like n-categories. So it remains to be seen *which* operads give theories that deserve to be called "n-category", and this is where the remaining components of this definition come in. This is analogous, at least in spirit, to the part of May's definition which involves deciding which operads are valid for parametrising weak composition; see Chapter 8.

Remark on algebraic theories

The fact that "monads give algebraic theories" is a standard idea of category theory. We might wonder if it is worth using all this abstract theory — if we dropped these conditions on W, how "unsensible" might the theory become? We will not go into this issue here.

3.1.3 What is an operad with contraction?

Globular operads

The actual definition of a globular operad is quite technical and involves various pieces of background theory that aren't really necessary for getting a feel for what's going on. Cartesian monads come into it, but we will postpone thinking about this for as long as possible. For now we just give a vague description, so that in the next section we can give an idea of what the definition of ω-category is going to look like.

The idea of a globular operad is to index cells by globular pasting diagrams, and then compose them by "sticking diagrams into each other" as in Section 3.1.1.

So the underlying data for a globular operad is a morphism of globular sets

$$\begin{tikzcd} A \arrow[d] \\ T1 \end{tikzcd}$$

We think of A as a globular set of cells, each lying over a globular pasting diagram as specified by the morphism in the diagram. This data is called a *collection*, and a morphism of collections is a commuting triangle; we write **Coll** for the category of collections and their morphisms, which is just the slice category $\mathbf{GSet}/T1$. The definition of operad looks like this:

> *A globular operad is a collection equipped with*
>
> - *operadic composition*
> - *operadic identity*
>
> *satisfying some axioms.*

Contractions

We can ask for a *contraction* to exist on a collection. The idea will be, given a collection of weak composites

$$\begin{tikzcd} W \arrow[d] \\ T1 \end{tikzcd}$$

a contraction ensures that "the weak composites are sufficiently like the strict ones". We express this by demanding that we can lift any cell in the bottom up to the top, provided we can lift its endpoints. (In fact Batanin only lifts identities, where Leinster lifts all cells.)

Then a *(globular) operad-with-contraction* is an operad with a contraction on its underlying collection.

3.1.4 The idea of the definition

We will be interested in the initial operad-with-contraction. We take the monad associated with this and define ω-categories to be algebras for this monad.

Remark *We also say "algebras for an operad", and this is the same*[1] *as "algebras for the associated monad", although there is a direct definition as well.*

The idea is that this initial object gives us the globular set W of "weak composites" that we were looking for in Section 3.1.2. It gives us just what we need and nothing more, i.e. we generate "freely" precisely what we need for an operad-with-contraction. (We discuss initial objects and free constructions further in a moment.)

Evidently, for different flavors of definition (Penon, Batanin, Leinster) we need different things, so when we generate freely we will get more things in some definitions than in others. We discuss this further in Section 3.5.1.

Categorical aside on initial objects and free things

We have an adjunction

$$\begin{tikzcd} \mathbf{Coll} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \textit{Operads-with-contraction} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

and we know from categorical arguments that left adjoints preserve initial objects. The initial object in **Coll** *is the empty globular set* $\emptyset$ *so the initial operad-with-contraction is* $F\emptyset$*, the "free operad-with-contraction on nothing". This gives us exactly what is needed for an operad with contraction, and nothing else.*

[1] In fact Leinster [70] has exhibited two non-isomorphic operads which induce isomorphic monads, but this will not affect us here.

3.2 Globular Operads

It is quite quick to give a very abstract definition of globular operad, but much harder to get a good intuitive feel for it. Drawing pictures is helpful but care must be taken — it is very easy (not in LaTeX) to draw ourselves pretty pictures that accidentally fail to be sufficiently general, and there are various degenerate situations it is important to remember.

3.2.1 Fast abstract definition

The most concise definition looks like this:

> *The category of collections can be given the structure of a monoidal category. A globular operad is a monoid in this monoidal category.*

Digesting this definition straight off might seem a bit like trying to jump on a Concorde as it flies past, so for the sake of any reader who isn't Superman, we will build up to it bit more intuitively.

3.2.2 A bit more intuitively

An operad is given by an underlying collection together with composition and identities. A collection gives a diagram

$$\begin{tikzcd} A \arrow[d] & & \cdots \arrow[r, shift left] \arrow[r, shift right] & A_{k+1} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A_k \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A_{k-1} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \cdots \\ T1 & = & \cdots \arrow[r, shift left] \arrow[r, shift right] & (T1)_{k+1} \arrow[r, shift left] \arrow[r, shift right] & (T1)_k \arrow[r, shift left] \arrow[r, shift right] & (T1)_{k-1} \arrow[r, shift left] \arrow[r, shift right] & \cdots \end{tikzcd}$$

(commuting serially), and the elements of A can be thought of as cells looking like:

0-cells

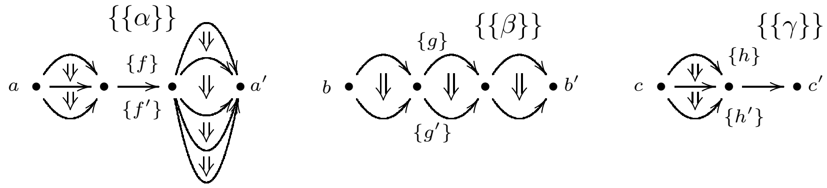
$$\bullet^{a} \qquad \bullet^{b} \qquad \bullet^{c}$$

1-cells

$$a \bullet \xrightarrow{\{f\}} \bullet\, a' \qquad b\ \underbrace{\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet}^{\{g\}}\, b' \qquad \overset{c}{\bullet}\ \text{degenerate}$$

Here we are using curly brackets to remind us that f and g are labelling the whole string of 1-cells and not just one individual one. This becomes crucial in the next dimension.

2-cells



being careful not to forget degenerate 2-cells such as

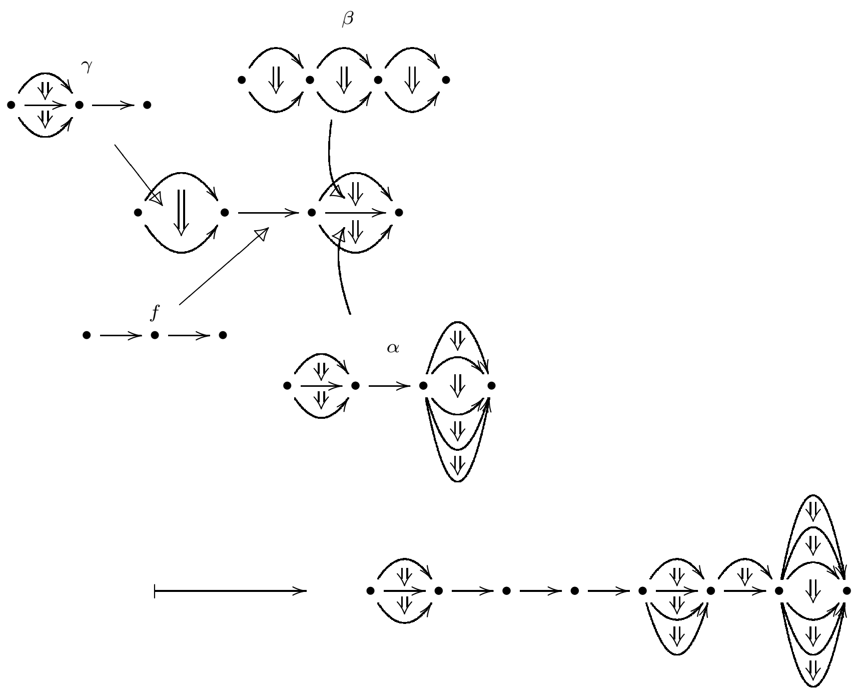
$$d \;\; \overset{\{k\}}{\bullet \longrightarrow \bullet \longrightarrow \bullet} \; d'$$

Now the double brackets indicate labels for the whole of each 2-cell; for example $\{\{\alpha\}\}$ is a 2-cell with globular source and target

$$a \;\; \overset{\{f\}}{\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet} \; a' \qquad \text{and} \qquad a \;\; \overset{\{f'\}}{\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet} \; a'$$

where as above f and f' are labelling the *whole string of 1-cells* not just one individual one. This is a bit hard to draw unambiguously.

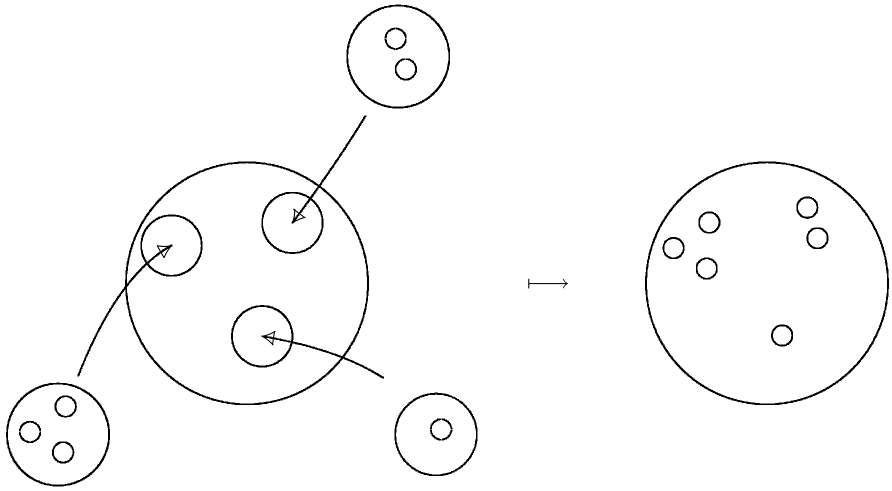
Here is an example of what operadic composition might look like:



Working out what to do with the labels is much more complicated (at least to notate).

Aside on the "little discs operad"

This kind of composition might remind you of composition in the little discs operad, if you've seen it:



It might also remind you of the opetopic composition in Chapter 4, if you happen to be reading this guide backwards.

3.2.3 Technically

An operad is going to be a monoid in the category **Coll** of collections. So it is given by a collection $\begin{array}{c} A \\ \downarrow \\ T1 \end{array}$ together with

- composition: $\left(\begin{array}{c} A \\ \downarrow \\ T1 \end{array}\right) \otimes \left(\begin{array}{c} A \\ \downarrow \\ T1 \end{array}\right) \longrightarrow \begin{array}{c} A \\ \downarrow \\ T1 \end{array}$, and

- unit: $\left(\begin{array}{c} 1 \\ \downarrow \\ T1 \end{array}\right) \longrightarrow \begin{array}{c} A \\ \downarrow \\ T1 \end{array}$

satisfying monoid axioms.

The tensor product $\begin{pmatrix} A \\ \downarrow \\ T1 \end{pmatrix} \otimes \begin{pmatrix} A \\ \downarrow \\ T1 \end{pmatrix}$ is the collection given by the left hand edge of the following diagram

$$\begin{tikzcd}
 & & & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
 & & TA \arrow[dl] \arrow[dr, "T!"'] & & A \arrow[dl] \\
 & T^2 1 \arrow[dl, "\mu"'] & & T1 & \\
T1 & & & &
\end{tikzcd}$$

Note that an element of this pullback is a pair

i) an element of A together with

ii) a T-configuration of elements of A

such that the T-configuration matches the underlying shape of the element of A given in (i) above. Or it is “a cell α together with a bunch of cells that can be glued into α”. The boundary labels of the cells we’re glueing in have to match up, but they don’t have to match the labels of the underlying cell α; these boundaries will have to be composed (at lower dimensions) as well.

Some reassurance: *Don’t worry if you didn’t follow that bit; it probably isn’t necessary in order to get a feel for the definition. The main thing is to bear those pictures of composition in mind.*

Note on technicalities

To check that this monoidal structure on **Coll** really works, we need to know that T is a “cartesian monad”. The reader may have seen “cartesian monads” mentioned in other expositions of operads. This is just to ensure that the monoidal structure on **Coll** really works, which is why we haven’t worried about it here.

3.2.4 Algebras for an operad

Recall our second answer to the question "Why operads?" (Section 3.1.2) was: to ensure that the pullback

$$\begin{tikzcd} & \text{“}WA\text{”} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\ W \arrow[dr] & & TA \arrow[dl] \\ & T1 & \end{tikzcd}$$

produced a monad by the assignation

$$A \mapsto WA.$$

We can now say:

> *An algebra for an operad is an algebra for its associated monad as above.*

3.2.5 Which operad do we want?

Having answered the question "Why operads?" and the question "What is an operad?" we now have to ask:

> **Question:** *What makes a particular operad a sensible choice for defining n-categories?*

Recall that we want to use operads to express weak composition of globular cells. We must be careful not to confuse the composition of cells in an n-category with the composition in an operad. The former will be obtained by putting some *extra structure* on our operad.

- Composition in an n-category will be given by requiring certain kinds of cells to be present in our operad.
- Composition in the operad will then generate all "compound" composites i.e. composites of composites applied repeatedly.

The question of what cells to require in our operad brings us to the subject of *contractions*, and so to the fork in the path where Leinster and Batanin diverge.

A general answer to the above question is:

> **Answer:** *It should at least*
>
> *i) have enough cells to give composition for an n-category, and*

ii) be sufficiently "like" $T1$ to be coherent.

Furthermore, we would prefer it to have a universal property, so that it is in some sense canonical.

The definitions of Leinster and Batanin arise from two different ways of satisfying these criteria. Batanin's does it with fewer cells; Leinster's does it with fewer definitions.

3.3 Leinster

We are going to look for an operad

$$\begin{tikzcd} L \arrow[d, "d"] \\ T1 \end{tikzcd}$$

and the idea is:

i) Throw in all the composites we want to specify in our theory.

ii) Throw in all the weak "compound" composites of these.

iii) Throw in coherence cells to mediate between the weak composites.

We begin by asking:

Question: *What sorts of composites do we want to specify in our theory of n-categories? Which ones are the basic ones we will use to generate all the others?*

Leinster's answer: *All of them, i.e. completely unbiased.*

That is, we must be able to lift every composite in $T1$ up to L. These strict composites will give our basic "generator" composites, and we will use operadic composition to generate compound composites of these.

We will perform this "lift" by means of a contraction on the map d. Moreover, this contraction will also ensure that L is sufficiently "like" $T1$, thereby fulfilling the two above requirements in one go.

3.3.1 Leinster's contractions

For an introductory discussion about contractions see Section 2.2.2. This is a more general kind of contraction than that of Penon and Batanin as we are now going to lift *all cells* where Penon and Batanin are (effectively) just lifting identities.

The definition of Leinster's contraction then says:

Given any two parallel k-cells $\alpha, \beta \in L$ *and a k-cell*

$$\theta : d\alpha \longrightarrow d\beta \in T1,$$

we get a contraction $(k+1)$*-cell*

$$[\alpha, \beta, \theta] : \alpha \longrightarrow \beta \in L$$

that is mapped to θ *by* d.

In particular, this gives us

i) a copy of every cell in $T1$, giving us all the basic unbiased composites that we want

ii) a mediating cell as before between any two weak composites in L lying over the same strict composite in $T1$, obtained by lifting the identity

iii) more "mediating cells" obtained by lifting non-identities — we can think of these as a cross between mediators and composites[2].

For examples of all of this, see Section 3.3.4, in which we look at an explicit construction of Leinster's operad in the first few dimensions.

3.3.2 The category of operads-with-contraction

We can now form the category **OWC** of "operads-with-contraction", with morphisms that preserve everything in sight. Here we are considering operads that have a *specified* contraction (unlike in Simpson's definition where the "composition maps" are just required to be *contractible*; see Chapter 5). This means that we can ask for an initial object in **OWC**. This is the universal property used to pick out the right sort of operad for use in Leinster's theory.

3.3.3 Leinster's definition

To sum up, we now gather up all the ingredients to make the following definition:

An ω-category *is an algebra for an initial operad-with-contraction.*

3.3.4 What does Leinster's operad look like?

To construct this operad we can proceed dimension by dimension (cf. the construction of Penon's monad, Section 2.3.2). At each stage we must have

i) a contraction, and

[2] Leinster's philosophy is that mediators and composites are really all part of the same notion, and so it makes sense for them to be produced with the same tool, namely contractions.

ii) operadic structure

so we "freely" throw in just enough cells to make this happen (see discussion on initial objects and free things, Section 3.1.4.) We illustrate the first few dimensions below.

0-cells

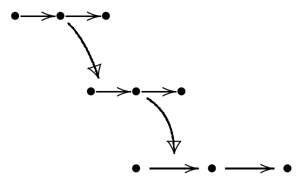
i) contraction: nothing happens here since there are no lower dimensions to consider needing contraction cells

ii) operadic structure: $\bullet$ from unit of operad

1-cells

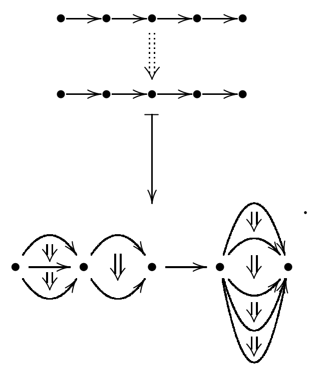
i) contraction: we must lift composites of all lengths from $(T1)_1$. That is, we get all 1-cells in $T1$ that start and end with $\bullet$ which is of course all of them. For example, $\bullet$, $\bullet \to \bullet$, $\bullet \to \bullet \to \bullet$ and $\bullet \to \bullet \to \bullet \to \bullet$ and so on.

ii) operadic structure: we get all sorts of operadic compositions like

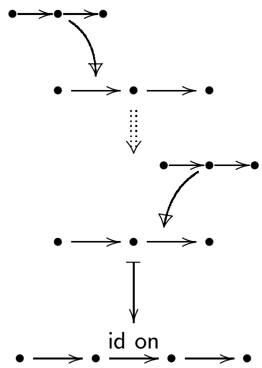


2-cells

i) contraction: we will exhibit each of the three cases mentioned in Section 3.3.1. First, we certainly get any 2-cell of $T1$ by lifting each one with its own source and target

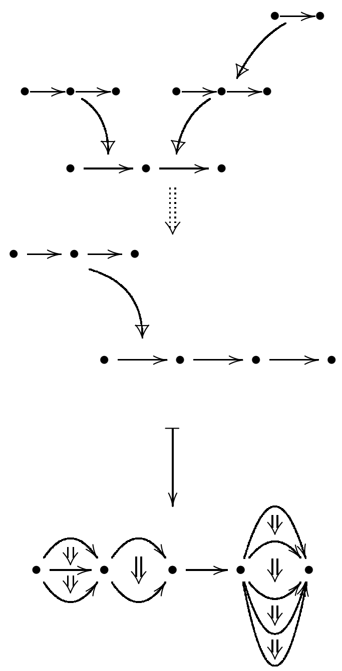


This works because we already have every 1-cell of $T1$. This will continue inductively at every dimension, so we will get every k-cell of $T1$ in our operad. Secondly, we get mediating cells as before by lifting identities, for example:



But thirdly, we also get to lift *any* 2-cell of $T1$ between *anything* of the correct length for the source and *anything* of the correct length for the target. i.e. we get any strict 2-cell pasting diagram together with some weak way of composing the source and some (possibly different) way of composing the target.

For example, the above 2-pasting diagram gets lifted in between *anything* of length 4 for the source and *anything* of length 4 for the target, such as in the diagram below.



ii) again, a whole ton of operadic composites.

This rather painstaking construction helps us see the subtle differences between Leinster, Batanin and Penon; see Sections 3.4.4 and 2.3.2.

3.4 Batanin

We now discuss Batanin's original form of the definition. We will look for an operad

$$\begin{tikzcd} K \arrow[d, "d"] \\ T1 \end{tikzcd}$$

and as before the idea is to throw in all the composites we want to specify in our theory, make weak "compound" composites of them using operadic composition, and throw in coherence cells to mediate between the weak composites.

> **Question:** *What makes an operad a sensible choice for defining n-categories?*
>
> **Answer:** *It should at least:*
>
> *i) have enough cells to give composition for an n-category,*
>
> *ii) be sufficiently "like" $T1$ to be coherent*
>
> *and have some kind of universal property.*

Now where Leinster uses unbiased contractions to meet both these requirements in one go, Batanin uses two notions: a (biased) contraction for the second, and a (biased) system of compositions for the first.

> **Question:** *What sorts of composites do we want to specify in our theory of n-categories?*
>
> **Leinster's answer:** *All of them.*
> **Batanin's answer:** *Just the binary ones.*

So we don't want to throw in *all* cells of $T1$, but only the binary ones. This is the role of the system of compositions. We then use a contraction (lifting only identities) and finally we can look for an initial object as before.

3.4.1 System of compositions

Composition for the eventual n-category is going to be given by some extra structure on the operad:

i) We demand a "system of compositions", giving all types of binary composite at all dimensions.

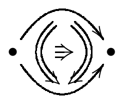
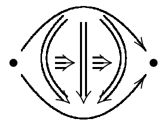
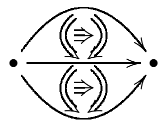
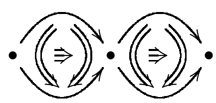
ii) Operad composition will then generate all the "compound" binary composites i.e. for binary composition applied repeatedly.

We follow the notation of [69]. We will be interested in a particular collection $\begin{array}{c} S \\ \downarrow \\ T1 \end{array}$ which we can think of as a sub-globular set of $T1$ containing just the *binary* composites. So S contains for each m

i) a basic boring k-cell called $\beta^m_m \in S(m)$, and

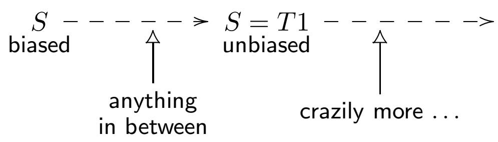
ii) all possible binary composites of that m-cell along p-cell boundaries. So for each $0 < p < m$ we have a composite $\beta^m_m \circ_p \beta^m_m$ and we write this as $\beta^m_p \in S$.

Example

	$m = 3$
β^3_3	
β^3_2	
β^3_1	
β^3_0	

So each set $S(m)$ has $m + 1$ elements. Also, we get a map $1 \longrightarrow S$ picking out the "boring cell" β^m_m at each dimension. (We need this boring cell to ensure that S really is a globular set; otherwise some cells in S wouldn't have a source and target in S.)

Remark *We could fiddle around with the notion of composition for our theory of n-categories by using a different S, e.g. we could go completely unbiased by putting $S = T1$, or we could go wild and demand that S have 42 cells of each dimension and a hundred ternary composites. The resulting theory would be very bizarre, but this example illustrates the point.*



Definition A *system of compositions* for an operad $\begin{array}{c}K\\ \downarrow\\ T1\end{array}$ is a morphism $S \longrightarrow K$ such that

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, hook] & & K \arrow[dl] \\ & T1 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1 \arrow[r] \arrow[dr, "\text{unit of operad}"'] & S \arrow[d] \\ & K \end{tikzcd}$$

commute.

In summary, the idea is that when we construct our operad freely, these are the formal composites we want to *start with* and use to generate all the weak ones.

3.4.2 Batanin's contractions

Batanin uses contractions to ensure that "K is sufficiently like $T1$". He uses the following flavour of contraction (lifting only identities):

> *Given any two parallel k-cells α and β with the same image under d, there must be a $(k+1)$-cell $\alpha \longrightarrow \beta$ that maps to the identity under d.*

We refer the reader to Section 2.2.2 for introductory discussion on contractions.

3.4.3 Batanin's definition

We construct a category **OCS** with

- objects: operads equipped with a system of compositions and specified contraction
- morphisms: preserve everything

and we seek an initial object to make the following definition:

> *An ω-category is an algebra for an initial object in* **OCS**.

(See Section 3.2.4 for the definition of an algebra.)

Technical remark: *This is Leinster's interpretation of Batanin in [69] but in fact it isn't what Batanin actually does in [9]. Batanin introduces a category of "contractible operads with a system of compositions", and because these operads are not equipped with a* specified *contraction, there can't be a genuinely initial object. Instead, Batanin constructs a specific operad and uses it for the definition of n-category. It is a "weakly initial object" in the sense that there exists a (not necessarily unique) map from it to any other object in the category. This is the best we can do if the operads aren't equipped with specified contractions; however, it seems that Batanin's operad is intended to look like an initial object in the category* **OCS** *as above.*

3.4.4 What does Batanin's operad look like?

To construct this operad we can proceed dimension by dimension (cf. Penon in Section 2.3.2, and Leinster in Section 3.3.4). This time, at each stage we must have

i) a contraction,

ii) a system of compositions, and

iii) operad structure.

We "freely" throw in just enough cells to make this happen. (Again, see discussion on initial objects and free things, Section 3.1.4.) We illustrate the first few dimensions below.

0-cells

i) contraction: nothing happens since there are no lower dimensions to consider needing contraction cells

ii) system of compositions: we must have $S(0) \longrightarrow K(0)$ and $S(0)$ has one boring 0-cell so we get one boring 0-cell in $K(0)$ that we denote as $\bullet$

iii) operadic structure: nothing new since the only cell we have so far is a unit, so nothing happens when we compose it

1-cells

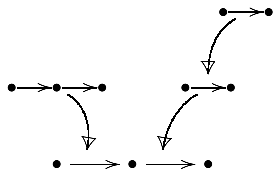
i) contraction: we get to lift the identity on $\bullet$ in $T1$. So we get

$$\begin{tikzcd} \bullet \arrow[r, "I_1"] & \bullet \quad \in K(1) \\ \text{id on } \bullet \quad \in T1(1) \arrow[from=1-1, to=2-1, mapsto, start anchor=east, end anchor=east, phantom] \end{tikzcd}$$

$$\bullet \xrightarrow{I_1} \bullet \quad \in K(1)$$
$$\Big\downarrow$$
$$\text{id on } \bullet \quad \in T1(1)$$

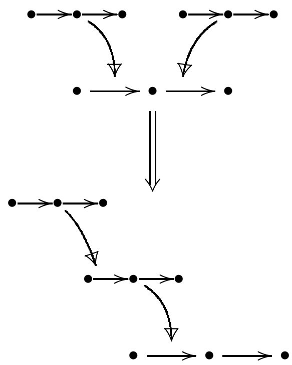
ii) system of compositions: we get $\bullet \to \bullet \to \bullet$ and $\bullet \to \bullet$ which has to be the boring operadic unit

iii) operadic structure: we get to compose these things operadically e.g.

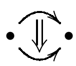
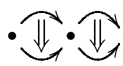
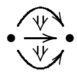


2-cells

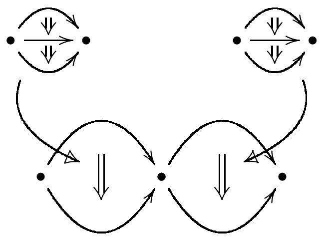
i) contraction: we get identities on everything and 2-cells between any two things with the same underlying length, for example



All of the cells obtained in this way lie over the appropriate identity cell in $T1$

ii) system of compositions: from our system of compositions we get the boring 2-cell  and  and 

iii) operadic structure: tons of cells from operadic composition like



3.5 Some informal comparison

In order to remain calm at this point, it is worth thinking about where each part of the n-category structure comes from in the definitions of Penon, Batanin and Leinster. We sum this up in the following table:

	Penon	**Batanin**	**Leinster**
identities	reflexivity	contraction	contraction
basic composites	magma structure	system of composition	contraction
compound composites	free magma structure	free operad structure	free operad structure
constraint	contractions	contraction	contraction
coherence	contraction	contraction	contraction

3.5.1 Operadic vs non-operadic

How is the non-operadic globular approach different from the operadic? When we say "different" we need to be clear what we mean, just as when we say "equivalent." We expect that Penon's approach is equivalent to the operadic — in some sense. But there is a precise sense in which the approaches are *different*. One way of thinking of this follows from our discussion in Section 3.1.2. We discussed the notion of using a globular set W to characterise "weak composites" in our theory in any way we want, forming labelled weak composites by the pullback on page 31:

$$\begin{tikzcd}
& \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & \\
W \arrow[dr] & & TA \arrow[dl] \\
& T1 &
\end{tikzcd}$$

As we remarked in Section 3.1.4 we now have a possible W for each of the globular definitions. If we write W_P, W_B, and W_L respectively, we get the following inter-relationship:

$$\begin{tikzcd}
W_P \arrow[r, hook] & W_B \arrow[r, hook] & W_L \\
\text{binary composites and biased contractions} \arrow[u] & & \\
& \text{need extra operadic composites} \arrow[uu] & \\
& & \text{need extra unbiased composites and unbiased contractions} \arrow[uuu]
\end{tikzcd}$$

So the monad action on a globular set A gives us three similarly related pullbacks

$$\begin{tikzcd}
 & & & \cdot \arrow[dddlll] \arrow[dddr] \arrow[d, dotted] & \\
 & & & \cdot \arrow[ddll] \arrow[ddr] \arrow[d, dotted] & \\
 & & & \cdot \arrow[dl] \arrow[dr] & \\
W_P \arrow[r, hook] \arrow[drrr] & W_B \arrow[r, hook] \arrow[drr] & W_L \arrow[dr] & & TA \arrow[dl] \\
 & & & T1 &
\end{tikzcd}$$

with maps induced as shown by the universal property of the pullbacks.

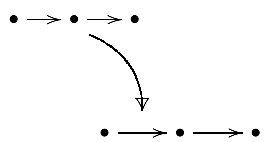
The comparison of the non-operadic with the operadic occurs via the (non-surjective) inclusion

$$W_P \hookrightarrow W_B$$

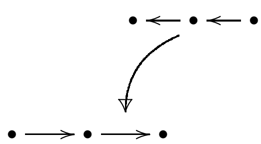
where W_B has "extra operadic composites". We will now give an example of the sort of composite that Batanin demands where Penon does not, drawing attention to as few technical details as we feel is possible. However, in order to see what is going on with this rather subtle difference, it is important to be quite clear about what our diagrams mean.

We will take this slowly, in three steps. In steps (i) and (ii) we warm up with some cells demanded in both theories; in step (iii) we find a cell demanded only by Batanin, building on step (i).

i) Both theories demand an "associator" by contraction. Let us write θ_1 for the cell



and θ_2 for the cell



.

Both of these lie over $\bullet \to \bullet \to \bullet \to \bullet$ in $T1_1$ so can be represented as $\bullet \to \bullet \xrightarrow{\{\theta_1\}} \bullet \to \bullet$ and $\bullet \to \bullet \xrightarrow{\{\theta_2\}} \bullet \to \bullet$ respectively. So by the contraction we get an "associator"

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r, "\{\theta_1\}"] \arrow[d, phantom, "\Downarrow a"] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r, "\{\theta_2\}"'] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

$$\Big\downarrow\!\!\!\!\!\!\top$$

$$\begin{array}{c}\text{id 2-cell on}\\ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet\end{array}$$

We can represent this as $\bullet \to \bullet \xrightarrow{\{\{a\}\}} \bullet \to \bullet$ with the double curly brackets[3] to remind us that a is lying over an identity 2-cell in $T1_2$.

ii) Now write ϕ_1 for

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[r] & \bullet \arrow[r] \arrow[dr, bend left] & \bullet & & \\
& & \bullet \arrow[r] & \bullet \arrow[r] \arrow[dr, bend left] & \bullet & \\
& & & & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

and ϕ_2 for

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[r] & \bullet \arrow[r] \arrow[dr, bend left] & \bullet & \bullet \arrow[r] & \bullet \arrow[r] \arrow[dl, bend right] & \bullet \\
& & \bullet \arrow[r] & \bullet \arrow[r] & \bullet &
\end{tikzcd}$$

similarly in both theories we get a "more general associator" a'

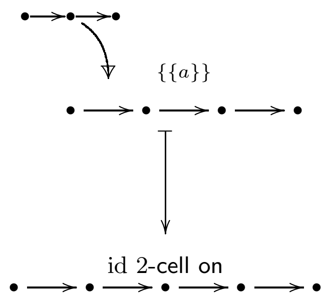
$$\bullet \longrightarrow \bullet \xrightarrow{\{\phi_1\}} \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \overset{a'}{\Longrightarrow} \quad \bullet \longrightarrow \bullet \xrightarrow{\{\phi_2\}} \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$\Big\downarrow\!\!\!\!\!\!\top$$

$$\begin{array}{c}\text{id 2-cell on}\\ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet\end{array}$$

[3] We don't pretend that this notation is very wonderful. These things are hard to draw.

iii) In Batanin's theory we also get an operadic composite



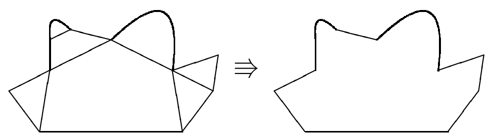
"composing on top of a constraint". (It looks like the thing we're composing on top of a is a 1-cell, but technically it's a degenerate/identity 2-cell.) The resulting cell has the same globular source and target as a' but is not the same as a'. This example is not something we find in Penon's theory. So this is an example of an "extra operadic composite".

Chapter 4

Opetopic

4.1 Introduction

The opetopic approach was first proposed by Baez and Dolan in [4]. The most striking feature of this definition is the underlying shapes of its cells. These shapes are called "opetopic" and look like



This is not just a whimsical artistic foible or even an arbitrary ideological decision — opetopic cells are this shape *in order to express composition.* That is, in this definition cells do not just play the role of "being cells"; they also directly give composition.

Historically, the definition followed Street's and can be thought of as a way of avoiding the complicated combinatorics of orienting simplices and calculating "admissible horns". Opetopic cells have orientation built into them. Note that the shapes are both more and less general than simplicial shapes — more, because they can have any number of "input faces", and less because they always have precisely one "output face".

The definition itself is a bit like a nerve condition where the underlying data is now an *opetopic set*, not a simplicial set.

Definition *An n-category is an opetopic set in which*

i) every niche has an n-universal filler, and

ii) every composite of n-universals is n-universal.

The first condition is like a horn filling condition, and the second is analogous to Street's condition: "if all but one face of a hollow cell is hollow [...] then the last face must also be hollow.” Here universality is analogous to hollowness, with a crucial difference that universality is an *inherent property* of certain cells in an opetopic set where hollowness is *extra structure* on a cell.

Note on non-algebraic approach

One key characteristic of this definition is that it consciously avoids being algebraic. Part of the ideology behind it is to get rid of equalities and uniqueness. So we will have a lot of “existence” demands but nothing given as actual *structure* on the underlying data.

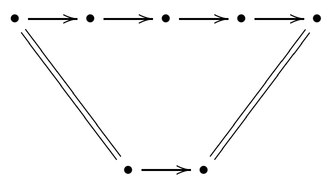
Note on technicalities

The technicalities involved in setting up this definition rigorously are quite severe. However, it is possible (and probably easier) to get a feel for this definition without mentioning the technical details *at all*. That is the approach we take here; the technicalities have been written up extensively elsewhere. See Section 4.5 for remarks on the literature.

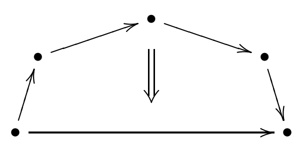
4.2 Opetopic cells

The aim of this section is to give a feel for what opetopic cells look like without even a hint of how they are technically constructed.

Opetopic shapes come from “what composition looks like”. For example, we can compose a string of 1-cells and get one 1-cell whose source and target are identified as shown:



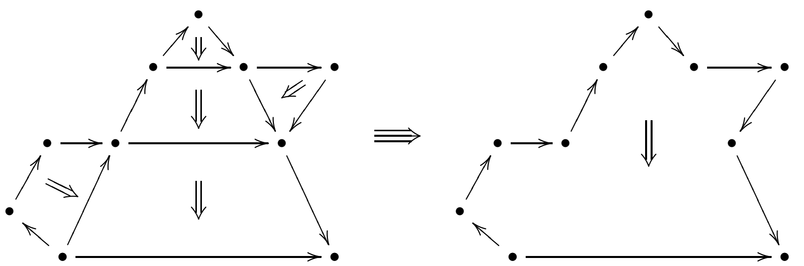
If we actually identify those endpoints in the diagram we just get:



We call this a 2-cell as its source(s) and target are 1-cells.

Next we need to think about what 2-cell composition (along boundary 1-

cells) would look like. Using the 2-cell shapes as above we get things like

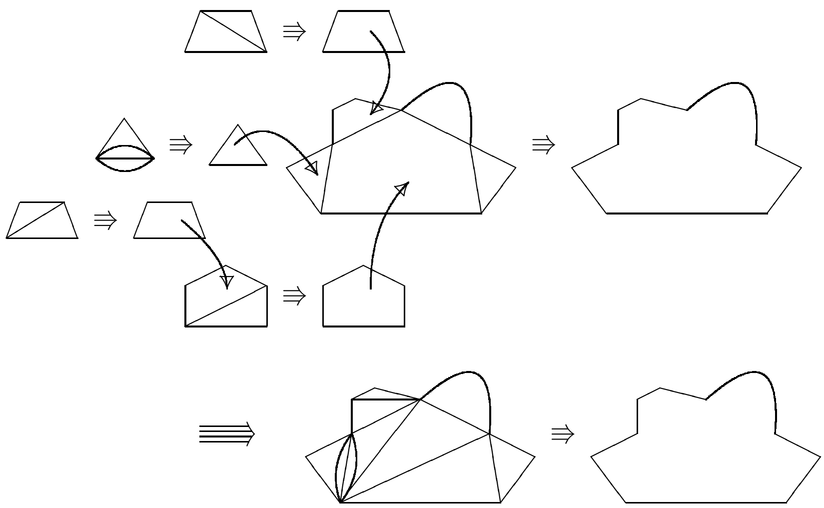


and we call this a 3-cell as its source(s) and target are 2-cells.

So the general principle for a k-cell is:

- the source is a picture of how to compose some $(k-1)$-cells
- the target is the shape of the result.

An example of a 4-cell is:



We often give up drawing lower-dimensional arrowheads when the directions are clear from the diagram.

Note on technical construction

Constructing these things as a piece of algebra is bit subtle and uses the language of multicategories. Multicategories are a bit like operads and the reader may well find the above 4-cell reminiscent of the operadic composites in Chapter 3. We will proceed, however, without the technical construction and hence will not need multicategories either.

4.3 The Definition

An n-category is an *opetopic set* in which

i) every *niche* has an *n-universal cell* in it, and

ii) every *composite* of n-universals is n-universal.

So we need to understand the terms in italics. This is the aim of the rest of this section.

Note

In [4] the dependence on n is not made explicit in conditions (i) and (ii). That is, the term "universal" is used rather than "n-universal", but the definition of "universal" uses a fixed n. We have found it clearer to emphasise this dependence on n, as this is the only way the n-dimensionality of the n-category comes in.

4.3.1 Opetopic sets

An opetopic set is a presheaf on the category $\mathcal{O}$ of opetopes. As usual, this can be thought of as a category of "underlying shapes". An opetopic set then gives a set of cells of each shape. This can also be thought of as giving "labels" to the blank opetopes.

Constructing the category of opetopes involves more technical details than we wish to include, and it is more illuminating to draw some pictures. The main point to note is that $\mathcal{O}$ has face maps but not degeneracies. The face maps tell us about the constituent lower-dimensional parts of a cell.

An opetopic set X consists of: for each $k \geq 0$ a set $X(k)$ of k-cells which are "labelled opetopes" e.g.

0-cells $\quad \bullet^{a} \qquad \bullet^{b} \qquad \bullet^{c}$

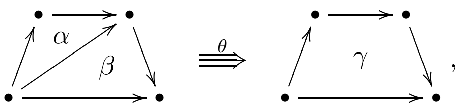
1-cells $\quad a \bullet \xrightarrow{f} \bullet\, b$

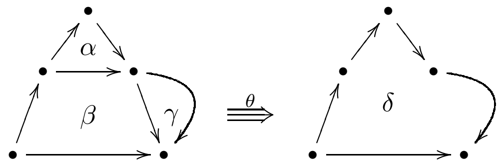
2-cells

$$\begin{tikzcd} & \bullet \arrow[dr, "f_2"] & \\ \bullet \arrow[ur, "f_1"] \arrow[rr, "f_3"'] & \arrow[u, phantom, "\Downarrow\alpha"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[r, "f_2"] & \bullet \arrow[dr, "f_3"] & \\ \bullet \arrow[ur, "f_1"] \arrow[rrr, "f_4"'] & & \arrow[ul, phantom, "\Downarrow\beta"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left, "f_1"] \arrow[r, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[d, phantom, "\Downarrow\alpha"] & \\ \bullet \arrow[rr, "f_3"'] & & \bullet \end{tikzcd}$$

... and for all lengths of source.

3-cells





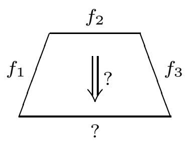
and so on.

Note that we often omit the lower dimensional labels when they are implied by the higher dimensions, but in all cases the constituent cells of all dimensions are elements of the opetopic set and so should have labels. We can think of the constituent $(k-1)$-cells of a k-cell as "faces".

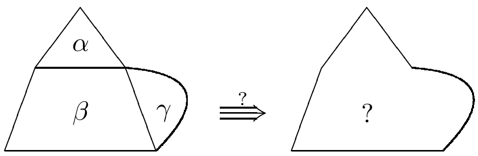
4.3.2 Niches

A k-niche is a "potential source" for a k-cell. It is a bit like a horn: it is a k-cell with no interior and one face missing. But here the missing face *must* be the target face. For example,

$a \bullet \xrightarrow{?} \bullet\ ?$ gives a "potential source" for a 1-cell



gives a "potential source" for a 2-cell



gives a "potential source" for a 3-cell.

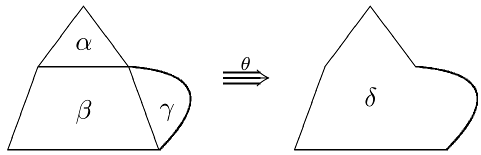
So a k-niche is a pasting diagram of $(k-1)$-cells, and can be thought of as "something that needs a composite".

4.3.3 Composites

Composition in an opetopic n-category is to be given by universal cells; we will define universal cells in the next section.

Definition *Given a universal cell α we say its target cell is a* composite *of its source cells.*

For example, in the diagram below

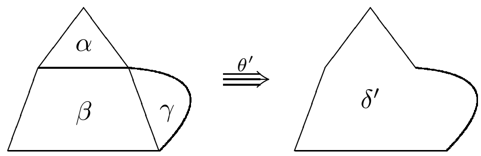


if θ is universal then we say δ is a composite of α, β, γ.

Recall in the definition of an n-category we demand that "*every niche has a universal filler*". So this can be interpreted as:

> *Everything that is composable does have a composite.*

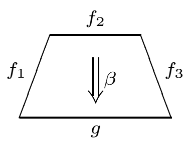
Note Composites are not unique in this theory. We might have another universal cell θ'



giving δ' as another composite of α, β, γ.

4.3.4 Universals

Universal cells are going to be a bit like pseudo-invertibles. Here are two ways of thinking about a universal cell β as below:



i) This cell translates into ordinary globular language as "a cell from the composite of f_3, f_2, f_1 to g" i.e.

$$f_3 \circ f_2 \circ f_1 \xrightarrow{\beta} g.$$

If β is *universal* we are saying that *this is an (internal) equivalence.* But the spirit of the opetopic definition then says that g should be considered as a perfectly good composite of f_3, f_2, f_1 as well.

ii) If β is universal it is like a "proof of the fact that g is a composite of f_3, f_2, f_1". The universal cell β witnesses the fact that g is a composite. There might be many composites and for each one there might be many proofs/witnesses. For a nice coherent structure, the different proofs should be "provably related", i.e. by 3-cells witnesses. There also might be different 3-cells giving these "proofs", but these should all be related by 4-cells, and so on.

The definition of universality

The full definition is by downward induction over dimension. Above n we have a uniqueness condition; below n we have a factorisation condition, where "factorisation" is given by universal cells at the dimension above. Note that this is why the definition can't do ω — the downward induction wouldn't have anywhere to start. (Makkai appears to have a definition of universality that *can* "do ω" [78].)

To get a feel for the idea of universality, we can think about how we define isomorphisms in an ordinary category.

- Usually we say:

  $a \xrightarrow{f} b$ *is an isomorphism if* $\exists\ b \xrightarrow{g} a$
  such that $fg = 1,\ gf = 1$

but we can't do this for opetopic shapes because the inverse might have to go from one cell to many cells, which is not allowed. (Remember, targets consist of only a single cell.)

- Another approach is by factorisation :

  $a \xrightarrow{f} b$ *is an isomorphism if* $\forall\ a \xrightarrow{h} c\ \exists!\ b \xrightarrow{g} c$

  such that $\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$ *commutes.*

The last method is the one we copy.

Remark on original Baez/Dolan definition

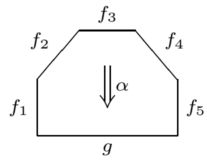
The definition of universality in [4] is more general as it is set up to be used for a range of definitions other than n-categories. We have boiled it down to leave only the cases needed for n-categories.

Definition (Sketch) *Let α be a k-cell.*

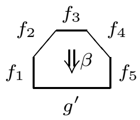
- *If $k > n$ then α is* n-universal *iff it is unique in its niche.*
- *If $k \leq n$ then α is* n-universal *iff*

  i) given any k-cell β with the same source, there exists a factorisation through α, and *ii) any factorisation through α is a universal factorisation.*

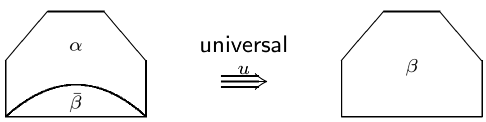
Rather than go through the technicalities of this (and the definition of "universal factorisation"), we now draw some pictures to illuminate the case $k = 2$.



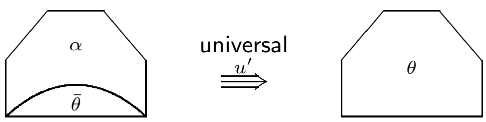
We examine what it means for α to be universal.

i) Given any 

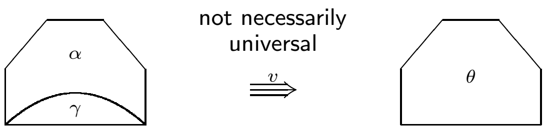
then by (i) we must have a factorisation



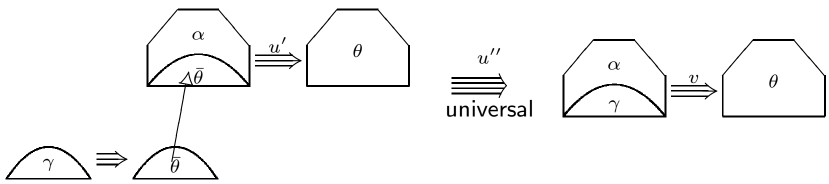
ii) *Any* factorisation through α, i.e.



must be a *universal factorisation.* This means, given any



there must exist a factorisation



and moreover, any factorisation of this form through u' must itself be a universal factorisation.

NB Factorisations of n-cells are defined to be universal if and only if they are unique, so this definition doesn't go on forever.

This definition is to ensure that we only call something a composite if it "really deserves it" and will give a sufficiently coherent structure at the end. For example we might imagine being given the *cells* of a bicategory but no other information about it. If we tried to put composition on it we could:

i) go very wrong by assigning "composites" that didn't deserve it, i.e. that didn't satisfy the coherence axioms, or

ii) find a perfectly good coherent composition that wasn't exactly the same as the one originally intended.

The use of universality in the opetopic definition *prevents* case (i) and *embraces* case (ii) — given many possibilities for coherent composition, we accept them all.

4.4 Comparison with classical bicategories

In order to shed more light on how this definition works we now sketch how opetopic 2-categories correspond to classical bicategories. It is worth first comparing the DSP trichotomy in each case:

	Opetopic	Classical
DATA	opetopic cells	globular cells
STRUCTURE	-	identities, composition, constraints
PROPERTIES	enough universals exist	coherence axioms

To get a classical bicategory from an opetopic one we can clearly start by taking the "globular shaped" cells from the opetopic set. However, we then have to find identities, composition and constraints, and check coherence. This is where the universals are needed. Here are some ways of thinking of it:

i) "An opetopic set is an n-category if enough universals exist to give us identities, composition and constraints coherently."

ii) "An opetopic set is an n-category if the globular cells alone can tell us all the *cells*, and the other shaped cells are only telling us about *behaviour*." Or, put another way

iii) "An opetopic set is an n-category if enough universals exist so that every cell can be distilled down to a globular cell."

The process of "distillation" is how we interpret an opetopic k-cell as an ordinary (globular) k-cell with just one $(k-1)$-cell at its source and target.

4.4.1 From opetopic to classical

Let X be an opetopic 2-category. We show how to construct a bicategory $\mathcal{B}$ from it. The beginning is easy enough:

- the 0-cells of $\mathcal{B}$ are the 0-cells of X
- the 1-cells of $\mathcal{B}$ are the 1-cells of X
- the 2-cells of $\mathcal{B}$ are the 2-cells of X with shape 

but from this point onwards an issue of choice arises. We will have to choose some universal cells in X. The whole process can be summed up in the following table.

Classical $\mathcal{B}$	**Opetopic X**
0-cells	0-cells
1-cells	1-cells
2-cells	globular shaped 2-cells
1-cell identities	*choice* of nullary universal 2-cells
1-cell composition	*choice* of binary universal 2-cells
2-cell identities	*unique* universal 3-cells
vertical 2-cell composition	*unique* universal 3-cells
horizontal 2-cell composition	*unique* factorisation of 2-cells
associators	*unique* factorisation of 2-cells
unit constraints	*unique* factorisation of 2-cells
axioms	follow from uniqueness of 2-cell factorisation

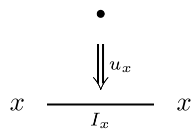
We spend the rest of this section going through this process in more detail; full details are written up in [27].

i) 1-cell identities

For each 0-cell $x \in X$ we have a universal 2-cell of shape

$$\begin{tikzcd} & \bullet \arrow[d, phantom, "\Downarrow"] & \\ x \arrow[rr, no head] & & x \end{tikzcd}$$

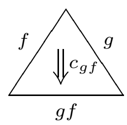
i.e. a "nullary" 2-cell, with zero 1-cells as the source. We pick one universal nullary 2-cell for each 0-cell, and call the target I_x



NB Crucially, we are picking the universal 2-cell and not just the 1-cell of its target. We will later see that this ensures that the structure we choose is coherent.

ii) 1-cell composition

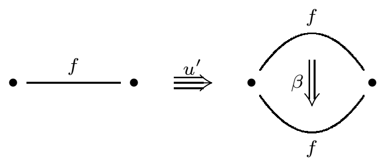
For each composable pair of 1-cells $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ we pick a universal 2-cell as below, and call the target gf



NB These are the only choices we have to make; the rest is now uniquely determined.

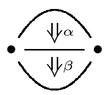
iii) 2-cell identities (no choice needed)

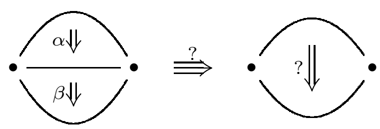
Again, we use nullary universals. At this dimension they look like



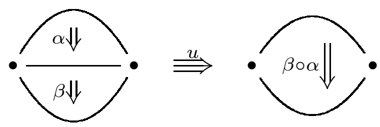
There is a unique 3-cell in this niche, so we call its target 1_f.

iv) Vertical 2-cell composition (no choice needed)

Each composable pair of 2-cells  gives a 3-niche



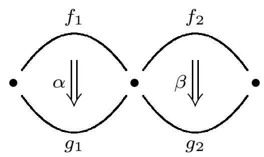
and there is a *unique* cell in this niche (since we have gone above n), so we call the target $\beta \circ \alpha$



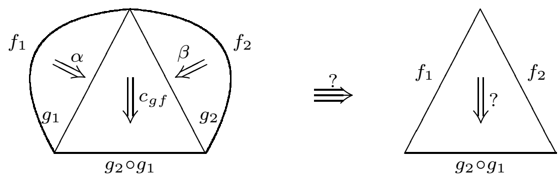
Note that although n is unique in its niche it still gives us important information by its target. This illustrates the fact that we do need 3-cells even for a 2-category.

v) Horizontal 2-cell composition (no choice needed)

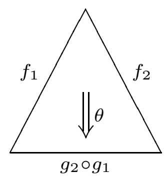
This now involves factorisations, but once the above choices have been made, this is uniquely determined. Given 2-cells



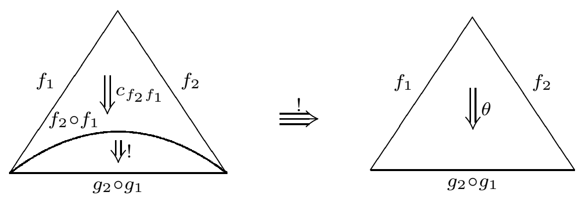
we have a 3-niche



so we have a unique 3-cell in it, with target



say. Now θ can be "distilled" back down to a globular cell by factoring through the chosen universal $c_{f_2 f_1}$ as shown:



and we call the unique factor $\beta * \alpha$

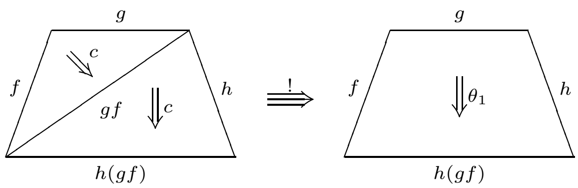
$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "f_2 \circ f_1"] \arrow[r, swap, "g_2 \circ g_1"] \arrow[r, phantom, shift left=2, "\Downarrow \beta * \alpha"] & \bullet \end{tikzcd}$$

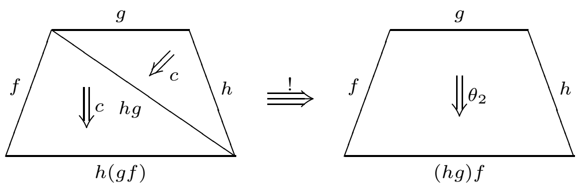
vi) Associators

Again, this makes use of factorisations, and is uniquely determined once we have made the above choices. For every composable string of 1-cells

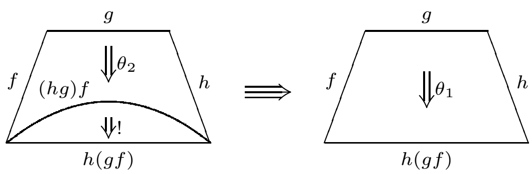
$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, "g"] & \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

we seek an invertible 2-cell $(hg)f \Rightarrow h(gf)$. Now we have unique (universal) 3-cells





and θ_1 and θ_2 are composites of universals, hence universal. So θ_1 must factor through θ_2 uniquely (by universality of θ_2):



and we call this factor
$$a_{hgf} : (hg)f \Rightarrow h(gf).$$

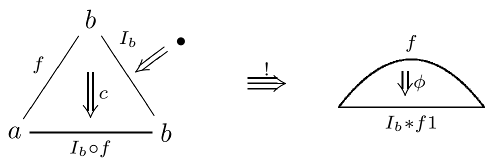
Moreover by universality of θ_1 we also get a unique 2-cell in the other direction:

$$h(gf) \Rightarrow (hg)f$$

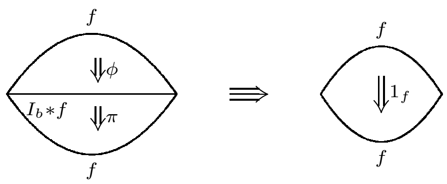
and we can show that these really are inverse to one another. We must also check naturality, which follows in a similar fashion to the associativity pentagon below (see viii).

vii) Unit constraints

Given $a \xrightarrow{f} b$ we need an invertible 2-cell $I_b \circ f \xrightarrow{r} f$. We use a unique 3-cell of the form



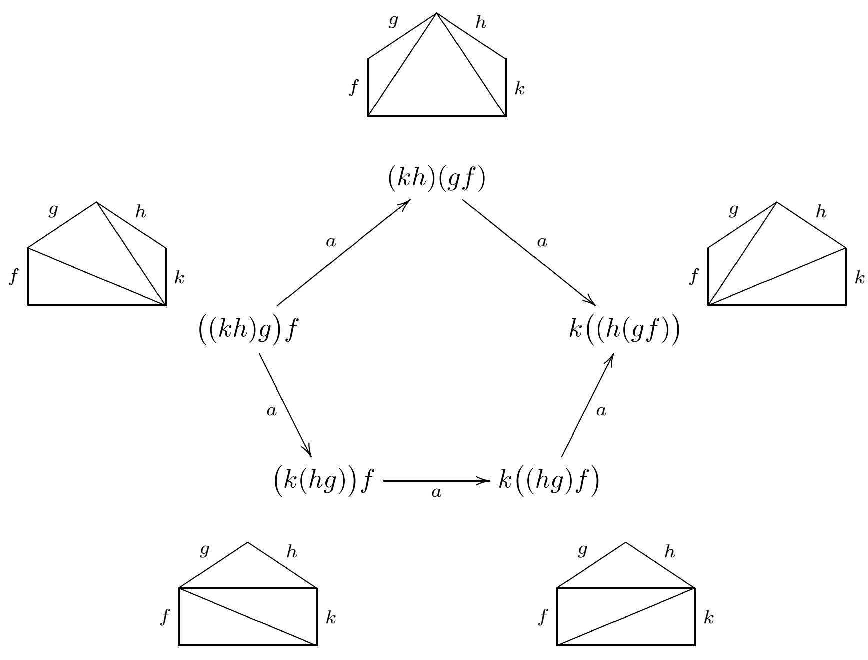
Now ϕ is going the wrong way, but we can get a factor using 1_f:



giving the invertible 2-cells we require.

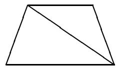
viii) Axioms

We must check the associativity pentagon:

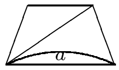


We use the following manipulations:

1) Associator substitution:
   If we have a sub-diagram

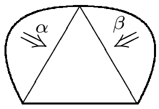
   

   we can substitute

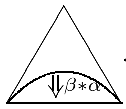
   

   where we have inserted the associator at the bottom.

2) Horizontal composite substitution:
   In a similar fashion we can replace

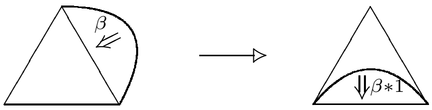
   

   by

   .

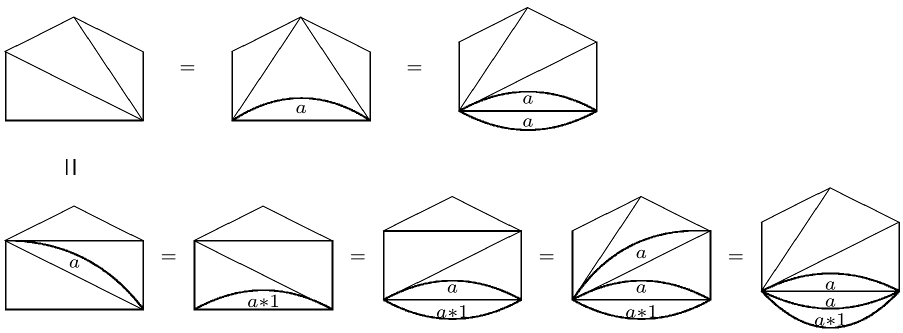
   An important degenerate case gives

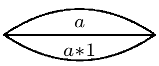
   and likewise on the left.

Note that these manipulations can be made rigorous.

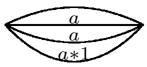
To check the pentagon, we then have a manipulation of the following form



and then by uniqueness of 2-cell composition and uniqueness of 2-cell factorisation, the components



and



must be equal. All other axioms follow similarly.

4.4.2 Category of opetopic n-categories

We form a category **Opic-n-Cat** by taking

- objects to be opetopic n-categories, and
- morphisms to be morphisms of underlying opetopic sets.

Remark This gives lax n-functors; if we demand that universals be preserved we get weak n-functors.

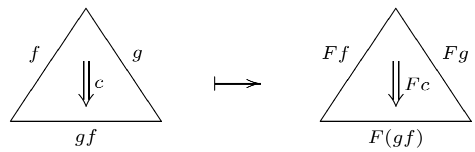
We can compare this with lax/weak functors of bicategories. To translate a morphism F of opetopic 2-categories into a functor of bicategories we need to find constraints

$$Fg \circ Ff \Rightarrow F(g \circ f)$$

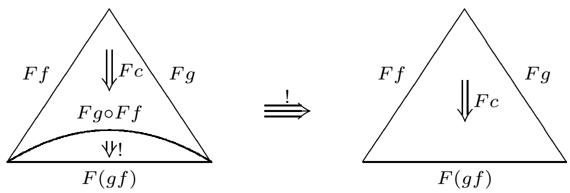
and

$$I_{FA} \Rightarrow FI_A.$$

We can examine the action of F on our chosen universal 2-cells:



and then we can "distil" this by factorisation to get



giving the constraint we require. And similarly for the unit constraint.

4.4.3 Opetopic 2-categories vs bicategories

We have an equivalence of categories

$$\textbf{Opic-2-Cat} \simeq \textbf{Bicat}$$

for both the lax and the weak functor cases. However, this is not canonical in either direction.

$\longrightarrow$ In this direction the issue is choosing universal 2-cells.
$\longleftarrow$ In this direction the issue is generating opetopic shaped cells.

4.5 Remarks on the literature

The story of the opetopic approach is a little complicated. The definition was first presented in [4]. Different approaches to the construction of opetopes were then proposed in [46] and [72]; these approaches were intended to give the same idea but were not *a priori* the same. Then in [33] a modification to the original Baez-Dolan approach was proposed, along lines which they had originally followed but then chose to abandon as they thought it would give the wrong notion of braided monoidal category. However, in [33] and [32] it is proved that this modification results in a precise equivalence between the three constructions of opetopes. Thus motivated, the modification is followed through into the full definition of n-category [27] and equivalence with the theory of classical bicategories is proved.

It must be stressed that the equivalence of the three approaches to opetopes is *not* proved with the aim of discarding two out of the three. On the contrary, the point is to benefit from the use of all three constructions simultaneously. We rely implicitly on this equivalence in all stages of the calculations. For this reason, rigorous exposition of the technicalities of this definition is a complicated matter, which we have carefully avoided here.

Chapter 5

Tamsamani and Simpson

Introduction

The idea behind the definitions of Tamsamani [107] and Simpson [93] is to generalise the notion of *nerve* of a category by *enriching* or *internalising*. The two definitions are explicitly and deliberately related: Simpson's is a reworking of Tamsamani's with the aim of simplification. The result is certainly a simpler definition, but at the cost of some generality in the resulting theory.

Given any category, we can construct its nerve, an "underlying simplicial set". We can then ask: when does a simplicial set arise as the nerve of a category? Not every simplicial set arises in this way, but it is quite straightforward to characterise those that do. Moreover, the nerve functor

$$N : \mathbf{Cat} \longrightarrow \mathbf{SSet}$$

is full and faithful — a functor between categories is given precisely by a morphism of their nerves.

So we could define a category as follows:

A category is a simplicial set satisfying the nerve condition.

The "nerve condition" tells us, essentially, that composition of cells can be coherently defined from the given data. We aim to generalise this to get a definition of the following form:

An n-category is an n-simplicial set satisfying the n-nerve condition.

The "n-nerve condition" should be a higher-dimensional analogue of the ordinary nerve condition; as usual with generalisations, the important starting point is to express the base case in a form conducive to generalisation. We discuss this further in Section 5.1.5.

For the underlying data, it is easy to *define* an n-simplicial set, but less straightforward to see what the *idea* behind the definition is, or what pictures we should be imagining. The motivation can be seen from the points of view of *enrichment*, or *internalisation*. We discuss these two related ideas in Sections 5.1.2 and 5.1.3. In Section 5.1.6 we discuss the "shortcut" Simpson uses to bypass some of the more complicated technicalities of Tamsamani's original approach. In Section 5.2.2 we indulge in a large amount of picture drawing, to give an idea of what sort of shapes arise from this definition. Only after all this intuitive build-up do we go into the actual technicalities of the definitions.

NB Note that we use the term "multisimplicial set" instead of n-simplicial set when we do not have a specific n in mind.

5.1 Intuitions

This pair of definitions can be thought of as generalising the notion of *nerve* using a generalisation of *enrichment*. This is the spirit of the definition, but formally the definition of multisimplicial sets looks more like a process of *internalisation*. This can be thought of as a useful technique for achieving enrichment, or else enrichment can be thought of as a special case of internalisation. Some readers may be more comfortable with one notion than another; we discuss both, and the relationship between them. First of all it helps to have a feel for the nerve of a category.

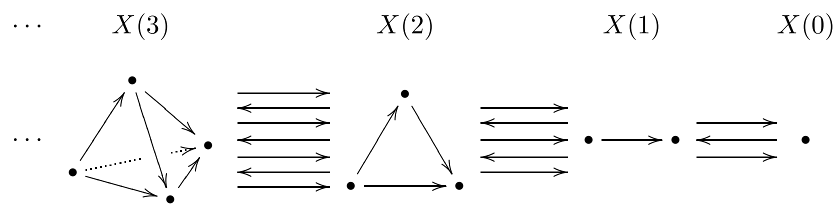
5.1.1 Nerves

Our aim is to construct the "underlying simplicial set" of a category. This simplicial set is going to capture all the information about the objects, morphisms, and coherent composition. We start by getting a geometric feel for simplicial sets.

What does a simplicial set X look like? A simplicial set is a functor

$$\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

so we have for each $k \geq 0$ a set $X(k)$ of cells together with various face and degeneracy maps. We think of the following picture:



and this suggests how to construct the nerve of a category. The set $X(k)$ tells us about k-ary composites of morphisms. So:

- $X(0)$ gives the objects ("nullary composites of morphisms")

- $X(1)$ gives the morphisms
- $X(2)$ tell us about composable pairs $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ together with their composite, i.e. commuting triangles
- $X(3)$ composable triples

— and so on.

It is worth noting that although the elements of $X(k)$ look k-dimensional, we use all dimensions[1] for the definition of a category:

$X(0), X(1)$	Data: objects and morphisms
$X(2)$	Structure: composition
$X(k) \quad k \geq 3$	Properties: asserting that associativity holds

We can now think about enriching or internalising this structure.

5.1.2 Enrichment

Enrichment is one way of building up dimensions. We can think of an n-category as a "category enriched in $(n-1)$-categories" (see Introduction, Section 1.1.3). This means that instead of having a *set* of morphisms $A \longrightarrow B$, we have an $(n-1)$-*category* of them; but we still have just a set of objects. There is a well-developed theory of enrichment [57], but unfortunately this theory is strict and so only gives us strict n-categories. Somehow we need to "enrich weakly". In the end, this is not explicit in the technicalities here, but this idea is one way to understand the spirit of the Tamsamani/Simpson definitions.

The definitions of Trimble and May take the enrichment approach much more explicitly; see Chapter 8.

Building up dimensions

The idea is to build up dimensions roughly as follows.

i) The starting point is the nerve of a category — a set of objects, a set of morphisms, and a set of k-ary composites for all $k \geq 2$.

ii) Next, a bicategory is a category somehow enriched in categories. So if a category is a simplicial set (with conditions), then a bicategory should be a "simplicial set enriched in simplicial sets" (with conditions). That is, we should have a set of objects but *simplicial* sets of morphisms. Let us call this a 2-enriched simplicial set.

[1] You don't actually need them all: a category can be defined as a presheaf-with-properties on the full subcategory

$$[0] \rightleftarrows [1] \; \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \; [2]$$

of Δ.

iii) Next, a 3-category should be a category enriched in bicategories, or a “simplicial set enriched in 2-enriched simplicial sets” (with conditions). That is, we have a set of objects but *2-enriched simplicial* sets of morphisms. We could call this a 3-enriched simplicial set.

— and so on.

Note that, because the nerve of a category gives sets of morphisms *and sets of k-ary composites*, these sets of composites will need enriching as well. So we want something like:

An n-enriched simplicial set X is given by

- *a set $X(0)$*
- *for all $k \geq 1$ an $(n-1)$-enriched simplicial set $X(k)$*

with suitable face and degeneracy maps.

In fact it is much simpler to express this as a special case of *internalisation.*

5.1.3 Internalisation

When we *enrich* a category in **Cat**, we replace each set of *morphisms* by a category of morphisms, but we leave the *objects* as they are. When we *internalise* a category in **Cat**, we also replace the set of *objects* by a category of objects. This gives an “internal category in **Cat**” or “category object in **Cat**”.

Unfortunately a category object in **Cat** is not a 2-category but a “double category”, in which the 2-cells look like

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \text{instead of} \qquad \begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \quad . \tag{5.1}$$

We can force a double category to be a 2-category by asserting that all morphisms in the category of objects must be identities. Effectively this collapses the sides of the square in (5.1) and ensures that we have globular, not cubical morphisms.

Schematically we have the following diagram showing how the processes of enrichment and internalisation are related:

$$\begin{tikzcd} \mathbf{Cat} \arrow[rr, dashed, "\text{enrichment}"] \arrow[dr, dashed, "\text{internalization}"'] & & \mathbf{2\text{-}Cat} \ . \\ & \mathbf{Double\text{-}Cat} \arrow[ur, dashed, "\text{non-cubical}"'] & \end{tikzcd}$$

We can copy this process for simplicial sets, and iterate the process of “internalising simplicial sets in simplicial sets”:

An n-simplicial set is a simplicial object in $(n-1)$-simplicial sets.

We will then need to impose a "non-cubical" condition on our n-simplicial sets to ensure that we do not end up with cubical n-categories; equivalently this is to ensure that we are only enriching morphisms and not objects. This is the first of two conditions we will impose on an n-simplicial set in order for it to be the nerve of an n-category; the second is the so-called "nerve condition". The following tables summarise the relationship between these notions:

		category	$\equiv$	simplicial set + nerve condition
		category object in $\mathcal{V}$	$\equiv$	simplicial set object in $\mathcal{V}$ + nerve condition
category enriched in $\mathcal{V}$	$\equiv$	category object in $\mathcal{V}$ + non-cubical condition	$\equiv$	simplicial set object in $\mathcal{V}$ + non-cubical condition + nerve condition

Now put $\mathcal{V} = (n-1)$-**Cat**:

category enriched in $(n-1)$-**Cat**	$\equiv$	category object in $(n-1)$-**Cat** + non-cubical condition	$\equiv$	simplicial object in $(n-1)$-**Cat** + non-cubical condition + nerve condition
$\vert\vert\vert$ n-category				$\vert\vert\vert$ simplicial object in $(n-1)$-**SSet** + non-cubical condition *at all dimensions* + nerve condition *at all dimensions*

The point of the final entry in the bottom right is that the conditions we impose "commute" with the induction process. That is, instead of demanding the necessary conditions at each stage of the induction, we can do the whole induction first and then demand the conditions on all dimensions at the same time. It turns out that the latter approach is much easier technically.

5.1.4 The definition of multisimplicial sets

Motivated by the above considerations, we have a definition of n-simplicial sets as "simplicial objects in $(n-1)$-**SSet**", that is:

*The category n-**SSet** of n-simplicial sets is the category of functors*

$$[\Delta^{\mathrm{op}},\ (n-1)\text{-}\mathbf{SSet}]$$

where we start by putting **1-SSet** = **SSet**.

There is one final move that makes the definition of multisimplicial set look quite unlike either enrichment or internalisation, but makes it rather more technically convenient. Observe that for $n = 2$ we can use cartesian closedness as follows:

$$\begin{array}{rcl} \mathbf{2\text{-}SSet} & = & [\Delta^{\mathrm{op}},\ [\Delta^{\mathrm{op}},\ \mathbf{Set}]] \\ & \cong & [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}},\ \mathbf{Set}] \\ & \cong & [(\Delta^2)^{\mathrm{op}},\ \mathbf{Set}] \end{array}$$

and in general we can do this n times to get

$$n\text{-}\mathbf{SSet} = [(\Delta^n)^{\mathrm{op}},\ \mathbf{Set}].$$

In Section 5.2.2 we will discuss how these two presentations give us two ways of picturing a multisimplicial set.

5.1.5 The nerve condition

In this section we will study the nerve condition for ordinary categories and indicate how to generalise the condition to characterise/define n-categories.

We must consider the following question:

Question: *When is a simplicial set the nerve of a category?*

The answer to this question is well understood, but we will give the answer several times, gradually working it into a form suitable for generalisation into higher dimensions.

Answer 1: For every pair of composable 1-cells $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ there must be a unique 2-cell

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet \end{tikzcd}$$

so that we have well-defined composition. And for every string of k composable 1-cells $\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \cdots \xrightarrow{f_k} \bullet$ there must be a unique k-cell, to ensure associativity.

Answer 2: There are isomorphisms

$$\begin{array}{ccc} \{ \text{ set of composable pairs } \} & \cong & \{ \text{ set of 2-cells } \} \\ \vdots & & \\ \{ \text{ set of composable strings of length } k \text{ } \} & \cong & \{ \text{ set of } k\text{-cells } \} \\ \vdots & & \end{array}$$

Answer 3: For each k there is a canonical function

$$\begin{array}{ccc} \{ \text{ set of } k\text{-cells } \} & \longrightarrow & \{ \text{ set of composable strings of length } k \text{ } \} \\ k\text{-cell} & \mapsto & \text{the } k \text{ 1-cells we're considering composing} \end{array}$$

This canonical function comes from the appropriate face maps; we will see later how to construct it precisely. For a simplicial set to be a nerve, this function must be an isomorphism for each k.

Generalisation into n dimensions.

To generalise this nerve condition we use the fact that a function between sets is really a 0-functor between 0-categories. So for the n-dimensional case we should be looking for a canonical $(n-1)$ functor

$$\begin{array}{c} (n-1)\text{-category} \\ \text{of } k\text{-cells} \end{array} \longrightarrow \begin{array}{c} (n-1)\text{-category of composable} \\ \text{strings of length } k \end{array}$$

We call these the *Segal maps for composition*. For an n-simplicial set to be the nerve of an n-category, we demand that each of these $(n-1)$-functors be an $(n-1)$-equivalence of $(n-1)$-categories.

5.1.6 Simpson's helpful shortcut

The approach outlined in the previous section is more or less what Tamsamani does. Simpson then simplifies the situation by taking the n-nerve condition on $(n-1)$-functors and turning it into a condition about the underlying morphism of $(n-1)$-simplicial sets. This can be thought of as an unravelling of Tamsamani's inductive definition, to achieve a more direct approach. There is a small price to pay, hence the slight loss of generality in Simpson's definition.

The idea of the shortcut can be illustrated in the 1-dimensional case: if we want to show that a functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence, how much of the category structure do we actually need? We can use the characterisation that says F is an equivalence if and only if it is

i) essentially surjective on objects, and

ii) full and faithful.

For condition (i) we need to know what the isomorphisms in $\mathcal{D}$ are, so this does require some knowledge of the category structure of $\mathcal{D}$. However, for condition (ii) we only need to know what the hom-sets are — we don't need to know about any of the category structure beyond that.

We can even get around (i) by demanding surjectivity "on the nose", so that we don't need to know about *composition* in the category at all. We do lose some generality, but certainly any such functor *will be* an equivalence. This is the essence of what Simpson's definition says, and why some generality is lost.

The final useful observation to make is that a functor between categories is precisely a map of the underlying nerves — functoriality is guaranteed by the simplicial set structure.

5.1.7 The shape of the definition

The definition then looks like this:

An n-category is an n-simplicial set such that

i) we have avoided becoming cubical, and

ii) Tamsamani: each Segal map for composition of m-cells is an $(n-m)$-equivalence of $(n-m)$-categories

Simpson: each Segal map for composition of m-cells is a "contractible" map of $(n-m)$-simplicial sets

Contractibility is what Simpson uses to characterise equivalence using only the underlying simplicial set structure.

5.2 Multisimplicial sets

We begin this section with the dry technical definition of a multisimplicial set and then move on to a big picture-drawing section.

5.2.1 The definition

Technically a *simplicial set* is a functor

$$\Delta^{\mathrm{op}} \to \mathbf{Set}$$

so we write

$$\mathbf{SSet} = \mathbf{1\text{-}SSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}].$$

An n-*simplicial set* is a functor

$$\Delta^{\mathrm{op}} \to (n-1)\text{-}\mathbf{SSet}$$

so by cartesian closedness we have

$$n\text{-}\mathbf{SSet} \;=\; [\Delta^{\mathrm{op}}, (n-1)\text{-}\mathbf{SSet}] \;\cong [(\Delta^n)^{\mathrm{op}},\ \mathbf{Set}].$$

Thus, an *n-simplicial set* can be thought of equivalently as a functor

$$\Delta^{\mathrm{op}} \longrightarrow (n-1)\text{-}\mathbf{SSet} \quad \text{or} \quad (\Delta^n)^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

The first way may be more intuitively clear but the second is useful as the induction has been unravelled, and it gives a very convenient way of writing everything down.

In fact there are many other presentations in between, as we progressively invoke closedness to "move" a Δ^{op} over to the domain. Furthermore, it depends *which ones* we have moved. So we have various alternatives of the form

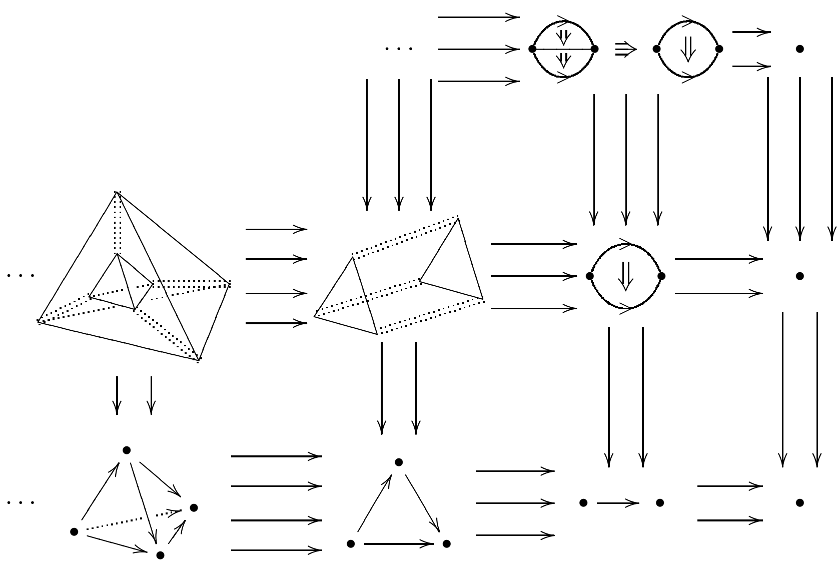
$$\big(\Delta^m\big)^{\mathrm{op}} \longrightarrow (n-m)\text{-}\mathbf{SSet}.$$

The idea is that if we fix m "coordinates" of an n-simplicial set, we will be left with an $(n-m)$-simplicial set. For example, in the 2-dimensional case, every row and column is itself a simplicial set. In the next section we will see how this is useful working out what the pictures should look like.

5.2.2 A long pictorial discussion of shapes

In this section we aim to give more intuition about what an n-simplicial set "looks like" — what pictures to have in mind and how the pictures correspond to the notation. Unfortunately it is a bit hard to draw the n-dimensional diagrams on a 2-dimensional piece of paper. Since we're only interested in n-simplicial sets satisfying the non-cubical condition, these are the only ones we will attempt to draw. (We will explain this non-cubical condition more in Section 5.3.2.)

A non-cubical 2-simplicial set might be depicted as follows:



where we are only drawing the underlying shapes of the elements, omitting all the labels.

If we write A for the presheaf $\left(\Delta^2\right)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ in question, then the positions in the above "grid" correspond to the presheaf notation as follows:

$$\begin{tikzcd}
 & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & A(1,2) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & A(0,2) \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\cdots A(3,1) \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A(2,1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left] \arrow[d, shift right] & A(1,1) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & A(0,1) \arrow[d, shift left] \arrow[d, shift right] \\
\cdots A(3,0) \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A(2,0) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A(1,0) \arrow[r, shift left] \arrow[r, shift right] & A(0,0)
\end{tikzcd}$$

where the arrows shown come from the face maps in Δ. These are the maps that give us the "clues" as to what these shapes "look" like — we should stress the fact that this is just a geometric interpretation (realisation) of what is technically only a combinatorial object. But we find it helpful to think of it in this pictorial way.

Explanation of the above pictures

i) The right hand column is "constant" — this comes from the non-cubical condition. See Section 5.3.2 for further explanation.

ii) $A(1,0)$ comes with two maps to $A(0,0)$ which we think of as source and target; hence we think of elements of $A(1,1)$ as arrows

$$a \xrightarrow{f} b\ .$$

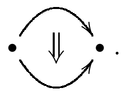
iii) $A(2,0)$ comes with three maps to $A(1,0)$ telling us there should be three constituent components $\longrightarrow$. Commuting conditions involving maps to $A(0,0)$ tell us that these should be arranged in a triangle shape as shown.

iv) We remarked in the previous section that every row and column is itself a simplicial set. So the column $A(1,-)$ (second from the right) should somehow resemble the "generic simplicial set" as seen in the row $A(-,0)$ (bottom row). See (v) and (vi) below.

v) $A(1,1)$ comes with several commuting squares given by

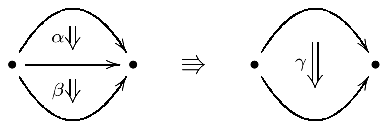
$$\begin{tikzcd}
A(1,1) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & A(0,1) \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
A(1,0) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A(0,0)
\end{tikzcd}$$

(commuting serially). Also the right hand side is degenerate ($s = t = 1$) by the non-cubical condition, and we can check that this means the whole thing reduces to globularity, hence the picture

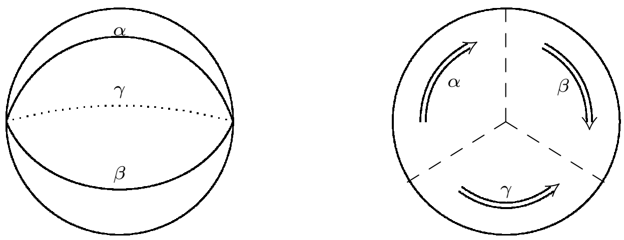


This can be thought of as an "arrow of 1-cells", as the 1-cell "$\longrightarrow$" is playing the role of "object" in the simplicial set that is the $A(1,-)$ column.

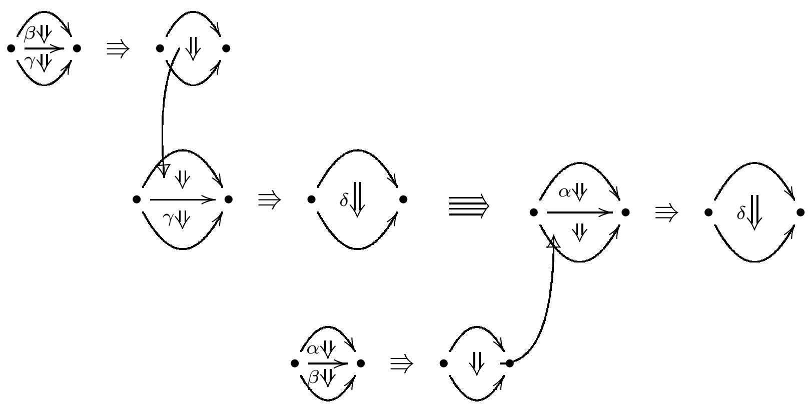
vi) $A(1,2)$ should therefore be a "triangle of arrows of 1-cells" since it is in the triangle position of the $A(1,-)$ simplicial set. If you have difficulty seeing how



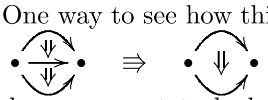
is a triangle of 2-cells, try drawing it on an orange. Score the skin as if you are dividing the orange into thirds and mark the thirds α, β, γ. Then look at the orange head on, where the three lines meet.

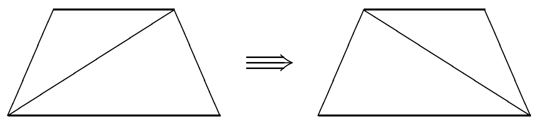


vii) $A(1,3)$ will be a tetrahedron of 2-cells. We can represent this as

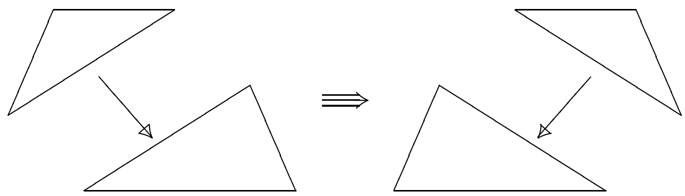


(5.2)

One way to see how this is a "tetrahedron of 2-cells" is to remember that  is a "triangle of 2-cells", and note that we can decompose a tetrahedron as



where the outer edges are identified. Or we could even decompose it to



making it look like much more like Diagram 5.2 above.

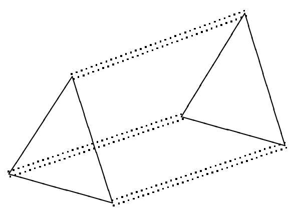
viii) $A(2,1)$ is an "arrow of triangles " — we are now in the simplicial set of the $A(2,-)$ column, where the objects are triangles. However, we are also in the simplicial set of the $A(-,1)$ row, so this could be thought of as a "triangle of 2-cells composing horizontally."

Observe also that we have five face maps

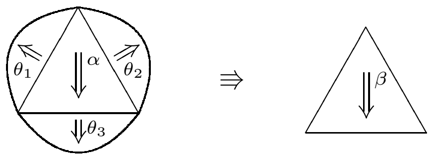
$$\begin{tikzcd} A(2,1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=1] \arrow[d, shift right=1] & A(1,1) \\ A(2,0) & \end{tikzcd}$$

telling us there should be two constituent triangles and three constituent globular 2-cells. We also have various commuting conditions telling us how the edges match up.

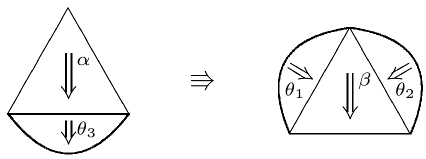
We can represent this as a "collapsed prism"



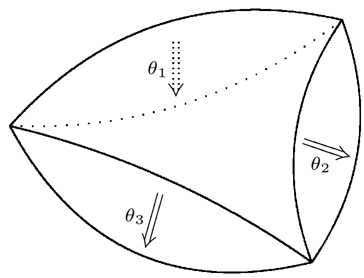
where the long equalities (shown between matching vertices of the triangles) collapse the rectangular faces into globular faces — the non-cubical condition at work. Another way to draw this is



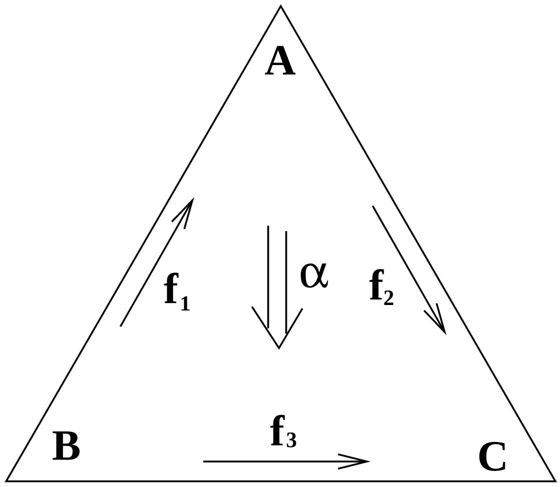
or

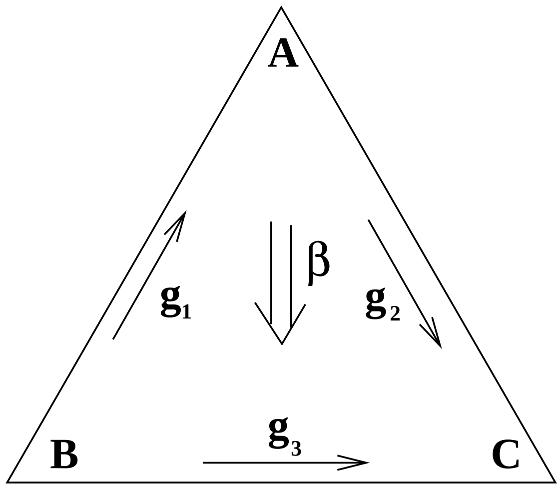


the latter of which might remind the reader of the process of finding horizontal composites of 2-cells in the opetopic theory. Here is an attempt at a drawing of the 3-dimensional figure



For an even better idea of the shape, cut out the triangles below.

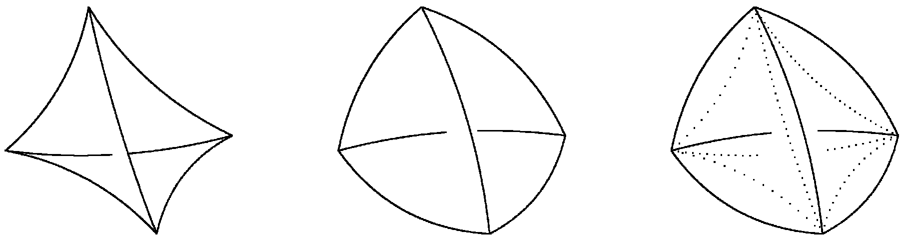




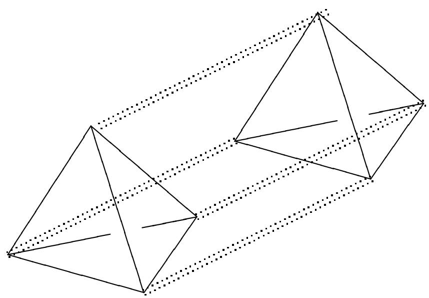
Staple the matching corners, and place a small object in between to give it some 3-dimensionality. Then observe the θ_i in the gaps between the edges

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \cdot \end{tikzcd}, \quad \begin{tikzcd} \cdot \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & \cdot \end{tikzcd}, \quad \begin{tikzcd} \cdot \arrow[r, bend left, "f_3"] \arrow[r, bend right, "g_3"'] \arrow[r, phantom, "\Downarrow\theta_3"] & \cdot \end{tikzcd}.$$

ix) $A(3,1)$ is already approaching the extremity of what we feel we can comfortably represent on 2-dimensional paper. We now need a “morphism of tetrahedra" or a "tetrahedron of 2-cells." The diagram shown in the grid on page 79 uses the above technique of identifying the matching vertices along the elongated equality signs. In this way, mediating between each matching face of the small and large tetrahedra we have a collapsed prism — an element of $A(2,1)$ (see viii). So we have one 2-cell $\Downarrow$ at each edge of a tetrahedron — hence a "tetrahedron of 2-cells." Here is an attempt at drawing the 3-dimensional figure with one concave and one convex tetrahedron;



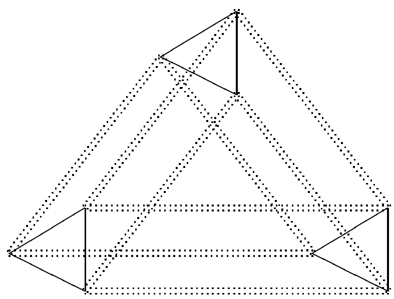
Sadly, we're cheating even like this — really this should be a 4-dimensional figure and we have no right to put one tetrahedron *inside* another. We could draw it like this



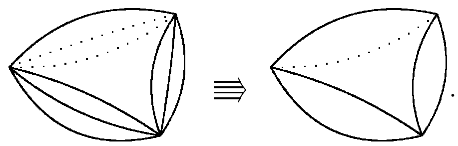
(5.3)

emphasising that the identification of those vertices goes "through" the tetrahedra, and hence we must be using a fourth dimension.

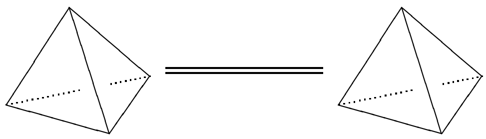
x) We can now attempt an element of $A(2,2)$ in the same spirit as (5.3) above: we make a triangle of triangles



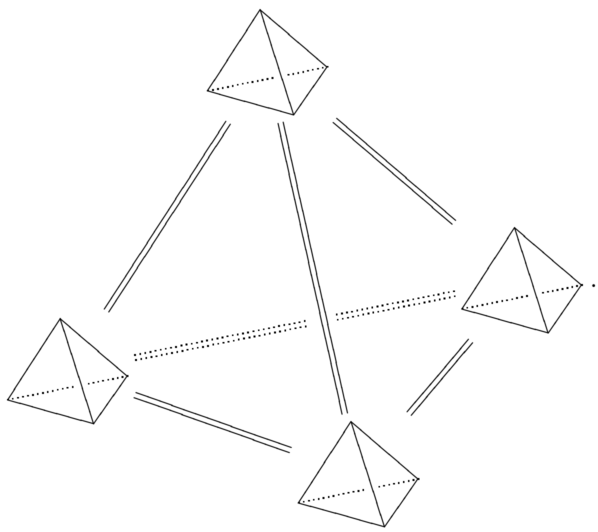
which collapses to



xi) Similarly we can attempt an element of $A(3, 3)$. Using the shorthand



for the diagram (5.3) above, we make a tetrahedron of tetrahedra



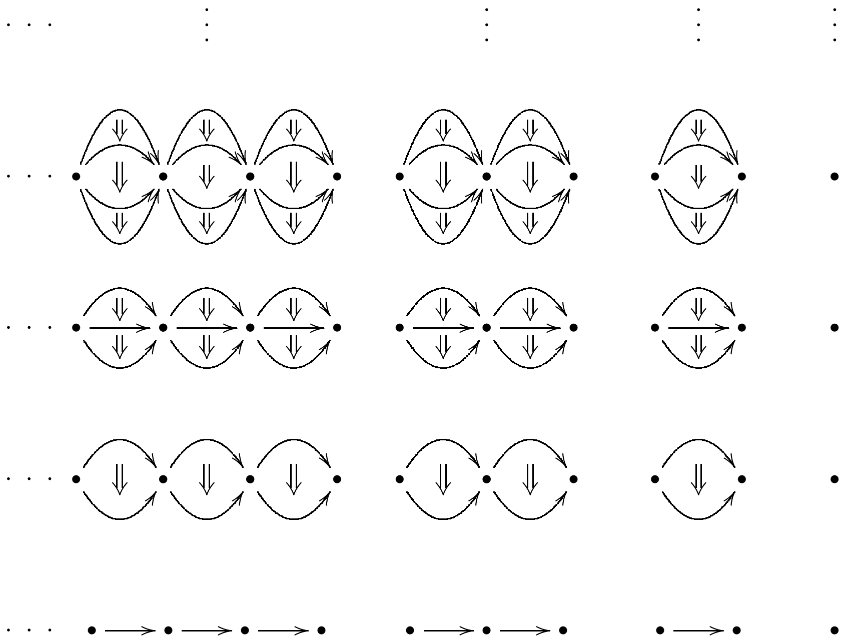
Another useful way of cheating

Finally, here is a way of cheating even further in our picture-drawing: we suppress all the "simplicial" parts of the diagrams, emphasising only the cells we are considering composing. We give some examples for $n = 3$, so the set of $A(k_1, k_2, k_3)$ refers to composition of

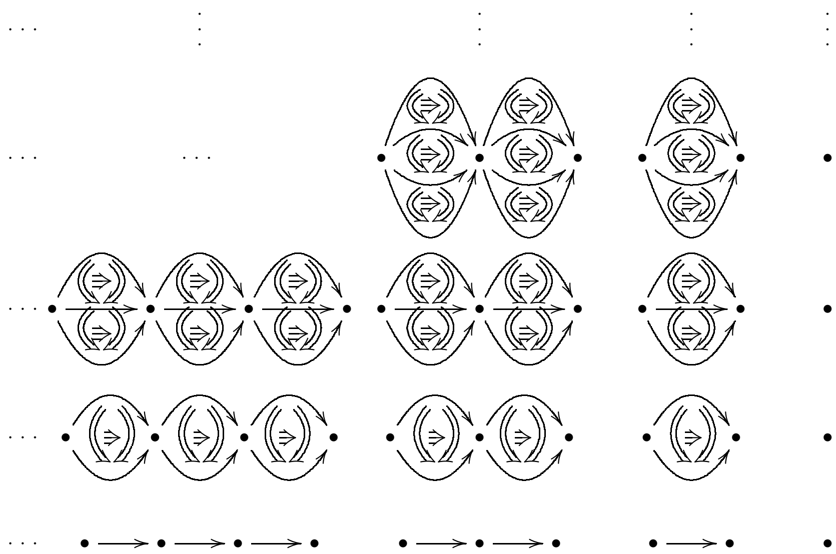
- k_1 1-cells end to end,
- k_2 2-cells end to end, and
- k_3 3-cells end to end.

We will draw the 3-dimensional grid as a series of cross-sections, one "face" at a time.

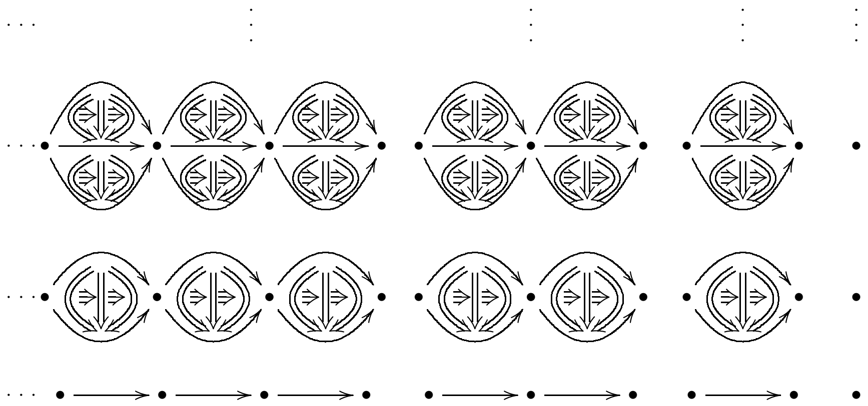
The first face $A(-,-,0)$:



The second face: $A(-,-,1)$:



The third face $A(-,-,2)$.



Note that this is for a functor

$$\begin{array}{ccccc} A & : & (\Delta^3)^{\text{op}} & \longrightarrow & \mathbf{Set} \\ & & (k_1,k_2,k_3) & \mapsto & A(k_1,k_2,k_3) \end{array}$$

and if we fix any two of the "coordinates" we get a functor

$$\Delta^{\text{op}} \longrightarrow \mathbf{Set}$$

i.e. every row, column and row-sticking-out-of-the-page is itself a simplicial set. Furthermore, fixing one coordinate we get that every plane either in the page or perpendicular to it is a (non-cubical) 2-simplicial set.

5.3 Simpson's definition

5.3.1 The definition

We now give the actual definition for the Simpson case and discuss how the formalities match up with the fuzzy intuitions we have given so far.

Definition *A weak n-category is a functor*

$$A\colon (\Delta^n)^{\text{op}} \longrightarrow \mathbf{Set}$$

such that for each $0 \le m \le n-1$, and $K = (k_1, k_2, \ldots, k_m) \in \Delta^m$

i) the functor $A(K,0,-)\colon (\Delta^{n-m-1})^{\text{op}} \longrightarrow \mathbf{Set}$ is constant and

ii) for each $k \in \Delta$ the "Segal map"

$$A(K,k,-) \longrightarrow A(K,1,-) \times_{A(K,0,-)} \cdots \times_{A(K,0,-)} A(K,1,-)$$

is contractible.

Condition (i) is the non-cubical condition discussed previously. Condition (ii) tells us that the Segal maps for composition are equivalences. The technicalities of the above definition will be explained in the sections that follow.

5.3.2 The non-cubical condition

Why does condition (i) mean we are not cubical? Note that $A(K, 0, -)$ is shorthand for

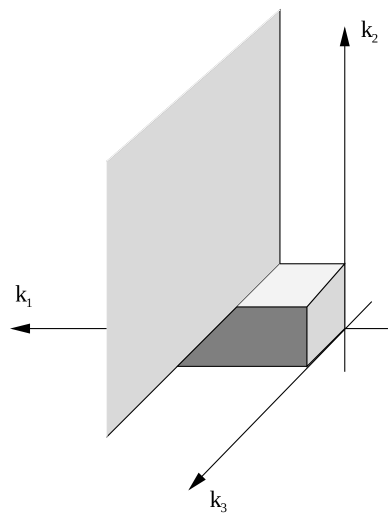
$$A(k_1, k_2, \ldots, k_m, \overset{\text{(m+1)th place}}{\overset{\downarrow}{0}}, \underbrace{-, -, \ldots, -}_{\substack{\text{n-m-1} \\ \text{blank spaces}}})$$

So we're looking at a k_i-length of i-cells for each i up to m, and no $(m+1)$-cells. The point is that if there are no $(m+1)$-cells in a pasting diagram then we can't have anything "interesting" at higher dimensions — only identities, so it should keep on being the same set. Hence no matter what lengths we plug in as the highest dimensional "coordinates", the set of cells remains constant.

We encourage the reader to fill in the coordinates of the 3-dimensional grid on page 87 to see this "stabilisation" condition at work.

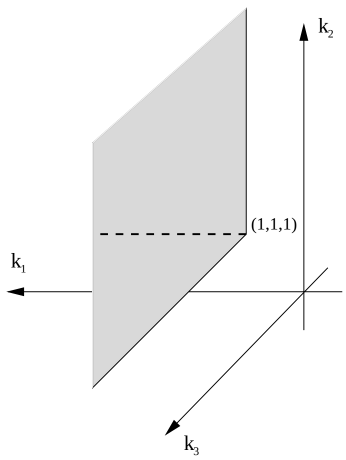
Aside for the curious

The reader might wonder what 3-dimensional geometrical shape this would look like if we actually omitted the parts that have "stabilised":



We're undecided about how illuminating this is, but it does satisfy our curiosity on the matter.

We could even try to draw the 4-dimensional case as a "movie" of 3-dimensional figures. The first would be as above, giving $A(-,-,-,0)$*. Then for* $A(-,-,-,1)$ *we have*



Note that the corner shown is $(1,1,1)$ *i.e. this figure does not touch any axis — it has been shaved off along every axis. Another way to think of it is that the protruding parts of the first diagram have been amputated.*

5.3.3 Segal maps for composition

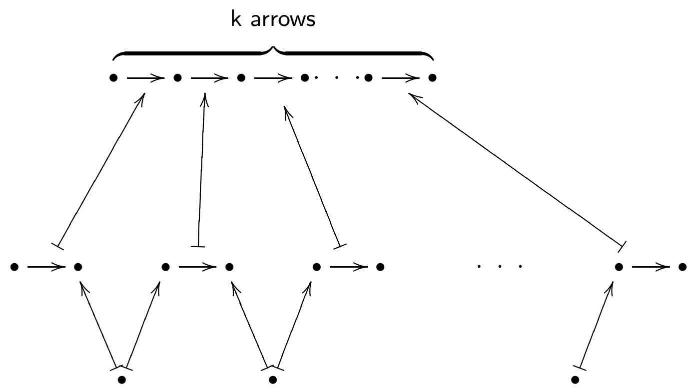
We mentioned earlier that the "Segal maps for composition" are canonical. This is because they are induced by a limit — a "wide pullback". Following the notation in [69], we write σ, τ for the two embeddings of 0 into 1, and

$$\iota_1, \ldots, \iota_k : 1 \longrightarrow k$$

for the k embeddings of 1 into k. Then the following diagram

$$\begin{tikzcd}
 & & & k & & & & \\
1 \arrow[urrr, "\iota_1"] & & 1 \arrow[ur, "\iota_2"] & & 1 \arrow[ul, "\iota_3"'] & \cdots & & 1 \arrow[ullll, "\iota_k"'] \\
 & 0 \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & 0 \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & & 0 \arrow[ur, "\sigma"'] &
\end{tikzcd} \tag{5.4}$$

commutes in Δ and can be thought of as representing the following situation:



So the index k is going to be the number of cells we are thinking about composing, at the relevant dimension. Now the "wide pullback"

$$X(1) \times_{X(0)} X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)$$

gives us k instances of *single* cells at the relevant dimension — the pullback condition ensures that this is a *composable* string, i.e. the single cells match end-to-end.

Thus, the image in X[2] of diagram (5.4) induces a canonical morphism to the wide pullback as below:

$$X(k) \longrightarrow X(1) \times_{X(0)} X(1) \times_{X(0)} \cdots \times_{X(0)} X(1).$$

Note that this is just a limit in **Set**.

Now, in the actual definition the description looks much more complicated because we have to do it for *every* shape of cell at *every* dimension. So first we fix the dimension $(m+1)$ of cells we're thinking about composing, and then we fix the actual shape of cell we're composing, i.e. the lengths of all of the lower dimensional cells involved — $(k_1, \ldots, k_m)$ in the formula. So we get a canonical map

[2] Note that technically X is a functor so perhaps we ought to say "the image *under* X". But if we're thinking of X as some kind of set of cells, it might sound more natural to think of the image *in* X, which is what we prefer.

k of the $(m+1)$-cells
we're interested in composing

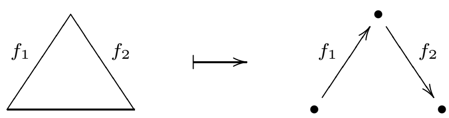
$$A(k_1, k_2, \ldots, k_m, k, -, -, \ldots, -)$$
$$\downarrow$$

1 of the $(m+1)$-cells
we're interested in composing

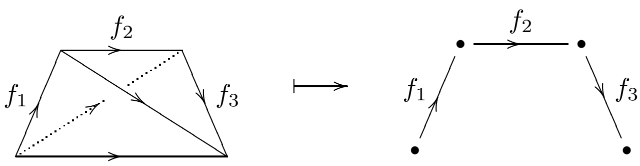
$$\underbrace{A(k_1, k_2, \ldots, k_m, 1, -, -, \ldots, -) \times_{A(k,0,-)} \cdots \times_{A(k,0,-)} A(k_1, k_2, \ldots, k_m, 1, -, -, \ldots, -)}_{k \text{ times}}$$

Some examples may help to illustrate the point.

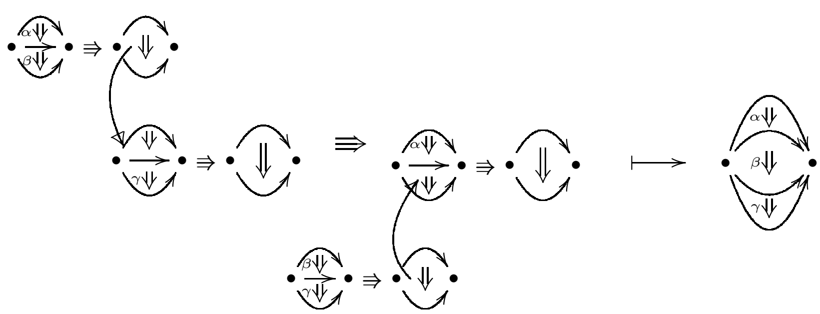
Example 1 $A(2,0) \longrightarrow A(1,0) \times_{A(0,0)} A(1,0)$



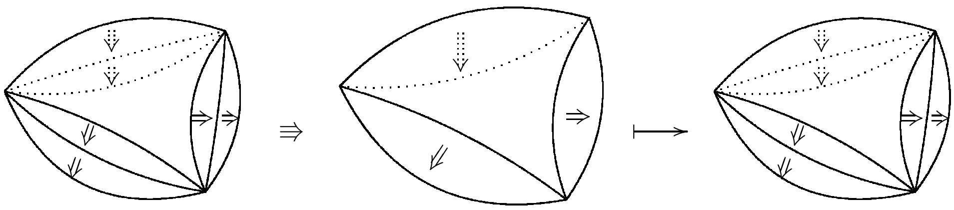
Example 2 $A(3,0) \longrightarrow A(1,0) \times_{A(0,0)} A(1,0) \times_{A(0,0)} A(1,0)$



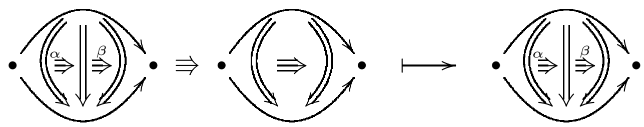
Example 3 $A(1,3) \longrightarrow A(1,1) \times_{A(1,0)} A(1,1) \times_{A(1,0)} A(1,1)$



Example 4 $A(2,2) \longrightarrow A(2,1) \times_{A(2,0)} A(2,1)$



Example 5 $A(1,1,2) \longrightarrow A(1,1,1) \times_{A(1,1,0)} A(1,1,1)$



5.3.4 Contractibility

We are going to demand that each Segal map for composition is *contractible*. The idea is that for every "pasting diagram" of cells, there should certainly *exist* a filler giving a composite — this will be a pre-image under the Segal map. But further, the "space of composites" lying over a pasting diagram should be contractible — that is, each Segal map should be contractible.

Question: Why don't we have to demand or ensure that

i) the hom$(n-r)$-structures really are $(n-r)$-categories,

ii) the Segal $(n-r)$-maps really are $(n-r)$-functors?

Answer:

i) — comes inductively from the fact that we place our condition on Segal maps at *all* dimensions.

ii) — is implied by (i), just as maps of nerves are automatically *functors* between the corresponding categories.

The globular part of a multisimplicial set

We might notice that elsewhere (e.g. in globular definitions, Chapters 2 and 3) contractibility is defined for maps of *globular* sets, whereas here we are considering maps of *multisimplicial* sets.

In fact, we can restrict our attention to the globular shapes here as well, and concentrate only on the map's action on the "globular part" of the multisimplicial set. The idea is:

> *The action of the map on* globular *shaped cells is the part that actually tells us what the* action *of the functor is; the rest is just to ensure functoriality.*

So we need to find the "globular part" of a multisimplicial set. An n-simplicial set has a n-globular set as its globular part. The globular cells are given by

$$A(I_p) \;=\; A(\underbrace{1,1,\ldots,1}_{p},\underbrace{0,0,\ldots,0}_{n-p}) \;\equiv\; \text{globular } p\text{-cells}$$

One 1-cell wide
One 2-cell high
One 3-cell out-of-page
$\vdots$
One p-cell

So this says that we are not composing any cells end-to-end.

Remark *The amount of restriction involved here makes it quite apparent how much more information is contained in an n-simplicial set than an n-globular set.*

The source and target maps can be defined as they "should" be for a globular set, and we can then define parallel cells as in the usual globular case, with either

i) x, y are 0-dimensional, otherwise

ii) x, y have the same source and target.

The definition of contractible map

The definition of "contractible" for a map ϕ proceeds just as in the globular theory in the case of Leinster (lifting everything, not just identities; see Section 3.3.1), with one extra condition for each extremity:

1) *0-dimensional extremity:*

   The map must be surjective on objects.

2) *The usual lifting property:*

   Given parallel p-cells x, x' and a p-cell $h\colon \phi x \longrightarrow \phi x'$
   there exists $g\colon x \longrightarrow x'$ such that $\phi g = h$.
   (Compare with the definition of *full*).

3) *n-dimensional extremity:*

   Given parallel n-cells x and x' such that $\phi x = \phi x'$ we must have $x = x'$. (Compare with the definition of *faithful*).

Note that we do not have to demand faithfulness at lower dimensions, as faithfulness at k dimensions is ensured by fullness at $(k+1)$ dimensions, together with the presence of degeneracies. This is why for the case $n = \omega$ in the globular theories there is no explicit demand of faithfulness; it can always be taken care of by fullness at the dimension above. We will discuss this further in Section 5.4.7.

Finally we emphasise that in this definition we demand only "contractibility" unlike the *specified contraction* demanded elsewhere (e.g. Leinster and Penon).

5.4 Tamsamani's definition

We now discuss what Tamsamani originally proposed. First recall the general shape of the definition.

An n-category is an n-simplicial set in which

i) we are not cubical, and

ii) for each m-cell, the composition $(n-m)$-map is an $(n-m)$-equivalence of $(n-m)$-categories.

Simpson simplifies condition (ii) by

1) looking at just the underlying maps of $(n-m)$-simplicial sets, and

2) using surjective equivalence.

Tamsamani on the other hand

1) checks that the underlying structures really are $(n-m)$-categories, and

2) uses *essential* surjectivity, i.e. surjectivity only up to "internal equivalence".

In order to understand Tamsamani's definition we have to define internal equivalence of m-cells. The method is to use higher dimensions to give an equivalence relation on m-cells which will be called "internal equivalence"; the difficult part is ensuring that the relation is actually an *equivalence* relation. The notion of *truncatability* is introduced to deal with this issue.

5.4.1 Internal vs external equivalence

We recall briefly the difference between "internal" and "external" equivalence:

- *external equivalence* is a relation on actual n-categories, and is given by n-functors satisfying certain properties

- *internal equivalence* is a relation on k-cells inside an n-category, and is given by $(k+1)$-cells satisfying certain properties

The two notions should coincide in an $(n+1)$-category of n-categories — the 0-cells are n-categories, and internal equivalence of those 0-cells should coincide with external n-equivalence of the n-categories in question (if only we knew what any of this meant).

5.4.2 Iterative approach to external equivalence

We will be modelling the definition of r-equivalence on the following characterisation of equivalence of categories

A functor is an equivalence if and only if

1) local behaviour: it is an isomorphism on homsets (i.e. full and faithful), and

2) on objects: it is essentially surjective on objects.

So we will seek an r-functor satisfying

1) local behaviour: it is an $(r-1)$-equivalence on hom-$(r-1)$-categories,

2) on objects: it is essentially surjective on 0-cells.

Note that to "unpack" this definition we only need to know

A. what is an isomorphism of sets (i.e. 0-equivalence of 0-categories), and

B. what is internal equivalence of k-cells for each $k \geq 0$ (in order to define essential surjectivity).

(A) is easy; for (B) we will iterate the *skeleton construction*.

5.4.3 The skeleton of a category

The skeleton construction can be thought of as a way of reducing the number of dimensions to think about. Given a category $\mathcal{C}$ we can "quotient out by isomorphisms" to produce the set X of isomorphism classes of objects. Equivalently this is the set of objects of the *skeleton* of $\mathcal{C}$. We get a quotient map

$$q\colon \mathrm{ob}\ \mathcal{C} \longrightarrow X$$

and the relation

$$a \cong b \in \mathcal{C} \iff q(a) = q(b).$$

This last condition holds as long as $\mathcal{C}$ *really was a category*; otherwise we might not get an actual equivalence relation, as we could have $a \xrightarrow{\sim} b \xrightarrow{\sim} c$ without $a \xrightarrow{\sim} c$. If we're doing this on simplicial sets, this amounts to the fact that we must start with a nerve.

Remark Quotienting is generally a violent act. Thus the skeleton construction is more of a *de*struction than a *con*struction.

5.4.4 Iterating the skeleton construction

Suppose we have an n-simplicial set. We will define internal equivalence of m-cells from the top down, by gradually quotienting our way down to the mth dimension. This iterative approach[3] can be described as follows:

- n-cells are internally equivalent if and only if they are equal
- provided the top dimension really is a nerve, we can perform the quotient on $(n-1)$-cells to get a notion of internal equivalence of $(n-1)$-cells
- provided this first quotient produced a nerve at the next dimension down, we can perform the quotient on $(n-2)$-cells to get a notion of internal equivalence of $(n-2)$-cells

— and so on.

We just need to demand that each stage of quotienting produces a nerve. This is the notion of truncatability.

5.4.5 Truncatability

Motivated by the discussion above, an n-simplicial set is called *truncatable* if (informally)

i) the top dimension is a nerve, and

ii) every stage of quotienting produces a nerve at the next dimension down.

So the issue is not simply whether we *can* truncate the top dimensions, but more subtlety, whether we can sensibly "sew up the ends" after this progressive beheading.

5.4.6 Wrong ways to prove coherence for bicategories

It is worth thinking about two obvious wrong ways to prove that every bicategory is biequivalent to a 2-category.

- **1st wrong way:** identify all $(hg)f$ with $h(gf)$, and all $I \circ f,\ f,\ f \circ I$

  Mistake: we *do* get something biequivalent
  but *don't* necessarily get a strict 2-category

  — because $(hg)f = h(gf)$ does not necessarily mean a_{hgf} is the identity.

[3] Note that technically we're not iterating the whole of the skeleton "construction" — we only find the objects and not the morphisms of the skeleton.

- **2nd wrong way:** quotient down to a category by turning all 2-cell isomorphisms into identities and ignoring all the others.

  Mistake: we *do* get a strict (in fact locally discrete) 2-category
  but *don't* get a biequivalence

  — because the quotient map is not faithful.

Nevertheless, the 2nd wrong way is useful for reducing dimensions. Another way to think of it is that we are literally performing a contraction:

- Elsewhere, we approach contraction from the lower dimensions and lift everything up.

- Here, we start at the higher dimensions and squash everything down.

The idea of internal equivalence of m-cells α and β is then the following:

> *We put $\alpha \sim \beta$ if, when we quotient all the way to m dimensions, α and β get identified.*

5.4.7 External equivalence of r-simplicial sets

We now put the above components together to arrive at the definition of external equivalence that we need. Consider a morphism of r-simplicial sets

$$\phi : X \longrightarrow Y$$

We say ϕ is an equivalence if

i) ϕ is essentially surjective on objects, i.e. surjective up to internal equivalence, and

ii) locally, ϕ is essentially surjective on k-cells for all k.

In both cases we demand that the essential pre-images be unique up to internal equivalence. These two conditions correspond to the two conditions given formally in [69].

NB Technically X and Y are functors $(\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, so ϕ is a natural transformation

$$\begin{tikzcd}
(\Delta^r)^{\mathrm{op}} \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathbf{Set}
\end{tikzcd}$$

Remark on Equivalence

It is worth thinking about why only (essential) surjectivity is required at each dimension, and not something corresponding to (essential) injectivity. This relates to the remark at the end of Section 5.3.4.

In 0 dimensions, an "equivalence" is an isomorphism of sets. A function

$$F : X \longrightarrow Y$$

is an isomorphism if and only if

i) it is surjective, and

ii) it is injective.

In 1-dimension, a functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

of categories is an equivalence if and only if

i) it is essentially surjective on objects

ii) it is locally surjective and injective on morphisms.

Question: *Why does injectivity only appear at the top dimension? That is, why don't we need to stipulate "essential injectivity" on objects?*

Answer: *Because "essential injectivity" would mean something like*

$$Fx \cong Fy \Rightarrow x \cong y$$

which follows from full-and-faithfulness.

Similarly, if we had 2-cells we could drop "injectivity on 1-cells" since it would be dealt with by full-and-faithfulness on 2-cells.

So we only ever need injectivity at the nth dimension, and only if there is no $(n+1)$th dimension to deal with it automatically. (If we had a headshrinking approach rather than a beheading approach we still would have an $(n+1)$th dimension that could serve this purpose, among others.)

Note that in Tamsamani's case, demanding that the pre-images be "unique up to internal equivalence" ensures injectivity at the top dimension, since internal equivalence at that level corresponds to equality. However, Tamsamani also demands this essential uniqueness at *all* dimensions; the above discussion suggests that at lower dimensions this condition is redundant.

5.4.8 Conclusion

Simpson slims down the definition by demanding surjectivity at every level until the top where we need injectivity as well. Simpson expresses this in terms of *contractibility* which plays the role of Tamsamani's use of *equivalence*.

Finally we remark that this contractibility tells us that "suitably coherent composition exists"; in Batanin's definition contractibility is verifying the coherence of a pre-existing composition which is given by another mechanism.

Chapter 6

Street

6.1 Introduction

Street's definition of weak ω-category [102] was the first to appear. The idea is, like Simpson/Tamsamani [93, 107], to generalise the nerve of a category, but here the underlying data is a simplicial set not a multisimplicial set. We will discuss the difference later. In fact, the definition itself is not the central aim of [102]. The aim is to construct the nerve of a strict ω-category, with n-cohomology as motivation. Like the nerve of an ordinary category, the nerve of an ω-category is to be a simplicial set.

In the closing paragraphs of [102], Street

i) *conjectures* a condition on simplicial sets to characterise those that arise as nerves in this way

ii) *observes* that in the case of (weak) bicategories, this condition is satisfied except for a certain "uniqueness" clause

iii) *suggests* removing the word "unique" from the condition and using the result as a notion of weak ω-category.

He goes on to prove the conjecture in a later paper [103]. The main part of the condition says:

Every admissible horn has a unique hollow filler.

A horn is, essentially, a simplicial cell with one face missing, and no interior. This is a weakened version of the much more straightforward condition for ω-groupoids that says "every horn has a unique filler." The difference is that we now have to allow for non-invertible cells which, unsurprisingly, makes the situation rather more complicated.

Street's method is highly combinatorial and it is quite easy for the geometric intuitions to be lost among the complicated (and seemingly magic) combinatorics. So it is our intention to bring out the intuitions and say rather little about how the actual combinatorics work out. Street himself is admirably candid in remarking that the conjecture was a matter of pattern-spotting in low-dimensional examples.

6.1.1 Why simplicial sets and not multisimplicial sets?

> *A beam of white light splits into a whole spectrum of colours as it passes through a prism. Street's approach is like a beam of white light where the multisimplicial approach is like a prism enabling us to see all the colours that make up the white light.*

In the multisimplicial approach each "multisimplicial shape" has one very specific role. Some shapes are for composition only, some are "genuine cells" and others are for associators or interchange, but the general idea is that each shape should only have one role to play. Using Δ^n and not just Δ gives us more shapes so that we *can* distinguish between those roles, like splitting out the colours of the rainbow.

In the simplicial approach on the other hand, any particular shape can play different roles for different dimensions of cell:

- a 2-dimensional shape gives
  i) 2-cells
  ii) composition for 1-cells
- a 3-dimensional shape gives
  i) 3-cells
  ii) composition for 2-cells
  iii) associators for 1-cells
- a 4-dimensional shape gives
  i) 4-cells
  ii) composition for 3-cells
  iii) associators for 2-cells
  iv) pentagons for 1-cells

So we have the following "diffraction diagram":

0-**cells**	1-**cells**	2-**cells**	3-**cells**	4-**cells**	5-**cells**
0-cells	1-cells	1-composition	1-associator	1-pentagon	1-hexagon
		2-cells	2-composition	2-associator	2-pentagon
			3-cells	3-composition	3-associator
				4-cells	4-composition
					5-cells

In fact the full multisimplicial picture is even more "diffracted" than this, giving an n-dimensional "rainbow" rather than the 2-dimensional one shown above.

The point is that the nerve functor

$$\mathbf{Cat} \longrightarrow \mathbf{SSet}$$

is very far from being surjective — simplicial sets have a great deal of expressive power left untapped when we use them to express mere categories. As k increases, k-cells become more and more redundant. In the nerve of a category 3-cells tell us about associativity, but 4-cells tell us nothing new. So there is plenty of "leftover space" available there to express the extra structure we want for 2-categories, 3-categories and so on.

The "beam of white light" is wonderfully compact but it does mean we have to do something else in order to see all the hidden "colours". In particular, we want to have a handle on the difference between "genuine cells" (as shown on the bottom diagonal edge of the "rainbow") and cells that are playing some sort of structure-giving role.

Street uses simplicial sets with certain cells picked out to play that structure-giving role. *A priori* these are just any old cells picked out and distinguished. The conditions then assert that these cells are actually suitable to play the structure-giving role that we want. Note that these cells are called "hollow" but hollowness is not a property that is inherent to them. We might just as well call them purple.

Remark on hollowness

We might think it is desirable to have "hollowness" as an inherent property rather than as added structure. This is one of the motivations for the use of "universal cells" in the opetopic definition; see Remark at the end of Section 6.1.4. In fact, Street also modified his original definition along these lines in a later paper [105].

6.1.2 Why this definition is "natural"

There are two more points worth noting that make this "beam of light" approach seem natural. We mention them briefly here and discuss them in more detail later.

Why it is natural: technically

Finding the nerve of a category is reduced, by abstract categorical arguments, to finding a functor

$$\Delta \longrightarrow \mathbf{Cat}$$

and making use of the Yoneda embedding to get

$$\begin{tikzcd} \Delta \arrow[rr, "\mathsf{Yoneda}"] \arrow[dr] & & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \arrow[dl, dotted, shift right] \arrow[dl, phantom, "\perp"] \\ & \mathbf{Cat} \arrow[ur, dotted, shift right, "\mathsf{nerve\ functor}"'] & \end{tikzcd}$$

One natural way to generalise this for ω-categories is to find a functor $\Delta \longrightarrow \omega\text{-}\mathbf{Cat}$ and get

$$\begin{tikzcd} \Delta \arrow[rr] \arrow[dr] & & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \arrow[dl, dotted, shift right] \arrow[dl, phantom, "\perp"] \\ & \omega\text{-}\mathbf{Cat} \arrow[ur, dotted, shift right, "\mathsf{nerve\ functor}"'] & \end{tikzcd}$$

Thus the nerve of an ω-category arises as a simplicial set, not a multisimplicial set.

Why it is natural: ideologically

Even Street's definition of strict ω-category is like a beam of white light compared with other definitions. The underlying data is a set, not a globular set — that is, cells of all dimensions have gone into the same set and the dimensions have vanished. The dimensionality is taken as, not an *a priori* property, but a consequence of the structure that is then given to this set. In the light of the succinct nature of this definition, the succinctness of Street's final definition seems quite natural.

6.1.3 What is all the complicated combinatorics about?

The original paper [102] is called "The algebra of oriented simplexes" and essentially the complicated combinatorics is all about the question of *orientation.*

Simplices can be thought of as a combinatorial tool very well suited for dealing with topological spaces. But the algebraic structure that most naturally arises is then not a category but a *groupoid*, since all morphisms are invertible. If we do not want every cell to be invertible we have to decide which way the morphisms are pointing.

- 1-cells: an arbitrary choice $\longrightarrow$ or $\longleftarrow$
- 2-cells: we have to decide how to orient the boundary as well as the middle

$$\begin{tikzcd}[column sep=small] & \bullet \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur] \arrow[rr] & {} & \bullet \end{tikzcd} \quad \text{or} \quad \begin{tikzcd}[column sep=small] & \bullet \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ \bullet \arrow[ur] \arrow[rr] & {} & \bullet \end{tikzcd} .$$

Note that at 2 dimensions another issue arises — the source part and target part of the boundary had better be well-defined composites. For example the following makes no sense in a 2-category:

$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur] \arrow[rr] & \Downarrow & \bullet \arrow[ul] \end{tikzcd} .$$

In general, it is not just a question of orienting everything, but also a question of checking that the chosen orientations give sources and targets that are well-defined composites. This is what Street calls "well-formedness".

The final complication arising from the non-invertibility of cells is about *which horns need fillers.* Without getting into too many technicalities yet, a horn is like a simplex with no interior, and one face missing. Here is a 2-dimensional (unoriented) horn

$$\begin{tikzcd} & \bullet \arrow[dr, dash, "g"] & \\ \bullet \arrow[ur, dash, "f"] \arrow[rr, dash, dotted] & & \bullet \end{tikzcd}$$

which may or may not have a 2-cell "filler" in it

$$\begin{tikzcd} & \bullet \arrow[dr, dash, "g"] & \\ \bullet \arrow[ur, dash, "f"] \arrow[rr, dash, "h"'] & \alpha & \bullet \end{tikzcd}$$

In a groupoid every such thing must have a filler. But if we give the 1-cells orientations, we will need fillers in some cases but not others. For example, we must have a filler

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, dotted] & & \bullet \end{tikzcd}$$

for composition, but not necessarily a filler

$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur, "f"] & & \bullet \arrow[ul, "g"'] \arrow[ll, dotted] \end{tikzcd}$$

unless f happens to be invertible. The nerve condition for ω-groupoids [22] simply says "*every horn has a unique filler*". But for ω-categories we need to say

> *Certain horns of the right kind need to have fillers.*

Working out which ones are the "right kind" is fiddly, and this is what is called "admissibility".

6.1.4 Summary of the definition

In summary, there are three pieces of combinatorial magic involved:

1) Well-formedness: how we detect that pasting diagrams actually paste properly

2) Orientation: using a system of orientations satisfying (1)

3) Admissibility: identifying the horns that need fillers according to the orientations given in (2)

The definition then looks like this:

A *strict* ω-category is given by a simplicial set with some cells distinguished as hollow, satisfying

i) all degenerate cells are hollow,

ii) if all but one face of a hollow cell is hollow, and if the horn formed by those faces is admissible, then the last face must also be hollow, and

iii) every admissible horn has a unique hollow filler.

A *weak* ω-category is as above but with the word "unique" removed from condition (iii).

Remarks on connection with the opetopic approach

The opetopic approach [4] was inspired/motivated by Street's paper. Opetopes can be seen as a way of avoiding the complicated combinatorics of orientation, by restricting to shapes with only one cell in the target. Whereas simplices arise *without* built-in orientation and so must have orientation imposed on them, opetopes are constructed with orientations *built in*, and hence well-formedness can also be built in; this can only happen since orientation already exists at the time of construction.

Furthermore, the idea of hollowness is turned into an inherent property of cells in an opetopic set: universality. Then the complication moves away from the problem of finding the horns/niches that require fillers; instead we have complicated conditions on which cells qualify as "structure-giving" fillers.

6.2 Motivation/Background

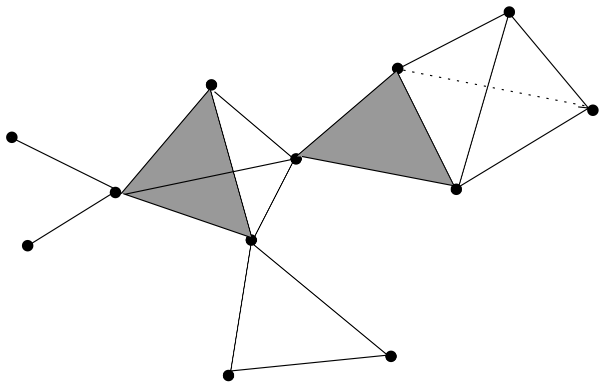
Rather than pluck the definition from thin air we aim to tell some of the story that leads to it. We hope to shed some light on the combinatorics even though we are unable to explain where the patterns really come from. We begin with a discussion about simplicial sets.

6.2.1 Geometric realisation and nerves of categories

Simplicial sets provide a useful means of studying topological spaces via an adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left] \end{tikzcd}.$$

When we draw pictures of simplicial sets such as

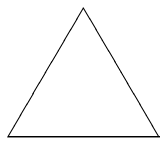


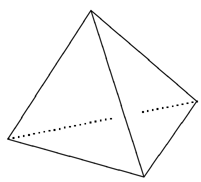
we are really drawing a topological space corresponding to it under *geometric realisation* — we are realising a simplicial set as an actual geometric shape.

In order to do this, all we have to do is decide what each individual simplicial object $\alpha \in \Delta$ should "look like"

0

1

2 

3 

and then we can stick them together.

Abstract categorical aside

Categorically what we have done above is define a functor

$$\Delta \longrightarrow \mathbf{Top}$$

and then used the Kan extension along the Yoneda embedding to induce the adjunction

$$\begin{tikzcd}
 & {[m]} \arrow[rr, mapsto] & & {\Delta(-,[m])} \\
 & \Delta \arrow[rr] \arrow[dr] & & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \arrow[dl, dotted, shift right, "F"'] \\
{[m]} \arrow[drr, mapsto] & & \omega\text{-}\mathbf{Cat} \arrow[ur, dotted, shift right, "G"'] \arrow[ur, phantom, "\perp"] & \\
 & & \Delta_m &
\end{tikzcd}$$

Then G is given by $GX([m]) = \mathbf{Top}(\Delta_m, X)$, the set of continuous maps from the generic "geometric" m-simplex to the space X, and F is found by using the facts that

- every presheaf is a colimit of representables,
- each representable $\Delta(-,[m])$ must be sent to Δ_m, and
- F is a left adjoint so must preserve colimits.

Copying this idea for categories

Whether we are thinking of the basic intuitive idea or the abstract categorical argument, the point is to get a well-behaved adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left, "G"] \end{tikzcd}$$

enabling us to use simplicial sets to study spaces. We can use the same argument with **Cat** instead of **Top**, to get an adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, shift left, "G"] \end{tikzcd}$$

enabling us to use simplicial sets to study *categories*. As above, we only need to define a functor

$$\Delta \longrightarrow \mathbf{Cat}$$

giving the "free category on a simplex". The free category on $[m]$ is simply the free category on m composable arrows

$$x_o \xrightarrow{f_1} x_1 \longrightarrow \;\cdots\; \xrightarrow{f_m} x_m\,.$$

The right adjoint G above gives the simplicial set whose m-cells are all the strings of m composable arrows in $\mathcal{C}$. This is called the *nerve* of $\mathcal{C}$, and G is the *nerve functor*.

Categorically, the reason this adjunction is useful is that although G is forgetful, it is not *too* forgetful: it is full and faithful. That is, a morphism in **Cat** corresponds precisely to a morphism in **SSet** of the underlying nerves. This means that we can construct a category **Nerve** $\cong$ **Cat** by taking the full subcategory of **SSet** whose objects are precisely those in the image of G.

So the key question is: what is the image of G? That is:

Which simplicial sets arise as the nerve of a category?

This question has a complete answer — the "nerve condition" which, essentially, ensures that every composable string of m arrows has a unique composite.

Copying this idea for ω-categories

The idea, then, is to replace **Cat** with ω-**Cat** in the following steps:

i) Construct a functor
$$\Delta \longrightarrow \omega\text{-}\mathbf{Cat}$$
giving the "free ω-category on m composable arrows".

ii) Use it to induce an adjunction
$$\begin{tikzcd} \mathbf{SSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \omega\text{-}\mathbf{Cat} \arrow[l, shift left, "G"] \end{tikzcd}$$
in which G should be full and faithful (which may be thought of as a justification that the functor chosen in (i) was a sensible/useful one).

iii) Find the image of the nerve functor G, that is, find a "nerve condition" with which to identify those simplicial sets that arise as the nerve of an ω-category.

iv) Weaken the nerve condition to produce a definition of weak ω-category.

6.2.2 Definition of strict ω-category

Although we are not going to go into technical details that require a technical definition of strict ω-category, we will discuss the technical approach briefly here in order to justify our earlier comments about the definition being a "beam of white light". Street uses the following definitions of category, 2-category and ω-category:

Definition *A* category *is given by* $(A, s, t, *)$ *where*

- A *is a set,*

- *s,t, $A \longrightarrow A$ are functions satisfying $ss = ts = s$, $tt = st = t$, and*
- *$* : \{(a,b) \in A \times A \mid s(a) = t(b)\} \longrightarrow A$ is a function satisfying $s(a*b) = sb$, $t(a*b) = ta$,*

with axioms for identities and associativity.

One striking feature of this is that there is no mention of objects except as identity arrows. The definition of 2-category makes this even more striking, as the underlying data is still only a set A (as compared with a more "diffracted" definition which begins with sets A_0, A_1, A_2 of 0-cells, 1-cells and 2-cells):

Definition *A* 2-category *is given by $(A, s_0, t_0, *_0, s_1, t_1, *_1)$ where $(A, s_0, t_0, *_0)$ and $(A, s_1, t_1, *_1)$ are categories, satisfying some axioms (globularity, interchange and so on).*

We can think of the following picture of $a \in A$

$$\begin{tikzcd} s_0(a) \arrow[rr, bend left, "s_1(a)"] \arrow[rr, bend right, "t_1(a)"'] \arrow[rr, phantom, "\Downarrow a"] & & t_0(a) \end{tikzcd}$$

and we have horizontal and vertical composition given by $*_0$, $*_1$ respectively. The identities for $*_1$ are called 1-cells and identities for $*_0$ are called 0-cells. Finally, we use the definition of 2-category to make the following definition of (strict) ω-category. The idea is that any pair of dimensions forms a 2-category.

Definition *A strict ω-category is given by $(A, (s_n, t_n, *_n)_{n \in \omega})$ where*

- *for each n, $(A, s_n, t_n, *_n)$ is a category, and*
- *for each $m < n$, $(A, s_m, t_m, *_m, s_n, t_n, *_n)$ is a 2-category.*

There are no further axioms — everything has been taken care of by the axioms for a 2-category. Identities for $*_n$ are called n-cells.

Remarks

i) Since identities are to be thought of as degenerate cells we see why, in this definition, "ordinary" (globular) n-cells will arise as degenerate cells — because ordinary cells are defined precisely by the identities on them.

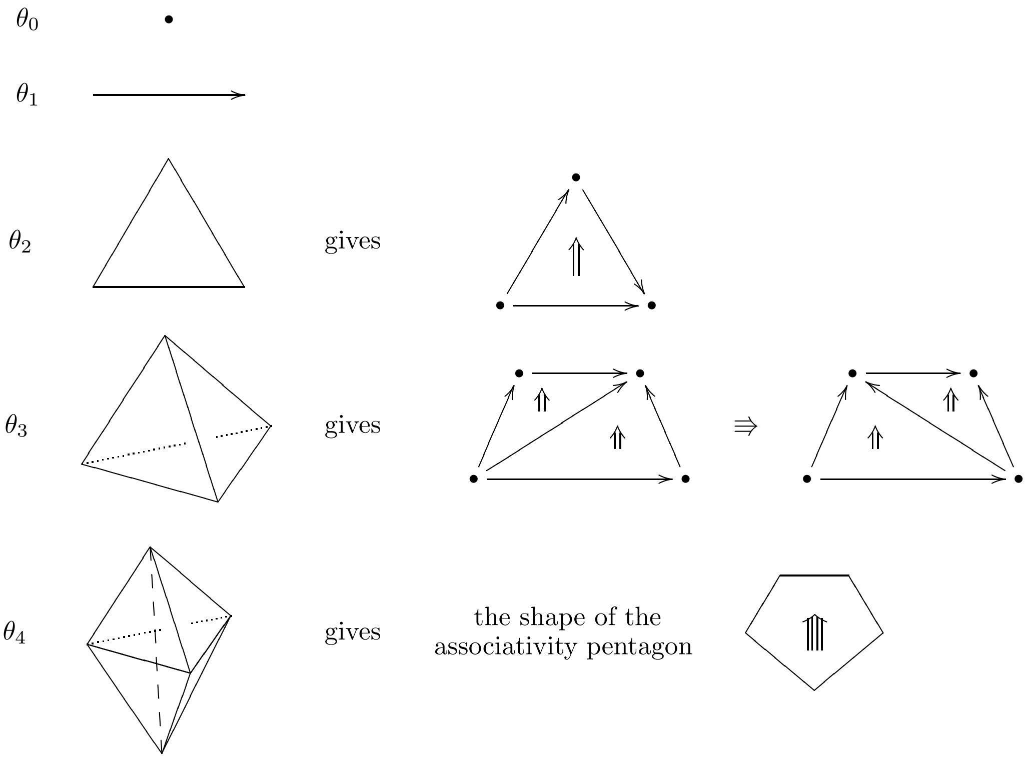
ii) Note that the above definition of ω-category includes the possibility of elements of A that are not n-cells for any $n \in \mathbb{N}$ — these may be thought of as "ω-cells". This differs from the notion that many people now think of as ω-category, which only has n-cells for *finite* n. Street restricts to this case later in the paper.

6.2.3 The functor $\Delta \longrightarrow \omega$-Cat

In this section we will explain the key ideas behind the functor $\Delta \longrightarrow \omega$-**Cat**. This functor

$$\begin{array}{ccc} \Delta & \longrightarrow & \omega\text{ -}\mathbf{Cat} \\ [m] & \mapsto & \theta_m \end{array}$$

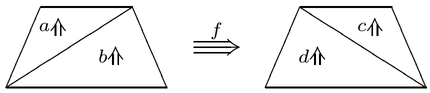
should give an ω-category θ with one m-cell in the "shape" of the m-simplex together with whatever lower-dimensional cells are needed to support it. We can illustrate the first few dimensions; we include orientations now but discuss them later.



Note that we are now running out of dimensions on the page so we have performed a useful "dimension shift" — in the pentagon, the edges are actually representing 3-cells.

Aside on "dimension shift"

We could have performed a dimension shift for the tetrahedron as well, using edges to represent 2-cells (i.e. cells of the previous dimension) which would yield a square



$$\begin{tikzcd} \bullet \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "f\Uparrow"] & \bullet \arrow[d, "d"] \\ \bullet \arrow[r, "a"'] & \bullet \end{tikzcd}$$

Staying at this level of "shift", θ_4 would then appear as a cube, one of whose faces is trivial, leaving five faces corresponding to the 5 edges of the usual pentagon.

Performing another dimension shift for θ_5, we get a hexagon; so each θ_4 has a $(n+1)$-gon representing the $(n+1)$ faces of an n-simplex. However, staying with the pentagons to build a 3-dimensional figure for θ_5, we get the the usual "4-cocycle condition" shape or 5-associahedron[1] (see also [98]).

We include a net of this figure that can be cut out and assembled (Appendix B). We encourage the reader to do this as the geometry and symmetries of the figure become much clearer than on a merely 2-dimensional piece of paper.

Notes on the 5-associahedron cut-out

The shapes A, B, C, D are dual to A', B', C', D' respectively. The square $E = E'$ is self dual. In [102] the orientation corresponds to

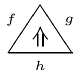
$$A, B, C, D \longrightarrow A', B', C', D' \ .$$

Note that the dimension shift here means that each face is representing a 4-dimensional shape, each edge a 3-dimensional shape and each vertex a 2-dimensional shape (pasting diagram). We have marked the 3-dimensional "source edge" with a dotted line and the target with a thick line. The 2-dimensional source vertex is marked  and the target .

It should be apparent from these diagrams that

- a free construction is needed at each dimension *before* the next dimension can be added, as we need formal composites of cells to be sources and targets of higher cells

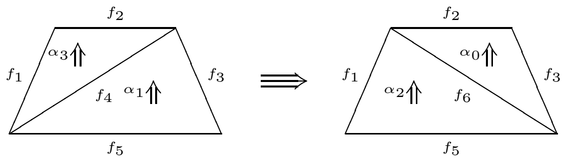
- it is important that orientations are picked consistently so that the formal composite is actually well-defined.

Once this is done, the nerve of an ω-category is constructed as the simplicial set whose "m-cells are instances of m-cells between composites of $(m-1)$-cells as described by θ_m". For example a 2-cell 

is to be a 2-cell $h \Rightarrow gf$ in the

[1] The "5" here is because we're considering the situation of composing 5 arrows.

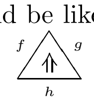
ω-category. A 3-cell



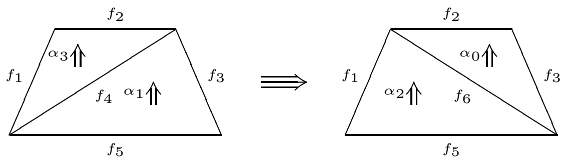
is a 3-cell which might be written linearly as $(f_3 *_0 \alpha_3) *_1 \alpha_1 \Rrightarrow (\alpha_0 *_0 f_1) *_2 \alpha_2$.

6.2.4 Hollow cells and admissible horns

Recall that the data for an ω-category is to be a "simplicial set with hollowness" that is, a simplicial set with some cells picked out and called hollow. Eventually these are supposed to be the structure-giving cells, so we need to decide when these cells do actually give enough structure. The answer will be "every admissible horn should have a hollow cell in it."

The idea is that hollow cells should be like equalities or identities, literally giving us compositional identities, e.g. 

being hollow tells us $gf = h$, and



being hollow tells us that

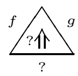
$$(f_3 *_0 \alpha_3) *_1 \alpha_1 = (\alpha_0 *_0 f_1) *_2 \alpha_2. \tag{6.1}$$

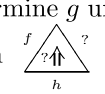
A *horn* for an m-cell gives us all the lower-dimensional data *except* for one $(m-1)$-dimensional face. The question is:

When does this partial data uniquely determine the missing cell?

This question says

When should this horn have a unique hollow m-cell in it?

The hollow m-cell will tell us how to "solve the equation". For example, given 

we know we must have a hollow cell telling us $gf = h$. However, if instead g is missing then the equation does not allow us to determine g unless f is already an identity (or invertible). This says that the horn 

only requires a hollow filler if f is already hollow.

Similarly we can consider the 3-cell equation (6.1) above:

- To determine α_0, we need α_2 and f_1 to be identities.
- To determine α_1, we need α_3 to be an identity.
- To determine α_2, we need α_4 to be an identity.
- To determine α_3, we need α_1 and f_3 to be identities.

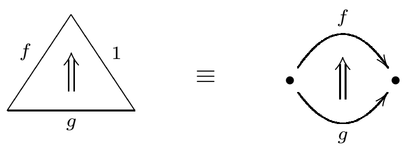
The question is: what is the general pattern here?

The key then is to express all these horns and faces in such a way that *pattern-spotting is made easier*. This is where all the combinatorics come in; everything is expressed as a string of indices, and then it is a matter of spotting patterns in the numbers. Street's conjecture in [102] is that the pattern he has spotted holds true at all dimensions.

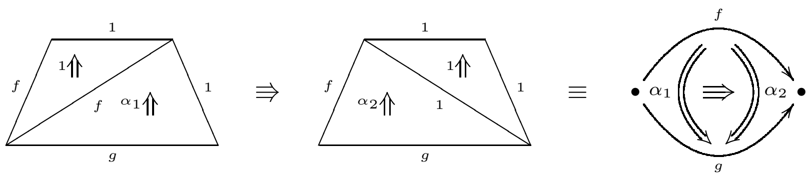
6.2.5 Where do globular cells come from?

Here are some examples of how globular cells arise as partly degenerate simplicial cells:

2 **dimensions**



3 **dimensions**



6.3 The actual definition

We now briefly go through the technicalities of the definition with notation as in [69].

6.3.1 Simplicial sets

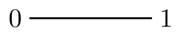
A simplicial set is a functor

$$A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

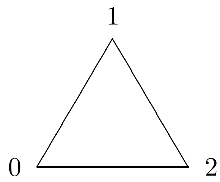
The category Δ has objects $[m] = \{0, \ldots, m\}$ for $m \geq 0$, together with order-preserving maps. So we can think of the underlying simplicial shapes as having vertices labelled by the indices $\{0, \ldots, m\}$ as in the following examples:

$m = 0$

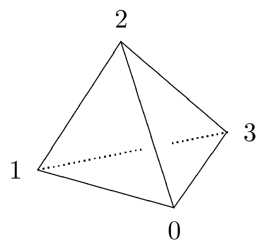
$m = 1$



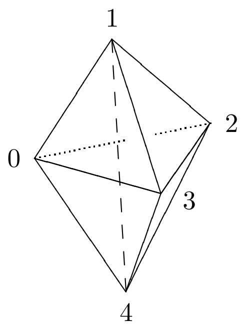
$m = 2$



$m = 3$



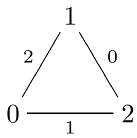
$m = 4$



Note that the 2-simplex has 3 constituent 1-simplices (faces). The 3-simplex (tetrahedron) has 4 constituent 2-simplices (triangles). The 4-simplex, thought of as a 4-dimensional figure, has 5 constituent tetrahedra — one at the top, one at the bottom, and three that "rotate" about the central dotted axis.

An m simplex has $(m + 1)$ faces, its constituent $(m - 1)$-simplices, and each one can be located by omitting one vertex of the m-simplex. Thus we say "ith face" for the face found by omitting the ith vertex. For example, the

(1-dimensional) faces of a 2-simplex are named as follows:



This notation is what enables the pattern-spotting in the end.

For *orientation*, Street uses the convention that the odd faces are in the source of a cell and the even ones in the target.

6.3.2 Maps in Δ

Faces

We're interested in maps

$$\delta_0, ..., \delta_m : [m-1] \longrightarrow [m]$$

where δ_i simply misses out i in the codomain. Then in the actual simplicial set A the map

$$A(\delta_i) : A[m] \longrightarrow A[m-1]$$

picks out the ith face of an m-cell, by missing out the ith vertex as discussed above.

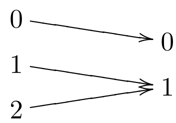
Degeneracies

Where faces are given by *injections*, degeneracies are given by *surjections*. Given $m' \leq m$, an order-preserving map

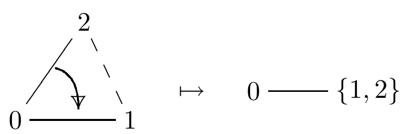
$$\sigma : [m] \longrightarrow [m-1]$$

can be thought of as identifying some points of $[m]$. For example the following map

$$\sigma : [2] \longrightarrow [1]$$



identifies 1 and 2. Geometrically this can be though of as "squashing" a triangle down to a line



In the simplicial set A, the map

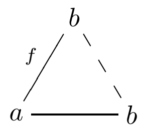
$$A\sigma : A[m-1] \longrightarrow A[m]$$

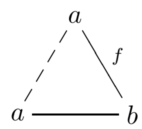
picks out, for every $(m-1)$-cell, an m-cell that is the higher-dimensional version of the $(m-1)$-cell obtained by inserting some degenerate faces. So for every 0-cell a we get a degenerate 1-cell

$$a \overset{a'}{\dashrightarrow} a$$

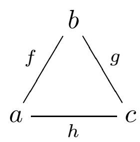
where $a = (A\sigma)a'$.

For every 1-cell $a \xrightarrow{f} b$ we get degenerate 2-cells

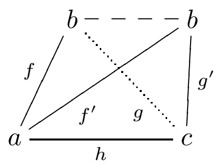




and for every 2-cell

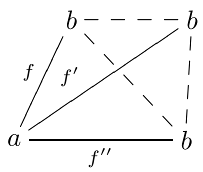


we get degenerate 3-cells



where we have inserted 2 degenerate triangles at the sides. We get even more degenerate tetrahedra from surjections

$$[3] \longrightarrow [1]$$



6.3.3 Horns and fillers

A *horn* is, essentially, an m-cell with no interior, and one face missing. In fact it is a whole simplicial set of lower-dimensional cells constituting this figure.

We write Λ^k_m for the horn which is an "m-cell with its kth face missing". For example, Λ^0_2 looks like

$$\begin{tikzcd} & 1 \arrow[dr, dash, "?"] & \\ 0 \arrow[ur, dash] \arrow[rr, dash] \arrow[ur, phantom, shift left=6, "?"] & & 2 \end{tikzcd}$$

A *horn in a simplicial set* A is given by a morphism $\Lambda^k_n \longrightarrow A$ (technically a natural transformation, since a simplicial set is technically a functor). Geometrically, this picks out the horn-shaped possibilities in A, e.g.

$$\begin{tikzcd} & b \arrow[dr, dash, "?"] & \\ a \arrow[ur, dash, "f"] \arrow[rr, dash, "h"'] \arrow[ur, phantom, shift left=6, "?"] & & c \end{tikzcd}$$

A *filler* for a horn $\Lambda^k_m a$ is an m-cell that "fills in" the missing parts of the horn. For the above horn we might have the following filler:

$$\begin{tikzcd} & b \arrow[dr, dash, "g"] & \\ a \arrow[ur, dash, "f"] \arrow[rr, dash, "h"'] \arrow[ur, phantom, shift left=6, "\alpha"] & & c \end{tikzcd}$$

6.3.4 Hollowness

A *simplicial set with hollowness* is a simplicial set A together with a subset of "hollow m-cells" $H_m \subseteq A[m]$ for each $m \geq 1$. In particular there are no hollow 0-cells. Note that there are no conditions at this point on which cells may or may not be hollow.

6.3.5 Orientation and admissible horns

This is where all the combinatorics comes in. We choose not to go into the details of exactly how to calculate which horns are admissible. The key is to think of a cell as a morphism oriented as follows

$$\text{odd-numbered faces} \rightarrowtail \text{even-numbered faces}$$

and then ask the question as discussed in Section 6.2.4:

Question: Given a missing cell, which of the other cells must be identities in order to determine it uniquely?

The combinatorial answer is found by examining the set of vertices of each face and testing to see if this set is "k-alternating".

Answer: Any face that is k-alternating must be an identity. So a horn is admissible if every k-alternating face is hollow.

Definition *A weak ω-category is a simplicial set with hollowness such that*

i) all degenerate cells are hollow,

ii) every admissible horn has a hollow filler, and

iii) if all but one face of a hollow cell is hollow, and if the horn formed by those faces is admissible, then all the faces must be hollow.

Chapter 7

Joyal

Introduction

Joyal's definition is given in an unpublished but well-known note [50]. It can be thought of as another generalisation of the nerve condition, though this may not be immediately apparent from the way the construction is actually effected. The definition ends up looking like this:

> *An ω-category is a cellular set in which every inner horn has a filler.*

Comparison with Street

We might compare this with Street's definition:

> *An ω-category is a simplicial set in which every admissible horn has a hollow filler.*

A *simplicial set* is a functor

$$\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

but a *cellular set* is a functor

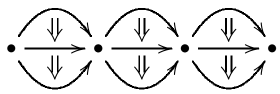
$$\Theta^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

where Θ is a category of certain shapes that are to play the role that simplices play for ordinary categories.

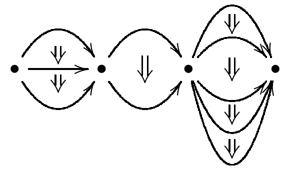
Street's scheme for generalisation is to use degenerate simplices for higher-dimensional cells, but Joyal introduces *more shapes* to deal with the higher-dimensional cells. This could be seen as a way of avoiding the complicated combinatorics of "admissible" horns — although Joyal's definition is also somewhat combinatorial, the condition for being an "inner horn" is rather easier to check than that for an "admissible horn". As usual, however, the complications have not vanished into thin air; they have been absorbed into the definition of the underlying category Θ and the definition of horn that goes with it. However, once we understand how to translate everything into "tree notation", the combinatorics of trees deals with all this quite neatly.

Comparison with Simpson/Tamsamani

Recall that Simpson/Tamsamani also introduce "more shapes" to deal with higher dimensions. However, their underlying category Δ^n is simpler to define (if not to draw); the difference is essentially that in Δ^n we are considering only pasting diagrams of "even depth" e.g.



not



In the Simpson/Tamsamani approach the latter must be obtained by means of degeneracies; for Joyal's definition we construct the latter kind of pasting diagram directly, right from the start.

7.1 Intuitions

The underlying data for this definition is to be a *cellular set*, that is a functor

$$\Theta^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

As with other definitions, this presheaf is to be interpreted as giving sets of cells with underlying shape in Θ. So we might ask what these shapes "look like".

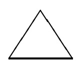
The objects of Θ are in fact globular pasting diagrams, but this does not capture the whole story — further important information is given by the *morphisms* of Θ. At 1-dimension, this situation coincides with Δ, so it is helpful to consider Δ for a moment. As usual, it helps to think about this "base case" in a particular way in order to see in what sense the generalisation is a natural one.

7.1.1 The category Δ

The objects of Δ are sets $\{0, 1, ..., n\}$. We can think of these as n-strings of (formal) arrows

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

but when we think of simplicial sets we usually draw a 2-simplex as



not just

$$\bullet \to \bullet \to \bullet\,.$$

This is because the face maps in Δ give us three 1-cell faces, not just two.

Now, to see how the generalisation to Θ arises naturally, we can think of the n-string of n arrows

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \cdots \longrightarrow \bullet$$

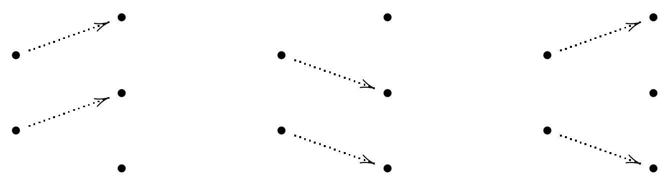
as a *1-pasting diagram.* To generalise to higher dimensions, we

i) use globular pasting diagrams of all dimensions (not just the 1-dimensional ones) as objects, and

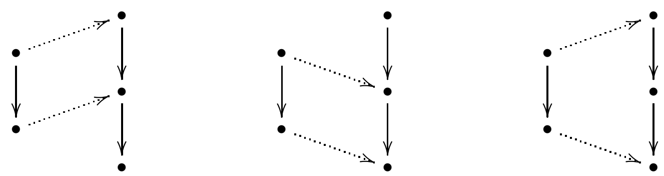
ii) consider that in order to draw more comprehensive pictures of the shapes for the presheaf, we should somehow "simplicialise" the globular pasting diagrams, to portray information about all the faces as well.

7.1.2 Faces

Recall that in Δ faces are given by injections. e.g. for $[2] \longrightarrow [3]$ we have three injections



If we think of $[n]$ as an n-string of arrows (here drawn vertically), this looks like



corresponding to

i) an instance of $\begin{array}{c}\bullet\\ \downarrow\\ \bullet\end{array}$ in $\begin{array}{c}\bullet\\ \downarrow\\ \bullet\\ \downarrow\\ \bullet\end{array}$, or

ii) a way of composing $\begin{array}{c}\bullet\\ \downarrow\\ \bullet\\ \downarrow\\ \bullet\end{array}$ to get $\begin{array}{c}\bullet\\ \downarrow\\ \bullet\end{array}$.

So we might draw the resulting simplicial shape as

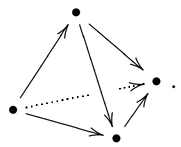
$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

showing the three "faces", rather than just the 1-pasting diagram

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

which only shows two. This is what we mean by "simplicialising" the pasting diagram. Similarly

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \text{becomes}$$

 .

Of course, higher-dimensional pasting diagrams become rather hard to draw in "simplicialised" form; this is where the useful combinatorics of Joyal/Batanin trees come in (see Section 7.1.4).

Finally, recall that all the injections in Δ can be generated by those of the form

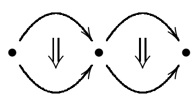
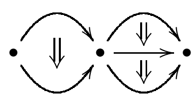
$$[i] \longrightarrow [i+1]$$

omitting one index. In terms of the strings of arrows, this corresponds to "composing" only two arrows at a time, i.e. a biased composition. The same can be done at higher dimensions. So we will be interested in faces ϕ of θ corresponding to

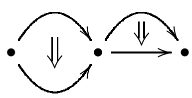
i) an instance of ϕ in θ (as a "sub-pasting-diagram") or

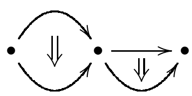
ii) a way of composing two constituent arrows of θ to get ϕ

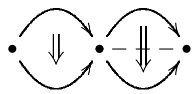
For example, given

$\phi \quad =$  $\qquad \theta \quad =$ 

we have three ways of getting ϕ as a face of θ:

i) 

ii) 

iii) 

In order to ensure that we have found a complete set of generators for the face maps, we introduce a notion of *volume* of a pasting diagram. Then the generators are given by faces ϕ of θ where

$$\mathrm{Vol}(\phi) = \mathrm{Vol}(\theta) - 1.$$

This is analogous to the use in Δ of injections

$$[i] \longrightarrow [i+1]$$

to generate face maps. Indeed, in the case of 1-pasting diagrams the two notions coincide.

Another way to think about faces is to consider the diagram giving the nerve construction

$$\begin{tikzcd}
\Theta \arrow[rr] \arrow[dr] & & {[\Theta^{\mathrm{op}}, \mathbf{Set}]} \arrow[dl, dotted, bend right=10] \\
& \mathbf{Str}\text{-}\omega\text{-}\mathbf{Cat} \arrow[ur, dotted, bend right=10] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

Since we want the map on the left to be full and faithful, we have a way of working out what the maps in Θ should be: we can generate strict ω-categories from the pasting diagrams in question and looking for ω-functors between them. In the previous example (with some added notation)

$$\begin{tikzcd}
x_0 \arrow[r, bend left=50, "f_1"] \arrow[r, bend right=50, "g_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & x_1 & x_2 \arrow[l, bend right=50, "f_2"'] \arrow[l, bend left=50, "g_2"] \arrow[l, phantom, "\beta\Downarrow"]
\end{tikzcd}
\qquad
\begin{tikzcd}
y_0 \arrow[r, bend left=50, "a_1"] \arrow[r, bend right=50, "c_1"'] \arrow[r, phantom, "\gamma\Downarrow"] & y_1 \arrow[r, bend left=60, "a_2"] \arrow[r, "b" description] \arrow[r, bend right=60, "c_2"'] \arrow[r, phantom, shift left=3, "\Downarrow\delta_1"] \arrow[r, phantom, shift right=3, "\Downarrow\delta_2"] & y_2
\end{tikzcd}$$

should give ω-categories with

0-cells	x_0, x_1, x_2	y_0, y_1, y_2
1-cells	$f_1,\ f_2, f_2 \circ f_1$	$a_1, a_2, a_2 \circ a_1$
	$g_1, g_2, g_2 \circ g_1$	$b, b \circ a_1, b \circ c_1$
	and identities	$c_1, c_2, c_2 \circ c_1$
		and identities
2-cells	$\alpha,\ \beta,\ \beta \circ \alpha$	$\gamma,\ \delta_1,\ \delta_2$
	and identities	$\delta_2 \circ \delta_1,\ \delta_1 * \gamma,\ \delta_2 * \gamma,\ (\delta_2 \circ \delta_1) * \gamma$
		and identities

We see that the 3 faces given earlier came from the following three possible assignations:

i) $\alpha \mapsto \gamma$
$\beta \mapsto \delta_1$

ii) $\alpha \mapsto \gamma$
$\beta \mapsto \delta_2$

iii) $\alpha \mapsto \gamma$
$\beta \mapsto \delta_2 \circ \delta_1$

the rest of the ω-functor action being determined by functoriality.

Note on disks

Note that, technically, Θ will be defined by its opposite, $\Theta^{\text{op}} = \mathbb{D}$, a category of "finite disks". This is because the morphisms arise more naturally in this direction when we use the combinatorics of trees.

7.1.3 Inner faces and inner horns

As for simplices, a horn is to be a "cell with no interior and one face missing". As in Street's definition, we will only require certain horns to have fillers: the "inner" horns. A filling condition can be thought of in terms of the following question:

> *Given all faces of a cell except one, when should we be able to determine the last face?*

The point is that some faces represent composition and some do not; only those representing composition should necessarily be determinable from the other faces.

Recall we observed in the previous section that a face ϕ of θ arises as

i) an instance of ϕ in θ as a sub-pasting-diagram, or

ii) a way of composing two constituent arrows of θ to get θ.

Only faces of type (ii) should be determinable from the other faces (by performing an appropriate composition), so only type (ii) will be called an "inner" face. A horn will be called "inner" if its missing face is an inner face; this is the kind of horn that needs a filler.

7.1.4 Trees

Trees are a useful combinatorial tool for handling higher-dimensional pasting diagrams that are too difficult to draw in a globular way.

Recall that a multisimplicial object can be specified quite simply by a string of "coordinates"

$$(i_1, i_2, i_3, ..., i_k)$$

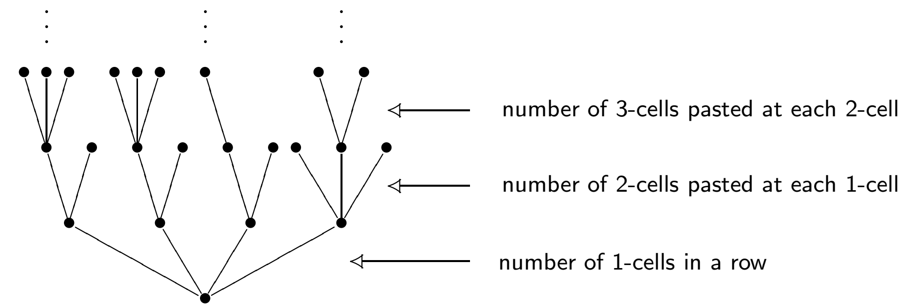
which is to be interpreted as a k-pasting diagram with

i_1 1-cells pasted end to end
i_2 2-cells pasted end to end *in each 1-cell position*
i_3 3-cells end to end *in each 2-cell position*
$\vdots$
i_k k-cells end to end *in each $(k-1)$-cell position.*

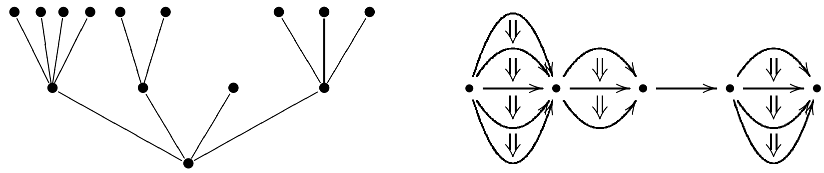
The situation is simplified by considering only "even depth" pasting diagrams as illustrated on page 122. Now, to allow for non-even depth diagrams as well, we need many more coordinates, giving:

i_1 1-cells pasted end to end

$i_{21}, ..., i_{2i_1}$ telling us how many 2-cells are pasted in each 1-cell position, and then a number of 3-cells to be pasted at each 2-cell, and so on. This is more easily represented as a tree:



Here is an example of a tree and its corresponding diagram:



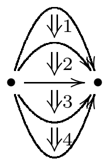
In fact, for Joyal's category $\mathbb{D}$ we introduce yet more combinatorics to help us easily establish which faces are "inner".

7.2 Technicalities

The combinatorics of globular pasting diagrams are made significantly easier (as compared with other shapes) by the fact that formal composites of k-cells along $(k-1)$-cell boundaries have a natural linear order on their constituents:

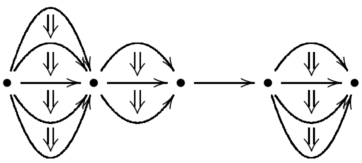
$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \bullet \xrightarrow{4} \bullet$$

or



etc.

These can then be built up into more complicated diagrams such as



This is done using a notion of "finite disk" which takes the form of a diagram of sets

$$\begin{tikzcd} D_k \arrow[r, "p_k"] & \cdots \arrow[l, bend left, "u_k"] \arrow[l, bend left=50, "v_k"'] & D_3 \arrow[r, "p_3"] & D_2 \arrow[l, bend left, "u_3"] \arrow[l, bend left=50, "v_3"'] \arrow[r, "p_2"] & D_1 \arrow[l, bend left, "u_2"] \arrow[l, bend left=50, "v_2"'] \arrow[r, "p_1"] & D_0 = 1 \arrow[l, bend left, "u_1"] \arrow[l, bend left=50, "v_1"'] \end{tikzcd} .$$

We can think of each D_m as giving the "set of m-cell positions" in the pasting diagram; for instance in the above example there are four 1-cell "positions". Each p_m then tells us in which $(m-1)$-cell position each m-cell lives. Continuing with the above example we have:

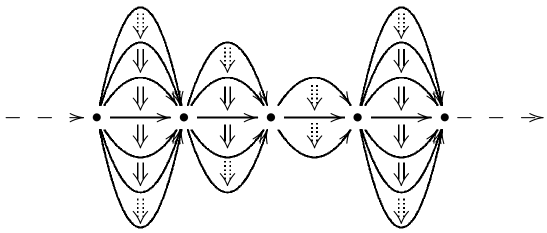
i) four 2-cells in the *first* 1-cell position

ii) two 2-cells in the *second* 1-cell position

iii) no 2-cells in the *third* 1-cell position

iv) three 2-cells in the *fourth* 1-cell position

However, we need more than this in order to deal with maps later: u_m and v_m give us formal "extremities" for each string of composites, as if telling us where the beginning and the end of each one is. We can think of it as



where dotted arrows give us the extremities at each dimension. In fact these are "finite disks"; to see why they are in any way disk-like it is helpful to consider an infinite disk example using the usual Euclidean balls.

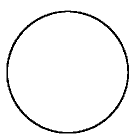
We are not going to give a complete account of the technical details.

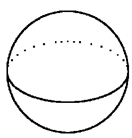
7.2.1 Euclidean balls as disks

To shed some light on the definition of disks, we consider the motivating example of Euclidean balls. In this case we have

D_0 =

D_1 = (unit interval)

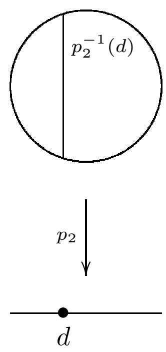
D_2 =  (unit disk)

D_3 =  (unit sphere)

We examine the part of the diagram involving D_1 and D_2

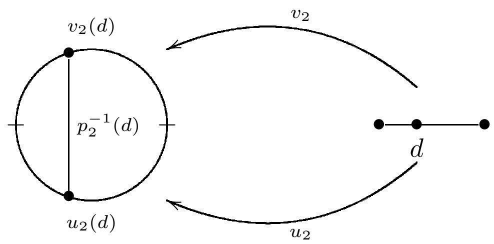
$$\begin{tikzcd} D_2 \arrow[r, "p_2"] & D_1 \arrow[l, bend left, "u_2"'] \arrow[l, bend left=60, "v_2"] \end{tikzcd} .$$

The map p_1 squashes the unit disk down to the unit interval as shown below:



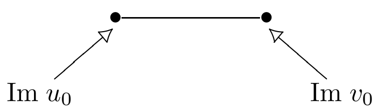
We observe that the pre-image of any point d has a *total order* on it; this will eventually give the total order for the string of composable cells in the "d position".

Now, u_2 and v_2 embed the interval into the disk "at the extremities"



We can formalise this using the total order on each fibre $p_2^{-1}(d)$ — we assert that $u_2(d)$ must be the least and $v_2(d)$ the greatest element on this fibre. (This is the first condition given in [69].)

Finally, we observe that u_2 and v_2 coincide *only at the extremities of the interval*, and these extremities are picked out by u_0 and v_0.



This gives the second condition listed in [69]:

$$u_m(d) = v_m(d) \iff d \in \mathrm{Im}(u_{m-1}) \text{ or } \mathrm{Im}(v_{m-1})$$

This is to ensure, eventually, that the source and target of a cell are always kept formally distinguishable.

7.2.2 Finite disks

Specifying the *interior* and *volume* of a disk is important because eventually we will have a special interest in maps which

- preserve the interior, and
- change the volume by only 1.

Finiteness refers to the "volume" of a disk; for us the volume will correspond to the total number of m-cell positions in the pasting diagram summed over all m, and this should of course be finite. *Dimension* will coincide with the dimension of a globular pasting diagram.

We have the following definitions:

i) The *interior* ιD_m is defined to be all of D_m except the "extremities", i.e.

$$\iota D_m = D_m \setminus \{\mathrm{Im}(u_m) \cup \mathrm{Im}(v_m)\}$$

ii) A disk D is said to be *finite* if

$$\coprod_{m>0} \iota D_m$$

is finite, i.e.

(a) each interior is finite, and

(b) there is only a finite number of non-empty interiors.

NB The "exterior" at each dimension is $\mathrm{Im}(u_m) \cup \mathrm{Im}(v_m)$ so its finiteness follows automatically.

iii) The *volume* of a finite disk is defined to be

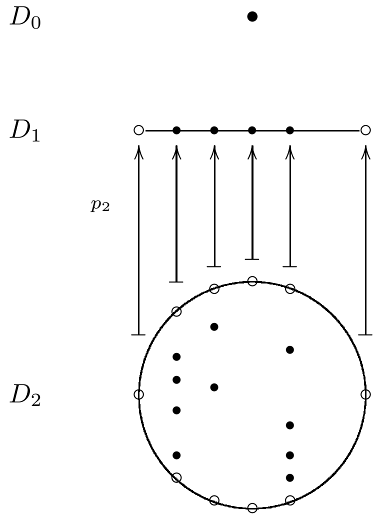
$$\left| \coprod_{m>0} \iota D_m \right|$$

i.e. the total number of elements in all dimensions of interior. (This makes more intuitive sense in tree notation as we will see in Section 7.2.4).

iv) The *dimension* of a disk is the largest m such that D_m has a non-empty interior. So it is the highest dimension where there is an actual cell.

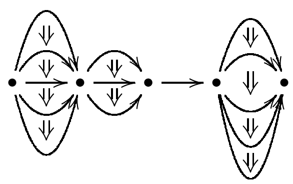
7.2.3 Finite disks and globular pasting diagrams

We will now see how a finite disk corresponds to a globular pasting diagram. An example of a finite disk is:



The line shows the unit interval; the dots $\bullet$ are the elements we are including in ιD_1 (a finite number of them); the dots $\circ$ show the extremities.

Now, in D_1 each $\bullet$ represents a 1-cell. In D_2 each vertical column of $\bullet$ represents a vertical string of 2-cells, so we have the diagram as before



Note that the important information is *how many elements are in each fibre* — it doesn't matter how spread out they are, for example. So in fact we take isomorphism classes of finite disks.

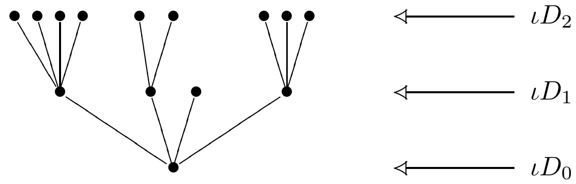
NB A morphism of finite disks preserves everything including, crucially, the ordering on each fibre. So isomorphism classes do give us precisely the information we want:

> *Finite disks are isomorphic if and only if they have the same number of elements in each fibre.*

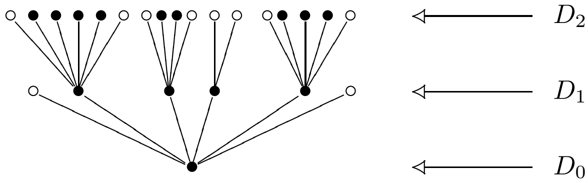
We write $\mathbb{D}$ for the skeleton of the category of finite disks (i.e. we pick one representative of each isomorphism class). Eventually we will be using $\Theta = \mathbb{D}^{\mathrm{op}}$.

7.2.4 Trees

Since the important information is how many elements are in each fibre, we can represent an element of $\mathbb{D}$ as a tree, where the number of edges above a node gives the number of elements in that fibre. e.g. for the above example



We can also include the extremities in the tree:



Note that we can now find the volume by simply counting the black dots • (ignoring the bottom one).

7.2.5 Faces and horns

Faces

Recall that in simplicial sets:

- *faces* come from injections in Δ, and
- *degeneracies* come from surjections in Δ.

For a cellular set

$$\Theta^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

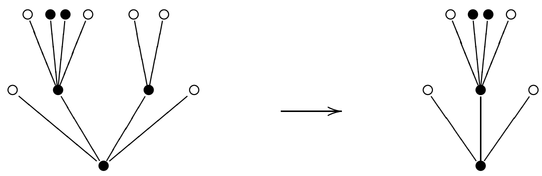
a technical duality arises since everything is easier to construct in $\mathbb{D} = \Theta^{\mathrm{op}}$. So faces will come from *surjections* in $\mathbb{D}$, and we might refer to these as "cofaces" (but in practice we mostly won't bother).

Surjections in $\mathbb{D}$ can be seen quite clearly in tree notation whereas in pasting diagram notation the duality can promote confusion. A tree map can be constructed from the ground up:

- the bottom node must map to the bottom node
- at the D_1 level each node must map to a node at the same level, order must be preserved, and exteriors must be preserved
- at the D_2 level, nodes in each fibre must map to the corresponding fibre, with order preserved, and exteriors of each fibre must be preserved

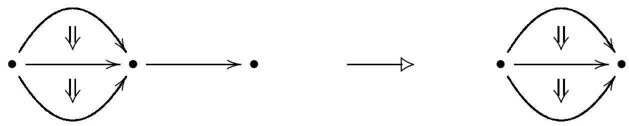
— and so on
Note that *interiors* do not necessarily have to be preserved; if they are, the map is called *inner*.

We can generate all surjections from those that reduce the volume by 1. For example,



Vol = 4 Vol = 3

corresponding to the surjection

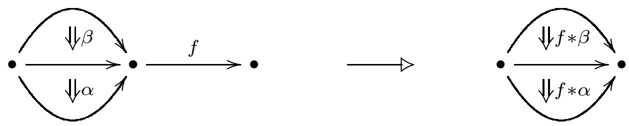


In general this can happen in precisely two ways:

i) interior is not preserved: an interior node • is identified with an exterior node ∘

ii) interior is preserved: precisely two interior nodes • are identified (as in the case above).

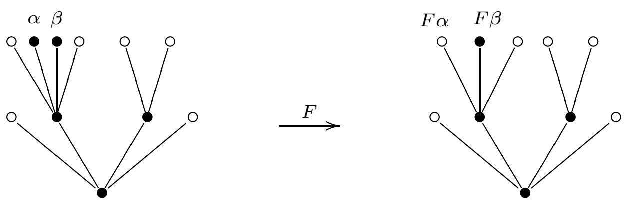
Case (i) corresponds to a pasting diagram embedding in another as a "sub-pasting-diagram"; case (ii) corresponds to some "composition" occurring in the pasting diagram. If we use labels in the example above we have



We encourage the reader to experiment with some trees and their corresponding pasting diagrams to get a feel for this correspondence.

Inner faces

An *inner face* is one arising as case (ii) above, i.e. the interior is preserved and some composition has occurred. An example of a non-inner face of the above pasting diagram is:



Vol = 4 Vol = 3

corresponding to the globular diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r, "f"] & \bullet \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M2, to=D, "\beta"] \end{tikzcd} \qquad \longrightarrow \qquad \begin{tikzcd} \bullet \arrow[r, ""{name=M, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r, "f"] & \bullet \arrow[Rightarrow, from=M, to=D, "F\beta"] \end{tikzcd}$$

Horns and fillers

Finally a (co)horn is a collection of "all (co)faces except one" for a given pasting diagram. For the above diagram we have

$$\begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M, above}] & \bullet \arrow[r, "f"] & \bullet \arrow[Rightarrow, from=U, to=M, "\alpha"] \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[r, ""{name=M, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r, "f"] & \bullet \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r, "f"] & \bullet \arrow[Rightarrow, from=U, to=D, "\beta\circ\alpha"] \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=M, "f*\alpha"] \arrow[Rightarrow, from=M2, to=D, "f*\beta"] \end{tikzcd}$$

of which only the last two are inner.

A filler for a horn in a cellular set $\Theta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is analogous to a filler in a simplicial set — a cell whose faces match those given by the horn in question. An *inner* (co)horn is one whose missing (co)face is inner. Thus, the notion of an inner horn is a way to determine which maps correspond to composition. This is analogous to Street's admissibility condition.

7.2.6 The definition

We have the following definition:

An ω-category *is a cellular set*

$$\Theta^{\mathrm{op}} \xrightarrow{X} \mathbf{Set}$$

in which every inner horn has a filler.

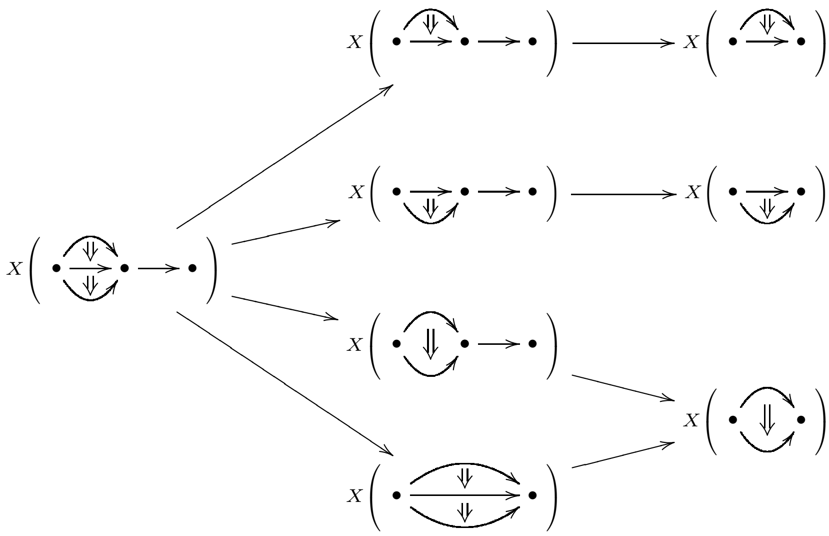
Recall that the maps described in the previous section were those of $\mathbb{D} = \Theta^{\mathrm{op}}$ (the directions here can get confusing). So for example a map in $\mathbb{D}$

$$\begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r] & \bullet \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M2, to=D] \end{tikzcd} \qquad \longrightarrow \qquad \begin{tikzcd} \bullet \arrow[r, ""{name=M, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r] & \bullet \arrow[Rightarrow, from=M, to=D] \end{tikzcd}$$

gives us a map in a cellular set X

$$X\left(\begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r] & \bullet \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M2, to=D] \end{tikzcd} \right) \qquad \longrightarrow \qquad X\left(\begin{tikzcd} \bullet \arrow[r, ""{name=M, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet \arrow[r] & \bullet \arrow[Rightarrow, from=M, to=D] \end{tikzcd} \right)$$

For this example we also have maps



Note that the bottom part of this diagram really does commute (check the trees if you are in doubt). This might make it look as though this situation is more strict than it ought to be — after all, this looks like the interchange law at work. However, this is another case of a non-algebraic definition in which the issue of constraints disappears (see Introduction).

In summary, for any pasting diagram α, $X(\alpha)$ can be thought of as a set of "coherent systems for composing up the diagram", witnessed by the diagram of morphisms with $X(\alpha)$ as domain.

Chapter 8

Trimble and May

Introduction

The definitions of Trimble [108] and May [81] take the "enrichment" point of view explicitly. Trimble's is almost disarmingly straightforward but relies heavily on the structure of topological spaces; May's can be thought of as a generalisation to a more abstract setting.

The two main components of this approach are

i) *enrichment*, for building up dimensions, and

ii) *parametrised weakening* of coherence for composition, using an operad.

Trimble's idea is: rather than try to understand "weak associativity" abstractly, we directly use the weakly associative composition of paths in topological spaces to parametrise composition in an n-category. To do this, he uses a certain naturally arising operad in topological spaces.

It may seem that this definition is immediately less general than the others. However, its use of a certain operad to parametrise composition does leave open the possibility of using other operads to get other, related, theories of n-categories with different notions of "weak composition". May's definition can be thought of as this very generalisation. The important question is then:

> *Which operads in which categories should qualify as sensible candidates for defining an n-category?*

Trimble's chosen operad E is certainly "contractible" (in the sense that each space $E(k)$ defining it is contractible). This means that we don't get any really unruly situations arising; but it also leaves enough room for manoeuvre.

May's idea is to replace **Top** by a category $\mathcal{B}$, and get an abstract notion of "contractible operad" in $\mathcal{B}$ analogous to the notion in **Top**. Then the remaining question is:

What structure does $\mathcal{B}$ need to have in order to be able to do this in a meaningful way?

We will demand:

i) enough structure to define an operad in $\mathcal{B}$ in the first place,

ii) a *Quillen model category* structure in order to have a suitable notion of "homotopy", and

iii) enough further structure to make the induction go through.

We will discuss this in Section 8.2.5 without going into full technical details of what we need and why.

8.1 Trimble

We begin by presenting Trimble's original definition before thinking about how it might generalise.

8.1.1 The idea

The definition is inductive and looks somewhat like the definition of an enriched category (see Section 8.1.2):

- a 0-category is a set
- an n-category A consists of
  i) objects: a set A_0
  ii) hom-$(n-1)$-categories: for all $a, a' \in A_0$ an $(n-1)$-category $A(a, a')$
  iii) k-ary composition: for all $k \geq 0$ and $a_0, \ldots, a_k \in A_0$ an $(n-1)$-functor

$$\Pi_{n-1}(E(k)) \times A(a_0, a_1) \times \cdots \times A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k)$$

  satisfying some axioms.

Question: What is going on with $\Pi_{n-1}(E(k))$?

Answer: The idea is that it is our way of parametrising the precise nature of the weakness of composition. It is like the "generic" $(n-1)$-category for weak k-ary composition in this theory.

The resulting n-category will have one k-ary composite *for each* object in $\Pi_{n-1}(E(k))$. The various k-ary composites will be related by higher-dimensional cells one dimension up, controlled by higher-dimensional cells in $\Pi_{n-1}(E(k))$ and so on.

Finally, what axioms do we demand? In fact E is an operad, and the axioms say:

> *The composition functions must behave properly with respect to the operad composition.*

Coherence then really comes from the coherence of $\Pi_{n-1}(E(k))$, not from the actual axioms themselves; the axioms ensure that this coherence can indeed be carried over.

Strictness

Note that if we replace E by the terminal operad 1 (which has $1(k) = 1$ for all k) we would get a *strict* n-category, as we would end up with precisely one k-ary composite for each k. Note that there will be an equivalence between the operads E and 1 of the form

$$\forall k \; E(k) \simeq 1(k) = 1.$$

That is, each space $E(k)$ will be contractible. This characterisation of E will be a crucial ingredient in the generalisation to May's definition; see Section 8.2.3.

8.1.2 Enrichment

The usual definition of enrichment looks like this:

> *Let $\mathcal{V}$ be a monoidal category. A $\mathcal{V}$-enriched category A consists of*
>
> *i) objects: a set A_0*
>
> *ii) hom-objects: for all $a, a' \in A_0$ an object $A(a,a') \in \mathcal{V}$*
>
> *iii) binary composition: for all $a_0, a_1, a_2 \in A_0$ a morphism in $\mathcal{V}$*
>
> $$A(a_0, a_1) \otimes A(a_1, a_2) \longrightarrow A(a_0, a_2)$$
>
> *iv) identities: for all $a \in A_0$ a morphism in $\mathcal{V}$*
>
> $$I \longrightarrow A(a,a)$$
>
> *satisfying* strict *associativity and unit laws.*

This gives a perfectly good way of defining *strict* n-categories inductively, by putting $\mathcal{V} = (n-1)$**-Cat**. However, if we want *weak* n-categories we will have to do something about the strictness in the above definition.

Perhaps the most obvious way to do this is to relax the axioms to make everything weak. However, this is going to be hard (if not technically impossible) to do in a general setting since a fully weakened notion of enrichment would use all n available dimensions — so we will need to know more about $\mathcal{V}$ than just a monoidal structure. Specifically, we will need to know more and more about the enrichment category $\mathcal{V}$ as we increase dimensions, so it will be hard to make a general definition of "weak enrichment" that will cover our needs.

Instead, a different approach is to weaken the notion of "composition" so that the whole notion of associativity more or less vanishes. This is Trimble's approach. Composition will be parametrised by the generic "weak associativity structures" $\Pi_{n-1}(E(k))$. Instead of having unique composition, we have one k-ary composite for every object of $\Pi_{n-1}(E(k))$. The higher-dimensional cells of $\Pi_{n-1}(E(k))$ will take care of the coherence. So, what is the operad E?

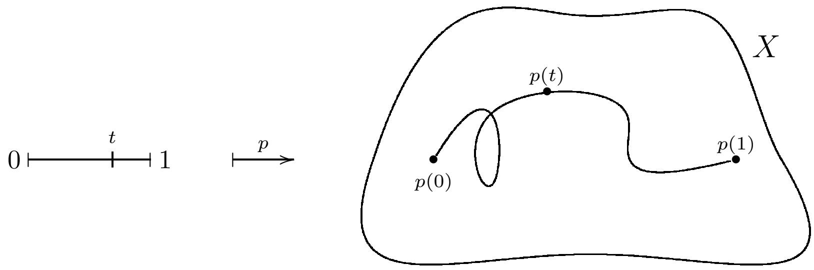
8.1.3 Paths and the operad E

E is a topological operad: so we have for each $k \geq 0$ a space $E(k)$ together with composition and identities. Each $E(k)$ can be though of as a space of k-ary operations.

The idea here is to make use of one of our leading examples of weakly associative composition: composition of paths in a space. A path in a space X is continuous map

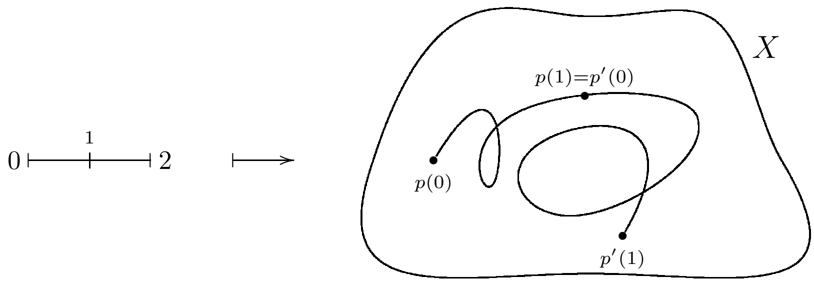
$$p : [0,1] \longrightarrow X$$

and is thought of as going from $p(0)$ to $p(1)$



Paths p and p' can be composed (if $p(1) = p'(0)$) but not canonically — the natural way of "sticking paths together" will give us a map

$$[0,2] \longrightarrow X$$



and we will need to turn this into a map

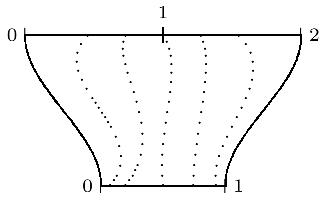
$$[0,1] \longrightarrow X.$$

We can do this by composing the above canonical "composite path" map $[0,2] \longrightarrow X$ with a continuous map

$$f\colon [0,1] \longrightarrow [0,2]$$

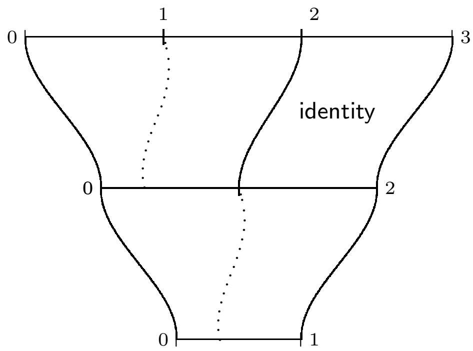
such that $f(0) = 0$ and $f(1) = 2$. However, there are rather a lot of those to choose from. We can go ahead and choose one, but in doing so we unleash all the associativity problems.

Choosing a map f as above is a bit like deciding "how fast" we are going to go along p and then p' (though actually it is more than that). This "scaling" can be drawn as

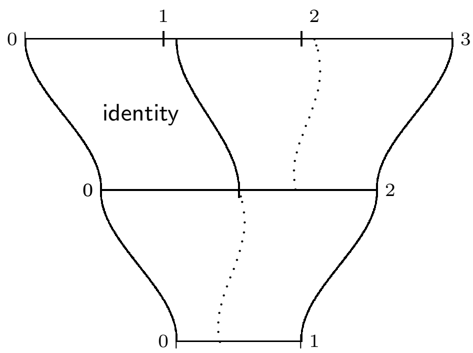


where the sides show that the endpoints are identified.

If we then wanted to compose with a third path we would get something like



or on the other side



and we see non-associativity emerging. However, we have a whole *space* of maps

$$[0,1] \longrightarrow [0,3]$$

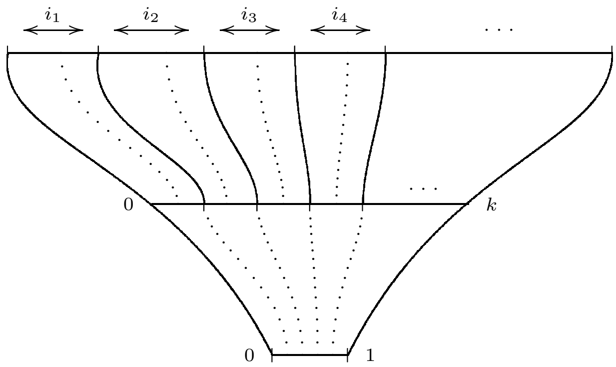
in which there will be a path mediating between the above two maps. This is the idea of the operad E. For each $k \geq 0$, $E(k)$ is defined to be the space of maps

$$[0,1] \longrightarrow [0,k].$$

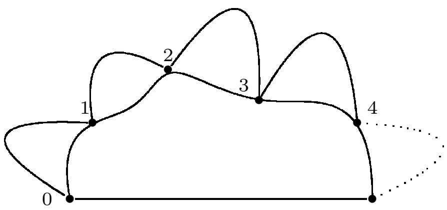
For composition, we need to map

$$E(i_1) \times \cdots \times E(i_k) \times E(k) \longrightarrow E(i_1 + \cdots + i_k)$$

and this will be by "substitution" as in the following diagram



NB If we actually identify the endpoints we get something like



which looks a bit like an opetope.

8.1.4 Action on path spaces

We can now use the operad E to keep a handle on our hitherto messy notion of path composition. Recall we said we could compose two paths $x_0 \text{———} x_1$ and $x_1 \text{———} x_2$ in X by using a "scaling" map

$$f\colon [0,1] \longrightarrow [0,2]$$

to record exactly how we scaled our double-length path back down to a unit-length path. This is just an element of $E(2)$ so we can express this as an action

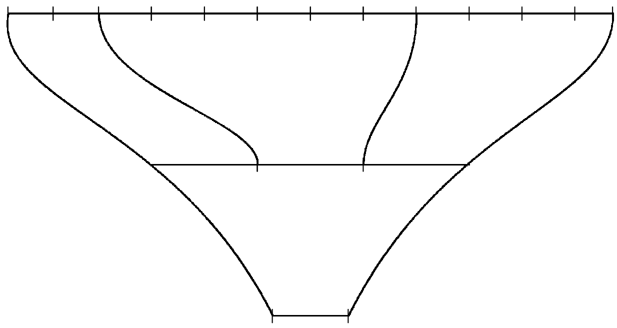
$$E(2) \times X(x_0,x_1) \times X(x_0,x_2) \longrightarrow X(x_0,x_2).$$

In general we can glue k paths together and scale them down again by using the map

$$E(k) \times X(x_0, x_1) \times \cdots \times X(x_{k-1}, x_k) \longrightarrow X(x_0, x_k),$$

giving one way of scaling for each object of $E(k)$. Note that each $X(a, b)$ is the *space* of paths from a to b and each of the above "actions" is a map of *spaces*, so we also have higher homotopies mediating coherently at all levels.

Finally, this is an *operad action* which means that the path composition interacts sensibly with operad composition, in the following sense. If we take a long string of paths we could glue them together (and scale) in two steps, glueing some sub-strings first, and then glueing the results afterwards:



This of course produces one big scaling map for the whole two-stage process — and it is precisely the scaling map produced by using the operad composition on the intermediate scaling maps. This is what it means to "interact sensibly" with the operad composition.

8.1.5 Weak composition for n-categories

The idea is to compose morphisms in n-categories by copying the above approach for composing paths in spaces. The difference now is that we can't have $E(k)$ acting directly on our "paths" (i.e. morphisms), because $E(k)$ is a space and our morphisms form $(n-1)$-categories. We fix this by turning each $E(k)$ into an $(n-1)$-category that captures the information about weak composition. This is the role of the functor

$$\Pi_{n-1} \colon \mathbf{Top} \longrightarrow (n-1)\text{-}\mathbf{Cat}$$

which we construct for each n.

NB When we generalise this for the May definition we will not have such a functor explicitly; rather, we will have a notion of one object acting on another object even when they live in different categories. So we do not explicitly need to turn each $E(k)$ into an $(n-1)$-category in order to define an action.

8.1.6 The induction

We will have to proceed by induction simultaneously for the definitions of Π_n and n-categories together, since the definition of Π_n needs the definition of n-**Cat**, which in turn needs the definition of Π_{n-1}.

The idea is that $\Pi_n(X)$ should be like the fundamental n-groupoid of X but *in the sense of the present definition*, i.e. composition is at all times parametrised by E. This parametrisation trick allows us to avoid all the usual technical difficulties with composition of homotopies. We never have to choose how fast to go along components of glued paths — we allow for every possible way and always remember exactly how we scaled back down to unit length.
We have:

0-cells	are	points of X
1-cells	are	paths in X
2-cells	are	homotopies between paths
$\vdots$		

And as usual, something violent has to happen at the top to force it to be an n-*dimensional* structure — we have to do some quotienting at the top dimension.

This produces for each $n \geq 1$ an $(n-1)$-category $\Pi_{n-1}(E(k))$ parametrising composition for n-categories.

8.1.7 The definition

The definition proceeds by induction starting from 0-**Cat** = **Set**. At each stage we will also need the following structure to make the induction go through:

i) n-**Cat** has finite products, and

ii) Π_n preserves finite products.

Definition *For $n \geq 1$ an* n-category *A consists of*

i) objects: a set A_0

ii) hom-$(n-1)$-categories: for all $a, a' \in A_0$ an $(n-1)$-category $A(a, a')$

iii) k-ary composition: for all $k \geq 0$ and $a_0, \ldots, a_k \in A_0$, an $(n-1)$-functor

$$\Pi_{n-1}\big(E(k)\big) \times A(a_0, a_1) \times \cdots \times A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k)$$

satisfying "compatibility" with the operadic composition of E as sketched in Section 8.1.4.

Definition *An* n-functor $A \xrightarrow{F} B$ *consists of* *i) on objects: a function* $F = F_0\colon A_0 \longrightarrow B_0$

ii) on morphisms: for all $a, a' \in A_0$ *an* $(n-1)$*-functor*

$$A(a,a') \longrightarrow B(Fa, Fa')$$

satisfying "functoriality" — for all $k \geq 0$ *and* $a_0, \ldots, a_k$ *the following diagram commutes:*

$$\begin{tikzcd}
\Pi_{n-1}\big(E(k)\big) \times A(a_0,a_1) \times \cdots \times A(a_{k-1},a_k) \arrow[r] \arrow[d] & A(a_0,a_k) \arrow[d] \\
\Pi_{n-1}\big(E(k)\big) \times B(Fa_0,Fa_1) \times \cdots \times B(Fa_{k-1},Fa_k) \arrow[r] & B(Fa_0,Fa_k)
\end{tikzcd}$$

For the sake of completeness we now also give the definition of

$$\Pi_n : \mathbf{Top} \longrightarrow n\text{-}\mathbf{Cat}$$

though we don't expect it to help clarify the idea.

Definition *Let* X *be a space. We define an* n*-category* $\Pi_n X = A$ *by*

i) objects: A_0 *is the underlying set of* X

ii) hom-$(n-1)$*-categories:* $A(x,x') = \Pi_{n-1}(X(x,x'))$ *— the* $(n-1)$*-category version of the path space*

iii) composition: we use the action of E *on path spaces and the fact that* Π_{n-1} *preserves products to make the following composite*

$$\begin{tikzcd}
\Pi_{n-1}\big(E(k)\big) \times X(x_0,x_1) \times \cdots \times X(x_{k-1},x_k) \arrow[d, "\sim"] \\
\Pi_{n-1}\big(E(k)\big) \times \Pi_{n-1}\big(X(x_0,x_1)\big) \times \cdots \times \Pi_{n-1}\big(X(x_{k-1},x_k)\big) \arrow[d, "\Pi_{n-1} \text{ of the action of } E \text{ on path spaces}"] \\
\Pi_{n-1}\big(X(x_0,x_k)\big)
\end{tikzcd}$$

The action of Π_n *on morphisms is defined in the obvious way.*

8.2 May

We now go on to show how May's definition can be seen as a natural generalisation of the Trimble approach, even if this is not how it originally arose.

8.2.1 The idea

The idea is to express the whole definition replacing **Top** by some "base category" $\mathcal{B}$ *with enough structure.* We can do this by examining Trimble's definition and restating it as abstract category theory. So we need to ask:

> *What, categorically, are the features of* **Top** *and the operad E that we exploited in Trimble's definition?*

The answer turns out to be quite tricky (to prove, if not to state) but we hope to make the idea clear without going into technicalities.

The following table sums up the (informal) correspondence between the various components in each definition, showing the result of generalisation into the more abstract setting:

Trimble	**May**
Top	$\mathcal{B}$
$\times$	$\otimes$ (symmetric) monoidal structure on $\mathcal{B}$
homotopy equivalence	weak equivalence in model category structure on $\mathcal{B}$
the operad E in **Top**	any A_∞-operad P in $\mathcal{B}$
Π_n : **Top** $\longrightarrow$ n-**Cat**	an "action" of $\mathcal{B}$ on n-**Cat**

There is one further subtlety in this generalisation: May's definition is "more enriched" in the sense that it takes a *starting point* that is already something enriched, where Trimble builds up from **Set** and **Cat** as usual.

Using the terminology of May, Trimble's inductive definition of n-category might be called an "E-category in $(n-1)$-**Cat**"; this means that it is *weakened* by E and *enriched* in $(n-1)$-**Cat**. Then we have the following inductive process:

	Trimble	**May**
0-**Cat**	**Set**	$\mathcal{B}$
1-**Cat**	**Cat**	P-categories in $\mathcal{B}$
$\vdots$	$\vdots$	$\vdots$
n-**Cat**	E-categories in $(n-1)$-**Cat**	P-categories in $(n-1)$-**Cat**
$\vdots$	$\vdots$	$\vdots$

So although the induction step looks the same, the starting point could be very different. (However, we could define **0-Cat** = **Set** and take the induction from there instead.)

We now have to ask the questions:

1) *What is the structure we need to start with in order to make this construction?*

2) *Is it automatically going to be given again at each step of the induction?*

We will discuss each part of the definition and then finally take an inventory in Section 8.2.5 of the structure we needed.

8.2.2 What will the induction look like?

We wish to define a notion of a category *enriched* in $\mathcal{V}$ and *weakened* by P. We will call this a "P-category in $\mathcal{V}$". (Note that this definition is also sketched in [71] and this construction is called "categorical P-algebras in $\mathcal{V}$".)

Here P is to be an operad in a base category $\mathcal{B}$. We can state the definition to look just like Trimble's inductive definition of n-category (Section 8.1.7):

Definition (Trimble) *An n-*category *A consists of*

i) objects: a set A_0

ii) hom $(n-1)$-categories: for all $a, a' \in A_0$ an $(n-1)$-category $A(a, a')$

iii) k-ary composition: for all $k \geq 0$ and $a_0, \ldots, a_k \in A_0$, an $(n-1)$-functor

$$\Pi_{n-1}\big(E(k)\big) \times A(a_0, a_1) \times \cdots \times A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k)$$

satisfying "compatibility" with the operadic composition of E.

So we have the following abstraction:

Definition (May) *A P-*category *A* in *$\mathcal{V}$ consists of*

i) objects: a set A_0

ii) hom-objects: for all $a, a' \in A_0$ an object $A(a, a')$ in $\mathcal{V}$

iii) k-ary composition: for all $k \geq 0$ and $a_0, \ldots, a_k \in A_0$, a morphism in $\mathcal{V}$

$$P(k) \odot A(a_0, a_1) \otimes \cdots \otimes A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k)$$

satisfying "compatibility" with the operadic composition of P.

Observe that $P(k)$ doesn't actually live in the same category as the rest of the above expression. Here $\odot$ denotes an "action" of $\mathcal{B}$ on $\mathcal{V}$ which gives, for all $B \in \mathcal{B}$ and $V \in \mathcal{V}$, an object $B \odot V \in \mathcal{V}$ in a suitably coherent way (see Section 8.2.5). Comparing the two definitions above it becomes apparent that this "action" has replaced the use of the functor

$$\Pi_{n-1} : \mathbf{Top} \longrightarrow (n-1)\text{-}\mathbf{Cat}.$$

We could demand a functor

$$\Pi : \mathcal{B} \longrightarrow \mathcal{V}$$

but we would lose some generality in doing so.

We write P-**Cat**-$\mathcal{V}$ for the category of P-categories in $\mathcal{V}$.

8.2.3 When is P a sensible choice of operad?

We have seen, in the case of Trimble, two possible operads that are *sensible* for this definition — E and 1. The issue here is not what works technically in the construction, but the following:

> **Question:** *When does an operad P give the sort of structure that deserves to be called "n-category"?*
>
> **Answer:** *Whenever each object $P(k)$ is suitably equivalent to 1.*

This might remind the reader of a similar question in Chapter 3. Note that this is an operad in the (monoidal) category $\mathcal{B}$, meaning that it has for each $k \geq 0$ an object $P(k) \in B$ together with suitably coherent composition morphisms in $\mathcal{V}$. Now if B is a *Quillen model category*, we have a good notion of "weak equivalence", and we have the following definition:

Definition *An* A_∞-operad *is an operad P in a model category $\mathcal{B}$ such that for all k, $P(k)$ is weakly equivalent to the unit object 1.*

For a sensible definition of n-category, we demand that P be an A_∞-operad.

8.2.4 The definition

The definition proceeds by induction as before. First, let $\mathcal{B}$ and $\mathcal{V}$ be *good enough* categories, and let P be an A_∞-operad in $\mathcal{B}$. Then we define n-categories (enriched in $\mathcal{B}$) as follows:

Definition

- 0-**Cat** $= \mathcal{B}$
- *for* $n \geq 1$, n-**Cat** $=$ *the category of P-categories in* $(n-1)$-**Cat**

8.2.5 What structure do we need?

We will not attempt to show that this induction goes through, but will finish with a brief discussion of what structure we need in $\mathcal{B}$ and $\mathcal{V}$ to make them "good enough".

At each stage of the inductive definition, the new enrichment category $\mathcal{V}$ is the previous category of P-categories in $\mathcal{V}$. So apart from deciding how much structure we need in $\mathcal{B}$ and $\mathcal{V}$ to make the above construction work, we also need to check that whatever structure we demanded in $\mathcal{V}$ will also be present in P-**Cat**-$\mathcal{V}$. Without making this at all rigorous, we make the following remarks:

- Structure required in $\mathcal{B}$
  - i) We need to be able to define an operad in $\mathcal{B}$, so we at least need a monoidal structure.
  - ii) We need to be able to define A_∞-operads in $\mathcal{B}$, so we need a model category structure on it.
- Structure required in $\mathcal{V}$
  - i) We need to have an "action" of $\mathcal{B}$ on $\mathcal{V}$; this may be expressed as $\mathcal{V}$ being *tensored* over $\mathcal{B}$.
  - ii) We need to have enough structure to ensure the same in P-**Cat**-$\mathcal{V}$.

Note that, if we examine the induction process in Trimble's definition, we should not expect the "action" to appear automatically at the next level up. That is, the inductive construction of Π_n involves the use of something very specific to the context, namely, a particular action of E on path spaces. It is not very obvious what we need to do in order to get something similar in the general setting.

In fact May assumes that $\mathcal{B}$ is closed symmetric monoidal, with Quillen model category structure cofibrantly generated and proper. $\mathcal{V}$ is assumed to be complete and cocomplete, and enriched, tensored and cotensored over $\mathcal{B}$. It appears that he has some broader ideological motivation than just making the definition "work"; indeed, there are various standard topological frameworks in which such structure might be considered a "bare minimum" starting point. However, it is not clear how all this structure is to be recreated at each stage of the induction process.

Appendix A

Classification tables

We have found it helpful to compile some tables collating the various ways of characterising definitions as discussed in the introduction and elsewhere. Of course, filling in boxes is a horribly crude way to sum up such subtle structures and in many cases we thought of several different justifiable ways of doing it. Nevertheless, we found it a useful exercise for clarifying our understanding so have included them here.

A.1 One-sentence summary of each definition

	slogan
Penon	An ω-category is an algebra for the monad induced by the adjunction $\mathbf{GSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{Q}$
Leinster	An ω-category is an algebra for an initial "operad with (more general) contraction".
Batanin	An ω-category is an algebra for an initial "operad with contraction and system of compositions".
Joyal	An ω-category is a cellular set in which every inner horn has a filler.
Street	An ω-category is a simplicial set with hollowness in which every admissible horn has a hollow filler and composites of hollow cells are hollow.
Opetopic	An n-category is an opetopic set in which every niche has a universal filler and composites of universals are universal.
Simpson Tamsamani	An n-category is a non-cubical n-simplicial set in which every Segal map is contractible.
Trimble	An n-category is a category weakly enriched in $(n-1)$-categories where weakness is parametrised by a certain topological operad describing path composition.
May	An n-category is a category weakly enriched in $(n-1)$-categories where weakness is parametrised by any contractible operad in a model category.

A.2 The Data-Structure-Properties trichotomy

	DATA	STRUCTURE	PROPERTIES
Penon	cells (and identities)	composition + mediators	sufficient mediation
Leinster	cells	composition + mediators	sufficient mediation
Batanin	cells	composition + mediators	sufficient mediation
Joyal	cells coherent composition	*none*	sufficient composition
Street	cells	composition	“sufficient and coherent” composition
Opetopic	cells + coherent composition	*none*	sufficient composition
Simpson Tamsamani	cells + coherent composition	*none*	sufficient composition
Trimble	cells + composition parametrised by E	hom-(n–1)-cats	coherence carries over from the operad E
May	cells + composition parametrised by some operad	hom-(n–1)-cats	the operad is coherent and coherence carries over from it

A.3 Other characteristics

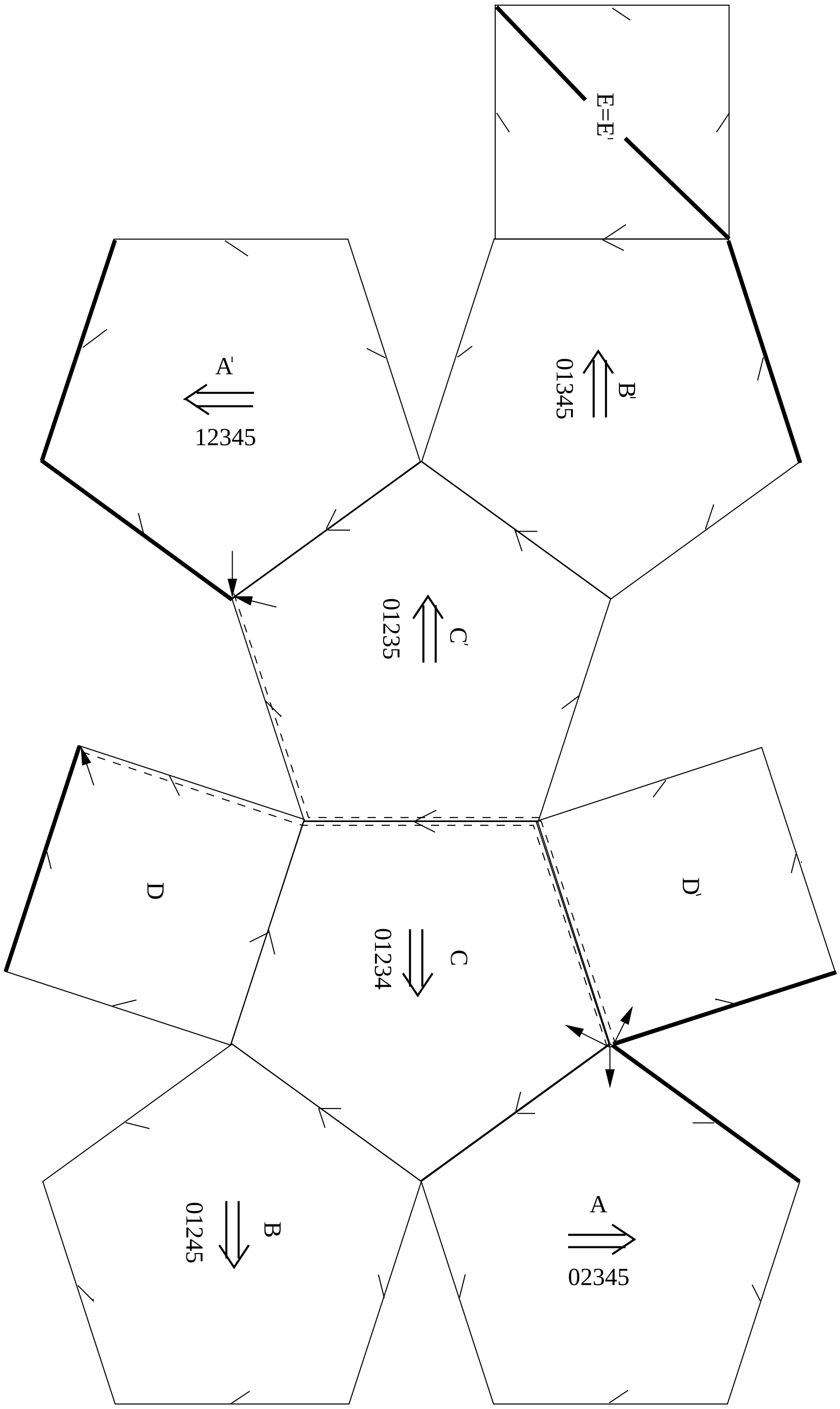
	idea	*cell shape*	*algebraic vs non-algebraic*	*coherence*
Penon	graph	globular	algebraic	contraction gives specified witnesses
Leinster	graph	globular	algebraic	contraction gives specified witnesses
Batanin	graph	globular	algebraic	contraction gives specified witnesses
Joyal	nerve	cellular (globular/simplicial)	non-algebraic	horn filling gives existence of witnesses
Street	nerve	simplicial	non-algebraic	horn filling gives existence of witnesses
Opetopic	nerve	opetopic	non-algebraic	horn filling gives existence of witnesses
Simpson Tamsamani	nerve/ enrichment	multisimplicial	non-algebraic	contractibility gives horn filling
Trimble	enrichment	not presheaf	algebraic	transferred from contractible operad
May	enrichment	not presheaf	algebraic	transferred from contractible operad

	bias?	*route*	*beheading vs headshrinking*	*technical inventory*
Penon	biased	higher then weaker — can do ω	beheading	monads
Leinster	unbiased	higher then weaker — can do ω	beheading	operads
Batanin	biased	higher then weaker — can do ω	beheading	operads
Joyal	biased	higher then weaker — can do ω	headshrinking	finite disks/trees
Street	biased	higher then weaker — can do ω	headshrinking	simplicial sets
Opetopic	unbiased	weaker then higher — can't do ω	headshrinking	(multicategories)
Simpson Tamsamani	biased	weaker then higher — can't do ω	beheading	simplicial sets
Trimble	unbiased	weaker then higher — can't do ω	beheading	topological spaces, operads
May	unbiased	weaker then higher — can't do ω	beheading	model categories, operads

Appendix B

A build-your-own 5-associahedron

See over.


E=E'
A'
12345
B'
01345
C'
01235
D
D'
C
01234
B
01245
A
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